Task-Structured Probabilistic I/O Automata

Ran Canetti*, †, Ling Cheung†, Dilsun Kaynar‡, Moses Liskov§, Nancy Lynch‡, Olivier Pereira¶, Roberto Segala∥

*IBM T.J. Watson Research Center, USA, †Radboud University of Nijmegen, the Netherlands, §MIT CSAIL, USA,
¶The College of William and Mary, USA, ¶Université Catholique de Louvain, Belgium, ∥Università di Verona, Italy

Abstract—Modeling frameworks such as Probabilistic I/O Automata (PIOA) and Markov Decision Processes permit both probabilistic and nondeterministic choices. In order to use such frameworks to express claims about probabilities of events, one needs mechanisms for resolving the nondeterministic choices. For PIOAs, nondeterministic choices have traditionally been resolved by schedulers that have perfect information about the past execution. However, such schedulers are too powerful for certain settings, such as cryptographic protocol analysis, where information must sometimes be hidden.

Here, we propose a new, less powerful nondeterminism-resolution mechanism for PIOAs, consisting of tasks and local schedulers. Tasks are equivalence classes of system actions that are scheduled by oblivious, global task sequences. Local schedulers resolve nondeterminism within system components, based on local information only. The resulting task-PIOA framework yields simple notions of external behavior and implementation, and supports simple compositionality results. We also define a new kind of simulation relation, and show it to be sound for proving implementation. We illustrate the potential of the task-PIOA framework by outlining its use in verifying an Oblivious Transfer protocol.

I. INTRODUCTION

The Probabilistic I/O Automata (PIOA) modeling framework [Seg95], [SL95] is a simple combination of I/O Automata [LT89] and Markov Decision Processes (MDP) [Put94]. As demonstrated in [LSS94], [SV99], [PSL00], PIOAs are well suited for modeling and analyzing distributed algorithms that use randomness as a computational primitive. In this setting, distributed processes use random choices to break symmetry, in solving problems such as choice coordination [R82] and consensus [B83], [AH90]. Each process is modeled as an automaton with random transitions, and an entire protocol is modeled as the parallel composition of process automata and automata representing communication channels.

This modeling paradigm combines nondeterministic and probabilistic choices in a natural way. Nondeterminism is used here for modeling uncertainties in the timing of events in highly unpredictable distributed environments. It is also used for modeling distributed algorithms at high levels of abstraction, leaving many details unspecified. This in turn facilitates algorithm verification, because results proved about nondeterministic algorithms apply automatically to an entire family of algorithms, obtained by resolving the nondeterministic choices in particular ways.

In order to formulate and prove probabilistic properties of distributed algorithms, one needs mechanisms for resolving the nondeterministic choices. In the randomized distributed setting, the most common mechanism is a perfect-information event scheduler, which has access to local state and history of all system components and has unlimited computation power. Thus, probabilistic properties of distributed algorithms are typically asserted with respect to worst-case, adversarial schedulers who can choose the next event based on complete knowledge of the past (e.g., [PSL00]).

One would expect that a similar modeling paradigm, including both probabilistic and nondeterministic choices, would be similarly useful for modeling cryptographic protocols. These are special kinds of distributed algorithms, designed to protect sensitive data when it is transmitted over unreliable channels. Their correctness typically relies on computational assumptions, which say that certain problems cannot be solved by an adversarial entity with bounded computation resources [Gol01]. However, a major problem with this extension is that the perfect-information scheduler mechanism used for distributed algorithms is too powerful for use in the cryptographic setting. A scheduler that could see all information about the past would, in particular, see “secret” information hidden in the states of non-corrupted protocol participants, and be able to “divulge” this information to corrupted participants, e.g., by encoding it in the order in which it schedules events.

In this paper, we present task-PIOAs, an adaptation of PIOAs, that has new, less powerful mechanisms for resolving nondeterminism. Task-PIOAs are suitable for modeling and analyzing cryptographic protocols; they may also be useful for other kinds of distributed systems in which the perfect information assumption is unrealistically strong.

Task-PIOAs: A task-PIOA is simply a PIOA augmented with a partition of non-input actions into equivalence classes called tasks. A task is typically a set of related actions, for example, all the actions of a cryptographic protocol that send a round 1 message. Tasks are units of scheduling; they are scheduled by simple oblivious, global task schedule sequences. We define notions of external behavior and implementation for task-PIOAs, based on the trace distribution semantics proposed by Segala [Seg95]. We define parallel composition in the obvious way and show that our implementation relation is compositional.

Canetti is supported by NSF CyberTrust Grant #430450; Cheung by DFG/NWO bilateral cooperation project 600.050.011.01 Validation of Stochastic Systems (VOSS); Kaynar and Lynch by DARPA/AFOSR MURI Award #FA9550-04-1-0121; Pereira by the Belgian National Fund for Scientific Research (FNRS); and Segala by MURST project Constraint-based Verification of reactive systems (CoVer).
We also define a new type of simulation relation, which incorporates tasks, and prove that it is sound for proving implementation relationships between task-PIOAs. This new relation differs from simulation relations studied earlier [SL95], [LSV03], in that it relates probability measures rather than states. In many cases, including our work on cryptographic protocols (see below), tasks alone suffice for resolving nondeterminism. However, for extra expressive power, we define a second mechanism, local schedulers, which can be used to resolve nondeterminism within system components, based on local information only. This mechanism is based on earlier work in [CLSV04].

Cryptographic protocols: In [CC+06a], we applied the task-PIOA framework to analyze an Oblivious Transfer (OT) protocol of Goldreich, et al. [GMW87]. That analysis required defining extra structure for task-PIOAs, in order to express issues involving computational limitations. Thus, we defined notions such as time-bounded task-PIOAs, and approximate implementation with respect to time-bounded environments. Details are beyond the scope of this paper, but we outline our approach in Section IV.

Adversarial scheduling: The standard scheduling mechanism in the cryptographic community is an adversarial scheduler, namely, a resource-bounded algorithmic entity that determines the next move adaptively, based on its own view of the computation so far. This is weaker than the perfect-information scheduler used for distributed algorithms, which have access to local state and history of all components and have unlimited computation power. Our task schedule sequences are essentially oblivious schedulers, which fix the entire schedule of tasks, nondeterministically, in advance. This formulation does not directly capture the adaptivity of adversarial schedulers.

Our solution is to separate scheduling concerns into two parts. We model the adaptive adversarial scheduler as a system component, for example, a message delivery service that can eavesdrop on the communications and control the order of message delivery. Such a system component has access to partial information about the execution: it sees information that other components communicate to it during execution, but not “secret information” that these components hide. On the other hand, basic scheduling choices are resolved by a task schedule sequence, chosen nondeterministically in advance. These tasks are equivalence classes of actions, independent of actual choices that are determined during the execution. We believe this separation is conceptually meaningful: The high-level adversarial scheduler is responsible for choices that are essential in security analysis, such as the ordering of message deliveries. The low-level schedule of tasks resolves inessential choices. For example, in the OT protocol, both the transmitter and receiver make random choices, but it is inconsequential which does so first.

Related work: The literature contains numerous models that combine nondeterministic and probabilistic choices (see [Sdv04] for a survey). However, few tackle the issue of partial-information scheduling, as we do. Exceptions include [CH05], which models local-oblivious scheduling, and [dA99], which uses partitions on the state space to obtain partial-information schedules. The latter is essentially within the framework of partially observable MDPs (POMDPs), originally studied in the context of reinforcement learning [KLA98]. All of these accounts neglect partial information aspects of (parameterized) actions, therefore are not suitable in a cryptographic setting. A version of local schedulers was introduced in [CLSV04].

Our general approach to cryptographic protocol verification was directly inspired by the Interactive Turing Machine (ITM) framework of [Can01]. There, participants in a protocol are modeled as ITMs and messages as bit strings written on input and output tapes. ITMs are purely probabilistic, and scheduling nondeterminism is resolved using predefined rules. In principle, this framework could be used to analyze cryptographic protocols rigorously, including computational complexity issues. However, complete analysis of protocols in terms of Turing machines is impractical, because it involves too many low-level machine details. Indeed, in the computational cryptography community, protocols are typically described using an informal high-level language, and proof sketches are given in terms of the informal protocol descriptions. We aim to provide a framework in which proofs in the ITM style can be carried out formally, at a high level of abstraction. Also, we aim to exploit the benefits of nondeterminism to a greater extent than the ITM approach.

Several other researchers have added features for computational cryptographic analysis to conventional abstract concurrency modeling frameworks such as process algebras and restricted forms of PIOAs [LMMS98], [PW00], [PW01], [MMS03]. These approaches again use less nondeterminism than we do: individual system components are purely probabilistic, and scheduling is determined by predefined rules. For example, in [LMMS98], a uniform distribution is imposed on the set of possible reductions for each term. In [MMS03], internal reductions are prioritized over external communications and several independence properties are assumed. In [PW01], scheduling is based on a distributed scheme wherein each system component schedules the next one, based on its own local information. None of the prior work separates high-level and low-level nondeterminism resolution, as we do.

Roadmap: Section II defines task-PIOAs, task schedules, composition, and implementation, and presents a compositional result. Section III presents our simulation relation and its soundness theorem. Section IV summarizes our OT protocol case study. Section V discusses local schedulers, and concluding discussions follow in Section VI. Further details appear in [CC+06b].

II. TASK-PIOAS

A. Basic PIOAs

We assume our reader is comfortable with basic notions of probability theory, such as \(\sigma\)-fields and (discrete) probability measures. A summary is provided in [CC+06b].

A probabilistic I/O automaton (PIOA) \(\mathcal{P}\) is a tuple \((Q, \tilde{q}, I, O, H, D)\) where: (i) \(Q\) is a countable set of states, with start state \(\tilde{q} \in Q\); (ii) \(I, O\) and \(H\) are countable and pairwise disjoint sets of actions, referred to as input, output and internal actions, respectively; and (iii) \(D \subseteq\)
experimenting with the idea that $\alpha$ is finite, then it ends with a state; and (ii) for every non-final $i$, there is a transition $(q_i, a_{i+1}, \mu) \in D$ with $q_{i+1} \in \text{supp}(\mu)$, where $\text{supp}(\mu)$ denotes the support of $\mu$. We write $\text{fstate}(\alpha)$ for $q_0$, and, if $\alpha$ is finite, we write $\text{lstate}(\alpha)$ for its last state. We use $\text{Frag}_P(\alpha)$ (resp., $\text{Frag}_P(\alpha)$) to denote the set of all (resp., all finite) execution fragments of $P$. An execution of $P$ is an execution fragment beginning from the start state $q$. $\text{Execs}(P)$ (resp., $\text{Execs}_P(\alpha)$) denotes the set of all (resp., finite) executions of $P$.

The trace of an execution fragment $\alpha$, written $\text{trace}(\alpha)$, is the restriction of $\alpha$ to the set of external actions of $P$. The symbol $\leq$ denotes the prefix relation on sequences, which applies in particular to execution fragments and traces.

Non-deterministic choices in $P$ are resolved by a scheduler, which is a function $\sigma: \text{Frag}_P(\alpha) \rightarrow \text{SubDisc}(D)$ such that $(q_i, a_i, \mu) \in \text{supp}(\sigma(\alpha))$ implies $i = \text{lstate}(\alpha)$. Here $\text{SubDisc}(D)$ denotes the set of discrete sub-probabilistic measures on $D$—that is, the measure of the entire space $D$ is required to be $1$. Thus, $\sigma$ decides (probabilistically) which transition (if any) to take after each finite execution fragment $\alpha$. A scheduler $\sigma$ and a finite execution fragment $\alpha$ generate a measure $\epsilon_{\sigma, \alpha}$ on the $\sigma$-field $\mathcal{F}_P$ generated by cones of execution fragments, where each cone $C_{\sigma'}$ is the set of execution fragments that have $\alpha'$ as a prefix. The measure of a cone, $\epsilon_{\sigma, \alpha}(C_{\sigma'})$, is defined recursively, as:

1. $0$, if $\alpha' \not\subseteq \alpha$ and $\alpha \not\subseteq \alpha'$.
2. $1$, if $\alpha' \subseteq \alpha$.
3. $\epsilon_{\sigma, \alpha}(C_{\sigma'})\mu_{\sigma(\alpha')}(a, q)$, if $\alpha'$ is of the form $a'' \alpha a q$ and $\alpha \subseteq a'\alpha''$. Here, $\mu_{\sigma(\alpha')}(a, q)$ is defined to be $\sigma(\alpha'(\alpha'))(\text{trace}(\sigma(\alpha'), a)\mu_{\text{state}(\alpha')}(a, q)$, that is, the probability that $\sigma(\alpha')$ chooses a transition labeled by $a'$ and that the new state is $q$.

Standard measure theoretic arguments ensure that $\epsilon_{\sigma, \alpha}$ is well-defined. We call the state $\text{state}(\alpha)$ the first state of $\epsilon_{\sigma, \alpha}$ and denote it by $\text{fstate}(\epsilon_{\sigma, \alpha})$. If $\alpha$ consists of the start state $\alpha$, only we call $\epsilon_{\sigma, \alpha}$ a probabilistic execution of $P$.

Let $\mu$ be a discrete probabilistic measure over $\text{Frag}_P(\alpha)$. We denote by $\epsilon_{\mu}$ the measure $\sum_{a} \mu(a)\epsilon_{\sigma, \alpha}$ and we say that $\epsilon_{\mu}$ is generated by $\sigma$ and $\mu$. We call the measure $\epsilon_{\sigma, \alpha}$ a generalized probabilistic execution fragment of $P$. If every execution fragment in $\text{supp}(\mu)$ consists of a single state, then we call $\epsilon_{\mu}$ a probabilistic execution fragment of $P$.

We note that the trace function is a measurable function from $\mathcal{F}_P$ to the $\sigma$-field generated by cones of traces. Thus, given a probability measure $\epsilon$ on $\mathcal{F}_P$, we define the trace distribution of $\epsilon$, denoted $\text{tdist}^*$($\epsilon$), to be the image measure of $\epsilon$ under trace. We denote by $\text{tdists}(\alpha)$ the set of trace distributions of (probabilistic executions of) $\alpha$.

B. Task-PIOAs

We now augment the PIOA framework with task partitions, our main mechanism for resolving nondeterminism.

Definition 2.1: A task-PIOA is a pair $\langle P, R \rangle$ where $\langle P \rangle = \langle Q, \tilde{q}, I, O, H, D \rangle$ is a PIOA (satisfying transition determinism and) $R$ is an equivalence relation on the locally-controlled actions ($O \cup H$). The equivalence classes of $R$ are called tasks. A task $T$ is enabled in a state $q$ if some $a \in T$ is enabled in $q$.

Unless otherwise stated, technical notions for task-PIOAs are inherited from those for PIOAs. Exceptions include the notions of probabilistic executions and trace distributions. For now, we impose the following action-determinism assumption, which implies that tasks alone are enough to resolve all nondeterministic choices. We will remove this assumption when we introduce local schedulers, in Section V.

- **Action determinism:** For every state $q \in Q$ and task $T \in R$, at most one action $a \in T$ is enabled in $q$.

A task schedule for $T$ is any finite or infinite sequence $\rho = T_1T_2 \ldots$ of tasks in $R$. A task schedule is static (or oblivious), in the sense that it does not depend on dynamic information generated during execution. Under the action-determinism assumption, a task schedule can be used to generate a unique probabilistic execution, and hence, a unique trace distribution of the underlying PIOA $P$. One can do this by repeatedly scheduling tasks, each of which determines at most one transition of $P$. Formally, we define an operation that "applies" a task schedule to a task-PIOA:

Definition 2.3: Let $\mathcal{T} = \langle \mathcal{P}, \mathcal{R} \rangle$ be an action-deterministic task-PIOA where $\mathcal{P} = \langle Q, \tilde{q}, I, O, H, D \rangle$. Given $\mu \in \text{Disc}(\mathcal{F}^*(\mathcal{P}))$ and a task schedule $\rho$. $\text{apply}(\mu, \rho)$ is the probability measure on $\mathcal{F}^*(\mathcal{P})$ defined recursively by:

1. $\text{apply}(\mu, \lambda) := \mu$. ($\lambda$ denotes the empty sequence.)
2. For $T \in R$, $\text{apply}(\mu, T)$ is defined as follows. For every $\alpha \in \mathcal{F}^*(\mathcal{P})$, $\text{apply}(\mu, T)(\alpha) := p_1 + p_2$, where:
   - $p_1 = \mu(\alpha' \eta q) i$ is of the form $\alpha' a q$, where $a \in T$ and $\text{state}(\alpha', a, \eta) \in D$; $p_1 = 0$ otherwise.
   - $p_2 = \mu(\alpha)$ if $T$ is not enabled in $\text{state}(\alpha)$; $p_2 = 0$ otherwise.
3. For $\rho$ of the form $\rho' \in \mathcal{R}$, $\text{apply}(\mu, \rho) := \text{apply}(\text{apply}(\mu, \rho'), T)$.
4. For $\rho$ infinite, $\text{apply}(\mu, \rho) := \lim_{i \to \infty} (\text{apply}(\mu, \rho_i))$, where $\rho_i$ denotes the length-$i$ prefix of $\rho$.

In Case (2) above, $p_1$ represents the probability that $\alpha$ is executed when applying task $T$ at the end of $\alpha'$. Because of
transition-determinism and action-determinism, the transition
\((I_{\text{state}}(\alpha'), \alpha, \eta)\) is unique, and so \(p_1\) is well-defined. The
term \(p_2\) represents the original probability \(\mu(\alpha)\), which is
relevant if \(T\) is not enabled after \(\alpha\). It is routine to check
that the limit in Case (4) is well-defined. The other two cases
are straightforward.

Next, we show that \(\text{apply}(\mu, \rho)\) is a generalized probabilis-
tic execution fragment generated by \(\mu\) and a scheduler for \(P\),
in the usual sense. Thus, a task schedule for a task-PIOA
is a special case of a scheduler for the underlying PIOA.

**Theorem 2.4:** Let \(T = (P, R)\) be an action-deterministic
task-PIOA. For each measure \(\mu\) on \(\text{Frags}^*(P)\) and task
schedule \(\rho\), there is scheduler \(\sigma\) for \(P\) such that \(\text{apply}(\mu, \rho)\)
is the generalized probabilistic execution fragment \(\epsilon_{\sigma, \rho}\).

Any such \(\text{apply}(\mu, \rho)\) is said to be a generalized prob-
abilistic execution fragment of \(T\). Probabilistic execution
fragments and probabilistic executions are then defined by
making the same restrictions as for basic PIOAs. We write
\(\text{tdist}(\mu, \rho)\) as shorthand for \(\text{tdist}(\text{apply}(\mu, \rho))\), the trace
distribution obtained by applying task schedule \(\rho\) starting from
the measure \(\mu\) on execution fragments. We write \(\text{tdist}(\rho)\)
for \(\text{tdist}(\text{apply}(\rho))\), the trace distribution obtained by
applying \(\rho\) from the unique start state. (Recall that the Dirac
measure for an element \(x, \delta(x)\), is the discrete probability
measure that assigns probability 1 to \(\{x\}\).) A trace distribu-
tion of \(T\) is any \(\text{tdist}(\rho)\). We use \(\text{tdists}(T)\) to denote the set
\(\{\text{tdist}(\rho) : \rho\) is a task schedule for \(T\}\). Finally, we define
composition of task-PIOAs:

**Definition 2.5:** Two task-PIOAs \(T_i = (P_i, R_i), i \in \{1, 2\}\),
are said to be \(\text{compatible}\) provided the underlying PIOAs are
compatible. In this case, we define their composition \(T_1 || T_2\)
to be the task-PIOA \((P_1 \cup P_2, R_1 \cup R_2)\).

It is easy to see that \(T_1 || T_2\) is in fact a task-PIOA. In
particular, since compatibility ensures disjoint sets of locally-
controlled actions, \(R_1 \cup R_2\) is an equivalence relation on the
locally-controlled actions of \(T_1 || T_2\). It is also easy to see
that action determinism is preserved under composition.
Note that, when two task-PIOAs are composed, no new
requirements are needed to schedule actions of the two
components—the tasks alone are enough.

**C. Implementation**

We now define the notion of external behavior for a task-
PIOA and the induced implementation relation between task-
PIOAs. Unlike previous definitions of external behavior, the
one we use here is not simply a set of trace distributions.
Rather, it is a mapping that specifies, for every possible
“environment” \(E\) for the given task-PIOA \(T\), the set of trace
distributions that can arise when \(T\) is composed with \(E\).

**Definition 2.6:** Let \(T\) be any task-PIOA and \(E\) be an
action-deterministic task-PIOA. We say that \(E\) is an envi-
ronment for \(T\) if (i) \(E\) is compatible with \(T\) and (ii) the
composition \(T || E\) is closed. Note that \(E\) may have output
actions that are not in the signature of \(T\).

**Definition 2.7:** The external behavior of \(T\), denoted by
\(\text{extbeh}(T)\), is the total function that maps each environment
\(E\) to the set of trace distributions \(\text{tdists}(T || E)\).

Thus, for each environment, we consider the set of trace
distributions that arise from all task schedules. Note that
these traces may include new output actions of \(E\), in addition
to the external actions already present in \(T\). Our definition of
implementation is influenced by common notions in the security
literature (e.g., [LMMS98], [Can01], [PW01]). Namely, the
implementation must “look like” the specification from the
perspective of every possible environment. The precise
notion of implementation is formulated in terms of inclusion
of sets of trace distributions for each environment.

An advantage of this style of definition is that it yields simple
compositionality results (Theorem 2.9).

**Definition 2.8:** Let \(T_1\) and \(T_2\) be comparable action-
deterministic task-PIOAs, that is, \(I_1 = I_2\) and \(O_1 = O_2\).
We say that \(T_1\) implements \(T_2\), written \(T_1 \preceq T_2\), if
\(\text{extbeh}(T_1)(E) \subseteq \text{extbeh}(T_2)(E)\) for every environment \(E\)
for both \(T_1\) and \(T_2\). In other words, we require \(\text{tdists}(T_1 || E) \subseteq
\text{tdists}(T_2 || E)\) for every \(E\).

The subscript 0 in the relation symbol \(\preceq\) refers to the
requirement that every trace distribution in \(\text{tdists}(T_1 || E)\)
must have an identical match in \(\text{tdists}(T_2 || E)\). For security
analysis, we also define another relation \(\preceq_{\text{neg,plt}}\), which
allows “negligible” discrepancies between matching trace
distributions [CC+06a].

**D. Compositionality**

Because external behavior and implementation are defined in
terms of mappings from environments to sets of trace
distributions, a compositionality result for \(\preceq\) follows easily:

**Theorem 2.9:** Let \(T_1\) and \(T_2\) be comparable action-
deterministic task-PIOAs such that \(T_1 \preceq T_2\), and let
\(T_3\) be an action-deterministic task-PIOA compatible with
each of \(T_1\) and \(T_2\). Then \(T_1 || T_3 \preceq T_2 || T_3\).

**Proof.** Let \(T_4 = (P_4, R_4)\) be any environment (action-
deterministic) task-PIOA for both \(T_1 || T_3\) and \(T_2 || T_3\). Fix
any task schedule \(\rho_1\) for \(T_1 || T_3\). Let \(\tau\) be the trace distribu-
tion of \(T_1 || T_3\) generated by \(\rho_1\). It suffices to show that \(\tau\) is also generated by some task schedule \(\rho_2\)
for \(T_2 || T_3\). Note that \(\rho_1\) is also a task schedule for \(T_1 || T_3\),
and that \(\rho_1\) generates the same trace distribution \(\tau\) in the
composed task-PIOA \(T_1 || (T_3 || T_4)\).

Now, \(T_3 || T_4\) is an (action-deterministic) environment task-
PIOA for each of \(T_1\) and \(T_2\). Since, by assumption, \(T_1 \preceq T_2\), we
infer the existence of a task schedule \(\rho_2\) for \(T_2 || (T_3 || T_4)\)
such that \(\rho_2\) generates trace distribution \(\tau\) in the task-PIOA
\(T_2 || (T_3 || T_4)\). Since \(\rho_2\) is also a task schedule for \(T_2 || T_3\)\\(T_4\)
and \(\rho_2\) generates \(\tau\), this suffices.

**III. SIMULATION RELATIONS**

We define a new simulation relation notion for closed,
action-deterministic task-PIOAs, and show that it is sound
for proving \(\preceq_{\text{neg,plt}}\). Our definition is based on three operations
involving probability measures: flattening, lifting, and expan-
sion. These have been previously defined, e.g., in [LSV03].

**A. Flattening, lifting, and expansion**

The flattening operation takes a discrete probability mea-
sure over probability measures and “flattens” it into a single
probability measure. Formally, let \(\eta\) be a discrete probability
measure on \(\text{Disc}(X)\). Then the flattening of \(\eta\), denoted by
\(\text{flatten}(\eta)\), is the discrete probability measure on \(X\) defined
by \(\text{flatten}(\eta) = \sum_{\mu \in \text{Disc}(X)} \eta(\mu)\mu\).

The lifting operation takes a relation \(R\) between two
domains \(X\) and \(Y\) and “lifts” it to a relation between discrete
measures over $X$ and $Y$. Informally speaking, a measure $\mu_1$ on $X$ is related to a measure $\mu_2$ on $Y$ if $\mu_2$ can be obtained by “redistributing” the probability masses assigned by $\mu_1$, in such a way that the relation $R$ is respected. Formally, the lifting of $R$, denoted by $\mathcal{L}(R)$, is a relation from $\text{Disc}(X)$ to $\text{Disc}(Y)$ defined by: $\mu_1 \mathcal{L}(R) \mu_2$ iff there exists a weighting function $w : X \times Y \rightarrow \mathbb{R}^{\geq 0}$ such that

1. For each $x \in X$ and $y \in Y$, $w(x,y) > 0$ implies $x R y$.
2. For each $x \in X$, $\sum_y w(x,y) = \mu_1(x)$.
3. For each $y \in Y$, $\sum_x w(x,y) = \mu_2(y)$.

Finally, the expansion operation takes a relation between discrete measures on two domains and returns a relation of the same kind that relates two measures whenever they can be decomposed into two $\mathcal{L}(R)$-related measures. Formally, let $R$ be a relation from $\text{Disc}(X)$ to $\text{Disc}(Y)$. The expansion of $R$, written $\mathcal{E}(R)$, is a relation from $\text{Disc}(X)$ to $\text{Disc}(Y)$ defined by: $\mu_1 \mathcal{E}(R) \mu_2$ iff there exist two discrete measures $\eta_1$ and $\eta_2$ on $\text{Disc}(X)$ and $\text{Disc}(Y)$, respectively, such that $\mu_1 = \text{flatten}(\eta_1)$, $\mu_2 = \text{flatten}(\eta_2)$, and $\eta_1 \mathcal{L}(R) \eta_2$.

We use expansions directly in our definition of simulation. Formally, $\mu_1 \mathcal{R} \mu_2$ means that it is possible to simulate from $\mu_2$ anything that can happen from $\mu_1$. Furthermore, $\mu_1 \mathcal{E}(R) \mu_2$ means that we can decompose $\mu_1$ and $\mu_2$ into pieces that can simulate each other, and so we can say that it is also possible to simulate from $\mu_2$ anything that can happen from $\mu_1$. This intuition is at the base of the proof of our soundness result (cf. Theorem 3.5).

B. Simulation relation definition

We need two more auxiliary definitions. The first expresses consistency between a probability measure over finite executions and a task schedule: informally, a measure $\epsilon$ over finite executions is said to be consistent with a task schedule $\rho$ if it assigns non-zero probability only to those executions that are possible under the task schedule $\rho$. We use this condition in order to avoid useless proof obligations in our definition of simulation relation.

Definition 3.1: Let $T = (\mathcal{P}, R)$ be a closed, action-deterministic task-PIOA, $\epsilon$ a discrete probability measure over finite executions of $\mathcal{P}$, and $\rho$ a finite task schedule for $T$. Then $\epsilon$ is consistent with $\rho$ provided that $\text{supp}(\epsilon) \subseteq \text{supp}(\text{apply}(\delta(\bar{q}), \rho))$, where $\bar{q}$ is the start state of $\mathcal{P}$.

For the second definition, suppose we have a mapping $c$ that, given a finite task schedule $\rho$ and a task $T$ of a task-PIOA $T_1$, yields a task schedule of another task-PIOA $T_2$.

Definition 3.2: Let $T_1 = (\mathcal{P}_1, R_1)$ and $T_2 = (\mathcal{P}_2, R_2)$ be two task-PIOAs, and let $c : (R_1^* \times R_1) \rightarrow R_2^*$ be a function that assigns a finite task schedule of $T_2$ to each finite task schedule of $T_1$ and task of $T_1$. Define full($c$) : $R_1^* \rightarrow R_2^*$ recursively as follows: full($c$)($\lambda$) := $\lambda$, and full($c$)($\rho T$) := full($c$)($\rho$) $\cup$ c($\rho, T$) (the concatenation of full($c$)($\rho$) and c($\rho, T$)).

We can now define our new notion of simulation for task-PIOAs and establish its soundness with respect to the $\leq_0$ relation. Note that our simulation relations do not just relate states to states, but rather, probability measures on executions to probability measures on executions. The use of measures on executions here rather than just executions is motivated by certain cases that arise in our OT protocol proof, e.g., cases where related random choices are made at different points in the low-level and high-level models (see Section III-D).

Definition 3.3: Let $T_1 = (\mathcal{P}_1, R_1)$ and $T_2 = (\mathcal{P}_2, R_2)$ be two comparable closed action-deterministic task-PIOAs. Let $R$ be a relation from $\text{Disc}(\text{Execs}^* (\mathcal{P}_1))$ to $\text{Disc}(\text{Execs}^* (\mathcal{P}_2))$, such that, if $\epsilon_1 R \epsilon_2$, then $\text{tdist}(\epsilon_1) = \text{tdist}(\epsilon_2)$. Then $R$ is a simulation from $T_1$ to $T_2$ if there exists $c : (R_1^* \times R_1) \rightarrow R_2^*$ such that the following properties hold:

1. Start condition: $\delta(\bar{q}_1) R \delta(\bar{q}_2)$.
2. Step condition: if $\epsilon_1 R \epsilon_2$, $\rho_1 \in R_1^*$, $\epsilon_1$ is consistent with $\rho_1$, $\epsilon_2$ is consistent with full($c$)($\rho_1$), and $T \in R_1$, then $\epsilon'_1 \in \mathcal{E}(R) \epsilon'_2$ where $\epsilon'_1 = \text{apply}(\epsilon_1, T)$ and $\epsilon'_2 = \text{apply}(\epsilon_2, c(\rho_1, T))$.

C. Soundness

Lemma 3.4: Let $T_1$ and $T_2$ be comparable closed action-deterministic task-PIOAs, $R$ a simulation from $T_1$ to $T_2$. Let $\epsilon_1$ and $\epsilon_2$ be discrete probability measures over finite executions of $T_1$ and $T_2$, respectively, such that $\epsilon_1 \mathcal{E}(R) \epsilon_2$. Then $\text{tdist}(\epsilon_1) = \text{tdist}(\epsilon_2)$.

The following theorem says that, for closed task-PIOAs, the existence of a simulation relation implies inclusion of sets of trace distributions. Our main soundness result for (not necessarily closed) task-PIOAs then follows as a corollary.

Theorem 3.5: Let $T_1$ and $T_2$ be comparable closed action-deterministic task-PIOAs. If there exists a simulation relation from $T_1$ to $T_2$, then $\text{tdists}(T_1) \subseteq \text{tdists}(T_2)$.

Proof. Let $R$ be the assumed simulation relation from $T_1$ to $T_2$. Let $\epsilon_1$ be the probabilistic execution of $T_1$ generated by $\bar{q}_1$ and a (finite or infinite) task schedule, $T_1, T_2, \ldots$. For each $i > 0$, define $\rho_i$ to be $c(T_1 \cdots T_{i-1}, T_1)$. Let $\epsilon_2$ be the probabilistic execution generated by $\bar{q}_2$ and $\rho_1 \rho_2 \cdots$. We claim that $\text{tdist}(\epsilon_1) = \text{tdist}(\epsilon_2)$, which suffices.

For each $j \geq 0$, let $\epsilon_{1,j} = \text{apply}(\bar{q}_1, T_1 \cdots T_j)$, and $\epsilon_{2,j} = \text{apply}(\bar{q}_2, \rho_1 \cdots \rho_j)$. Then for each $j \geq 0$, $\epsilon_{1,j} \leq \epsilon_{1,j+1}$ and $\epsilon_{2,j} \leq \epsilon_{2,j+1}$; moreover, $\lim_{j \rightarrow \infty} \epsilon_{1,j} = \lim_{j \rightarrow \infty} \epsilon_{2,j} = \epsilon_2$. Also, for every $j \geq 0$, $\text{apply}(\epsilon_{1,j}, T_{j+1}) = \epsilon_{1,j+1}$ and $\text{apply}(\epsilon_{2,j}, T_{j+1}) = \epsilon_{2,j+1}$. Observe that $\epsilon_{1,0} = \delta(\bar{q}_1)$ and $\epsilon_{2,0} = \delta(\bar{q}_2)$. The start condition for a simulation relation and a trivial expansion imply that $\epsilon_{1,0} \mathcal{E}(R) \epsilon_{2,0}$. Then by induction, using the definition of a simulation relation in proving the inductive step (this uses a series of lemmas; see [CC+06b] for details), we show that, for each $j \geq 0$, $\epsilon_{1,j} \mathcal{E}(R) \epsilon_{2,j}$. Then, by Lemma 3.4, for each $j \geq 0$, $\text{tdist}(\epsilon_{1,j}) = \text{tdist}(\epsilon_{2,j})$. Now, $\text{tdist}(\epsilon_1) = \lim_{j \rightarrow \infty} \text{tdist}(\epsilon_{1,j})$, and $\text{tdist}(\epsilon_2) = \lim_{j \rightarrow \infty} \text{tdist}(\epsilon_{2,j})$. Since for each $j \geq 0$, $\text{tdist}(\epsilon_{1,j}) = \text{tdist}(\epsilon_{2,j})$, we conclude that $\text{tdist}(\epsilon_1) = \text{tdist}(\epsilon_2)$, as needed. □

Corollary 3.6: Let $T_1$ and $T_2$ be two comparable action-deterministic task-PIOAs. Suppose that, for every environment $\mathcal{E}$ for both $T_1$ and $T_2$, there exists a simulation relation $R$ from $T_1[|\mathcal{E}]$ to $T_2[|\mathcal{E}]$. Then $T_1 \leq_0 T_2$.

1 It would be nice to simplify this definition so that it involves measures on states instead of measures on executions, but we don’t yet know how to do this.
D. Example: Trapdoor vs. Rand

The following example, from our OT proof, was a key motivation for generalizing prior notions of simulation relations. We consider two closed task-PIOAs, Trapdoor and Rand. Rand chooses a number randomly and outputs it. Trapdoor, on the other hand, first chooses a random number, then applies a known permutation to the chosen number, and then outputs the result. (The name Trapdoor refers to the type of permutation that is used in the OT protocol.)

More precisely, Rand has output actions report(k), k ∈ [n] = {1, ..., n} and internal action choose. It has tasks Report = {report(k) : k ∈ [n]}, and Choose = {choose}. Its state contains one variable zval, which assumes values in [n] ∪ {⊥}, initially ⊥. The choose action is enabled when zval = ⊥, and has the effect of setting zval to a number in [n], chosen uniformly at random. The report(k) action is enabled when zval = k, and has no effect on the state (so it may happen repeatedly). See Figure 1.

\[
\begin{array}{c}
\text{choose} \\
\text{\hspace{1em} z = 1 \quad \text{report(1)}} \\
\text{\hspace{1em} z = 2 \quad \text{report(2)}} \\
\text{\hspace{1em} \vdots} \\
\text{\hspace{1em} z = n \quad \text{report(n)}} \\
\end{array}
\]

Fig. 1. Task-PIOA Rand

Trapdoor has the same actions as Rand, plus internal action compute. It has the same tasks as Rand, plus the task Compute = {compute}. Trapdoor’s state contains two variables, y and z, each of which takes on values in [n] ∪ {⊥}, initially ⊥. The choose action is enabled when y = ⊥, and sets y to a number in [n], chosen uniformly at random. The compute action is enabled when y ≠ ⊥ and z = ⊥, and sets z := f(y). The report(k) action behaves exactly as in Rand. See Figure 2.

\[
\begin{array}{c}
\text{choose} \\
\text{\hspace{1em} y = 1 \quad \text{compute} \quad z = f(1) \quad \text{reportf(1)}} \\
\text{\hspace{1em} y = 2 \quad \text{compute} \quad z = f(2) \quad \text{reportf(2)}} \\
\text{\hspace{1em} \vdots} \\
\text{\hspace{1em} y = n \quad \text{compute} \quad z = f(n) \quad \text{reportf(n)}} \\
\end{array}
\]

Fig. 2. Task-PIOA Trapdoor

We wanted to use a simulation relation to prove that \(\text{tdists}(\text{Trapdoor}) \subseteq \text{tdists}(\text{Random})\). In doing so, we decided that the steps that define z should correspond in the two automata, which meant that the choose steps of Trapdoor, which define y, should map to no steps of Rand. Then, between the choose and compute in Trapdoor, a randomly-chosen value would appear in the y component of Trapdoor’s state, but no such value would appear in the corresponding state of Rand. Therefore, the simulation relation would have to relate a probability measure on states of Trapdoor to a single state of Rand.

We were able to express this correspondence using a simulation relation of our new kind: If \(\epsilon_1\) and \(\epsilon_2\) are discrete measures over finite execution fragments of Trapdoor and Rand, respectively, then we defined \((\epsilon_1, \epsilon_2) \in R\) exactly if the following conditions hold: (i) For every \(s \in \supp(\text{state}(\epsilon_1))\) and \(u \in \supp(\text{state}(\epsilon_2))\), \(s.z = u.z\). (ii) For every \(u \in \supp(\text{state}(\epsilon_2))\), if \(u.z = ⊥\) then either \text{state}(\epsilon_1).y is everywhere undefined or else it is the uniform distribution on \([n]\). The task correspondence mapping \(c\) is defined by: \(c(p, \text{Choose}) = λ, c(p, \text{Compute}) = \text{Choose}, c(p, \text{Report}) = \text{Report}\).

IV. Application to Security Protocols

In [CC+06a], we use the task-PIOAs of this paper to model and analyze the Oblivious Transfer (OT) protocol of Goldreich et al. [GMW87]. In the OT problem, two input bits \((x_0, x_1)\) are submitted to a Transmitter Trans and a single input bit \(i\) to a Receiver Rec. After engaging in an OT protocol, Rec should output only the single bit \(x_i\). Rec should not learn the other bit \(x_{1-i}\), and Trans should not learn \(i\); moreover, an eavesdropping adversary should not, by observing the protocol messages, be able to learn anything about the inputs or the progress of the protocol. OT has been shown to be “complete” for multi-party secure computation, in the sense that, using OT as the only cryptographic primitive, one can construct protocols for securely realizing any functionality.

The protocol of [GMW87] uses trap-door permutations (and hard-core predicates) as an underlying cryptographic primitive. It uses three rounds of communication: First, Trans chooses a random trap-door permutation \(f\) and sends it to Rec. Second, Rec chooses two random numbers \((y_0, y_1)\) and sends \((z_0, z_1)\) to Trans, where \(z_i\) for the input index \(i\) is \(f(y_i)\) and \(z_{1-i} = y_{1-i}\). Third, Trans applies the same transformation to each of \(z_0\) and \(z_1\) and sends the results back as \((h_0, b_1)\). Finally, Rec decodes and outputs the correct bit. The protocol uses cryptographic primitives and computational hardness in an essential way. Its security is inherently only computational, so its analysis requires modeling computational assumptions.

Our analysis follows the trusted party paradigm of [GMW87], with a formalization that is close in spirit to [PW00], [Can01]. We first define task-PIOAs representing the real system (RS) (the protocol) and the ideal system (IS) (the requirements). In RS, typical tasks include “choose random \((y_0, y_1)\)”, “send round 1 message”, and “deliver round 1 message”, as well as arbitrary tasks of incompletely-specified environment and adversary automata. Note that these tasks do not specify exactly what transition occurs; e.g., the send task does not specify the message contents—these are chosen by Trans, based on its own internal state.

Then we prove that RS implements IS. The proof consists of four cases, depending on which parties are corrupted. In the two cases where Trans is corrupted, we can show that RS implements IS unconditionally, using \(≤_0\). In the cases where Trans is not corrupted, we can show implementation only in a “computational” sense, namely, (i) for resource-bounded adversaries, (ii) up to negligible differences, and

---

2Actually, in [CC+06a], we prove only one case—when only \(R\) is corrupted. We prove all four cases in [CC+05], but using a less general definition of task-PIOAs than the one used here and in [CC+06a], and with non-branching adversaries.
(iii) under computational hardness assumptions. Modeling these aspects requires additions to the task-PIOA framework of this paper, namely, defining a time-bounded version of task-PIOAs, and defining a variation, $\leq_{\text{neg.p.t.}}$, on the $\leq_0$ relation, which describes approximate implementation with respect to polynomial-time-bounded environments. Similar relations were defined in [LMMS98], [PW01]. Our simulation relations are also sound with respect to $\leq_{\text{neg.p.t.}}$. We also provide models for the cryptographic primitives (trap-door functions and hard-core predicates). Part of the specification for such primitives is that their behavior should look “approximately random” to outside observers; we formalize this in terms of $\leq_{\text{neg.p.t.}}$.

The correctness proofs proceed by levels of abstraction, relating each pair of models at successive levels using $\leq_{\text{neg.p.t.}}$. In the case where only $\text{Rec}$ is corrupted, all but one of the relationships between levels are proved using simulation relations as defined in this paper (and so, they guarantee $\leq_0$).

The only exception relates a level in which the cryptographic primitive is used, with a higher level in which the use of the primitive is replaced by a random choice. Showing this correspondence relies on our $\leq_{\text{neg.p.t.}}$-based definitions of the cryptographic primitive, and on composition results for time-bounded task-PIOAs. Since this type of reasoning is isolated to one correspondence, the methods of this paper in fact suffice to accomplish most of the work of verifying OT.

Each of our system models, at each level, includes an explicit adversary component automaton, which acts as a message delivery service that can eavesdrop on communications and control the order of message delivery. The behavior of this adversary is arbitrary, subject to general constraints on its capabilities. In our models, the adversary is the same at all levels, so our simulation relations relate the adversary states at consecutive levels directly, using the identity function.

This treatment allows us to consider arbitrary adversaries without examining their structure in detail (they can do anything, but must do the same thing at all levels).

Certain patterns that arise in our simulation relation proofs led us to extend earlier definitions of simulation relations [SL95], [LSV03], by adding the expansion capability and by corresponding measures to measures: (i) We often correspond random choices at two levels of abstraction—for instance, when the adversary makes a random choice, from the same state, at both levels. We would like our simulation relation to relate the individual outcomes of the choices at the two levels, matching up the states in which the same result is obtained. Modeling this correspondence uses the expansion feature. (ii) The Trapdoor vs. Random example described in Section III occurs in our OT proof. Here, the low-level system chooses a random $y$ and then computes $z = f(y)$ using a trap-door permutation $f$. The higher level system simply chooses the value of $z$ randomly, without using value $y$ or permutation $f$. This correspondence relates measures to measures and uses expansion. (iii) In another case, a lower-level system chooses a random value $y$ and then computes a new value by XOR’ing $y$ with an input value. The higher level system just chooses a random value. However, XOR’ing any value with a random value yields the same result as just choosing a random value. This correspondence relates measures to measures and uses expansion.

V. Local Schedulers

With the action-determinism assumption, our task mechanism is enough to resolve all nondeterminism. However, action determinism limits expressive power. Now we remove this assumption and add a second mechanism for resolving the resulting additional nondeterminism, namely, a local scheduler for each component task-PIOA. A local scheduler for a given component can be used to resolve nondeterministic choices among actions in the same task, using only information about the past history of that component. Here, we define one type of local scheduler, which uses only the current state, and indicate how our results for the action-deterministic case carry over to this setting.

Our notion of local scheduler is simply a “sub-automaton”: We say that task-PIOA $T' = (P', R')$ is a sub-task-PIOA of task-PIOA $T = (P, R)$ provided that all components are identical except that $D' \subseteq D$, where $D$ and $D'$ are the sets of discrete transitions of $P$ and $P'$, respectively. Thus, the only difference is that $T'$ may have a smaller set of transitions. A local scheduler for a task-PIOA $T$ is any action-deterministic sub-task-PIOA of $T$. A probabilistic system is a pair $M = (T, S)$, where $T$ is a task-PIOA and $S$ is a set of local schedulers for $T$. A probabilistic execution of a probabilistic system $M = (T, S)$ is defined to be any probabilistic execution of any task-PIOA $S \in S$.

If $M_1 = (T_1, S_1)$ and $M_2 = (T_2, S_2)$ are two probabilistic systems, and $T_1$ and $T_2$ are compatible, then their composition $M_1 || M_2$ is the probabilistic system $(T_1 || T_2, S)$, where $S$ is the set of local schedulers for $T_1 || T_2$ of the form $S_1 || S_2$, for some $S_1 \in S_1$ and $S_2 \in S_2$.

If $M = (T, S)$ is a probabilistic system, then an environment for $M$ is any environment (action-deterministic task-PIOA) for $T$. If $M = (T, S)$ is a probabilistic system, then the external behavior of $M$, $\text{extbeh}(M)$, is the total function that maps each environment task-PIOA $E$ for $M$ to the set $\bigcup_{S \in S} \text{tdists}(S[|E|])$. Thus, for each environment, we consider the set of trace distributions that arise from two choices: of a local scheduler of $M$ and of a global task schedule $\rho$. If $M_1 = (T_1, S_1)$ and $M_2 = (T_2, S_2)$ are comparable probabilistic systems (i.e., $T_1$ and $T_2$ are comparable), then $M_1$ implements $M_2$, written $M_1 \leq_0 M_2$, provided that $\text{extbeh}(M_1)(E) \subseteq \text{extbeh}(M_2)(E)$ for every environment (action-deterministic) task-PIOA $E$ for both $M_1$ and $M_2$. We obtain a sufficient condition for implementation of probabilistic systems, in which each local scheduler for the low-level system always corresponds to the same local scheduler of the high-level system.

Theorem 5.1: Let $M_1 = (T_1, S_1)$ and $M_2 = (T_2, S_2)$ be two comparable probabilistic systems. Suppose that $f$ is a total function from $S_1$ to $S_2$ such that, for every $S_1 \in S_1$, $S_1 \leq f(S_1)$. Then $M_1 \leq M_2$.

We also obtain a compositionality result for probabilistic systems. The proof is similar to that of Theorem 2.9, for the action-deterministic case.

Theorem 5.2: Let $M_1$, $M_2$ be comparable probabilistic systems such that $M_1 \leq_0 M_2$, and let $M_3$ be a probabilistic system compatible with each of $M_1$ and $M_2$. Then $M_1 || M_3 \leq_0 M_2 || M_3$. 
VI. CONCLUSIONS

We have extended the traditional PIOA model with a task mechanism, which provides a systematic way of resolving nondeterministic scheduling choices without using information about past history. We have provided basic machinery for using the resulting task-PIOA framework for verification, including a compositional trace-based semantics and a new kind of simulation relation. We have proposed extending the framework to allow additional nondeterminism, resolved by schedulers that use only local information. We have illustrated the utility of these tools with a case study involving analysis of an Oblivious Transfer cryptographic protocol.

Although our development was motivated by concerns of cryptographic protocol analysis, partial-information scheduling is interesting in other settings. For example, some distributed algorithms work with partial-information adversarial schedulers, although the problems they solve are provably unsolvable with perfect-information adversaries [Cha96], [Asp03]. Also, partial-information scheduling is realistic for modeling large distributed systems, in which basic scheduling decisions are made locally, and not by any centralized mechanism.

Many questions remain in our study of task-PIOAs: Our notion of implementation, \( \leq_s \), is defined by considering all environments; can we characterize \( \leq_s \) using a small subclass of environments? Can our simulation relation notion be simplified without sacrificing soundness or applicability? Also, our notion of local schedulers needs further development.

It remains to consider more applications of task-PIOAs, for cryptographic protocol analysis and for other applications. A next step in cryptographic protocol analysis is to formulate and prove protocol composition results like those of [PW00], [Can01] in terms of task-PIOAs. Finally, we would like to model perfect-information schedulers, as used for analyzing randomized distributed algorithms, using task-PIOAs.

Acknowledgments: We thank Frits Vaandrager for collaboration in early stages of this project, and Michael Backes, Anupam Datta, Ralf Kuesters, John Mitchell, Birgit Pfitzmann, and Andre Scedrov for technical discussions that helped us in clarifying our ideas and their connections to other work in analysis of cryptographic protocols. We thank Silvio Micali for impressing upon us the importance of adaptive adversarial schedulers in the cryptographic setting. We thank Sayan Mitra both for technical discussions and for help in producing the paper.

REFERENCES


