AUSTRIAN JOURNAL OF STATISTICS
Volume 35 (2006), Number 2&3, 131–141

Student's t-Statistic under Unimodal Densities

Harrie Hendriks[1], Pieta C. IJzerman-Boon[2], and Chris A. J. Klaassen[3]

[1]Radboud University, Nijmegen, The Netherlands
[2]NV Organon, Oss, The Netherlands
[3]Universiteit van Amsterdam, Amsterdam, The Netherlands

Abstract: Consider an unknown distribution with a symmetric unimodal density and the induced location-scale family. We study confidence intervals for the location parameter based on Student's t-statistic, and we conjecture that the uniform distribution is least favorable in that it leads to confidence intervals that are largest given their coverage probability, provided the nominal confidence level is large enough. This conjecture is supported by an argument based on second order asymptotics in the sample size and on asymptotics in the length of the confidence interval, by a finite sample inequality, and by simulation results.

Keywords: Unimodality, Symmetric Distribution, Confidence Interval, Finite Sample.

1 Student's t-Statistic

Consider a location-scale family of symmetric distributions with the mean as location and the standard deviation as scale parameter. We are interested in confidence intervals for the location parameter based on observations that will be viewed as realizations of independent and identically distributed (i.i.d.) random variables from this location-scale family. We will adopt the following notation.

Let $X_1, \dots, X_n$ be i.i.d. random variables with distribution function

$$P(X_i \leq x) = F_{\mu,\sigma}(x) = F\left(\frac{x-\mu}{\sigma}\right), \quad x \in \mathbb{R}\,,\ i = 1, \dots, n\,,$$

where $F = F_{0,1}$ is standardized such that

$$\mathrm{E}_{F_{\mu,\sigma}} X_i = \mu\,, \quad \mathrm{var}_{F_{\mu,\sigma}} X_i = \sigma^2$$

hold.

We denote the sample mean $\sum_i X_i/n$ by $\bar{X}_n$ and the variance $\sum_i (X_i - \bar{X}_n)^2/(n-1)$ by S_n^2 with $S_n \geq 0$. Note that the statistic

$$T_n(\mu) = \frac{\sqrt{n}(\bar{X}_n - \mu)}{S_n} \tag{1}$$

is the location and scale invariant Student's t-statistic. If F equals the standard normal distribution function Φ, then under $\Phi((\cdot - \mu)/\sigma)$ the t-statistic $T_n(\mu)$ has a Student t-distribution with $n-1$ degrees of freedom. It is well known that this distribution is called after Student, pseudonym of William Sealy Gosset, who was chemist and Brewer-in-Charge of the Experimental Brewery of Guinness' Brewery. Gosset determined the density of the t-statistic from (1) under normality as

$$\frac{\Gamma(n/2)}{\sqrt{\pi(n-1)}\,\Gamma((n-1)/2)}\left(1+\frac{t^2}{n-1}\right)^{-n/2}, \quad t \in \mathbb{R}, \tag{2}$$

in Section III of his paper Student (1908).

With $t_{n-1}(p)$ denoting the p-th quantile of this distribution we obtain

$$I_t(X_1,\dots,X_n) = [\bar{X}_n - tS_n/\sqrt{n}, \bar{X}_n + tS_n/\sqrt{n}] \tag{3}$$

with $t = t_{n-1}(1-\alpha/2)$ as a confidence interval for the location parameter μ with coverage probability $1-\alpha$, provided the underlying distribution of the random variables generating the observations is normal. This confidence interval is the classical, standard method for constructing confidence intervals for location parameters. However, quite often the observations may not be viewed as stemming from normal random variables and consequently, one is not sure then about the true coverage probability

$$P_{F_{\mu,\sigma}}\left(\mu \in I_t(X_1,\dots,X_n)\right) = P_F\left(0 \in I_t(X_1,\dots,X_n)\right)$$

and one is not even sure if $1-\alpha$ is still a valid confidence level. The question poses itself: 'How should t be chosen such that the interval $I_t(X_1,\dots,X_n)$ has confidence level at least $1-\alpha$ under all F from a given class of distributions?' We will discuss this question for the class of distributions with a symmetric unimodal density under the complicating assumption that the sample size n be small. In practice, sample sizes as small as 3 are not rare. Nevertheless, we will start with asymptotics, as n, the sample size, tends to ∞.

2 Asymptotics

By the Central Limit Theorem and the Law of Large Numbers we have

$$T_n(\mu) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$$

as $n \to \infty$, regardless of the underlying distribution function $F_{\mu,\sigma}$ (with finite and positive variance). This asymptotic normality means that asymptotically the coverage probability of $I_t(X_1,\dots,X_n)$ equals $\Phi(t) - \Phi(-t)$, or

$$\lim_{n\to\infty} P_{F_{\mu,\sigma}}\left(\mu \in I_t(X_1,\dots,X_n)\right) = 2\Phi(t) - 1, \quad \mu \in \mathbb{R},\ \sigma > 0,\ t > 0,$$

irrespectively of the underlying distribution function F with finite variance. Although this result implies that a confidence interval with approximate confidence level $1-\alpha$ may be constructed by choosing $t = \Phi^{-1}(1-\alpha/2)$ in $I_t(X_1,\dots,X_n)$, the confidence level might be quite misleading for finite sample sizes n. A better approximate confidence interval might be obtained by application of Edgeworth expansions for the distribution function of T_n. In fact, this yields the following approximation, with φ denoting the standard normal density.

Theorem 2.1 *Let F be symmetric and nonsingular in the sense that the part of F that is absolutely continuous with respect to Lebesgue measure, does not vanish. If F has finite fourth moment, then $\kappa(F) = E_F X^4 (E_F X^2)^{-2} - 3$, the kurtosis (excess), is well defined and*

$$P_{F_{\mu,\sigma}}(\mu \in I_t(X_1, \ldots, X_n)) = 2\Phi(t) - 1 + \frac{t}{6n}\{\kappa(F)(t^2 - 3) - 3t^2 - 3\}\varphi(t) + o(\tfrac{1}{n}) \tag{4}$$

holds uniformly in t. In other words, if the t-statistic confidence interval $I_t(X_1, \ldots, X_n)$ has coverage probability

$$P_{F_{\mu,\sigma}}(\mu \in I_t(X_1, \ldots, X_n)) = 1 - \alpha\,, \quad \mu \in \mathbb{R}\ \sigma > 0\,,$$

under F, then

$$t = \Phi^{-1}(1 - \tfrac{\alpha}{2})\{1 + \tfrac{1}{12n}[-\kappa(F)(\{\Phi^{-1}(1 - \tfrac{\alpha}{2})\}^2 - 3) + 3\{\Phi^{-1}(1 - \tfrac{\alpha}{2})\}^2 + 3]\} + o(\tfrac{1}{n})\,. \tag{5}$$

Proof Straightforward application of the Theorem of Hall (1987) yields (4). Note that Hall's T_0 equals $\sqrt{1 + 1/(n-1)}T_n$ and hence his y has to be replaced by $t + (2n)^{-1}t$. Since this expansion (4) is uniform in t the second statement is implied by it. □

Let us assume

$$\alpha < 2(1 - \Phi(\sqrt{3})) = 0.0832\,. \tag{6}$$

Then, the right-hand side of (5) is decreasing in $\kappa(F)$. Consequently, the confidence interval $I_t(X_1, \ldots, X_n)$ has coverage probability at least $1 - \alpha$ for all symmetric F with finite fourth moment, if t satisfies (5) with $\kappa(F)$ minimal. Under all such distributions F the kurtosis excess $\kappa(F)$ equals at least -2, i.e.

$$\kappa(F) = \mathrm{E}_F X^4 (\mathrm{E}_F X^2)^{-2} - 3 \geq -2\,, \tag{7}$$

in view of $(\mathrm{E}_F X^2)^2 \leq \mathrm{E}_F X^4$. A generalization of this inequality to possibly asymmetric distributions is given by Karl Pearson (1916). Equality is attained in (7) if X^2 is degenerate, i.e. if X is Bernoulli. However, in many applications this is not a natural distribution. Moreover, if n is small, $n < 1 - \log\alpha/\log 2$, then a bounded confidence interval for μ based on Student's t-statistic does not exist for this distribution. In fact, if $P(X = 1) = P(X = -1) = 1/2$ holds, then $P(T_n = \infty) = P(X_1 = \cdots = X_n) = 2^{1-n} > \alpha$.

In the next Section we will restrict attention to symmetric distributions F with unimodal density and we will determine the (distribution with) minimal value of $\kappa(F)$ within this class of unimodal distributions.

3 As $n \to \infty$ the Uniform Distribution is Least Favorable

Our discussion will be based on the following inequality.

Theorem 3.1 *Let F be a distribution with a symmetric unimodal density. If F has finite fourth moment, then the kurtosis proper of F equals at least $9/5$, which implies that the kurtosis excess satisfies*

$$\kappa(F) \geq -6/5\,. \tag{8}$$

Equality holds here, iff F is uniform.

Proof Let f be the symmetric unimodal density of F. Then f may be written as a mixture of symmetric uniform densities. This means that there exists a probability distribution G on $(0, \infty)$ such that

$$f(x) = \int_0^\infty \tfrac{1}{2y} \mathbf{1}_{(-y,y)}(x) dG(y) \tag{9}$$

holds. By this Khintchin representation of f we obtain for $k = 1$ or 2

$$\begin{aligned} \mathrm{E}_F X^{2k} &= 2 \int_0^\infty x^{2k} \int_0^\infty \tfrac{1}{2y} \mathbf{1}_{(-y,y)}(x) dG(y) dx \\ &= \int_0^\infty \int_0^y \tfrac{1}{y} x^{2k} dx dG(y) = \tfrac{1}{2k+1} \mathrm{E}_G Y^{2k} \end{aligned}$$

and hence

$$\mathrm{E}_F X^4 (\mathrm{E}_F X^2)^{-2} = \tfrac{9}{5} \mathrm{E}_G Y^4 (\mathrm{E}_G Y^2)^{-2} \geq \tfrac{9}{5}$$

by Cauchy-Schwarz as in (7). Note that equality holds iff Y^2 is degenerate, i.e. iff X is uniformly distributed.

A related inequality for possibly asymmetric unimodal distributions has been given by Klaassen, Mokveld, and van Es (2000). □

Together with (5) inequality (8) shows that asymptotically to second order, the uniform distribution is least favorable in the class of symmetric unimodal distributions for constructing confidence intervals based on Student's t-statistic at large confidence levels. More precisely, at fixed n and $\alpha < 0.0832$ such a confidence interval is largest if the underlying distribution is uniform. At first sight, this might seem surprising since e.g. the normal distribution itself has heavier tails than the uniform distribution. However, (5) seems to imply that in some sense, heavier tails in the underlying distribution of the observations result in less heavy tails for the distribution of the t-statistic and vice versa. This conjecture and evidence for it have been around for many years; see for example p. 645 of Benjamini (1983). In fact, (5) itself is an asymptotic version of this conjecture. For finite sample size, the situation that interests us most, this conjectured phenomenon has been formulated precisely and proved for $t \to \infty$ by Zwet (1964b, 1964a). We will prove in the next section that the uniform distribution is least favorable among the symmetric unimodal distributions in that sense too.

4 As $t \to \infty$ the Uniform Distribution is Least Favorable

In this section the relative tail behavior of $P_F(0 \notin I_t(X_1, \ldots, X_n))$ with respect to $P_H(0 \notin I_t(X_1, \ldots, X_n))$ as $t \to \infty$, will be discussed for H being the uniform distribution function.

Theorem 4.1 *Let F and H be two symmetric unimodal distributions, and assume that they are symmetric about zero. If H is uniform, then*

$$\lim_{t \to \infty} \frac{P_F(0 \notin I_t(X_1, \ldots, X_n))}{P_H(0 \notin I_t(X_1, \ldots, X_n))} \leq 1 \tag{10}$$

holds.

Proof By symmetry, the left-hand side of (10) equals

$$\lim_{t\to\infty} \frac{P_F(T_n \geq t)}{P_H(T_n \geq t)} = \frac{R_n(F)}{R_n(H)} = 2^n n \int_0^\infty x^{n-1} f^n(x)dx \tag{11}$$

with

$$R_n(F) = \lim_{t\to\infty} \frac{P_F(T_n \geq t)}{P_\Phi(T_n \geq t)}$$

introduced and studied by Hotelling (1961); cf. his (3.2) and (3.6). According to Theorem 6.2.1 of Zwet (1964b) $R_n(H) \geq R_n(F)$ holds for F and H symmetric about 0, if $F^{-1}(H(x))$ is convex in $x \in (0,\infty)$. This proves the theorem, since for H uniform, this convexity of $F^{-1}(H)$ is equivalent to convexity of $F^{-1}(u)$ in $u \in (\frac{1}{2}, 1)$, which in turn is equivalent to unimodality of f.

An alternative proof may be based on (11) and the representation (9). Indeed, we have

$$\begin{aligned}\int_0^\infty x^{n-1} f^n(x)dx &= \int_0^\infty x^{n-1} \left\{\int_0^\infty \tfrac{1}{2y}\mathbf{1}_{(0,y)}(x)dG(y)\right\}^n dx \\ &\leq \int_0^\infty \int_0^\infty x^{n-1}(2y)^{-n}\mathbf{1}_{(0,y)}(x)dxdG(y) = 2^{-n}n^{-1}\,.\end{aligned}$$

□

Studying the limit behavior of the coverage probability of the Student t-interval $I_t(X_1, \ldots, X_n)$ as $t \to \infty$ with n fixed, we have seen that within the class of symmetric unimodal distributions the uniform is least favorable since its coverage probability is converging to 1 at the smallest possible speed. A strengthened version of this result is presented in the next section.

5 For $t \geq n-1$ the Uniform Distribution is Least Favorable

The argument in the preceding section is based on a geometric interpretation of the t-statistic by Hotelling (1961), which has also been exploited by Efron (1969). The strongest version of this type of result is by Benjamini (1983). His main theorem is that for $t \geq n-1$ and for symmetric distributions F and G with

$$\frac{F^{-1}(u)}{G^{-1}(u)} \text{ nondecreasing in } u \in [\tfrac{1}{2}, 1) \tag{12}$$

the inequality

$$\frac{P_F(0 \notin I_t(X_1,\ldots,X_n))}{P_G(0 \notin I_t(X_1,\ldots,X_n))} \leq 1 \tag{13}$$

holds. As a simple corollary to this result we have the following.

Theorem 5.1 *Let F and H be unimodal distributions symmetric about 0. If H is uniform, then*

$$\frac{P_F(0 \notin I_t(X_1,\ldots,X_n))}{P_H(0 \notin I_t(X_1,\ldots,X_n))} \leq 1\,, \quad t \geq n-1\,,$$

holds.

Proof In view of (13) it suffices to show that F is more stretched than a uniform distribution in the sense of (12); this is the terminology of Benjamini (1983). Since the quantile function of a uniform distribution symmetric around 0 is a multiple of $u \mapsto u - \frac{1}{2}$, this means that it suffices to show

$$\frac{F^{-1}(u)}{u - \frac{1}{2}} \leq \frac{F^{-1}(v)}{v - \frac{1}{2}}\,, \quad \tfrac{1}{2} < u < v < 1\,. \tag{14}$$

The unimodality of F yields convexity of $F^{-1}(u)$, $\frac{1}{2} \leq u < 1$, and hence for $\frac{1}{2} < u < v < 1$

$$F^{-1}(u) = F^{-1}\left(\frac{v-u}{v-\frac{1}{2}}\frac{1}{2} + \frac{u-\frac{1}{2}}{v-\frac{1}{2}}v\right) \leq \frac{u-\frac{1}{2}}{v-\frac{1}{2}}F^{-1}(v)\,,$$

which is (14). □

Together with the asymptotics, both as $n \to \infty$ for t fixed and as $t \to \infty$ for n fixed, this result supports the claim that the uniform distribution is least favorable. Section 6 adds some numerical evidence to this claim.

6 Simulation Results

In our Monte Carlo study we have simulated the behavior of confidence intervals based on Student's t-statistic under the uniform, triangular, Laplace, Cauchy, and normal distribution for sample sizes $n = 2, 3, 4, 5, 6, 11$, and 21. This has been done by simulating 10^8 realizations of the t-statistic in each of these 35 cases. We report the p-quantiles of the resulting empirical distributions of the t-statistic for $p = 0.9$, 0.95, and 0.975, corresponding to $(1-\alpha)$-confidence intervals with $\alpha = 0.2$, 0.1, and 0.05, respectively. In analogy with the usual way of reporting a sample mean together with its sample standard error, i.e. in analogy with the usual t-statistic confidence interval(!), we also report between brackets in the tables half the length of the $\Phi(1) - \Phi(-1) \approx 0.68$-confidence interval based on the above empirical of the corresponding p-quantile. These results show that the third digit in our simulated quantiles is very reliable.

The unimodal distributions used in these simulations, are ordered according to the kurtosis (excess) $\kappa(F)$ as follows

uniform	$-6/5$
triangular	$-3/5$
normal	0
Laplace	3
Cauchy	∞

Straightforward computation shows that for $n = 2$ and $p \geq 3/4$ we have

$$t_{n-1}(p) = t_1(p) = \tfrac{1}{2}(1-p)^{-1} - 1$$

under the uniform distribution, yielding in Table 1 the exact values 4, 9, and 19, respectively, for $n = 2$. An analytic calculation as in Perlo (1933) shows that the exact values for $n = 3$ are 2.073664, 3.589439, and 5.741739.

Table 1: Uniform distribution, estimated p-th quantiles of the t-statistic

n	$p = 0.9$	$p = 0.95$	$p = 0.975$
2	3.9998(.0010)	9.0009(.0030)	18.9982(.0085)
3	2.0731(.0004)	3.5881(.0008)	5.7400(.0016)
4	1.6707(.0003)	2.6314(.0005)	3.8531(.0009)
5	1.5199(.0002)	2.2577(.0004)	3.1465(.0006)
6	1.4554(.0002)	2.0737(.0003)	2.7913(.0005)
11	1.3572(.0002)	1.8177(.0002)	2.2724(.0003)
21	1.3171(.0002)	1.7244(.0002)	2.1002(.0003)

Table 2: Triangular distribution, estimated p-th quantiles of the t-statistic

n	$p = 0.9$	$p = 0.95$	$p = 0.975$
2	3.1384(.0007)	6.5498(.0021)	13.2634(.0059)
3	1.9043(.0003)	2.9574(.0006)	4.3999(.0011)
4	1.6556(.0002)	2.3911(.0004)	3.2321(.0006)
5	1.5420(.0002)	2.1670(.0003)	2.8331(.0005)
6	1.4780(.0002)	2.0409(.0003)	2.6217(.0004)
11	1.3678(.0002)	1.8183(.0002)	2.2492(.0003)
21	1.3218(.0002)	1.7252(.0002)	2.0932(.0002)

Table 3: Laplace distribution, estimated p-th quantiles of the t-statistic

n	$p = 0.9$	$p = 0.95$	$p = 0.975$
2	2.5011(.0005)	5.0027(.0016)	10.0066(.0045)
3	1.7299(.0002)	2.4583(.0004)	3.4764(.0008)
4	1.5834(.0002)	2.1321(.0003)	2.7317(.0005)
5	1.5135(.0002)	1.9981(.0003)	2.4902(.0004)
6	1.4717(.0002)	1.9248(.0002)	2.3657(.0003)
11	1.3847(.0002)	1.7861(.0002)	2.1490(.0003)
21	1.3371(.0002)	1.7180(.0002)	2.0529(.0002)

In Table II on page 337 of Hotelling (1961) $t_2(0.975)$ has been computed as 3.48, in accordance with our simulations in Table 3.

Hotelling (1961) derived $t_2(p)$ for the Cauchy distribution as (cf. his (6.41))

$$t_2(p) = (1 - 3\mathrm{tg}^2(\pi(2p-1)/6))^{-1/2}\,, \quad p \geq \tfrac{1}{2} + \tfrac{3}{\pi}\,\mathrm{arctg}\tfrac{1}{2}\,,$$

which leads to $t_2(0.975) = 2.9412$ in line with our results in Table 4 (note that $t_2(0.995) = 6.46$ and not 3.69 as in (6.42) and Table II of Hotelling (1961)).

For the sake of validation of our simulation method we also compiled the Table 5, which might be compared to the Table 6 with exact results. Indeed, our simulation results support our claim that the uniform distribution is least favorable within the class of symmetric unimodal distributions and that heavier tails (in terms of $\kappa(F)$) result in smaller quantiles for the t-statistic, at least for $\alpha = 0.05$.

Table 4: Cauchy distribution, estimated p-th quantiles of the t-statistic

n	$p = 0.9$	$p = 0.95$	$p = 0.975$
2	2.0825(.0004)	4.0779(.0012)	8.1157(.0036)
3	1.5342(.0002)	2.1265(.0003)	2.9412(.0007)
4	1.4241(.0002)	1.8558(.0002)	2.3551(.0004)
5	1.3769(.0002)	1.7485(.0002)	2.1475(.0003)
6	1.3506(.0001)	1.6908(.0002)	2.0417(.0003)
11	1.3018(.0001)	1.5881(.0002)	1.8618(.0002)
21	1.2793(.0001)	1.5422(.0002)	1.7849(.0002)

Table 5: Normal distribution, estimated p-th quantiles of the t-statistic

n	$p = 0.9$	$p = 0.95$	$p = 0.975$
2	3.0767(.0007)	6.3098(.0020)	12.7031(.0054)
3	1.8857(.0003)	2.9202(.0006)	4.3023(.0011)
4	1.6378(.0002)	2.3530(.0004)	3.1819(.0006)
5	1.5334(.0002)	2.1319(.0003)	2.7764(.0005)
6	1.4757(.0002)	2.0151(.0003)	2.5707(.0004)
11	1.3721(.0002)	1.8125(.0002)	2.2281(.0003)
21	1.3253(.0002)	1.7246(.0002)	2.0857(.0003)

Table 6: Normal distribution, exact p-th quantiles of the Student t-statistic

n	$p = 0.9$	$p = 0.95$	$p = 0.975$
2	3.077684	6.313752	12.7062
3	1.885618	2.919986	4.302653
4	1.637744	2.353363	3.182446
5	1.533206	2.131847	2.776445
6	1.475884	2.015048	2.570582
11	1.372184	1.812461	2.228139
21	1.325341	1.724718	2.085963
∞	1.281552	1.644854	1.959964

7 Relation to the Literature and some History

Besides deriving the t-distribution, Gosset in his paper Student (1908) already started the discussion about the behavior of the t-statistic under nonnormality. On his page 19 he writes: 'If the distribution is not normal, the mean and the standard deviation of a sample will be positively correlated, so that although both will have greater variability, yet they will tend to counteract each other, a mean deviating largely from the general mean tending to be divided by a larger standard deviation.' Subsequently he states to believe that in many practical situations distributions are sufficiently close to normal to have reliable results via the student distribution (2), even for small sample sizes. Nevertheless, he suggested Fisher, who in Fisher (1915) proved the validity of (2) in a mathematically rigorous way, to also study the distribution of the t-statistic for uniformly distributed random variables. See Lehmann (1999) for a beautiful analysis of the discussions between Gosset, Fisher, and Egon Pearson on the Student t-statistic, its distribution, its robustness, and the ideas on testing hypotheses.

Over the years, many authors have derived asymptotic results as $n \to \infty$. Based on an asymptotic expansion of the variance Geary (1936) notes that deviation from the Student distribution is larger for asymmetric distributions and he studies the behavior of Student's t under such distributions. Chung (1946) is the first to prove an Edgeworth expansion for the distribution of the t-statistic, but Hall (1987) does so under minimal moment conditions. Gayen (1949) computes correction terms on the Student density (2) based on skewness and kurtosis of the underlying nonnormal distribution.

Many results have been derived for finite n as well. Benjamini (1983) studies the conservatism of the t-test and confidence interval (3) with the value of t equal to the quantile of the t-distribution from (2). He proves that this interval is conservative for $t \geq n - 1$ if the underlying distribution is stretched in the sense of (12) with respect to the normal distribution. His study of (13) is based on a geometric interpretation of the t-statistic as in Hotelling (1961). Efron (1969) presents Hotelling's approach to the geometry of the Student statistic and discusses its behavior under so-called orthant symmetry. Hotelling (1961) derives the distribution of the t-statistic in some non-normal cases, and he presents and studies (11). Hotelling's geometric interpretation has predecessors in Laderman (1939) and Rider (1929), and actually dates back to page 509 of Fisher (1915).

Hyrenius (1950) also notes that the behavior of the t-statistic under non-normal distributions varies considerably and that this shows that characterizing distributions via a couple of parameters doesn't suffice to predict the finite sample behavior of the t-statistic. Bowman et al. (1977) reviews the literature up to 1977 and discusses several approximating procedures for the performance of the t-statistic under nonnormality, these procedures depending on characteristics of the underlying distribution.

In contrast, Edelman (1990a) presents a bound on the tail probability of Student's t-statistic valid for all symmetric distributions, where the observations need not be identically distributed; his bound does not depend on these distributions. Furthermore, Edelman (1990b) proves an inequality that yields a confidence interval for the location parameter of a unimodal distribution based on just one observation. In this inequality the uniform distribution attains equality, that is, it is least favorable.

Our approach differs somewhat from most of the literature in that we don't try to identify the distribution for which the normal theory t-confidence interval is conservative or liberal, but we try to identify a least favorable distribution in a natural class of distributions, namely the class of symmetric unimodal distributions.

8 Conclusion

Consider the location problem with an unknown symmetric distribution. Confidence intervals for the location parameter may be based on Student's t-statistic. If the underlying distribution of the observations is known to have a unimodal density this can be done in a conservative way by assuming uniformity of the underlying distribution, if (6) holds. In other words, if the t-value in (3) is chosen according to the third column of Table 1 then the coverage probability will be at least $2p - 1 = 1 - \alpha = 0.95$, *whatever the unknown symmetric unimodal distribution of the observations.* This claim is supported by the Edgeworth expansion of Section 2 and its consequence of Section 3, by the finite sample analysis of the far tails in Section 4, by the finite sample analysis of the moderately far tails in Section 5, and by the Monte Carlo results of Section 6. However, to *prove* this claim we need Theorem 5.1 under a much weaker condition than $t \geq n - 1$.

Standard approaches like the bootstrap (i.e. estimating F) and empirical Edgeworth expansion (i.e. estimating $\kappa(F)$ in (4)) are based completely on asymptotic considerations as $n \to \infty$. Consequently, for small sample sizes these techniques are not reliable whereas for large sample sizes the gain in efficiency as compared to our recommendation of using the uniformity assumption, is not dramatic; compare Tables 1 and 6 and note that for $n = 21$ the relative difference in interval length is less than 0.7%.

Acknowledgements

We would like to thank Erich Lehmann for his comments on an earlier version of this manuscript and for many references. This research was triggered by a problem suggested by Rolf Muijlwijk of the NMi Van Swinden Laboratorium, the former Dutch Legal Metrology Service, to the 33rd European Study Group with Industry, Leiden, September 14-18, 1998.

References

Benjamini, Y. (1983). Is the t test really conservative when the parent distribution is longtailed? *Journal of the American Statistical Association*, *78*, 645-654.

Bowman, K. O., Beauchamp, J. J., and Shenton, L. R. (1977). The distribution of the t-statistic under non-normality. *International Statistical Reviews*, *45*, 233-242.

Chung, K. L. (1946). The approximate distribution of Student's statistic. *Annals of Mathematical Statistics*, *17*, 447-465.

Edelman, D. (1990a). An inequality of optimal order for the tail probabilities of the t statistic under symmetry. *Journal of the American Statistical Association*, *85*, 120-122.

Edelman, D. (1990b). A note on uniformly most powerful two-sided tests. *The American Statistician*, *3*, 219-220.

Efron, B. (1969). Student's t-test under symmetry conditions. *Journal of the American Statistical Association*, *64*, 1278-1302.

Fisher, R. A. (1915). Frequency distribution of the values of the correlation coefficient in samples from an indefinitely large population. *Biometrika*, *10*, 507-521.

Gayen, A. K. (1949). The distribution of 'Student's' t in random samples of any size drawn from non-normal universes. *Biometrika*, *36*, 353-369.

Geary, R. C. (1936). The distribution of "Student's" ratio for non-normal samples. *Supplement Journal of the Royal Statistical Society*, *3*, 178-184.

Hall, P. (1987). Edgeworth expansion for Student's t statistic under minimal moment conditions. *Annals of Probability*, *15*, 920-931.

Hotelling, H. (1961). The behavior of some standard statistical tests under nonstandard conditions. In J. Neyman (Ed.), *Proceedings of the fourth berkeley symposium on mathematical statistics and probability* (Vol. I, p. 319-359). Berkeley: University of California.

Hyrenius, H. (1950). Distribution of 'Student'-Fisher's t in samples from compound normal functions. *Biometrika*, *37*, 429-442.

Klaassen, C. A. J., Mokveld, P. J., and van Es, A. J. (2000). Squared skewness minus kurtosis bounded by 186/125 for unimodal distributions. *Statistics and Probability Letters*, *50*, 131-135.

Laderman, J. (1939). The distribution of "Student's" ratio for samples of two items drawn from non-normal universes. *Annals of Mathematical Statistics*, *10*, 376-379.

Lehmann, E. L. (1999). "Student" and small-sample theory. *Statistical Science*, *14*, 418-426.

Pearson, K. (1916). Mathematical contributions to the theory of evolution, XIX; Second supplement to a memoir on skew variation. *Philos. Trans. Roy. Soc. London Ser. A*, *216*, 429-457.

Perlo, V. (1933). On the distribution of Student's ratio for samples drawn from a rectangular distribution. *Biometrika*, *25*, 203-204.

Rider, P. (1929). On the distribution of the ratio of mean to standard deviation in small samples from non-normal universes. *Biometrika*, *21*, 124-143.

Student. (1908). The probable error of a mean. *Biometrika*, *6*, 1-25.

Zwet, W. van. (1964a). Convex transformations: A new approach to skewness and kurtosis. *Statist. Neerlandica*, *18*, 433-441.

Zwet, W. van. (1964b). *Convex transformations of random variables* (Mathematical Centre Tracts 7 ed.). Amsterdam: Mathematical Centre.

Authors' addresses:

Pieta C. IJzerman-Boon
NV Organon
Molenstraat 110
5342 CC Oss, The Netherlands
Tel. +31-412-663243
E-mail: Pieta.IJzerman@organon.com

Harrie Hendriks
Department of Mathematics
Radboud University Nijmegen
P.O. Box 9010
6500 GL Nijmegen, The Netherlands
Tel. +31-24-3652868
E-mail: hhendr@sci.kun.nl

Chris A. J. Klaassen (Corresponding author)
Korteweg - de Vries Institute for Mathematics and Institute for Business and Industrial Statistics
Universiteit van Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam, The Netherlands
Tel. +31-20-5255010
E-mail: chrisk@science.uva.nl