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The nature of magnetic fields and the flux quantum

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Abstract. The understanding of electromagnetic phenomena is based on the historic landmark *A Treatise on Electricity & Magnetism* of 1873 by James Clerk Maxwell. The form in which the theory is most commonly expressed is in the form of the vector equations introduced by Gibbs.

There is another more general formulation due to Élie Cartan, in terms of external differential forms. We will develop the description of electromagnetic phenomena using these concepts. It is a pre-metric description of conservation laws, that will highlight the basic properties of the electromagnetic phenomena. We will in particular demonstrate the fundamental role of the flux quantum $\Phi_0 = h/2e$, and point to ways to calibrate magnetic fields.

1. Introduction

The endeavors of Michael Faraday and James Clerk Maxwell and many other brilliant scientists, culminated in the comprehensive description of electromagnetic phenomena in the historic landmark *A Treatise on Electricity & Magnetism* published in 1873 [1].

Already before the turn of the century people started realizing that the Maxwell equations were not invariant for Galilean transformations appropriate in Euclidean geometry but rather “Lorentz-invariant”, with the proposed Fitzgerald-Lorentz contraction to explain the constancy of the speed of light observed in the Michelson-Morley experiment.

We now understand that they obey the rules of special relativity as so masterly explained to us by Albert Einstein, and electromagnetic phenomena have to be considered as events in space-time, or Minkowski space. Yet, in the laboratory we tend to take a pragmatic point of view and we consider magnetic fields (that we generate at considerable costs) as entities by themselves, not always aware of their relativistic character. In this paper we will give some food for thought regarding the nature of magnetic fields.

2. Maxwell’s approach to the description of electromagnetic phenomena

The equations of electromagnetic theory by Maxwell were written out in all their components and derivatives, that is unlike the formulation used in modern textbooks which is based on the vector algebra developed by J.W. Gibbs in the 1880’s (see the introduction in Ref. [2]). The vector notation allows a compact formulation; the Maxwell-Faraday equations:

$$\begin{aligned}\nabla \times \mathbf{E} &= -\partial\mathbf{B}/\partial t \\ \nabla \cdot \mathbf{B} &= 0\end{aligned}$$

and the Maxwell-Ampère equations:

$$\begin{aligned}\nabla \times \mathbf{H} &= \partial \mathbf{D} / \partial t + \mathbf{j} \\ \nabla \cdot \mathbf{D} &= \rho\end{aligned}$$

The Maxwell equations are Lorentz invariant, meaning that they can only be understood in the context of special relativity. The Maxwell equations also determine most of what we observe: As Richard Feynman has stated in his famous lectures on Quantum Electrodynamics at UCLA “Most of the phenomena you are familiar with involve the interaction of light and electrons - all of chemistry and biology, for example. The only phenomena that are not covered by this theory are phenomena of gravitation and nuclear phenomena; everything else is contained in this theory.” [3].

That we do experience that the universe and our close environment obey special relativity is therefore no surprise: the interaction of physical entities is basically through electromagnetic forces (interaction through electromagnetic waves, interaction of electrons with the nucleus and other atoms, even the resistance offered by “solid objects” when touched, and the behaviour of gases, liquids and solids). In short: the “world” around us is experienced through electromagnetic interactions, which do obey special relativity.

3. A pre-metric formulation of the theory of electro-magnetism

While Maxwell was completing his chef-d’oeuvre, Grassmann developed geometrical analysis, also around 1850, and Hamilton discovered the quaternions, four-dimensional complex numbers. Later, in the early 1900’s, Cartan developed the theory of differential forms based on the outer product of the Grassmann algebra [4]. The formulation of physics laws using exterior differential forms can be found in the famous book *Gravitation* from 1973 [5]; in particular, chapter 4 of that book and a more recent publication [2] elucidate the rules of using the operators. In the language of differential forms, the Maxwell equations become very compact:

$$\begin{aligned}\mathbf{d} \text{Faraday} &= 0 \\ \mathbf{d} \text{Maxwell} &= \mathbf{j}\end{aligned}$$

The **Faraday** and **Maxwell** are antisymmetric second rank tensors, so-called 2-forms, where **Faraday** = $\mathbf{F} = \frac{1}{2} F_{\alpha\beta} \mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta$, and its dual $\star \mathbf{F} = \mathbf{Maxwell} = \frac{1}{2} M_{\mu\nu} \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu$; the symbol \wedge is the wedge product, quite similar to the familiar outer product in \mathbb{R}^3 , \star is the hodge operator (which acting on a p -form will result in its dual, a $(n - p)$ -form with $n = 4$ for space-time), and \mathbf{d} the gradient operator (which acting on a p -form will result in a $(p + 1)$ -form, and to zero when acting on a 4-form). The concepts of form and exterior derivative are metric-free [5], and the formulation is therefore pre-metric.

The entities that are important are the 4-mass \mathbf{m} and 4-charge \mathbf{q} (4-forms), the mass-flow \mathbf{s} and charge-flow \mathbf{j} (3-forms) and electromagnetic field **Maxwell** (2-form), and their inductions **Faraday** = $\star \mathbf{Maxwell}$ (2-form), the kinetic moment $\mathbf{p} = \star \mathbf{s}$ and current $\mathbf{l} = \star \mathbf{j}$ (1-forms), and the scalar mass and charge fields $m = \star \mathbf{m}$ and $q = \star \mathbf{q}$.

4. Magnetic fields in space-time

The Universe has curved space due to gravitational forces, and the differential forms allow formulation of the physics laws in curved space; around a static spherically symmetric center of attraction of heavy mass m the appropriate choice of basis vectors would be: $\omega^0 = \sqrt{1 - 2m/r} \mathbf{d}t$, $\omega^1 = (1/\sqrt{1 - 2m/r}) \mathbf{d}r$, $\omega^2 = r \mathbf{d}\theta$ and $\omega^3 = r \sin(\theta) \mathbf{d}\phi$ [5]. Locally, however, space-time is flat and consequently special relativity is valid, an appropriate set of basis vectors is then: $\mathbf{e}^0 = \mathbf{d}t$ (or rather $\mathbf{e}^0 = c \mathbf{d}t$), $\mathbf{e}^1 = \mathbf{d}x$, $\mathbf{e}^2 = \mathbf{d}y$ and $\mathbf{e}^3 = \mathbf{d}z$.

The **Faraday** has 6 independent components: $\mathbf{F} = B_x \mathbf{d}y \wedge \mathbf{d}z + B_y \mathbf{d}z \wedge \mathbf{d}x + B_z \mathbf{d}x \wedge \mathbf{d}y - E_x \mathbf{d}t \wedge \mathbf{d}x - E_y \mathbf{d}t \wedge \mathbf{d}y - E_z \mathbf{d}t \wedge \mathbf{d}z$, the magnetic term is space-like, the electric time-like. This demonstrates the unity of electric and magnetic fields. Neither one by itself, \vec{E} or \vec{B} , is a frame-independent, geometric entity. But merged into a single entity, **Faraday**, they acquire a meaning and significance that transcends coordinates and reference frames (Ref. [5] chapter 3). Remarkably in the **Maxwell** = $D_x \mathbf{d}y \wedge \mathbf{d}z + D_y \mathbf{d}z \wedge \mathbf{d}x + D_z \mathbf{d}x \wedge \mathbf{d}y + H_x \mathbf{d}t \wedge \mathbf{d}x + H_y \mathbf{d}t \wedge \mathbf{d}y + H_z \mathbf{d}t \wedge \mathbf{d}z$ the magnetic term is time-like, and the electric term space-like.

5. Lorentz invariance and construction of the metric

Now consider the Proca Lagrangian **Proca** = **Faraday** \wedge **Maxwell**. The expression $\mathbf{F} \wedge \mathbf{M}$ evaluates to $(\vec{B} \cdot \vec{H} - \vec{E} \cdot \vec{D}) \mathbf{d}t \wedge \mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z$. The Lorentz invariant scalar may be simplified for the vacuum: $(\vec{B} \cdot \vec{H} - \vec{E} \cdot \vec{D}) = (\vec{B} \vec{\mu}^{-1} \vec{B} - \vec{E} \vec{\epsilon} \vec{E}) \approx (B^2 - \mu_0 \epsilon_0 E^2) / \mu_0 = (B^2 - c^2 E^2) / \mu_0$.

The exterior derivative of the action $\mathbf{F} \wedge \mathbf{M} = F_{\alpha\beta} M_{\mu\nu} \mathbf{d}\Omega$ is trivially zero in 4-dimensional space-time. For $\mathbf{j} \neq 0$ this means that there is a preferred direction (usually taken as the time) and that $\det(\vec{F})$ should be zero. This is equivalent to the statement that only events can be observed on the light-cone, and hence special relativity is implied. *Lorentz invariance is pre-metric and topological in character.*

The metric can be constructed by experiment: one can measure distance by sending a beam of light (through either flat or curved space-time) and wait for the reflection after some time Δt , the distance is then defined as $r = \frac{1}{2} c \Delta t$ and simultaneity can be defined as $r^2 / c^2 - t^2 = 0$, or more generally $c^2 t^2 - r^2 = \mathbf{r} \vec{g} \mathbf{r} = 0$ with the familiar Lorentz metric \vec{g} .

It can be shown that the conservation laws $\mathbf{d}\mathbf{m}=0$ and $\mathbf{d}\mathbf{q}=0$ lead to basic laws of physics such as Newton's law, conservation of angular momentum, fluid dynamics and solid mechanics and to the Maxwell equations: mechanics and electromagnetic theory are very strongly intertwined.

According to the Poincaré lemma, when $\mathbf{d}\mathbf{m}=0$ there exists a 3-form \mathbf{P} , the canonical momentum ($\mathbf{P} = \Omega \mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z + P_x \mathbf{d}t \wedge \mathbf{d}y \wedge \mathbf{d}z + P_y \mathbf{d}t \wedge \mathbf{d}z \wedge \mathbf{d}x + P_z \mathbf{d}t \wedge \mathbf{d}x \wedge \mathbf{d}y$). Likewise when $\mathbf{d}\mathbf{q}=0$ there exists a 3-form \mathbf{A} , the electromagnetic momentum, associated with the familiar vector potential \vec{A} ($\mathbf{A} = \Phi \mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z + A_x \mathbf{d}t \wedge \mathbf{d}y \wedge \mathbf{d}z + A_y \mathbf{d}t \wedge \mathbf{d}z \wedge \mathbf{d}x + A_z \mathbf{d}t \wedge \mathbf{d}x \wedge \mathbf{d}y$). This illustrates that one should distinguish canonical momentum and kinetic momentum. The unit of momentum is the Planck constant h , and thus the unit of vector potential is $h/e \propto \Phi_0$.

Both \mathbf{P} and \mathbf{A} are similar to the mass-flow \mathbf{s} or charge-flow \mathbf{j} . The familiar minimal substitution $\mathbf{s} = (\mathbf{P} - \mathbf{A})$ is a freedom of choice, gauge invariance. In [6] we have elaborated a generalization of the gauge invariance and introduced $\star \mathbf{\Pi} = e \star (\mathbf{A} - \mathbf{j})$, which we coined the "Janner gauge". It shows that angular momentum and magnetic fields are very closely related.

6. Outlook

The present realization of the SI system of units is very well described in the paper by Kovalevsky and Quinn [8]. In particular they emphasize that when in the 19th Century the need appeared to introduce electromagnetic forces and express them in terms of the mechanical units (cm, g and s), inconsistencies appeared that were not resolved with the introduction suggested by Giorgi of the Ampère (or Ohm) as additional unit, as now implemented in the MKSA-system of units. The Metre and Second are both determined using the transition between atomic energy levels, which in their turn reflect Bohr quantisation, *i.e.* the conservation of flux quanta enclosed by the electron orbits. The Metre and Second are also intimately linked through the velocity of light which has been fixed with no uncertainty (leaving uncertainty in the duration of the unit of time). The Second is the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the ¹³³Cs atom, while the Metre is the length of the path travelled by light in vacuum during a time interval of $1/c=1/299,792,458$ of a second. The SI system of units is both inconsistent and overcomplete!

Some units have already been defined using fundamental constants, and there is a lot of discussion in the community of metrology to redefine the kilogram in terms of fundamental constants [9, 10]:

- (i) The Second and the Metre are based on quantum transitions on Bohr quantised orbits, *i.e.* defined in terms of the flux quantum Φ_0 ;
- (ii) The Ampère is defined using the Volt and the Ohm:
 - The Volt is defined by the Josephson effect, *i.e.* the flux quantum Φ_0 ,
 - The Ohm is defined by the Quantum Hall effect, *i.e.* the fine structure constant $\alpha \propto e/\Phi_0$.

To this date the Kilogram is an artefact, and equal to the mass of the international prototype; however, the levitation balance and Watt balance [11] clearly show that even heavy mass can be measured in terms of counting flux quanta Φ_0 .

We suggest that from the many possible combinations of fundamental constants suggested [12], the set flux quantum and electron charge $[\Phi_0, e]$ should be chosen as the (complete) basis of both the mechanical and electromagnetic units. There are only two quantum-mechanically independent fundamental constants since the Planck constant $h \propto e \times \Phi_0$ and the coupling constant $\alpha \propto e/\Phi_0$. There is a lot of research in Metrology world wide checking the constancy and consistency of these relations [9].

The flux quantum Φ_0 is a key unit in physics and the magnetic field describing its density is a mysterious bonding force of space-time. A way of looking at it is to imagine that electrons are localized real charges, a *local* quantity, and that their mutual interaction is a *global* connection. In the language of QED, this connection is exchange of photons, but we tend to believe it is more appropriate to imagine exchange of flux quanta, and that the magnetic flux quantum Φ_0 is a mechanical entity and a physical reality!

The subject of this article was the fundamental role of the flux quantum as a base unit for the SI system and the connection to the nature of the “Magnetic field”. We believe that indeed the time has come to redefine the kilogram, but then in terms of the flux quantum $\Phi_0 = h/2e$, and using measurements in magnetic fields.

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- [1] Maxwell J C 1873 *Treatise on Electricity and Magnetism*; reprint of 3rd edition of 1891, 1954 (Dover Publications, New York).
- [2] Lindell I V 2004 *Differential forms in Electromagnetics* (John Wiley & Sons Inc., Hoboken NJ).
- [3] Feynman R P 1985 *QED: the Strange Theory of Light and Matter*; reprint 1990 (Penguin Books, London).
- [4] Cartan H 1970 *Formes Différentielles* (Hermann, Paris).
- [5] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (W.H. Freeman and Company, New York).
- [6] Peters C M E E, Perenboom J A A J, van der Grinten J G M and Zijl W 2006 About electromagnetic and mechanics laws of physics I. Consequences for theoretical metrology, submitted for publication.
- [7] Goldstein H 1950 *Classical Mechanics*, 1950 (Addison Wesley) 12th edition p. 5-8 and p. 176-178.
- [8] Kovalevsky J and Quinn T J 2004 The international system of units (SI) *C.R. Physique* **5** 799-811.
- [9] Piquemal F, Bounouh A, Devoille L, Feltin N, Thevenot O and Tracon G 2004 Fundamental electrical standards and the quantum metrological triangle, *C.R. Physique* **5** 857-879.
- [10] Mills I A, Mohr P J, Quinn T J, Taylor B N, Williams E R 2005 Redefinition of the kilogram: a decision whose time has come *Metrologia* **42** 71-80.
- [11] Schwitz W, Jeckelmann B and Richard P 2004 Towards a new kilogram definition based on a fundamental constant *C.R. Physique* **5** 881-892.
- [12] Wilczek F 2005 On absolute units, I: Choices *Physics Today* **October 2005** 12-13; Wilczek F 2006 On absolute units, II: Challenges and responses *Physics Today* **January 2006** 10-11.