Copulas and Extreme Values

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Contents

Introduction

1 Insurance

Stochastics can be seen as the mathematical way to describe the uncertainties we face in life. This is not a book on sociology, but one can safely assume that most people are and have been risk-averse at least some of the time. A lot of human behaviour will very likely stem from this aversity to risk, but certainly the existence of insurances and even insurance companies are a proof of this. The way most of these insurances work, is by spreading the risks over a lot of people, so that all share the pain if one of them gets into bad luck. There are various ways to achieve this, for example by the commitment to help someone after bad luck has befallen her/him. Another way is if all participants fill a common reserve, from which bad luck is compensated to the victim. One of the properties of the second approach is that people have to contribute to the insurance even before bad luck has befallen (one of) them. This can be a disadvantage (i.e. paying without anything actually having gone wrong). But it can also be an advantage: If bad luck befalls one of them, she/he is fairly certain to receive compensation immediately. Another advantage of a common reserve over commitment to help is that the risks are not only spread over several people, but also over time. But with this second approach things also start to get a bit more difficult. Here we can get questions like "How large should this reserve be?" ("Is it solvent?") and "With what speed are we going to fill it?" ("Is it profitable?") .

The second question deals with the problem that in the long run the reserve should stay (at least partially) filled by regular donations by its participants. This question is not dealt with in this book, but I would like to mention that it is often troubled by matters like "If we ask less then 20 euro per month from our participants we will go broke in 30 years due to some future catastrophe, but if we ask more than 15 euro per month from our participants, most will join another insurance and we will go broke in 30 days.” .

In this book we try to give several mathematical approaches to the first question. It is not spread over time, like the other question, but more focussed on one point in time. What the participants in a common reserve typically want is that the reserve is large enough to compensate for bad luck if it occurs. Therefore we shall consider the maximum damage caused in a single stroke of bad luck, since we want to know how large the reserve has to be to compensate the damage in this situation. Obviously, if the reserve can ’survive’ this maximum damage, it can survive any damage.

This maximum damage can be caused in many ways. For instance if one of the participants suffers from severe misfortune, or if many participants suffer from bad luck at the same time, or both. But to answer the question with a number, we need to model the situation and then do some mathematics. Although some of the results in this book have a purely theoretical importance (for instance the Weibull-case), most of them can be used to help solve these problems for different models.

2 Modeling

In order to do mathematics, we need to translate reality into mathematical expressions. Concessions have to be made, but we try to incorporate as much relevant properties of ”bad luck” as we can handle mathematically. In this modeling we have to make some choices. In the following chapters we will try to explain some of the choices, but here we would like to defend the general ones. In this book we will typically look at a set (portfolio) of m random variables (risks). We take into account that they are dependent, although we will only consider certain kinds of dependency. At first we will look at dependency as given by Archimedean copulas (see 2.1), whereas later (in Chapter 3) we expand this to more general dependencies. But even then concessions were made, since there remain copulas (dependencies) that don’t fulfill the conditions of that chapter. We will also assume that the individual bad luck distributions are the same in most part of this book, although we loosen this constraint in Chapter 2. And lastly we only look at distributions that lie in one of the domains of attraction of the Fisher-Tippett Theorem. This is not as strange as it might seem, since we are looking at the maximum damage that might occur. In this book we look at extreme damages that might occur in a single period of time; e.g. a year. But the results could be used with the Fisher-Tippett Theorem to say something about the maximum damage that occurs over the course of many years. So the two main modeling choices are to take (Archimedean) copulas for the dependence structure and to take the individual distributions to be from one of the domains of attraction of the Fisher-Tippett Theorem. Therefore we will say a bit more about these two subjects.

2.1 Copulas

A way to describe the full dependence structure of dependent random variables is the so-called copula approach. Copulas are simply a convenient way to describe joint distributions of two or more random variables. They were introduced in the seminal paper by Sklar [25], who showed that every finite dimensional probability law has a copula function associated with it that describes the dependency of its marginal distributions. His ideas can be traced back to Fréchet, see e.g. [16]. We give the mathematical definition of a copula as well as an example below (standard copula literature is e.g. Joe [19] and Nelsen [23]). For an extensive discussion of copula methods the reader is referred to Dall'Aglio-Kotz-Salinetti's book [10], in particular Schweizer [24] therein.

The idea behind copulas is that the dependence structure of a finite family of random variables is completely determined by their joint distribution function.

Definition 2.1 *For $m \geq 2$. An m-dimensional copula is an m-dimensional distribution function on $[0,1]^m$, with marginals that are uniformly distributed on $[0,1]$.*

The concept of copulas is to separate a multivariate distribution function into two parts, one describing the dependence structure and the other one describing marginal behaviour. Moreover, as stated in Sklar's theorem, all distribution functions with continuous marginals have a copula associated with them and vice versa:

Theorem 2.2 (Sklar [25]) *For a given joint distribution function F with continuous marginals $F_1, \ldots, F_m$ there exists a unique copula C satisfying*

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_m(x_m)). \tag{2.1}$$

Conversely, for a given copula C and marginals $F_1, \ldots, F_m$ we have that (2.1) defines a distribution with marginals F_i.

Note that the copula of a random vector $(X_1, \ldots, X_m)$ is invariant under strictly increasing transformations of that vector (cf. Nelson [23]).

Just to give a feeling what copulas are about, we present here two very simple ones. First, the copula denoting independence:

Definition 2.3 (The product copula)

$$\Pi : [0,1]^m \to [0,1] \ : \ (x_1, \dots, x_m) \to \prod_{i=1}^{m} x_i \ . \tag{2.2}$$

The copula associated with complete positive dependence:

Definition 2.4 (The comonotone copula)

$$M : [0,1]^m \to [0,1] \ : \ (x_1, \dots, x_m) \to \min(\{x_1, \dots, x_m\}) \ . \tag{2.3}$$

It is clear that these functions have uniform marginals (i.e. $\Pi(x_1, 1, \dots, 1) = M(x_1, 1, \dots, 1) = x_1$) and one can see that Π is the distribution function of the uniform distribution on $[0,1]^m$, whereas M is the distribution function of the uniform distribution on the diagonal $x_1 = x_2 = \ldots = x_m$.

In this book we mainly focus on a special family of copulas, the Archimedean ones:

Definition 2.5 *Choose $m \geq 2$. Let $\phi : [0,1] \to [0,\infty]$ be strictly decreasing, convex and such that $\phi(0) = \infty$ and $\phi(1) = 0$. Define for $x_i \in [0,1], i = 1, \dots, m$:*

$$C^\phi(x_1, \dots, x_m) \stackrel{def.}{=} \phi^{-1}\left(\sum_{i=1}^{m} \phi(x_i)\right) . \tag{2.4}$$

The function ϕ is called generator *of C^ϕ.*

In the case $m = 2$ this definition automatically implies that C^ϕ is a copula. In the case $m \geq 3$, a further assumption is required for C^ϕ to be a copula: If for all k and $x > 0$ the $k-$th derivative of the inverse of $\phi, \frac{d^k}{dx^k}\phi^{-1}(x)$, exists and satsifies

$$(-1)^k \frac{d^k}{dx^k}\phi^{-1}(x) \geq 0, \tag{2.5}$$

then C^ϕ is a distribution function, and hence a copula (cf. [22]). Copulas of this type will be called (strict) Archimedean copulas. Except for Chapter 3, we restrict ourselves to Archimedean copulas. The following two thoughts lead to this choice. Firstly, it is much easier for a statistician to estimate an Archimedean copula than to estimate a general copula. This because one only

has to estimate the univariate generator instead of the multivariate copula. To improve the ability to estimate the copula further, we shall even restrict ourselves to generators ϕ that are regularly varying at 0^+ with index $-\alpha$.
We recall here the definition of regular variation (a standard reference on regular variation is Bingham-Goldie-Teugels [6]):

Definition 2.6 *A function f is called* regularly varying *at some point x^- (or x^+, respectively)* with index $\alpha \in \mathbb{R}$ *if for all $t > 0$*

$$\lim_{s \uparrow x} \frac{f(st)}{f(s)} = t^\alpha, \tag{2.6}$$

(or $\lim_{s \downarrow x} \frac{f(st)}{f(s)} = t^\alpha$, respectively).

The second thought leading to our choice for Archimedean copulas is that the set is large enough to contain some interesting copulas that are actually used by insurance companies. Look for instance at one of the best studied Archimedean copulas: the *Clayton copula* with parameter $\alpha > 0$. It is generated by $\phi(t) = t^{-\alpha} - 1$ and takes the form

$$C^{Cl,\alpha}(x_1, \ldots, x_m) \stackrel{def.}{=} (x_1^{-\alpha} + \ldots + x_m^{-\alpha} - m + 1)^{-1/\alpha}. \tag{2.7}$$

An interesting property of this distribution, is that the limit $\alpha \to 0$ leads to independence, while $\alpha \to \infty$ leads to comonotonicity, i.e. complete positive dependence. This holds for all Archimedean copulas we look at, but only near the origin.

For more examples we refer to Joe [19] and Nelsen [23].

2.2 Extreme values

While we use copulas to describe the dependencies between the random variables, we shall use marginal distributions inspired by extreme value theory. In this book we shall frequently use that the marginal distributions are of either Fréchet, Weibull or Gumbel type. This is motivated by the Fisher-Tippett theorem, a limit theorem for the distribution of the (weighted) maximum of a sequence of i.i.d. random variables. One can think of it as an analogy to the CLT, but with respect to the maximum of a sequence of instead of the sum. We restate it here without proof (it can be found in [13]).

Theorem 2.7 (Fisher-Tippett [13]) *Let $X_1, X_2, X_3, \dots$ be a sequence of i.i.d. random variables. If there exist norming constants $c_n > 0$, $d_n \in \mathbb{R}$ and some non-degenerate distribution function H such that*

$$\frac{max(X_1, \dots, X_n) - d_n}{c_n} \overset{distr.}{\to} H, \tag{2.8}$$

then H is one of the following distribution functions:

$$\text{Fréchet: } \Phi_\alpha(x) \overset{def.}{=} \begin{cases} 0 & \text{if } x \le 0; \\ \exp(-x^{-\alpha}) & \text{if } x > 0, \end{cases} \text{ for some } \alpha > 0. \tag{2.9}$$

$$\text{Weibull: } \Psi_\alpha(x) \overset{def.}{=} \begin{cases} \exp(-(-x)^{\alpha}) & \text{if } x \le 0; \\ 1 & \text{if } x > 0, \end{cases} \text{ for some } \alpha > 0. \tag{2.10}$$

$$\text{Gumbel: } \Lambda(x) \overset{def.}{=} \exp\left(-e^{-x}\right) \tag{2.11}$$

This theorem is very appropriate for the question we are looking at: How large should the reserve be? It says something about the behaviour of the maximum of a sequence of random variables. Just think of the individual random variables as the losses per month. Then, with the use of this theorem, we can easily determine the probability that a certain reserve will be sufficient over a large number of months. And this becomes very appropriate for our model in combination with the results of Chapter 1.

Definition 2.8 (Maximum Domain of Attraction) *If the scaled (as in the theorem) maximum of the X_i's indeed converges to a non-degenerate distribution function as in* (2.9)*, then their common distribution function F is said to be in the Maximum Domain of Attraction of the Fréchet-distribution. This is noted as:*

$$F \in MDA(\Phi_\alpha)$$

Likewise, if the scaled maximum converges to a distribution function as in (2.10) *or* (2.11)*, it lays in the Maximum Domain of Attraction of the Weibull- resp. Gumbel-distribution, noted as*

$$F \in MDA(\Psi_\alpha), F \in MDA(\Lambda)$$

As it turns out, these Maximum Domains of Attraction can be characterized in the following way:

Theorem 2.9 *For a distribution function F the following holds:*

- $$F \in MDA(\Phi_\alpha) \Leftrightarrow$$
  $x \to 1 - F(x)$ is regularly varying in ∞ with index $-\alpha$ for some $\alpha < \infty$.

- $$F \in MDA(\Psi_\alpha) \Leftrightarrow$$
  $\exists c < \infty$ such that $x \to 1 - F(c - 1/x)$ is regularly varying in ∞ with index $-\alpha$.

- $$F \in MDA(\Lambda) \Leftrightarrow$$
  $\exists c \leq \infty$, there exists some positive function $x \to a(x)$ such that for all $t \in \mathbb{R}$:
  $$\lim_{x \uparrow c} \frac{1 - F(x + ta(x))}{1 - F(x)} = e^{-t}. \tag{2.12}$$

Since the Fisher-Tippett Theorem is about maxima, the MDA says something about the right tail of the distributions. In this book, however, we model the losses as negative numbers. This means that we rather look at the left tails and the minima, even though we still speak of maximum loss. This choice means that we talk about F rather than $\overline{F} = 1 - F$. The Fisher-Tippett Theorem and its domains of attraction, combined with our choice for negative numbers to model losses, leads us to the following: In this book, if we refer to a distribution F as being of Fréchet, Weibull or Gumbel type, we mean that $x \to F(-x)$ is in the Maximum Domain of Attraction of respectively Φ_α, Ψ_α or Λ, as defined above.

3 Structure of the chapters

This book is structured as follows: In Chapter 1 we look at the limit ratio

$$\lim_{u \to \infty} \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^{m} X_i \leq -u \right), \tag{3.1}$$

where the random variables are dependent with an Archimedean copula with regularly varying generator. Also the marginal distributions of the random variables are the same. Most parts of this chapter were published in [2]. This chapter is the starting point of the present work both in time (it was the first result we got) and in theory (the later chapters are either generalizations (Chapter 2 and Chapter 3) of Chapter 1 or make use of its results (Chapter 4 and Chapter 5)).

Chapter 2 and Chapter 3 generalize part of Chapter 1. Chapter 2 loosens the requirements on the marginal distributions of the random variables. Most importantly it allows these marginal distributions to be different from each other. Chapter 3 extends the results of Chapter 1 to other than just Archimedean copulas.

Finally, Chapter 4 and Chapter 5 deal with expectation, rather than probability. These chapters both use the results of Chapter 1. Chapter 4 was inspired by the discussion about risk-measures. In this chapter we investigate the Expected Shortfall, which has some advantages over Value-at-Risk as a risk measure. This chapter was also largely published as [3]. Chapter 5 looks at the Esscher Premium, which in that chapter is presented as a modification of the Expected Shortfall, so that it picks up diversification effects.

Throughout all chapters we give examples, to clarify the results, as well as to show how the results can be used.

Chapter 1

Value-at-Risk

1 Introduction

Worldwide, regulators look for new methods to calculate solvency requirements for insurance companies (Europe, Switzerland, Australia, Canada, revision of the US RBC, etc.). It is generally understood that the new methods should consider all risks and that risk-adjusted solvency capitals should be calculated. Usually the risks are classified into different categories. In each category one is then able to analyze the risks (e.g. using an analytical approach). The main difficulty comes in when one tries to aggregate the different (dependent) categories and when one tries to quantify the diversification between the different categories. In this chapter we give a partial answer to such questions: Consider m identically distributed dependent risks $X_1, \dots, X_m$, then we will see that the probability of a large aggregate loss of $\sum_{i=1}^m X_i$ scales like the probability of a large individual loss of X_1, times a proportionality factor. This factor depends on the dependence strength and the tail behaviour of the individual loss and is different for the cases where the tail behaviour lies in the domain of attraction of the Fréchet, the Weibull or the Gumbel distribution (see [13]). E.g. in the Fréchet case we see

$$\mathbb{P}\left(\sum_{i=1}^m X_i \leq -u\right) \sim q_m \cdot \mathbb{P}\left(X_1 \leq -u\right), \qquad \text{as } u \to \infty, \tag{1.1}$$

i.e. the constant q_m quantifies the diversification effect between the dependent risks. In general we give a formula for q_m that can be calculated numerically.

For $m = 2$ we give explicit formulas for q_m.

By diversification effect we mean the following: Suppose one needs an amount of money d in reserve to be able to pay claim X_1 with probability p. Now suppose further that one has m similar risks $X_1, \ldots, X_m$ in one portfolio and one needs an amount of money D in reserve to be able to pay the aggregate claim $\sum_{i=1}^m X_i$ with probability p. (Note that this is the Value-at-Risk approach.) The diversification effect is now defined as $1 - (D/md)$. It is the part of the reserves that can be saved by putting several risks into one portfolio. In Section 3 the diversification effect is defined more precisely (and calculated) for an example.

The modelling of stochastic dependencies has shown to be particularly important in extreme value theory, where a profound knowledge of the complete dependence structure of the underlying random variables is needed to come to the right conclusions. In particular, it was understood in recent research (see e.g. Embrechts-McNeil-Straumann [14], Frees-Valdez [17], Juri-Wüthrich [21]) that simple measures of dependence such as the correlation coefficient are insufficient to cover the full range of possible consequences of dependent events.

Many applications of copulas to actuarial sciences can be found in literature, as e.g. Carrière-Frees-Valdez [8]. Many authors have tried to find upper und lower bounds for expressions like formula (1.1) (see e.g. Dhaene-Denuit [12], Denuit-Genest-Marceau [11], Bäuerle-Müller [5] and Cossette-Denuit-Marceau [9]). We choose a different approach: instead of finding bounds, we rather analyze the asymptotic properties. We find some universal behaviour (weak convergence theorems) that enables us to analyze different classes of models. The dependence structure is described using the copula framework. Successful steps in this direction have been undertaken e.g. by Wüthrich [27] or Juri-Wüthrich [20].

The first of these two papers is also the starting point for our investigations. There one sees that the extreme value behaviour of a sum of correlated, identically distributed random variables – where the correlation comes from a copula – scales like the extreme value behaviour of one variable with the same distribution. The aim of this chapter is twofold: On the one hand we give a different proof for Wüthrich's result, on the other hand we also derive properties of the proportionality factor.

The chapter is organized as follows. In Section 2 we give our main result, the asymptotic behaviour (1.1), moreover we provide the properties of the limiting constant q_m for $m = 2$. In Section 3 we give a practical example. Finally, in Section 4 we prove our results.

2 An extreme value theorem and corollaries

In Wüthrich [27] an extreme-value theorem is proven, that basically states that the extreme value behaviour of a sum of dependent random variables with identical marginals scales like the extreme value behaviour of one such variable. The formula for the limiting proportionality constant is rather complicated though. Below we give an alternative proof that leads to a more transparent description of the limiting constants and allows to analyze properties of these constants.

2.1 Main theorem

The main theorem of extreme value theory states that extreme value behaviour of a sequence of i.i.d. random variables is either degenerate or in exactly one of the following three classes (see the introduction or [13], Theorem 3.2.3.): Fréchet, Weibull or Gumbel, i.e. there are essentially three different types of marginal behaviours. Their characterizations are given in the following theorem.

Let c denote the left end-point of the one-dimensional distribution F, where appropriate (i.e., in the Weibull and Gumbel case).

Theorem 2.1 *Let $m \geq 2$, $\alpha, \beta > 0$, there are constants $q_m^F(\alpha,\beta), q_m^W(\alpha,\beta), q_m^G(\alpha)$ such that following holds true: Assume $X = (X_1, \ldots, X_m)$ has real-valued identically distributed random components, with continuous marginal*

$$F(x) = \mathbb{P}(X_i \leq x)$$

and X has Archimedean copula C^ϕ, where ϕ is regularly varying at 0^+ with index $-\alpha$. Then

a) (The Fréchet case) If F is regularly varying at $-\infty$ with index $-\beta$, then

$$\lim_{u\to\infty} \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^m X_i \leq -u\right) = q_m^F(\alpha,\beta), \qquad \textit{with} \tag{2.1}$$

$$q_m^F(\alpha,\beta) = \lim_{\varepsilon\downarrow 0} \int_{\sum_i 1/x_i \geq 1, x_1 \leq 1/\varepsilon} \frac{d^m}{dx_1 \ldots dx_m} \left(\sum_{i=1}^m x_i^{-\alpha\beta}\right)^{-1/\alpha} dx_1 \ldots dx_m \ . \tag{2.2}$$

b) (The Weibull case) If there is a $c > -\infty$ such that $s \mapsto F(c - 1/s)$ is regularly varying at $-\infty$ with index $-\beta$, then

$$\lim_{u\to\infty} \frac{1}{F(c+1/u)} \mathbb{P}\left(\sum_{i=1}^m X_i \leq mc + 1/u\right) = q_m^W(\alpha,\beta), \qquad \textit{with} \tag{2.3}$$

$$q_m^W(\alpha,\beta) = \lim_{\varepsilon\downarrow 0} \int_{\sum_i x_i \le 1, x_1 \le 1/\varepsilon} \frac{d^m}{dx_1 \dots dx_m} \left(\sum_{i=1}^m x_i^{-\alpha\beta} \right)^{-1/\alpha} dx_1 \dots dx_m \ . \tag{2.4}$$

c) (The Gumbel case) If there is a $c \ge -\infty$ *and a positive function* $s \mapsto a(s)$ *such that for* $t \in \mathbb{R}$ *one has*

$$\lim_{u\downarrow c} F(u + ta(u))/F(u) = e^t,$$

then

$$\lim_{u\downarrow c} \frac{1}{F(u)} \mathbb{P}\left(\sum_{i=1}^m X_i \le mu \right) = q_m^G(\alpha) \cdot e^{-\frac{1}{m}}, \qquad \textit{with} \tag{2.5}$$

$$q_m^G(\alpha) = \int_{\sum_i x_i \le 1} \frac{d^m}{dx_1 \dots dx_m} \left(\sum_{i=1}^m e^{-x_i\alpha} \right)^{-1/\alpha} dx_1 \dots dx_m \ . \tag{2.6}$$

Remarks 2.2 Note the following

- The parameter α plays the role of the dependence strength. It is essentially a measure for the dependence in the tails (compare to the tail dependence results in Juri-Wüthrich [20], Theorem 3.9).

- For analyzing the asymptotic behaviour of $\sum_{i=1}^m X_i$ we only need to know the marginals X_i and the "dependence strength" α. I.e. with Theorem 2.1 we can avoid explicitly choosing the dependence structure (copula), which is a notoriously difficult object (see also Embrechts-McNeil-Straumann [14]), but still obtain appropriate asymptotic results. This is common in extreme value theory, the asymptotic results divide into different classes/distributions where one only needs to estimate certain parameters.

- The limiting distributions found in the formulas for the constants: (2.2), (2.4) and (2.6) have Clayton copula, this is not surprising in view of the results presented in Juri-Wüthrich [20].

2.2 Properties of the limiting constants for m=2

The new characterizations of the limiting constants $q_m^F(\alpha,\beta)$, $q_m^W(\alpha,\beta)$ and $q_m^G(\alpha)$ still look complex. Nevertheless, they allow explicit calculations in $m=2$ and they have nice monotonicity properties (presented below).

Definition 2.3 *For $\alpha \neq 0$ and $y \geq 0$ define:*

$$f_\alpha(y) \overset{def.}{=} (1+y^\alpha)^{-1/\alpha-1}\,. \tag{2.7}$$

Then we can prove

Lemma 2.4 *For $\alpha > 0$, $f_\alpha(y)$ is a probability density on $[0,\infty)$.*

Theorem 2.5 *(Fréchet case) For $\alpha > 0$ and $Y_\alpha \sim f_\alpha$ we have*

$$q_2^F(\alpha,\beta) = 1 + \mathbb{E}\left(f_{-1/\beta}(Y_\alpha)\right) = 1 + \mathbb{E}\left(\left(1+Y_\alpha^{-1/\beta}\right)^{\beta-1}\right). \tag{2.8}$$

Moreover:

- $q_2^F(\alpha,\beta)$ *is strictly increasing in* β.
- *For* $\beta > 1$, $q_2^F(\alpha,\beta)$ *is strictly increasing in* α.
- $q_2^F(\alpha,1) = 2$
- *For* $\beta < 1$, $q_2^F(\alpha,\beta)$ *is strictly decreasing in* α.
- $\lim_{\alpha\to\infty} q_2^F(\alpha,\beta) = 2^\beta$ *as well as* $\lim_{\alpha\downarrow 0} q_2^F(\alpha,\beta) = 2$.

Remarks 2.6

- The behaviour of $q_2^F(\alpha,\beta)$ is illustrated in Figure 1.1.
- For $\beta > 1$ there is a "positive" diversification effect, i.e. $q_2^F(\alpha,\beta)$ is strictly increasing in the dependence strength α. At first sight it might seem confusing, that this does not hold true for $\beta \leq 1$. One interpretation for this phenomenon is that for $\beta \leq 1$ we have no finite mean of the marginals, i.e. a large aggregate loss is typically not generated by multiple (dependent) individual large losses, but one very large individual loss. So the less dependent these losses are, the more risk that at least one of the individual losses will exceed the threshold. Therefore in this case dependency actually reduces the risk of a large aggregate loss, since the more dependent the individual losses, the smaller the probability that at least one of the individual losses is very large.

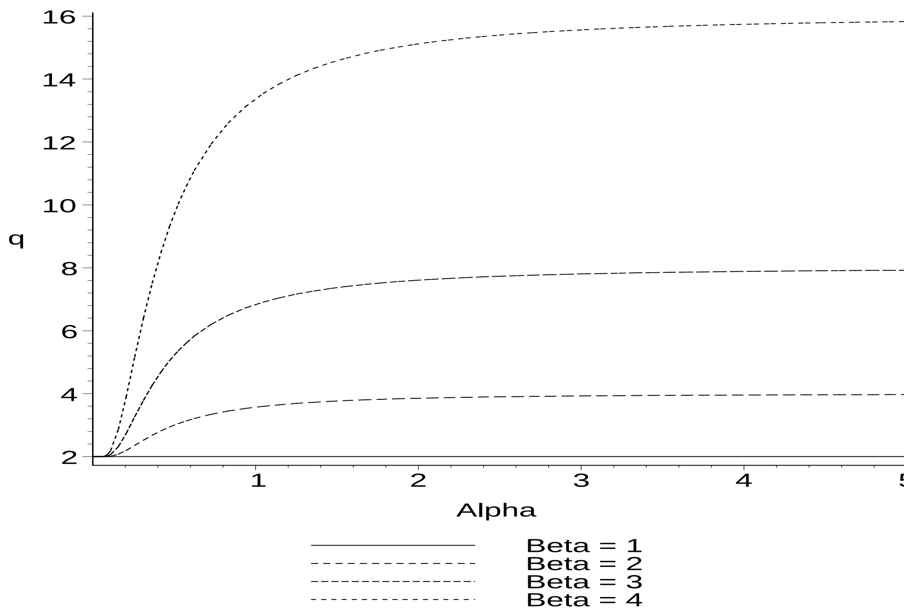


Figure 1.1: $q = q_2^F(\alpha, \beta)$ as a function of α, for different β's.

- For $\beta \in \mathbb{N} \setminus \{0\}$ we have

$$q_2^F(\alpha, \beta) = \sum_{k=0}^{\beta} \binom{\beta}{k} \frac{\Gamma\left(\frac{\beta-k}{\alpha\beta} + 1\right) \Gamma\left(\frac{k}{\alpha\beta} + 1\right)}{\Gamma(1 + 1/\alpha)}. \tag{2.9}$$

Theorem 2.7 *(Weibull case) For $\alpha > 0$ and $Y_\alpha \sim f_\alpha$ we have*

$$q_2^W(\alpha, \beta) = \mathbb{E}\left(\left(1 + Y_\alpha^{1/\beta}\right)^{-\beta-1}\right). \tag{2.10}$$

The limiting constant $q_2^W(\alpha, \beta)$ is strictly increasing in α and strictly decreasing in β. Moreover $q_2^W(\alpha, \beta) \leq 1$ for all $\alpha, \beta > 0$, and it holds

$$\lim_{\alpha \to \infty} q_2^W(\alpha, \beta) = 2^{-\beta} \quad \text{as well as} \quad \lim_{\alpha \downarrow 0} q_2^W(\alpha, \beta) = 0.$$

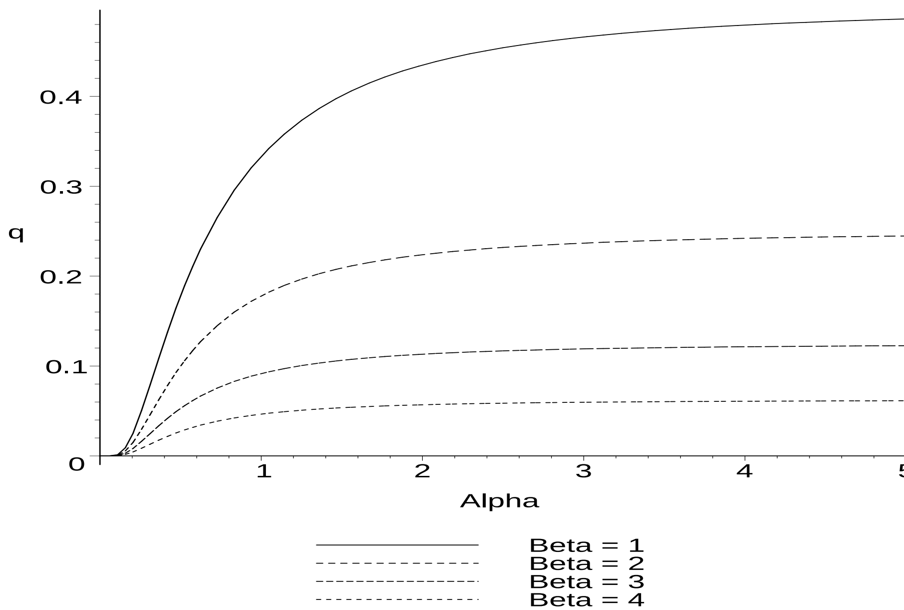


Figure 1.2: $q = q_2^W(\alpha, \beta)$ as a function of α, for different β's.

Remark 2.8 *The behaviour of $q_2^W(\alpha, \beta)$ is shown in Figure 1.2, for different α and β. Again we have decreasing diversification for increasing α (for the notion of diversification effect see the introduction).*

Theorem 2.9 *(Gumbel case) For $\alpha > 0$ we have*

$$q_2^G(\alpha) = e^{1/2} \cdot \frac{\Gamma^2(1 + 1/(2\alpha))}{\Gamma(1 + 1/\alpha)} = e^{1/2}\left(\frac{q_2^F(\alpha, 2)}{2} - 1\right) . \qquad (2.11)$$

Furthermore $q_2^G(\alpha)$ is strictly increasing in α and

$$\lim_{\alpha \to 0} q_2^G(\alpha) = 0 \qquad and \qquad \lim_{\alpha \to \infty} q_2^G(\alpha) = e^{1/2}. \qquad (2.12)$$

And the behaviour of $q_2^G(\alpha)$ is shown in Figure 1.3, as a function of α.

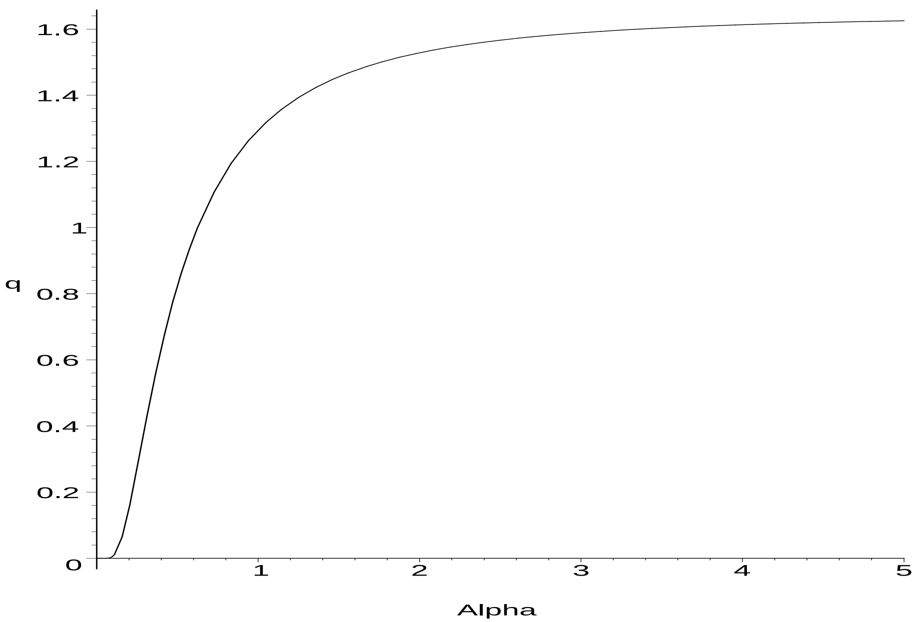


Figure 1.3: $q = q_2^G(\alpha)$ as a function of α.

2.3 Conclusions

We find that for m identically and continuously distributed risks $X_1, \dots, X_m$, the probability to suffer a large loss by their sum scales like the probability to suffer a large loss by just one of them. In formulas (for the Fréchet case)

$$\mathbb{P}\left(\sum_{i=1}^{m} X_i \leq -u\right) \sim q_m(\alpha) \cdot \mathbb{P}\left(X_1 \leq -u\right), \qquad \text{as } u \to \infty \tag{2.13}$$

Moreover, the constant $q_m(\alpha)$ describes the diversification effect: the larger the dependence strength α the smaller the diversification effect (Weibull, Gumbel and Fréchet case for $\beta > 1$ — for the notion of diversification effect see the introduction).

The limiting constant q_m only depends on the choice of the marginals and on the choice of the dependence strength α, i.e. we do not need to specify the whole dependence structure (i.e. the copula) to apply our results. As soon as we can estimate α and the marginals we can apply our theorems to estimate asymptotic quantiles, of course this is a major simplification of the problem (an example is presented in the next section).

3 An example

We model two motor liability portfolios X_1 and X_2. Our goal is to merge them to one big portfolio, and we want to measure the diversification effect we can expect by merging the two portfolios.

Assume X_1 and X_2 have Archimedean copula generated by a regularly varying function with index $-\alpha$ at 0^+ ($\alpha > 0$). Moreover assume that $-X_1$ and $-X_2$ have translated Pareto marginals with translation $v_1 = 880$ and $v_2 = 820$, i.e. $Y_i = -(X_i + v_i)$ is Pareto distributed with $\theta = 80$ and $\beta = 3$: for $i = 1, 2$.

$$\mathbb{P}(X_i \leq x) = \mathbb{P}(X_i + v_i \leq x + v_i) = \left(\frac{\theta}{-(x+v_i)}\right)^{\beta} \qquad \text{for } x \leq -(\theta + v_i). \tag{3.1}$$

Choose $p = 99.5\%$ and define Value-at-Risk

$$\mathrm{VaR}_{X_i} \stackrel{def.}{=} -\sup\{x;\ \mathbb{P}(X_i \geq x) \geq p\} + \mathbb{E}(X_i). \tag{3.2}$$

Hence we have

	portfolio 1	portfolio 2
translation v_i	880	820
mean $\mathbb{E}(-X_i)$	1000	940
variational coefficient	6.9%	7.3%
VaR_{X_i}	347.8	347.8

As shown one can easily calculate quantiles for solvency purposes. The main difficulty is to calculate solvency requirements for two such aggregated portfolios. We use Theorem 2.1 and find for u large ($v = v_1 + v_2$)

$$\mathbb{P}(X_1 + X_2 \leq -u) = \mathbb{P}(X_1 + X_2 + v \leq -u + v) \sim q_2^F(\alpha, \beta) \left(\frac{\theta}{u - v} \right)^{\beta}. \quad (3.3)$$

Define $\mathrm{VaR}_{X_1+X_2}(\alpha)$ as in (3.2). Hence the Value-at-Risk of $X_1 + X_2$ is now a function of the dependence strength α and can be approximated by (3.3). We obtain

$$\mathrm{VaR}_{X_1+X_2}(\alpha) \approx V_{X_1+X_2}(\alpha) \stackrel{def.}{=} \left(\theta \left(\frac{q_2^F(\alpha,\beta)}{1-p} \right)^{1/\beta} + v \right) + \mathbb{E}[X_1 + X_2]. \quad (3.4)$$

Since we have a nice expression for $q_2^F(\alpha, \beta)$ (Theorem 2.5), we can numerically approximate the Value-at-Risk for different α (see Figure 1.4), and thus the decrease in Value-at-Risk when diversifying a portfolio, i.e. the diversification effect, which is defined as $1 - V_{X_1+X_2}(\alpha)/(\mathrm{VaR}_{X_1} + \mathrm{VaR}_{X_2})$ where $\mathrm{VaR}_{X_1} + \mathrm{VaR}_{X_2}$ corresponds to total positive dependence (see Figure 1.5). In this picture one can see that our approximation is not sharp for small α, but this is not bad, since one can calculate the VaR directly for independent portfolios ($\alpha = 0$). In the tabular at the end of this section we use this direct method for $\alpha = 0$ only.

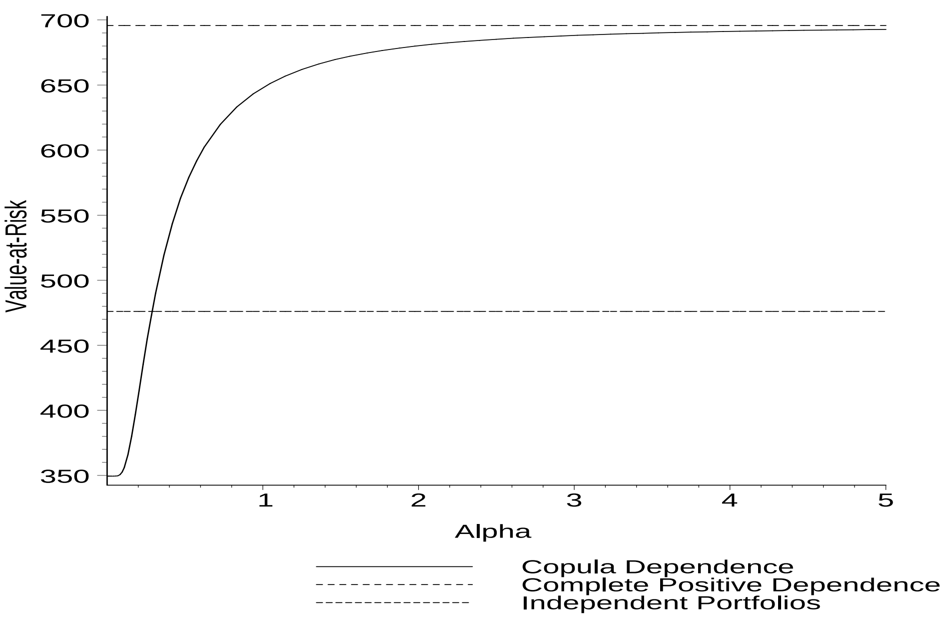


Figure 1.4: Assymptotic Value-at-Risk $V_{X_1+X_2}(\alpha)$ for different α, compared to independent portfolios and comonotonic portfolios.

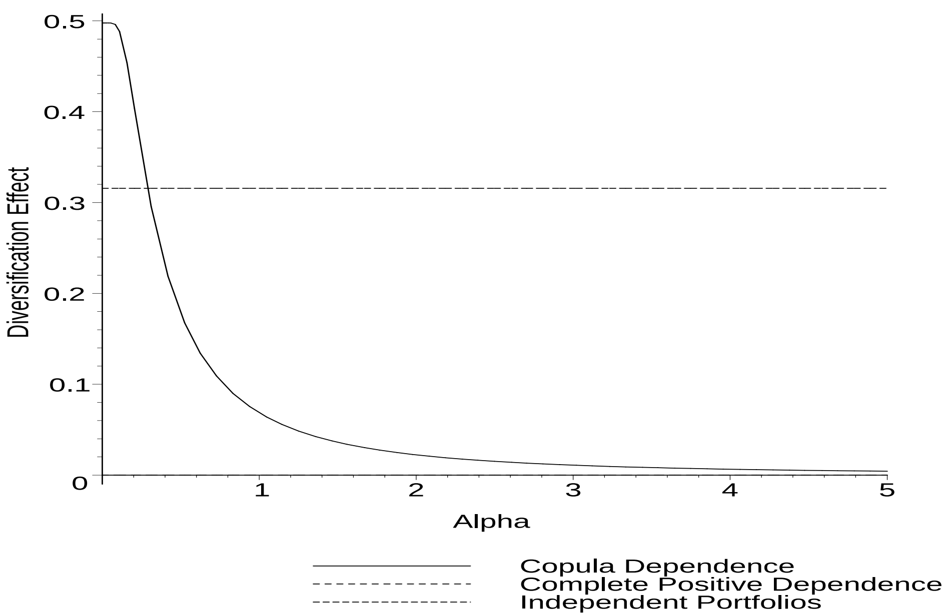


Figure 1.5: Diversification effect as a function of α, compared to independent portfolios and comonotonic portfolios.

α	*indep.*	0.5	1.0	1.5	2.0	3.0	4.0	∞
$-E[X_1+X_2]$	1940	1940	1940	1940	1940	1940	1940	1940
$V_{X_1+X_2}(\alpha)$	476.1	571.8	648.0	670.5	680.2	688.1	691.2	695.7
Div.eff.(α)	31.6%	17.8%	6.9%	3.6%	2.2%	1.1%	0.6%	0%

4 The proofs

In this section we provide the proofs to the statements in the previous sections.

4.1 Proof of the extreme value theorem

As announced above we give a new proof of Theorem 2.1. We work out the details for the Fréchet case and indicate where the proofs in the Weibull and Gumbel case differ.

Fréchet case

Lemma 4.1 *(Fréchet) Let $m \geq 2, \alpha > 0$ and $\beta > 0$. Let $X = (X_1, \ldots, X_m)$ have Archimedean copula C^ϕ, where ϕ is a regularly varying function at 0^+ with index $-\alpha$. Moreover assume that all X_i have the same, continuous marginal $F(x)$ that is regularly varying at $-\infty$ with index $-\beta$. Furthermore, let $\varepsilon \in (0,1)$, $x_1 \in (0, 1/\varepsilon)$ and $x_2, \ldots, x_m > 0$. Then:*

$$\lim_{u\to\infty} \mathbb{P}\left(X_i \leq -u/x_i, i = 1, \ldots, m \mid X_1 \leq -\varepsilon u\right) = \left(\sum_{i=1}^m x_i^{-\alpha\beta}\right)^{-1/\alpha} \varepsilon^\beta \ . \quad (4.1)$$

Proof. Since ϕ and F are regularly varying, the following holds: For every $\delta > 0$ there is an u_0 such that for all i and $u > u_0$:

$$F(-u/x_i) \leq (x_i+\delta)^\beta F(-u), \qquad \text{and} \qquad (\varepsilon+\delta)^{-\beta} F(-u) \leq F(-\varepsilon u), \quad (4.2)$$

and $F(-u)$ is so close to 0 that :

$$\phi((x_i+\delta)^\beta F(-u)) \geq ((x_i+\delta)^\beta+\delta)^{-\alpha}\phi(F(-u)), \text{ and} \quad (4.3)$$

$$\sum_{i=1}^m ((x_i+\delta)^\beta+\delta)^{-\alpha}\phi\circ F(-u) \leq \phi\left(\left(\sum_{i=1}^m ((x_i+\delta)^\beta+\delta)^{-\alpha}-\delta\right)^{-1/\alpha} F(-u)\right). \quad (4.4)$$

Now we show the upper bound:

$$
\begin{aligned}
&\limsup_{u\to\infty} \mathbb{P}\left(X_i \le -u/x_i, i = 1,\dots,m \mid X_1 \le -\varepsilon u\right) \\
&\quad = \limsup_{u\to\infty} \frac{\phi^{-1}\left(\sum_{i=1}^m \phi \circ F(-u/x_i)\right)}{F(-\varepsilon u)} \\
&\quad \le \limsup_{u\to\infty} \frac{\phi^{-1}\left(\sum_{i=1}^m \phi\left((x_i+\delta)^\beta F(-u)\right)\right)}{(\varepsilon+\delta)^{-\beta}F(-u)} \\
&\quad \le \limsup_{u\to\infty} \frac{\phi^{-1}\left(\sum_{i=1}^m ((x_i+\delta)^\beta+\delta)^{-\alpha}\phi\circ F(-u)\right)}{(\varepsilon+\delta)^{-\beta}F(-u)} \\
&\quad \le \frac{\left(\sum_{i=1}^m ((x_i+\delta)^\beta+\delta)^{-\alpha}-\delta\right)^{-1/\alpha}}{(\varepsilon+\delta)^{-\beta}}, \qquad (4.5)
\end{aligned}
$$

where for the first inequality we applied (4.2), for the second inequality we applied (4.3), and for the third inequality we applied (4.4). Since this holds for all $\delta > 0$, we get the upper bound. The lower bound is proven similarly (take $-\delta$ instead of $+\delta$). ■

Note that

$$
G_\varepsilon^{\alpha,\beta}(x_1,\dots,x_m) \overset{def.}{=} \left(\sum_{i=1}^m x_i^{-\alpha\beta}\right)^{-1/\alpha} \varepsilon^\beta \qquad (4.6)
$$

is a distribution function on $(0,1/\varepsilon)\times(0,\infty)^{m-1}$. Let $g_\varepsilon^{\alpha,\beta}$ be its density function and define:

$$
\begin{aligned}
G(\varepsilon) &\overset{def.}{=} \varepsilon^{-\beta}\int_{\sum_i 1/x_i\ge 1, x_1\le 1/\varepsilon} g_\varepsilon^{\alpha,\beta}(x_1,\dots,x_m)dx_1\dots dx_m \\
&= \int_{\sum_i 1/x_i\ge 1, x_1\le 1/\varepsilon} \frac{d^m}{dx_1\dots dx_m}\left(\sum_{i=1}^m x_i^{-\alpha\beta}\right)^{-1/\alpha} dx_1\dots dx_m \qquad (4.7)
\end{aligned}
$$

Since $G(\varepsilon)$ is increasing for $\varepsilon \downarrow 0$, one can define $G(0) = \lim_{\varepsilon\downarrow 0} G(\varepsilon) \le \infty$.

Proof of Theorem 2.1 a). The key idea is to connect

$$
\mathbb{P}(\sum_{i=1}^m X_i \le -u | X_1 \le -\varepsilon u) \text{ with } \mathbb{P}(X_i \le -u/x_i, i = 1,\dots,m \mid X_1 \le -\varepsilon u)
$$

in the following way:

$$\lim_{u\to\infty} \mathbb{P}(\sum_{i=1}^m X_i \le -u | X_1 \le -\varepsilon u) = \varepsilon^\beta G(\varepsilon).$$

This is done by taking random vector $Y^{(u)} = (Y_1^{(u)}, \ldots, Y_m^{(u)})$ with distribution function

$$H(x_1, \ldots, x_m) \stackrel{def.}{=} \mathbb{P}(X_i \le -u/x_i, i = 1, \ldots, m \mid X_1 \le -\varepsilon u)$$

and random variables $Y_1, \ldots, Y_m$ with distribution function $G_\varepsilon^{\alpha,\beta}(x_1, \ldots, x_m)$. From Lemma 4.1 it follows that $(Y_1^{(u)}, \ldots, Y_m^{(u)})$ converges in distribution to $(Y_1, \ldots, Y_m)$, as $u \to \infty$, and thus

$$\mathbb{P}(\sum_{i=1}^m 1/Y_i^{(u)} \ge 1) = \mathbb{P}(\sum_{i=1}^m X_i \le -u | X_1 \le -\varepsilon u)$$

converges (again as $u \to \infty$) to

$$\mathbb{P}(\sum_{i=1}^m 1/Y_i \ge 1) = \int_{\sum_i 1/x_i \ge 1, x_1 \le 1/\varepsilon} g_\varepsilon^{\alpha,\beta}(x_1, \ldots, x_m) dx_1 \ldots dx_m = \varepsilon^\beta G(\varepsilon).$$

For the lower bound we see that

$$\begin{aligned} \liminf_{u\to\infty} \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^m X_i \le -u\right) &\ge \liminf_{u\to\infty} \frac{F(-\varepsilon u)}{F(-u)} \mathbb{P}\left(\sum_{i=1}^m X_i \le -u \middle| X_1 \le -\varepsilon u\right) \\ &= \liminf_{u\to\infty} \frac{F(-\varepsilon u)}{F(-u)} \varepsilon^\beta G(\varepsilon) = G(\varepsilon), \end{aligned} \tag{4.8}$$

where we used again that F is regularly varying. Since $\varepsilon > 0$ was arbitrary

$$\liminf_{u\to\infty} \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^m X_i \le -u\right) \ge G(0). \tag{4.9}$$

For the upper bound choose $\varepsilon < 1/m$. Then

$$\begin{aligned}
&\limsup_{u\to\infty} \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^m X_i \le -u\right) \\
&= \limsup_{u\to\infty} \left(\frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^m X_i \le -u, X_1 \le -\varepsilon u\right) \right. \\
&\qquad \left. + \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^m X_i \le -u, X_1 > -\varepsilon u\right)\right).
\end{aligned}$$

For the first term we have:

$$\limsup_{u\to\infty} \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^m X_i \le -u, X_1 \le -\varepsilon u\right) = G(\varepsilon). \tag{4.10}$$

For the second term:

$$\begin{aligned}
&\limsup_{u\to\infty} \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^m X_i \le -u, X_1 > -\varepsilon u\right) \\
&\le \limsup_{u\to\infty} \frac{1}{F(-u)} \sum_{i=2}^m \mathbb{P}\left(X_i \le -u/m, X_1 > -\varepsilon u\right) \\
&= \limsup_{u\to\infty} \frac{m-1}{F(-u)} \mathbb{P}\left(X_2 \le -u/m, X_1 > -\varepsilon u\right) \\
&= \limsup_{u\to\infty} \frac{m-1}{F(-u)} \left(\mathbb{P}(X_2 \le -u/m) - \mathbb{P}\left(X_2 \le -u/m, X_1 \le -\varepsilon u\right)\right) \\
&= (m-1)\left(m^{\beta} - (m^{-\alpha\beta} + \varepsilon^{\alpha\beta})^{-1/\alpha}\right),
\end{aligned} \tag{4.11}$$

where in the last equation we repeatedly use the fact that ϕ and F are regularly varying. Since $\varepsilon \in (0, 1/m)$ we let $\varepsilon \downarrow 0$ and arrive at:

$$\limsup_{u\to\infty} \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^m X_i \le -u\right) \le G(0), \tag{4.12}$$

which is the upper bound we claimed. This finishes the proof of Theorem 2.1.
■

Weibull case

The Weibull case is very similar. Lemma 4.1 is replaced by the following lemma:

Lemma 4.2 *(Weibull) Let $m \geq 2, \alpha > 0$ and $\beta > 0$. Let $X = (X_1, \dots, X_m)$ have Archimedean copula C^ϕ, where ϕ is a regularly varying function at 0^+ with index $-\alpha$. Let all X_i have the same, continuous marginal $F(x)$ such that there is a constant c such that $s \mapsto F(c - 1/s)$ is regularly varying at $-\infty$ with index β. Furthermore, let $\varepsilon \in (0,1)$, $x_1 \in (0, 1/\varepsilon)$ and $x_2, \dots, x_m \geq 0$ Then:*

$$\lim_{u \to \infty} \mathbb{P}\left(X_i \leq c + x_i/u, i = 1, \dots, m \mid X_1 \leq c + 1/\varepsilon u\right) = \frac{\left(\sum_{i=1}^m x_i^{-\alpha\beta}\right)^{-1/\alpha}}{\varepsilon^{-\beta}} . \tag{4.13}$$

Proof of Lemma 4.2 and Theorem 2.1 b).
The proof of Lemma 4.2 follows, mutatis mutandis, the lines of the proof of Lemma 4.1 in the Fréchet case. The only change for the proof of Theorem 2.1 is that now we take $Y_1^{(u)}, \dots, Y_m^{(u)}$ with distribution function

$$H^*(x_1, \dots, x_m) \stackrel{def.}{=} \mathbb{P}(X_i \leq c + x_i/u, i = 1, \dots, m \mid X_1 \leq c + 1/\varepsilon u),$$

and this time

$$\mathbb{P}(\sum_{i=1}^m X_i \leq mc + 1/u \mid X_1 \leq c + 1/\varepsilon u) = \mathbb{P}(\sum_{i=1}^m Y_i^{(u)} \leq 1),$$

such that

$$\begin{aligned} &\lim_{u \downarrow c} \mathbb{P}\left(\sum_{i=1}^m X_i \leq mc + 1/u \middle| X_1 \leq c + \frac{1}{\varepsilon u}\right) \\ &= \int_{\sum_i x_i \leq 1, x_1 \leq 1/\varepsilon} g_\varepsilon^{\alpha,\beta}(x_1, \dots, x_m) dx_1 \dots dx_m \end{aligned}$$

where $g_\varepsilon^{\alpha,\beta}(x_1, \dots, x_m)$ again is the density function associated with $G_\varepsilon^{\alpha,\beta}(x_1, \dots, x_m)$. Thus in this case $q_m^W(\alpha, \beta) = \lim_{\varepsilon \downarrow 0} G^*(\varepsilon)$, where

$$G^*(\varepsilon) \stackrel{def.}{=} \varepsilon^{-\beta} \int_{\sum_i x_i \leq 1, x_1 \leq 1/\varepsilon} g_\varepsilon^{\alpha,\beta}(x_1, \dots, x_m) dx_1 \dots dx_m . \tag{4.14}$$

This finishes the proofs in the Weibull case. ■

Gumbel case

Eventually, the Gumbel case is slightly different.

Lemma 4.3 *(Gumbel) Let* $m \geq 2, \alpha > 0$ *. Let* $X = (X_1, \ldots, X_m)$ *have Archimedean copula* C^ϕ*, where* ϕ *is a regularly varying function at* 0^+ *with index* $-\alpha$*. Let all* X_i *have the same, continuous marginal* $F(x)$ *such that there is a constant* c *and a positive function* $s \mapsto a(s)$ *such that* $\lim_{u \downarrow c} F(u+ta(u))/F(u) = e^t$*, for all* $t \in \mathbb{R}$*. Furthermore, let* $\varepsilon \in (0,1)$*,* $x_1 \in (-\infty, 1/\varepsilon)$ *and* $x_2, \ldots, x_m \in \mathbb{R}$ *Then:*

$$\lim_{u \downarrow c} \mathbb{P}\left(X_i \leq u + x_i a(u), i = 1, \ldots, m \mid X_1 \leq u + a(u)/\varepsilon\right) = e^{-1/\varepsilon} \left(\sum_{i=1}^m e^{-\alpha x_i}\right)^{-1/\alpha}. \tag{4.15}$$

Proof. Again the proof follows the proof of Lemma 4.1. But this time we have to change more. Again for $\delta > 0$, we need Gumbel-case variants for inequalities (4.2), (4.3) and (4.4). In the Gumbel case, we need u_0 such that for all i and $u < u_0$: instead of (4.2):

$$F(u+x_i a(u)) \leq e^{(x_i+\delta)} F(u), \qquad \text{and} \qquad e^{\left(\frac{1}{\varepsilon}-\delta\right)} F(u) \leq F(u+a(u)/\varepsilon), \tag{4.16}$$

instead of (4.3) and (4.4) we now need $F(u)$ to be so close to 0 such that:

$$\phi(e^{(x_i+\delta)} F(u)) \geq (e^{(x_i+\delta)} + \delta)^{-\alpha} \phi(F(u)), \text{ and} \tag{4.17}$$

$$\sum_{i=1}^m \left(e^{(x_i+\delta)} + \delta\right)^{-\alpha} \phi \circ F(u) \leq \phi\left(\left(\sum_{i=1}^m \left(e^{(x_i+\delta)} + \delta\right)^{-\alpha} - \delta\right)^{-1/\alpha} F(u)\right). \tag{4.18}$$

With these equations the Gumbel-equivalent of (4.5) becomes:

$$\begin{aligned}
&\limsup_{u \to \infty} \mathbb{P}\left(X_i \leq u + x_i a(u), i = 1, \ldots, m \mid X_1 \leq u + a(u)/\varepsilon\right) \\
&\quad \leq \limsup_{u \to \infty} \frac{\phi^{-1}\left(\sum_{i=1}^m \phi\left(e^{x_i+\delta} F(-u)\right)\right)}{e^{1/\varepsilon-\delta} F(-u)} \\
&\quad = \left(\left(\sum_{i=1}^m (e^{x_i+\delta} + \delta)^{-\alpha}\right) - \delta\right)^{-1/\alpha} e^{\delta - 1/\varepsilon}. \qquad (4.19)
\end{aligned}$$

With $\delta \downarrow 0$ and a similar lower bound this proves the lemma.
■

Proof of Theorem 2.1 c). For the proof of the last part of Theorem 2.1 we take $Y_1^{(u)}, \ldots, Y_m^{(u)}$ with distribution function

$$H^{\heartsuit}(x_1, \ldots, x_m) \stackrel{def.}{=} \mathbb{P}(X_i \le u + x_i a(u), i = 1, \ldots, m \mid X_1 \le u + a(u)/\varepsilon)$$

and $Y_1, \ldots, Y_m$ with distribution function

$$G_\varepsilon^{\heartsuit\ \alpha}(x_1, \ldots, x_m) \stackrel{def.}{=} e^{-1/\varepsilon}\Big(\sum_{i=1}^m e^{-x_i\alpha}\Big)^{-1/\alpha}. \tag{4.20}$$

Then, if $g_\varepsilon^{\heartsuit\ \alpha}$ denotes the density of $G_\varepsilon^{\heartsuit\ \alpha}$

$$\begin{aligned}\lim_{u\downarrow c} \mathbb{P}\left(\sum_{i=1}^m X_i \le mu + a(u) \mid X_1 \le u + a(u)/\varepsilon\right) &= \lim_{u\downarrow c} \mathbb{P}\left(\sum_{i=1}^m Y_i^{(u)} \le 1\right) \\ &= \int_{\sum_i x_i \le 1, x_1 \le 1/\varepsilon} g_\varepsilon^{\heartsuit\ \alpha}(x_1, \ldots, x_m) dx_1 \ldots dx_m, \end{aligned} \tag{4.21}$$

and thus $q_m^G(\alpha) = \lim_{\varepsilon\downarrow 0} G^{\heartsuit}(\varepsilon)$, where

$$G^{\heartsuit}(\varepsilon) \stackrel{def.}{=} e^{1/\varepsilon} \int_{\sum_i x_i \le 1, x_1 \le 1/\varepsilon} g_\varepsilon^{\heartsuit\ \alpha}(x_1, \ldots, x_m) dx_1 \ldots dx_m\ . \tag{4.22}$$

Just as in the Weibull-case the limit is already reached as soon as $\varepsilon \le 1$, thus:

$$q_m^G(\alpha) = \int_{\sum_i x_i \le 1} \frac{d^m}{dx_1 \ldots dx_m} \left(\sum_{i=1}^m e^{-x_i\alpha}\right)^{-1/\alpha} dx_1 \ldots dx_m. \tag{4.23}$$

We have now proved that

$$\lim_{u\downarrow c} 1/F(u)\mathbb{P}\left(\sum_{i=1}^m X_i \le mu + a(u)\right) = q_m^G(\alpha). \tag{4.24}$$

The proof of Theorem 2.1 (Gumbel case) now follows using the transformation $v = u + a(u)/m$:

$$\begin{aligned}
&\lim_{v \downarrow c} 1/F(v)\mathbb{P}\left(\sum_{i=1}^{m} X_i \le mv\right) \\
&= \lim_{u \downarrow c} 1/F(u + a(u)/m)\mathbb{P}\left(\sum_{i=1}^{m} X_i \le mu + a(u)\right) \\
&= \lim_{u \downarrow c} \frac{F(u)}{F(u + a(u)/m)} 1/F(u)\mathbb{P}\left(\sum_{i=1}^{m} X_i \le mu + a(u)\right) \\
&= \frac{1}{e^{\frac{1}{m}}} \cdot q_m^G(\alpha) = q_m^G(\alpha) \cdot e^{-\frac{1}{m}}. \qquad (4.25)
\end{aligned}$$

■

4.2 Limiting constants in the case m=2

The integrals in (2.2), (2.4) and (2.6), although numerically calculable, look rather unappealing. But we are able to calculate them more explicitely for the case where $m = 2$. This basically works because one can rewrite the m-time integral as a $m-1$-time integral, although that integral loses the little beauty the m-time integral had. But in the case of a double integral we can turn it into a single integral that can be made more manageable with some rewriting. Sadly, this doesn't help for the cases where $m > 3$, so until a better method is found, in that case the numerical approach seems best.

Fréchet marginals

Let us choose $(Z_1, Z_2) \sim G_\varepsilon^{\alpha,\beta}$ (see formula (4.6)). Choose $\varepsilon < 1$. Then we can compute $G(\varepsilon)$:

$$\begin{aligned}
G(\varepsilon) &= \varepsilon^{-\beta}\mathbb{P}\left(\frac{1}{Z_1} + \frac{1}{Z_2} \ge 1\right) \\
&= \varepsilon^{-\beta}\left(\mathbb{P}(Z_1 \le 1) + \mathbb{P}\left(\frac{1}{Z_1} + \frac{1}{Z_2} \ge 1, 1 < Z_1 \le 1/\varepsilon\right)\right) \qquad (4.26) \\
&= 1 + \varepsilon^{-\beta}\mathbb{P}\left(Z_2 \le \frac{Z_1}{Z_1 - 1}, 1 < Z_1 \le 1/\varepsilon\right).
\end{aligned}$$

Inserting the densities we obtain

$$G(\varepsilon) = 1+\beta\int_1^{1/\varepsilon}\left(x_1^{-\alpha\beta}+\left(\frac{x_1}{x_1-1}\right)^{-\alpha\beta}\right)^{-1/\alpha-1} x_1^{-\alpha\beta-1}dx_1$$
$$= 1+\beta\int_1^{1/\varepsilon} x_1^{\beta-1}\left(1+(x_1-1)^{\alpha\beta}\right)^{-1/\alpha-1} dx_1. \qquad (4.27)$$

Since the function under the integral is of order $x_1^{-(1+\alpha\beta)}$ as $x_1\to\infty$, which is in $\mathcal{L}^1$, we can let $\varepsilon\to 0$ and we find

$$q_2^F(\alpha,\beta) = G(0) = 1+\beta\int_1^{\infty} x_1^{\beta-1}\left(1+(x_1-1)^{\alpha\beta}\right)^{-1/\alpha-1} dx_1. \qquad (4.28)$$

To analyze the integral, we first substitute $x_1-1\mapsto z$, and then $z^\beta\mapsto y$ to obtain:

$$q_2^F(\alpha,\beta) = 1+\beta\int_0^{\infty}(z+1)^{\beta-1}\left(1+z^{\alpha\beta}\right)^{-1/\alpha-1} dz \qquad (4.29)$$
$$= 1+\int_0^{\infty}\left(1+y^{-1/\beta}\right)^{\beta-1}(1+y^{\alpha})^{-1/\alpha-1} dy\ .$$

Hence we have separated the term into a product of two terms, one only depending on α, the other one only depending on β. Moreover these terms have the same structure. Hence, if we define $f_\alpha(y)$ as above we arrive at

$$q_2^F(\alpha,\beta) = 1+\int_0^{\infty} f_{-1/\beta}(y)\cdot f_\alpha(y)dy. \qquad (4.30)$$

Proof of Lemma 2.4. Choose $0\le c_1<c_2\le\infty$. Then

$$\int_{c_1}^{c_2} f_\alpha(y)dy = \frac{-1}{\alpha}\int_{c_1^{-\alpha}}^{c_2^{-\alpha}}(1+z)^{-1/\alpha-1}dz \qquad (4.31)$$
$$= \left(1+c_2^{-\alpha}\right)^{-1/\alpha}-\left(1+c_1^{-\alpha}\right)^{-1/\alpha},$$

where in the first step we applied the substitution $y^\alpha\mapsto z^{-1}$. Letting $c_1\to 0$ and $c_2\to\infty$ we find that f_α is indeed a probability density function on $[0,\infty)$. ■

As a direct result we now see that

$$q_2^F(\alpha,\beta) = 1+\mathbb{E}\left(f_{-1/\beta}(Y_\alpha)\right)\ , \qquad (4.32)$$

which is the first statement of Theorem 2.5.

For an absolutely continuous random variable Y_α with density function f_α we can compute

$$H(c;\alpha) \overset{def.}{=} \mathbb{P}(Y_\alpha \geq c) = 1 - \left(1 + c^{-\alpha}\right)^{-1/\alpha}. \tag{4.33}$$

Now

$$\frac{dH(c;\alpha)}{d\alpha} = -\frac{1}{\alpha^2}\left(1+c^{-\alpha}\right)^{-1/\alpha-1}\left(\left(1+c^{-\alpha}\right)\log\left(1+c^{-\alpha}\right) - c^{-\alpha}\log c^{-\alpha}\right). \tag{4.34}$$

For the last term in the above expression we know that

$$(1+x)\log(1+x) - x\log x = \log(1+x) + x\log(1+1/x) > 0,$$

since both last terms are positive for all $x > 0$. This implies that

$$\frac{dH(c;\alpha)}{d\alpha} < 0 \qquad \text{for all } c \in (0,\infty). \tag{4.35}$$

Moreover $\lim_{c\to 0}\frac{dH(c;\alpha)}{d\alpha} = 0$. Hence $H(c;\alpha) = \mathbb{P}(Y_\alpha \geq c) = 1 - (1+c^{-\alpha})^{-1/\alpha}$ is strictly decreasing in α for all $c > 0$. We are now ready to prove Theorem 2.5:

Proof of Theorem 2.5. Fix $\beta > 1$ and $0 < \alpha_1 < \alpha_2$. Now (with (4.32))

$$\begin{aligned}
& q_m^F(\alpha_1,\beta) - q_m^F(\alpha_2,\beta) \\
&= \left(\mathbb{E}\left(\left(1+Y_{\alpha_1}^{-1/\beta}\right)^{\beta-1}\right) - \mathbb{E}\left(\left(1+Y_{\alpha_2}^{-1/\beta}\right)^{\beta-1}\right)\right) \qquad (4.36)\\
&= \int_0^\infty \mathbb{P}\left(\left(1+Y_{\alpha_1}^{-1/\beta}\right)^{\beta-1} > x\right) - \mathbb{P}\left(\left(1+Y_{\alpha_2}^{-1/\beta}\right)^{\beta-1} > x\right) dx \\
&= \int_1^\infty \mathbb{P}\left(Y_{\alpha_1} < \left(x^{1/(\beta-1)}-1\right)^{-\beta}\right) - \mathbb{P}\left(Y_{\alpha_2} < \left(x^{1/(\beta-1)}-1\right)^{-\beta}\right) dx\ .
\end{aligned}$$

Using (4.33)-(4.35) we see that this last term is always negative, implying that $q_2^F(\alpha_1,\beta) - q_2^F(\alpha_2,\beta) < 0$ for $\alpha_1 < \alpha_2$, hence that $\alpha \mapsto q_2^F(\alpha,\beta)$ is a strictly increasing function for $\beta > 1$. Analogously for $\beta < 1$

$$\begin{aligned}
& q_m^F(\alpha_1,\beta) - q_m^F(\alpha_2,\beta) \\
&= \int_0^1 \mathbb{P}\left(Y_{\alpha_1} > \left(x^{1/(\beta-1)}-1\right)^{-\beta}\right) - \mathbb{P}\left(Y_{\alpha_2} > \left(x^{1/(\beta-1)}-1\right)^{-\beta}\right) dx > 0.
\end{aligned}$$

Hence $\alpha \mapsto q_2^F(\alpha,\beta)$ is a strictly decreasing function for $\beta < 1$. The case $\beta = 1$ is clear.

Next we prove that $q_2^F(\alpha,\beta)$ is strictly increasing in β. Write $y = (z^{1/\beta}-1)^\beta$, then

$$\begin{aligned} q_2^F(\alpha,\beta) &= 1 + \int_0^\infty \left(1+y^{-1/\beta}\right)^{\beta-1} (1+y^\alpha)^{-1/\alpha-1}\, dy \qquad (4.37)\\ &= 1 + \int_1^\infty \left(1+(z^{1/\beta}-1)^{\alpha\beta}\right)^{-1/\alpha-1} dz \\ &= 1 + \int_1^\infty \left(1+\exp\left\{\alpha\beta\log(z^{1/\beta}-1)\right\}\right)^{-1/\alpha-1} dz. \end{aligned}$$

Define $h(\beta;z) = \beta\log(z^{1/\beta}-1)$.

$$\begin{aligned} \frac{dh(\beta;z)}{d\beta} &= \frac{1}{z^{1/\beta}-1}\left((z^{1/\beta}-1)\log(z^{1/\beta}-1) - z^{1/\beta}\log z^{1/\beta}\right) \\ &< 0 \qquad \text{for } z > 1. \qquad (4.38) \end{aligned}$$

Hence $h(\cdot;z)$ is strictly decreasing for all $z > 1$, which implies that $q_2^F(\alpha,\beta)$ is strictly increasing in β. This finishes the proof of the first part of Theorem 2.5.

■

Weibull marginals

Proof of Theorem 2.7. For the Weibull case recall that using (4.14) and (4.6) we can compute for $m = 2$ and $\varepsilon < 1$:

$$\begin{aligned} G^*(\varepsilon) &= \varepsilon^{-\beta} \int_{x_1+x_2\le 1, x_1\le 1/\varepsilon} g_\varepsilon^{\alpha,\beta}(x_1,x_2)dx_1dx_2 \\ &= \varepsilon^{-\beta} P[Z_1+Z_2 \le 1] \\ &= \varepsilon^{-\beta} \int_0^1 \left(\frac{d}{dx_2} G_\varepsilon^{\alpha,\beta}(x_1,x_2)\right)_{x_1=0}^{1-x_2} dx_2 \\ &= \beta \int_0^1 x_2^{\beta-1} \left(\left(\frac{1-x_2}{x_2}\right)^{-\alpha\beta} + 1\right)^{-1/\alpha-1} dx_2. \qquad (4.39) \end{aligned}$$

Substituting $y = (1-x_2)/x_2$ and $z = y^{-\beta}$ we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0} G^*(\varepsilon) &= \beta \int_0^\infty (y+1)^{-1-\beta} \left(y^{-\alpha\beta}+1\right)^{-1/\alpha-1} dy \\ &= \int_0^\infty \left(1+z^{1/\beta}\right)^{-1-\beta} (z^\alpha+1)^{-1/\alpha-1}\, dz \ , \end{aligned} \tag{4.40}$$

which proves

$$q_2^W(\alpha,\beta) = E\left((1+Y_\alpha^{1/\beta})^{-\beta-1}\right) .$$

Now, similarly to (4.36), fix $\beta > 1$ and $0 < \alpha_1 < \alpha_2$, then

$$\begin{aligned} & q_2^W(\alpha_1,\beta) - q_2^W(\alpha_2,\beta) \\ &\quad = \left(\mathbb{E}\left(\left(1+Y_{\alpha_1}^{1/\beta}\right)^{-\beta-1}\right) - \mathbb{E}\left(\left(1+Y_{\alpha_2}^{1/\beta}\right)^{-\beta-1}\right)\right) \qquad (4.41) \\ &\quad = \int_0^\infty \mathbb{P}\left(\left(1+Y_{\alpha_1}^{1/\beta}\right)^{-\beta-1} > x\right) - \mathbb{P}\left(\left(1+Y_{\alpha_2}^{1/\beta}\right)^{-\beta-1} > x\right) dx \\ &\quad = \int_0^1 \mathbb{P}\left(\left(1+Y_{\alpha_1}^{1/\beta}\right)^{-\beta-1} > x\right) - \mathbb{P}\left(\left(1+Y_{\alpha_2}^{1/\beta}\right)^{-\beta-1} > x\right) dx \\ &\quad = \int_0^1 \mathbb{P}\left(Y_{\alpha_1} < \left(x^{-1/(\beta+1)}-1\right)^\beta\right) - \mathbb{P}\left(Y_{\alpha_2} < \left(x^{-1/(\beta+1)}-1\right)^\beta\right) dx \ . \end{aligned}$$

Again,using (4.33)-(4.35) we see that this last integrand is always negative, implying that $\alpha \mapsto q_2^F(\alpha,\beta)$ is a strictly increasing function for all β. This finishes the proof of the first part of Theorem 2.7. ■

Gumbel marginals

Proof of Theorem 2.9. For the Gumbel case, we can perform similar calculations using (4.20) and (4.21):

$$
\begin{aligned}
G^{\heartsuit}(\varepsilon) &= e^{1/\varepsilon}\int_{x_1+x_2\le 1, x_1\le 1/\varepsilon} g_{\varepsilon}^{\heartsuit\alpha}(x_1,x_2)dx_1dx_2 \\
&= e^{1/\varepsilon}P[Z_1+Z_2\le 1] = e^{1/\varepsilon}P[Z_2\le 1-Z_1] \\
&= e^{1/\varepsilon}\int_{-\infty}^{1/\varepsilon}\left(\frac{d}{dx_1}G_{\varepsilon}^{\heartsuit\alpha}(x_1,x_2)\right)_{x_2=-\infty}^{1-x_1} dx_1 \\
&= \int_{-\infty}^{1/\varepsilon} e^{-\alpha x_1}\left(e^{-\alpha x_1}+e^{-\alpha(1-x_1)}\right)^{-1/\alpha-1} dx_1 \\
&= \int_{-\infty}^{1/\varepsilon} e^{x_1}\left(1+e^{-\alpha(1-2x_1)}\right)^{-1/\alpha-1} dx_1. \qquad (4.42)
\end{aligned}
$$

Letting $\varepsilon \to 0$ and substituting $e^{-(1-2x_1)} = y$ we obtain

$$
G^{\heartsuit}(0) = \frac{e^{1/2}}{2}\int_0^{\infty} y^{-1/2}\left(1+y^{\alpha}\right)^{-1/\alpha-1} dy. \qquad (4.43)
$$

Now we see the first part of Theorem 2.9. The second part follows from Theorem 2.5 since $q_2^F(\alpha,2) = 2(1+e^{-1/2}q_2^G(\alpha))$. This finishes the proof of Theorem 2.9. ■

α-Limiting behaviour

It remains to analyze the limiting behaviour as stated at the end of Theorems 2.5 and 2.7. This is fairly straightforward and most of the work goes into showing that limit and integral can be interchanged. For the $\alpha \to \infty$ limit in Theorem 2.5 we use monotone convergence. To justify this we first look at $d/d\alpha f_\alpha(y)$. Differentiating shows that

$$
\frac{d}{d\alpha} f_\alpha(y) > 0 \iff (1+y^\alpha)\log(1+y^\alpha) > (1+\alpha)y^\alpha \log y^\alpha\ . \qquad (4.44)
$$

We see that for $y \in [0,1)$ the right-hand side is negative and the left-hand side is positive, so $f_\alpha(y)$ is increasing in α for $y \in [0,1)$. For the interval $[1,\infty)$ we have to do some extra work: Let $\varepsilon > 0$. Now for all α such that

$$
\alpha^2 - \alpha > \frac{2\log 2}{\log(1+\varepsilon)}
$$

and for all $y \geq 1+\varepsilon$ it holds that

$$\alpha^2 - \alpha > \frac{2\log 2}{\log(1+\varepsilon)} \geq \frac{2\log 2}{\log y} \ .$$

This leads to

$$\begin{aligned} \alpha \log y^\alpha &> 2\log 2 + \alpha \log y = \log 4y^\alpha > \log\left(y^\alpha + 2 + y^{-\alpha}\right) \\ &= \log\left(y^\alpha + 1\right) + \log\!\left(y^{-\alpha}+1\right) > y^{-\alpha}\log\left(y^\alpha+1\right) + \log\left(y^{-\alpha}+1\right) \\ &= \left(1+y^{-\alpha}\right)\log\left(y^\alpha+1\right) - \log y^\alpha, \qquad (4.45) \end{aligned}$$

which in turn can be rewritten as

$$(1+\alpha)y^\alpha \log y^\alpha > (1+y^\alpha)\log\left(1+y^\alpha\right) \ . \qquad (4.46)$$

This, together with (4.44) implies that $f_\alpha(y)$ is decreasing in α on $[1+\varepsilon, \infty)$, for sufficiently large α. So we now can use monotone convergence on $[1+\varepsilon, \infty)$. Here we remark that $f_\alpha(y)$ is bounded from above by 1 and that $f_{-1/\beta}(y)$ is bounded on $[1, 1+\varepsilon]$ by a constant c.

$$\begin{aligned} &\limsup_{\alpha\to\infty} q_2^F(\alpha,\beta) \overset{(4.30)}{=} \limsup_{\alpha\to\infty}\left(1 + \int_0^\infty f_{-1/\beta}(y)\cdot f_\alpha(y)dy\right) \\ = \ & 1 + \int_0^1 f_{-1/\beta}(y)\cdot \lim_{\alpha\to\infty} f_\alpha(y)dy + \int_{1+\varepsilon}^\infty f_{-1/\beta}(y)\cdot \lim_{\alpha\to\infty} f_\alpha(y)dy \\ & + \limsup_{\alpha\to\infty}\int_1^{1+\varepsilon} f_{-1/\beta}(y)\cdot f_\alpha(y)dy \\ \leq \ & 1 + \int_0^1 f_{-1/\beta}(y)\cdot 1 \ dy + \int_{1+\varepsilon}^\infty f_{-1/\beta}(y)\cdot 0 \ dy + \lim_{\alpha\to\infty}\int_1^{1+\varepsilon} c\cdot 1 \ dy \\ = \ & 1 + \left[(1+y^{1/\beta})^\beta\right]_0^1 + c\varepsilon = 2^\beta + c\varepsilon \ . \qquad (4.47) \end{aligned}$$

From these calculations one can also see that

$$\liminf_{\alpha\to\infty} q_2^F(\alpha,\beta) \geq 2^\beta.$$

For the $\alpha \to 0$ limit let $\varepsilon, \beta > 0$. Since

$$\lim_{y\to\infty} f_{-1/\beta}(y) = \lim_{y\to\infty}(1+y^{-1/\beta})^{\beta-1} = 1 \ , \qquad (4.48)$$

there is an $y_{\varepsilon,\beta}$ such that for all $y > y_{\varepsilon,\beta}$: $|f_{-1/\beta}(y) - 1| < \varepsilon$. Then:

$$\begin{aligned}
q_2^F(\alpha,\beta) &\overset{(4.30)}{=} 1 + \int_0^\infty f_{-1/\beta}(y) \cdot f_\alpha(y)dy \\
&\leq 1 + \int_0^{y_{\varepsilon,\beta}} f_{-1/\beta}(y) \cdot f_\alpha(y)dy + \int_{y_{\varepsilon,\beta}}^\infty (1+\varepsilon) f_\alpha(y)dy \qquad (4.49)\\
&\overset{(4.31)}{=} 1 + \int_0^{y_{\varepsilon,\beta}} f_{-1/\beta}(y) \cdot f_\alpha(y)dy + \left(1 - \left(1 + y_{\varepsilon,\beta}^{-\alpha}\right)^{-1/\alpha}\right)(1+\varepsilon) \ .
\end{aligned}$$

As both $\lim_{\alpha\downarrow 0} (1+x^\alpha)^{-1/\alpha} = 0$ and $\lim_{\alpha\downarrow 0} (1+x^{-\alpha})^{-1/\alpha} = 0$ for all $x > 0$ by dominated convergence we arrive at

$$\lim_{\alpha\downarrow 0} \int_0^{y_{\varepsilon,\beta}} f_{-1/\beta}(y) \cdot f_\alpha(y)dy = \int_0^{y_{\varepsilon,\beta}} f_{-1/\beta}(y) \cdot \left(\lim_{\alpha\downarrow 0} f_\alpha(y)\right) dy = 0 \ .$$

The last three equations yield

$$\limsup_{\alpha\downarrow 0} q_2^F(\alpha,\beta) \leq 2 + \varepsilon. \qquad (4.50)$$

Likewise

$$\liminf_{\alpha\downarrow 0} q_2^F(\alpha,\beta) \geq 2 - \varepsilon, \qquad (4.51)$$

which what is claimed in Theorem 2.5.

The considerations leading to the α-limits in Theorem 2.7 are almost the same: For the $\alpha \to \infty$–limit we can take 1 for c and the integral becomes:

$$\begin{aligned}
&\lim_{\alpha\to\infty} q_2^W(\alpha,\beta) = \lim_{\alpha\to\infty} \left(\int_0^\infty f_{1/\beta}(y) \cdot f_\alpha(y)dy\right) \\
= \ & \int_0^1 f_{1/\beta}(y) \cdot \lim_{\alpha\to\infty} f_\alpha(y)dy + \int_{1+\varepsilon}^\infty f_{1/\beta}(y) \cdot \lim_{\alpha\to\infty} f_\alpha(y)dy \\
&+ \lim_{\alpha\to\infty} \int_1^{1+\varepsilon} f_{1/\beta}(y) \cdot f_\alpha(y)dy \\
= \ & \int_0^1 f_{1/\beta}(y) \cdot 1 \ dy + \int_{1+\varepsilon}^\infty f_{1/\beta}(y) \cdot 0 \ dy + O\left(\lim_{\alpha\to\infty} \int_1^{1+\varepsilon} 1 \ dy\right) \\
= \ & \left[(1+y^{-1/\beta})^{-\beta}\right]_0^1 + o(\varepsilon) = 2^{-\beta} + O(\varepsilon) \ . \qquad (4.52)
\end{aligned}$$

And this becomes $2^{-\beta}$ as $\varepsilon \downarrow 0$. For the $\alpha \to 0$–limit we remark that (compare (4.48))

$$\lim_{y\to\infty} f_{1/\beta}(y) = \lim_{y\to\infty} (1+y^{1/\beta})^{-\beta-1} = 0 \tag{4.53}$$

and if we now take $y_{\varepsilon,\beta}$ such that

$$f_{1/\beta}(y) < \varepsilon \ , \ \forall y > y_{\varepsilon,\beta}, \tag{4.54}$$

we see that

$$\begin{aligned}\limsup_{\alpha\downarrow 0} q_2^W(\alpha,\beta) &\leq \limsup_{\alpha\downarrow 0}\left(\int_0^{y_{\varepsilon,\beta}} f_{1/\beta}(y)\cdot f_\alpha(y)dy + \int_{y_{\varepsilon,\beta}}^{\infty} \varepsilon f_\alpha(y)dy\right)\\ &= \int_0^{y_{\varepsilon,\beta}} f_{-1/\beta}(y)\cdot\left(\lim_{\alpha\downarrow 0} f_\alpha(y)\right)dy + \limsup_{\alpha\downarrow 0}\left(1+\left(1+y_{\varepsilon,\beta}^{-\alpha}\right)^{-1/\alpha}\right)\varepsilon = \varepsilon.\end{aligned}$$

Eventually (2.12) follows immediately from (2.11).

Chapter 2

Different marginals

In this chapter we shall look at the case where the random variables have different marginals. The copula remains unchanged, however.

1 Introduction

The restrictions we allowed ourselves in Chapter 1 are rather strict. Only very rarely one does encouter such nice conditions. Even in our own example in section 3 of Chapter 1 we did not use portfolios (random variables) with identical marginal distributions. However, even the translation trick we used there is often not sufficient to apply the result of the previous chapter. In this chapter we shall loosen some of the restrictions. We shall still use an Archimedean copula with a regularly varying generator, but we shall allow for more differences in the marginal distributions. There are two ideas that make this possible. Firstly, we shall see that the limiting behaviour is determined by the marginals with the heaviest tails. Secondly, we do not need all marginals to have the same behaviour, as long as they (or at least the 'heaviest' ones) behave more or less the same in the limit. To be more precise: If their distribution functions behave similarly up to a certain constant in the limit. When we have shown that the restrictions of the theorem can be loosened, we shall revisit the example from Chapter 1. We shall also show that this theorem should be used with care, when using it to determine Value-at-Risk.

2 A theorem about different marginals

For the Fréchet case we see the following:

Theorem 2.1 *Let $m \geq 2$ and $\alpha, \beta > 0$. Furthermore let $X = (X_1, \dots, X_m)$, $X_i \leq 0 \forall i$ have marginals $F_1, \dots, F_m$ such that F_1 is regularly varying at $-\infty$ with index $-\beta$, and let $c_1 := 1$, $c_2 \geq \dots \geq c_m$ s.t.*

$$\lim_{u \to \infty} \frac{F_i(-u)}{F_1(-u)} = c_i \tag{2.1}$$

exists for all $i \in \{2, \dots, m\}$ and s.t. X has Archimedean copula C^ϕ, where ϕ is regularly varying at 0^+ with index $-\alpha$. Finally let D be the largest number s.t. $c_D > 0$.

Then:

$$\lim_{u \to \infty} \frac{1}{F_1(-u)} \mathbb{P}\left(\sum_{i=1}^m X_i \leq -u \right) = q_D^F(\alpha, \beta), \qquad \textit{with} \tag{2.2}$$

$$q_D^F(\alpha,\beta) = \lim_{\varepsilon \downarrow 0} \int_{\sum_i 1/x_i \geq 1, x_1 \leq 1/\varepsilon} \frac{d^D}{dx_1 \dots dx_D} \left(\sum_{i=1}^D \left(c_i x_i^\beta \right)^{-\alpha} \right)^{-1/\alpha} dx_1 \dots dx_D \;. \tag{2.3}$$

This result is a bit different from the result from the previous chapter in that it does not depend on the last $m - D$ random variables and because of the constants. But:

Remark 2.2 *If $c_2 = \dots = c_D = 1$, then the constant q_D^F is actually the same as the constant in the previous chapter, with D random variables.*

3 The proof

The proof is largely the same as in the previous chapter, but with some exceptions and most notably by showing that the last $m - D$ random variables do not matter in the limit.

Lemma 3.1 *(Fréchet) Let $m \geq 2$ and $\alpha, \beta > 0$. Furthermore let $X = (X_1, \dots, X_m)$, $X_i \leq 0 \forall i$ have marginals $F_1, \dots, F_m$ such that F_1 is regularly varying at $-\infty$*

with index $-\beta$, and let $c_1 := 1$, $c_2 \geq \ldots \geq c_m$ s.t.

$$\lim_{u\to\infty} \frac{F_i(-u)}{F_1(-u)} = c_i \tag{3.1}$$

exists for all $i \in \{2, \ldots, m\}$ and s.t. X has Archimedean copula C^ϕ, where ϕ is regularly varying at 0^+ with index $-\alpha$. Let D be the largest number s.t. $c_D > 0$. Furthermore, let $\varepsilon \in (0,1)$, $x_1 \in (0, 1/\varepsilon)$ and $x_2, \ldots, x_m > 0$. Then:

$$\lim_{u\to\infty} \mathbb{P}\left(X_i \leq -u/x_i, i = 1, \ldots, D \mid X_1 \leq -\varepsilon u\right) = \left(\sum_{i=1}^{D} \left(c_i x_i^\beta\right)^{-\alpha}\right)^{-1/\alpha} \varepsilon^\beta \ . \tag{3.2}$$

Proof. Since ϕ and F_1 are regularly varying, the following holds: For every $\delta > 0$ there is an u_0 such that for all $i \in \{1, \ldots, D\}$ and $u > u_0$:

$$\begin{aligned} F_1(-u/x_i) \leq (x_i + \delta)^\beta F_1(-u), \qquad (\varepsilon + \delta)^{-\beta} F_1(-u) \leq F_1(-\varepsilon u), \\ \frac{F_i(-u)}{F_1(-u)} \leq c_i + \delta \qquad \text{and} \end{aligned} \tag{3.3}$$

$F(-u)$ is so close to 0 that :

$$\begin{aligned} &\phi((c_i + \delta)(x_i + \delta)^\beta F(-u)) \\ &\quad \geq ((c_i + \delta)(x_i + \delta)^\beta + \delta)^{-\alpha} \phi(F(-u)), \text{ and} \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\sum_{i=1}^{D} ((c_i + \delta)(x_i + \delta)^\beta + \delta)^{-\alpha} \phi \circ F(-u) \\ &\quad \leq \phi\left(\left(\sum_{i=1}^{D} ((c_i + \delta)(x_i + \delta)^\beta + \delta)^{-\alpha} - \delta\right)^{-1/\alpha} F(-u)\right). \end{aligned} \tag{3.5}$$

Now we show the upper bound:

$$
\begin{aligned}
&\limsup_{u\to\infty} \mathbb{P}\left(X_i \le -u/x_i, i=1,\ldots,D \mid X_1 \le -\varepsilon u\right) \\
&\quad = \limsup_{u\to\infty} \frac{\phi^{-1}\left(\sum_{i=1}^{D} \phi\circ F_i(-u/x_i) + \sum_{j=D+1}^{m} \phi(\mathbb{P}(X_j<\infty))\right)}{F_1(-\varepsilon u)} \\
&\quad = \limsup_{u\to\infty} \frac{\phi^{-1}\left(\sum_{i=1}^{D} \phi\circ F_i(-u/x_i)\right) + \sum_{j=D+1}^{m} \phi(1)}{F_1(-\varepsilon u)} \\
&\quad = \limsup_{u\to\infty} \frac{\phi^{-1}\left(\sum_{i=1}^{D} \phi\circ F_i(-u/x_i)\right) + \sum_{j=D+1}^{m} 0}{F_1(-\varepsilon u)} \\
&\quad = \limsup_{u\to\infty} \frac{\phi^{-1}\left(\sum_{i=1}^{D} \phi\left(\frac{F_i(-u/x_i)}{F_1(-u/x_i)} F_1(-u/x_i)\right)\right)}{F_1(-\varepsilon u)} \\
&\quad \le \limsup_{u\to\infty} \frac{\phi^{-1}\left(\sum_{i=1}^{D} \phi\left((c_i+\delta) F_1(-u/x_i)\right)\right)}{F_1(-\varepsilon u)} \\
&\quad \le \limsup_{u\to\infty} \frac{\phi^{-1}\left(\sum_{i=1}^{D} \phi\left((c_i+\delta)(x_i+\delta)^{\beta} F(-u)\right)\right)}{(\varepsilon+\delta)^{-\beta} F_1(-u)} \\
&\quad \le \limsup_{u\to\infty} \frac{\phi^{-1}\left(\sum_{i=1}^{D} ((c_i+\delta)(x_i+\delta)^{\beta}+\delta)^{-\alpha} \phi\circ F(-u)\right)}{(\varepsilon+\delta)^{-\beta} F(-u)} \\
&\quad \le \frac{\left(\sum_{i=1}^{D} ((c_i+\delta)(x_i+\delta)^{\beta}+\delta)^{-\alpha} - \delta\right)^{-1/\alpha}}{(\varepsilon+\delta)^{-\beta}},
\end{aligned}
\tag{3.6}
$$

where for the first two inequalities we applied (3.3), for the third inequality we applied (3.4), and for the fourth inequality we applied (3.5). Since this holds for all $\delta > 0$, we get the upper bound. The lower bound is proven similarly, by taking $-\delta$ instead of $+\delta$ ■ Note that

$$
G_{\varepsilon}^{\alpha,\beta}(x_1,\ldots,x_D) \overset{def.}{=} \left(\sum_{i=1}^{D} \left(c_i x_i^{\beta} \right)^{-\alpha} \right)^{-1/\alpha} \varepsilon^{\beta} \tag{3.7}
$$

is a distribution function on $(0,1/\varepsilon)\times(0,\infty)^{D-1}$. Let $g_\varepsilon^{\alpha,\beta}$ be its density function and define:

$$\begin{aligned} G(\varepsilon) &\stackrel{def.}{=} \varepsilon^{-\beta}\int_{\sum_i 1/x_i\geq 1, x_1\leq 1/\varepsilon} g_\varepsilon^{\alpha,\beta}(x_1,\dots,x_D)dx_1\dots dx_D \\ &= \int_{\sum_i 1/x_i\geq 1, x_1\leq 1/\varepsilon} \frac{d^D}{dx_1\dots dx_D}\left(\sum_{i=1}^{D}\left(c_i x_i^\beta\right)^{-\alpha}\right)^{-1/\alpha} dx_1\dots dx_D. \quad (3.8)\end{aligned}$$

Since $G(\varepsilon)$ is increasing for $\varepsilon\downarrow 0$, one can define $G(0)=\lim_{\varepsilon\downarrow 0}G(\varepsilon)\leq\infty$.

Proof of Theorem 2.1. Once again we connect

$$\mathbb{P}\left(\sum_{i=1}^m X_i\leq -u|X_1\leq -\varepsilon u\right) \text{ with } \mathbb{P}(X_i\leq -u/x_i, i=1,\dots,D \mid X_1\leq -\varepsilon u)$$

in the following way:

$$\lim_{u\to\infty}\mathbb{P}(\sum_{i=1}^m X_i\leq -u|X_1\leq -\varepsilon u)=\varepsilon^\beta G(\varepsilon).$$

And again we take random variables $Y_1^{(u)},\dots,Y_D^{(u)}$ with distribution function

$$H(x_1,\dots,x_D)\stackrel{def.}{=}\mathbb{P}(X_i\leq -u/x_i, i=1,\dots,D \mid X_1\leq -\varepsilon u)$$

and random variables $Y_1,\dots,Y_D$ with distribution function $G_\varepsilon^{\alpha,\beta}(x_1,\dots,x_D)$. From Lemma 3.1 it follows that $(Y_1^{(u)},\dots,Y_D^{(u)})$ converges in distribution to $(Y_1,\dots,Y_D)$, as $u\to\infty$, and thus

$$\mathbb{P}(\sum_{i=1}^D 1/Y_i^{(u)}\geq 1)=\mathbb{P}(\sum_{i=1}^D X_i\leq -u|X_1\leq -\varepsilon u)$$

converges (again as $u\to\infty$) to

$$\mathbb{P}(\sum_{i=1}^D 1/Y_i\geq 1)=\int_{\sum_i 1/x_i\geq 1, x_1\leq 1/\varepsilon} g_\varepsilon^{\alpha,\beta}(x_1,\dots,x_D)dx_1\dots dx_D=\varepsilon^\beta G(\varepsilon).$$

For the lower bound we see that

$$\begin{aligned}
&\liminf_{u\to\infty} \frac{1}{F_1(-u)}\mathbb{P}\left(\sum_{i=1}^{m} X_i \le -u\right) \\
&\ge \liminf_{u\to\infty} \frac{1}{F_1(-u)}\mathbb{P}\left(\sum_{i=1}^{D} X_i \le -u\right) \\
&\ge \liminf_{u\to\infty} \frac{F_1(-\varepsilon u)}{F_1(-u)}\mathbb{P}\left(\sum_{i=1}^{D} X_i \le -u \,\middle|\, X_1 \le -\varepsilon u\right) \\
&= \liminf_{u\to\infty} \frac{F_1(-\varepsilon u)}{F_1(-u)}\varepsilon^{\beta} G(\varepsilon) \;=\; G(\varepsilon), \qquad (3.9)
\end{aligned}$$

where we used again that F_1 is regularly varying. Since $\varepsilon > 0$ was arbitrary

$$\liminf_{u\to\infty} \frac{1}{F_1(-u)}\mathbb{P}\left(\sum_{i=1}^{D} X_i \le -u\right) \ge G(0). \qquad (3.10)$$

For the upper bound choose $\varepsilon < 1/D$, $1 > \gamma > 0$. Then

$$\begin{aligned}
&\limsup_{u\to\infty} \frac{1}{F_1(-u)}\mathbb{P}\left(\sum_{i=1}^{m} X_i \le -u\right) \\
&= \limsup_{u\to\infty} \left(\frac{1}{F_1(-u)}\mathbb{P}\left(\sum_{i=1}^{m} X_i \le -u, X_1 \le -\varepsilon u\right)\right. \\
&\qquad\qquad \left. + \frac{1}{F_1(-u)}\mathbb{P}\left(\sum_{i=1}^{m} X_i \le -u, X_1 > -\varepsilon u\right)\right) \\
&\le \limsup_{u\to\infty} \left(\frac{1}{F_1(-u)}\mathbb{P}\left(\sum_{i=1}^{D} X_i \le -(1-\gamma)u, X_1 \le -\varepsilon u\right)\right. \\
&\qquad\qquad + \frac{1}{F_1(-u)}\mathbb{P}\left(\sum_{i=D+1}^{m} X_i \le -\gamma u, X_1 \le -\varepsilon u\right) \\
&\qquad\qquad \left. + \frac{1}{F_1(-u)}\mathbb{P}\left(\sum_{i=1}^{m} X_i \le -u, X_1 > -\varepsilon u\right)\right).
\end{aligned}$$

For the first term we have (as $\gamma \downarrow 0$):

$$\limsup_{u\to\infty} \frac{1}{F_1(-u)} \mathbb{P}\left(\sum_{i=1}^{D} X_i \le -u, X_1 \le -\varepsilon u\right) = G(\varepsilon). \tag{3.11}$$

For the second term we see:

$$\begin{aligned}
& \limsup_{u\to\infty} \frac{1}{F_1(-u)} \mathbb{P}\left(\sum_{i=D+1}^{m} X_i \le -\gamma u, X_1 \le -\varepsilon u\right) \\
\le\ & \limsup_{u\to\infty} \frac{1}{F_1(-u)} \sum_{i=D+1}^{m} \mathbb{P}\left(X_i \le -\frac{\gamma}{m-D}u, X_1 \le -\varepsilon u\right) \\
=\ & \limsup_{u\to\infty} \frac{1}{F_1(-u)} \sum_{i=D+1}^{m} \phi^{-1}\left(\phi(F_1(-\varepsilon u)) + (\sum_{i=2}^{D}\phi(1)) + \right. \\
& \qquad \left. \phi(F_i(-\frac{\gamma}{m-D}u) + (\sum_{k=D+1,k\neq i}^{m} \phi(1))\right) \\
=\ & \limsup_{u\to\infty} \frac{1}{F_1(-u)} \sum_{i=D+1}^{m} \phi^{-1}\left(\phi(F_1(-\varepsilon u)) + \phi(F_i(-\frac{\gamma}{m-D}u)\right) \\
=\ & \limsup_{u\to\infty} \frac{1}{F_1(-u)} \cdot \\
& \sum_{i=D+1}^{m} \phi^{-1}\left(\varepsilon^{\alpha\beta}\phi(F_1(-u)) + \left(\left(\frac{\gamma}{m-D}\right)^{-\beta}\frac{F_i(-u)}{F_1(-u)}\right)^{-\alpha}\phi(F_1(-u))\right) \\
=\ & \limsup_{u\to\infty} \sum_{i=D+1}^{m} \left(\varepsilon^{\alpha\beta} + \left(\left(\frac{\gamma}{m-D}\right)^{-\beta}\frac{F_i(-u)}{F_1(-u)}\right)^{-\alpha}\right)^{-1/\alpha} \\
=\ & 0,
\end{aligned} \tag{3.12}$$

where in the last equality we used the fact that

$$\lim_{u\to\infty} \frac{F_i(-u)}{F_1(-u)} = 0,$$

for all $i \in \{D+1, \ldots, m\}$.

For the third term:

$$
\begin{aligned}
&\limsup_{u\to\infty} \frac{1}{F_1(-u)} \mathbb{P}\left(\sum_{i=1}^m X_i \le -u, X_1 > -\varepsilon u\right) \\
&\le \limsup_{u\to\infty} \frac{1}{F_1(-u)} \sum_{i=2}^m \mathbb{P}\left(X_i \le -u/m, X_1 > -\varepsilon u\right) \\
&= \limsup_{u\to\infty} \frac{1}{F_1(-u)} \sum_{i=2}^m \left(\mathbb{P}(X_i \le -u/m) - \mathbb{P}\left(X_i \le -u/m, X_1 \le -\varepsilon u\right)\right) \\
&= \lim_{u\to\infty} \sum_{i=2}^m \frac{F_i(-u/m)}{F_1(-u)} \\
&- \liminf_{u\to\infty} \sum_{i=2}^m \frac{\mathbb{P}\left(X_i \le -u/m, X_1 \le -\varepsilon u\right)}{F_1(-u)} \\
&= \lim_{u\to\infty} \sum_{i=2}^m \frac{F_i(-u/m)}{F_1(-u/m)} \frac{F_1(-u/m)}{F_1(-u)} \\
&- \liminf_{u\to\infty} \sum_{i=2}^m \frac{\phi^{-1}\left(\phi\circ F_1(-\varepsilon u) + \phi\circ F_i(-u/m)\right)}{F_1(-u)} \\
&= \sum_{i=2}^m c_i m^\beta - \liminf_{u\to\infty} \sum_{i=2}^m \frac{\phi^{-1}\left(\phi\circ F_1(-\varepsilon u) + \phi\left(\frac{F_i(-u/m)}{F_1(-u/m)} F_1(-u/m)\right)\right)}{F_1(-u)} \\
&= \sum_{i=2}^D c_i m^\beta \\
&- \liminf_{u\to\infty} \sum_{i=2}^m \frac{\phi^{-1}\left(\varepsilon^{\alpha\beta}\phi\circ F_1(-u) + \left(\frac{F_i(-u/m)}{F_1(-u/m)}\right)^{-\alpha} m^{-\alpha\beta}\phi\circ F_1(-u)\right)}{F_1(-u)} \\
&= \sum_{i=2}^D c_i m^\beta - \liminf_{u\to\infty} \sum_{i=2}^m \left(\varepsilon^{\alpha\beta} + \left(\frac{F_i(-u/m)}{F_1(-u/m)}\right)^{-\alpha} m^{-\alpha\beta}\right)^{-1/\alpha} \\
&= \sum_{i=2}^D c_i m^\beta - \liminf_{u\to\infty} \sum_{i=2}^D \left(\varepsilon^{\alpha\beta} + c_i^{-\alpha} m^{-\alpha\beta}\right)^{-1/\alpha} .
\end{aligned}
$$

First we let $\gamma \downarrow 0$ and then, since $\varepsilon \in (0, 1/m)$, we let $\varepsilon \downarrow 0$ and arrive at:

$$\limsup_{u\to\infty} \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^{m} X_i \leq -u\right) \leq G(0), \tag{3.13}$$

which is the upper bound we claimed. This finishes the proof of Theorem 2.1. ■

4 Examples

In this section we would like to revisit the example from Chapter 1 and look at a second example, which serves as a warning against the wrong use of the results of this chapter.

4.1 Chapter 1 revisited

Armed with Theorem 2.1, we look again at our two motor liability portfolios from the previous chapter. Again we name them X_1 and X_2 and merge them to one big portfolio, but this time we do not translate the portfolios, but directly use our theorem. Just like in Chapter 1 we assume X_1 and X_2 have Archimedean copula generated by a regularly varying function with index $-\alpha$ at 0^+ ($\alpha > 0$). Moreover, we assume that $-X_1$ and $-X_2$ have translated Pareto marginals with translation $V_1 = 880$ and $V_2 = 820$, i.e. $Y_i = -(X_i + V_i)$ is Pareto distributed with $\theta = 80$ and $\beta = 3$: for $i = 1, 2$

$$\mathbb{P}(X_i \leq x) = \mathbb{P}(X_i + V_i \leq x + V_i) = \left(\frac{\theta}{-(x + V_i)}\right)^{\beta} \qquad \text{for } x \leq -(\theta + V_i). \tag{4.1}$$

Again we choose $p = 99.5\%$ and look at the Value-at-Risk

$$\mathrm{VaR}_{X_i} \overset{def.}{=} -\sup\{x;\ \mathbb{P}(X_i \geq x) \geq p\} + E[X_i]. \tag{4.2}$$

We shall repeat the table from the previous chapter:

	portfolio 1	portfolio 2
translation V_i	880	820
mean $\mathbb{E}(-X_i)$	1000	940
variational coefficient	6.9%	7.3%
VaR_{X_i}	347.8	347.8

But here we diverge from what we did in the previous chapter. We shall no longer use individual translations of the random variables. First we look at the case where we translate both portfolios by the same amount, before applying Theorem 2.1. We shall translate them by 850 (the average of V_1 and V_2). Let's define $Z_i := X_i + 850$ and $F_i := F_{Z_i}$. (Note here that (Z_1, Z_2) has the same copula as (X_1, X_2).) Then we see that for $t > 0$:

$$\lim_{u\to\infty} \frac{F_1(-tu)}{F_1(-u)} = \lim_{u\to\infty} \frac{\mathbb{P}(X_1 + 850 \leq -tu)}{\mathbb{P}(X_1 + 850 \leq -u)}$$
$$= \lim_{u\to\infty} \frac{\mathbb{P}(X_1 + 880 \leq 30 - tu)}{\mathbb{P}(X_1 + 880 \leq 30 - u)} = \lim_{u\to\infty} \frac{\left(\frac{80}{tu-30}\right)^{\beta}}{\left(\frac{80}{u-30}\right)^{\beta}} = t^{-\beta}. \tag{4.3}$$

Now we look at the ratio of F_1 and F_2:

$$c_2 := \lim_{u\to\infty} \frac{F_2(-u)}{F_1(-u)} = \lim_{u\to\infty} \frac{\mathbb{P}(X_2 + 850 \leq -u)}{\mathbb{P}(X_1 + 850 \leq -u)}$$
$$= \lim_{u\to\infty} \frac{\mathbb{P}(X_2 + 820 \leq -30 - u)}{\mathbb{P}(X_1 + 880 \leq 30 - u)} = \lim_{u\to\infty} \frac{\left(\frac{80}{u+30}\right)^{\beta}}{\left(\frac{80}{u-30}\right)^{\beta}} = 1. \tag{4.4}$$

It is very nice to get a ratio of one, because now we can use Theorem 2.5 of the previous chapter for our calculations below. But first we see that for large u:

$$\mathbb{P}(X_1 + X_2 \leq -u)$$
$$= \mathbb{P}(Z_1 + Z_2 \leq -u + 1700) \sim q_2^F(\alpha,\beta) \left(\frac{\theta}{u - 1700 - 30}\right)^{\beta}. \tag{4.5}$$

Notice the difference between this result and (3.3) of Chapter 1. To further compare the results, we shall again look at $\mathrm{VaR}_{X_1+X_2}(\alpha)$ as defined in (4.2). Hence the Value-at-Risk of $X_1 + X_2$ is can now be approximated by (4.5). We obtain

$$\mathrm{VaR}_{X_1+X_2}(\alpha) \approx V(\alpha) \overset{def.}{=} \left(\theta \left(\frac{q_2^F(\alpha,\beta)}{1-p}\right)^{1/\beta} + 1730\right) + \mathbb{E}(X_1 + X_2). \tag{4.6}$$

Since we can use Theorem 2.5 of the previous chapter for $q_2^F(\alpha,\beta)$, we can numerically approximate the Value-at-Risk for different α and get the following table (where we again calculate the result for $\alpha = 0$ directly):

α	*indep.*	0.5	1.0	1.5	2.0	3.0	4.0	∞
$-\mathbb{E}(X_1+X_2)$	1940	1940	1940	1940	1940	1940	1940	1940
$V(\alpha)$	476.1	601.8	678.0	700.5	710.2	718.1	721.2	725.7
Div.eff.(α)	31.6%	13.5%	2.5%	−0.7%	−2.1%	−3.2%	−3.7%	−4.3%

These results are not too bad, but for large α not as sharp as the previous chapter (a negative diversification effect is impossible with total dependence). So be careful when using this theorem for calculating Value-at-Risk.

4.2 Warning

To make things worse, Theorem 2.1 can easily be used in an improper manner. For instance, let's look at a large portfolio that consists of m smaller portfolios $\{X_1,\dots,X_m\}$. Assume these portfolios are dependent according to some Archimedean copula with a generator function ϕ that is regularly varying at 0^+ with index $-\alpha$ for some $\alpha>0$. Furthermore, suppose these portfolios have the same marginal distribution function, which is regularly varying at $-\infty$ with index $-\beta$ for some $\beta>1$. Then, for some probability p, define $V_p = VaR_{X_1} (= VaR_{X_2} = \dots = VaR_{X_m})$. But now we shall misuse the theorem. Instead of calculating the VaR of the entire portfolio using Chapter 1 (which is the proper thing to do), we use a trick and the result of this chapter. We fabricate a fictious portfolio X_0 that is regularly varying at $-\infty$ with index $\varepsilon-\beta$ for some very small $\varepsilon>0$. Then we pretend this fictious portfolio to be co-dependent with the other portfolios through a $(m+1)$-dimensional copula that has the same generator ϕ. Because of the structure of Archimedean copulas, this can be done without penalty. Then we can remark that

$$\lim_{u\to\infty}\frac{F_{X_i}}{F_{X_0}} = 0\ ,$$

for all $i\in\{1,\dots,m\}$. Now we can (mis)use Theorem 2.1 and get

$$\lim_{u\to\infty}\frac{1}{F_0(-u)}\mathbb{P}\left(\sum_{i=0}^{m} X_i \le -u\right) = q_1^F(\alpha,\beta-\varepsilon) = 1. \qquad (4.7)$$

So now, using this method to calculate VaR, we can state that

$$VaR_{\sum_{i=0}^{m} X_i} = VaR_{X_0}\ .$$

Since X_0 was only a product of our imagination, we can choose ε conveniently close to 0, so that VaR_{X_0} is close to V_p. This would mean that the VaR of the entire portfolio is as large as the VaR of one of the smaller portfolios, which is of course false.
So be careful when D is much smaller than m and when in those cases the ratio 0 between the marginal distributions is reached only very slowly.

Chapter 3

Non-Archimedian copulas

In Chapter 1 we have described the behaviour of the sum of m dependent random variables in the situation where the dependence structure of the risks can be described by an Archimedean copula. However, the Archimedean assumption was more than actually was needed. The aim of this chapter is to relax in the Archimedean setting and to still obtain a result of the following type. Consider m identically distributed dependent risks $X_1, \ldots, X_m$. Then, under appropriate conditions, we obtain results of the following type

$$\mathbb{P}\left(\sum_{i=1}^{m} X_i \leq -u\right) \sim q_m \cdot \mathbb{P}\left(X_1 \leq -u\right), \qquad \text{as } u \to \infty. \tag{0.1}$$

Here, the constant q_m quantifies the diversification effect between the dependent risks. Moreover it shows how the m-dimensional case is related to the one-dimensional case and how this relation changes with increasing m. For $m = 2$ we give explicit formulas for q_d, which give the connection between the diversification effect and the dependence strength.

1 The theorem

Let $m \in \mathbb{N}$ and $X_1, \ldots, X_m$ be random variables with (marginal) distribution functions $F_1, \ldots, F_m$. These will be the random variables that model the risks mentioned in the introduction. The goal is to derive extreme value theorems for their sum and to compare it to the extreme value behaviour of one of their marginals. To this end we will define the so–called lower-tail dependence coefficient.

Definition 1.1 *The lower-tail dependence coefficient* λ *of* $X_1, \ldots, X_m$ *is defined as*

$$\lambda \stackrel{def.}{=} \lim_{x \downarrow 0} \mathbb{P}(X_2 < F_2^{-1}(x), \ldots, X_m < F_m^{-1}(x) | X_1 < F_1^{-1}(x)) \ , \tag{1.1}$$

if this limit exists.

Note here that if the dependence structure of $(X_1, \ldots, X_m)$ is given by a copula C, then

$$\lambda = \lim_{x \downarrow 0} \frac{C(x, x, \ldots, x, x)}{x} \ . \tag{1.2}$$

For the rest of the text we assume the following.

Assumption 1.2 *The random vector* $(X_1, \ldots, X_m)$ *has a symmetric distribution (i.e. its distribution is invariant under permutations of the* X_i*) with copula* C*, marginals* F *and lower-tail dependence coefficient* $\lambda > 0$*. For simplicity assume that* $X_1, \ldots, X_m \leq 0$*. Furthermore there exists a measure* ν *on* $[0, \infty)^m$ *such that for* $g : [0, \infty)^m \mapsto [0, \infty) : (x_1, \ldots, x_m) \mapsto \nu((0, x_1] \times \ldots \times (0, x_m])$ *the following holds*

$$\lim_{u \downarrow 0} \frac{C(x_1 u, \ldots, x_m u)}{u} = \lambda \cdot g(x_1, \ldots, x_m), \qquad x \in [0, \infty), \tag{1.3}$$

and

$$\lim_{x \to \infty} \lambda g(1, x, \ldots, x) = 1 \ . \tag{1.4}$$

Remark 1.3 This second equation (1.4) states that if one of the random variables takes on an extreme value, the others will also have a more or less extreme value.

The theorem we present here splits into three parts. Just like in Chapter 1, each of these parts represents one of three types of tail-behaviour: Fréchet, Weibull or Gumbel (see also [13]).

Theorem 1.4 *Let* $m \geq 2$*,* $\beta > 0$ *and let* $(X_1, \ldots, X_m)$ *be a random vector that satisfies Assumption 1.2 with identical marginal distributions* F*. Then there are constants* q_g^F*,* q_g^W *and* q_g^G *such that*

a) *(The Fréchet case) If the marginals F are regularly varying at $-\infty$ with parameter $-\beta < 0$. Then:*

$$\lim_{u\to\infty} \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^{m} X_m \le -u\right) = q_g^F. \tag{1.5}$$

b) *(The Weibull case) If there is a $c \ge -\infty$ such that $s \to F(c - 1/s)$ is regularly varying at $-\infty$ with parameter $-\beta < 0$, then*

$$\lim_{u\downarrow c} \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^{m} X_m \le mc + 1/u\right) = q_g^W. \tag{1.6}$$

c) *(The Gumbel case) If there is a $c \ge -\infty$ and a positive function $s \to a(s)$ such that for $t \in \mathbb{R}$ one has*

$$\lim_{u\downarrow c} F(u + ta(u))/F(u) = e^t.$$

Then:

$$\lim_{u\downarrow c} \frac{1}{F(u)} \mathbb{P}\left(\sum_{i=1}^{m} X_m \le mu + a(u)\right) = q_g^G. \tag{1.7}$$

But if the awkward condition (1.4) is not fulfilled, the Gumbel part of the theorem still holds, and for the Fréchet part we can say the following:

Theorem 1.5 *Let $m \ge 2$, $\beta > 0$ and let $(X_1, \dots, X_m)$ be a random vector that satisfies Assumption 1.2, except for* (1.4), *with identical marginal distribution functions F that are regularly varying at $-\infty$ with parameter $-\beta < 0$. Then there is a constant q_g^F such that*

$$q_g^F \le \liminf_{u\to\infty} \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^{m} X_m \le -u\right) \le \limsup_{u\to\infty} \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^{m} X_m \le -u\right)$$
$$\le q_g^F + \lim_{x\to\infty} m(m-1)^{1-\beta} \left(1 - \lambda g\left(1, x, \dots, x\right)\right) \ . \tag{1.8}$$

The constant q_g^F is the same in the two theorems. These theorems will be proved in the following section. From this proof explicit values of q_g^F, q_g^W and q_g^G follow.

Proposition 1.6 *The constants q_g^F, q_g^W and q_g^G from* Theorem 1.4 *(and* 1.5*) are given by*

$$q_g^F = \lambda\nu\left(\left\{(x_1^\beta,\ldots,x_m^\beta) : \sum_{i=1}^m 1/x_i \geq 1\right\}\right), \tag{1.9}$$

$$q_g^W = \lambda\nu\left(\left\{(x_1^\beta,\ldots,x_m^\beta) : \sum_{i=1}^m x_i \geq 1\right\}\right) \text{ and} \tag{1.10}$$

$$q_g^G = \lambda\nu\left(\left\{(e^{x_1},\ldots,e^{x_m}) : \sum_{i=1}^m x_i \geq 1\right\}\right). \tag{1.11}$$

Remark 1.7 One might hope that knowledge of the lower-tail dependency coefficient λ is sufficient to be able to calculate q_g^F, q_g^W and q_g^G. However, it turns out that this is not the case; two copulas can have the same lower-tail dependency coefficient λ, but different limiting constants. (See Example 3.1.)

2 The proofs

As the theorem was split into three parts, so is the proof. First we prove the theorem in the Fréchet case and then we will deal with the Weibull and Gumbel case, insofar their proof is different from the Fréchet case.

Lemma 2.1 *g, as defined in Assumption 1.2, is uniformly continuous.*

Proof. Let $\varepsilon > 0$, $x_1,\ldots,x_m,y_1,\ldots,y_m > 0$ such that $\sum_{i=1}^m |x_i - y_i| < \varepsilon$, then

$$|g(x_1,\ldots,x_m) - g(y_1,\ldots,y_m)| = \lim_{u\downarrow 0}\left|\frac{C(x_1u,\ldots,x_mu)}{u} - \frac{C(y_1u,\ldots,y_mu)}{u}\right|$$
$$\stackrel{(*)}{\leq} \lim_{u\downarrow 0}\frac{\sum_{i=1}^m |ux_i - uy_i|}{u} = \sum_{i=1}^m |x_i - y_i| < \varepsilon, \tag{2.1}$$

where in $(*)$ we use [23], Theorem 2.10.7 .

■

2.1 The Fréchet case

The main idea is to first relate

$$\mathbb{P}(X_1 \leq -u) \qquad \text{to} \qquad \mathbb{P}(X_1 \leq -u/x_1,\ldots,X_m \leq -u/x_m)$$

and then afterwards to study the relation between

$$\mathbb{P}(X_1 \le -u/x_1, \dots, X_m \le -u/x_m) \qquad \text{and} \qquad \mathbb{P}(\sum_{i=1}^{m} X_m \le -u)\ .$$

The combinations of these two relations will give the assertion of the theorem.

Lemma 2.2 *(Fréchet) Assume $(X_1, \dots, X_m)$ satisfies Assumption 1.2 with identical marginals F which are regularly varying at $-\infty$ with parameter $-\beta < 0$. Furthermore, let $\varepsilon \in (0,1)$, $x_1 \in (0, 1/\varepsilon)$ and $x_2, \dots, x_m \ge 0$. Then:*

$$\lim_{u\to\infty} \mathbb{P}\left(X_1 \le -u/x_1, \dots, X_m \le -u/x_m \mid X_1 \le -\varepsilon u\right) = \varepsilon^\beta \lambda g\left(x_1^\beta, \dots, x_m^\beta\right)\ . \tag{2.2}$$

Proof of Lemma 2.2. Using the continuity of g we see

$$\begin{aligned}
&\lim_{u\to\infty} \mathbb{P}\left(X_1 \le -u/x_1, \dots, X_m \le -u/x_m \mid X_1 \le -\varepsilon u\right) \qquad (2.3)\\
&= \lim_{u\to\infty} \frac{\mathbb{P}\left(X_1 \le -u/x_1, \dots, X_m \le -u/x_m\right)}{\mathbb{P}\left(X_1 \le -\varepsilon u\right)}\\
&= \lim_{u\to\infty} \frac{C\left(F(-u/x_1), \dots, F(-u/x_m)\right)}{F(-\varepsilon u)}\\
&= \lim_{u\to\infty} \frac{C\left(\frac{F(-u/x_1)}{F(-u)} \cdot F(-u), \dots, \frac{F(-u/x_m)}{F(-u)} \cdot F(-u)\right)}{F(-u)} \cdot \frac{F(-u)}{F(-\varepsilon u)}\\
&\overset{(*)}{=} \lambda g\left(x_1^\beta, \dots, x_m^\beta\right) \varepsilon^\beta,
\end{aligned}$$

where in (*) we use that F is regularly varying at $-\infty$, equation (1.3) and the continuity of g. ∎

Definition 2.3 *In order to continue with the proof we need to define some measures:*
$\nu_\varepsilon^{\lambda,\beta}$ *on* $(0, 1/\varepsilon] \times (0,\infty)^{m-1}$. *For* $A \subset (0, 1/\varepsilon] \times (0,\infty)^{m-1}$ *(measurable)*

$$\nu_\varepsilon^{\lambda,\beta}(A) \overset{def.}{=} \varepsilon^\beta \lambda \nu\left(\left\{(x_1^\beta, \dots, x_m^\beta) : (x_1, \dots, x_m) \in A\right\}\right) \tag{2.4}$$

and μ_u *on* $(0, 1/\varepsilon) \times (0,\infty)^{m-1}$

$$\mu_u(A) \overset{def.}{=} \mathbb{P}\left(\left(\frac{-u}{X_1}, \dots, \frac{-u}{X_m}\right) \in A \,\middle|\, X_1 \le -\varepsilon u\right)\ . \tag{2.5}$$

Definition 2.4 *Furthermore, define the following function* $H(\varepsilon) : (0,\infty) \to [0,\infty)$

$$H(\varepsilon) \overset{def.}{=} \varepsilon^{-\beta}\nu_\varepsilon^{\lambda,\beta}\left(\left\{(x_1,\ldots,x_m) \in (0,1/\varepsilon]^m : \sum_{i=1}^m 1/x_i \geq 1\right\}\right).$$

Remark 2.5 Note here that $\varepsilon^{-\beta}$ and ε^{β} cancel each other, and that thus $H(\varepsilon)$ is decreasing in ε .

Lemma 2.6 *For* $\varepsilon > 0$ *it holds*

$$\lim_{u\to\infty} \mathbb{P}\left(\sum_{i=1}^m X_i \leq -u,\ X_n \leq -\varepsilon u\ \forall n \in \{1,\ldots,m\}\middle| X_1 \leq -\varepsilon u\right) = \varepsilon^\beta H(\varepsilon)\ . \tag{2.6}$$

Proof. From Lemma 2.2 we see that on

$$\{(0,a_1) \times \ldots \times (0,a_m) : (a_1,\ldots,a_m) \in (0,1/\varepsilon) \times (0,\infty)^{m-1}\},$$

μ_u converges to $\nu_\varepsilon^{\lambda,\beta}$ as u goes to infinty. But then we have that $\lim_{u\to\infty}\mu_u(B) = \nu_\varepsilon^{\lambda,\beta}(B)$ for all bounded subsets B of $(0,1/\varepsilon) \times (0,\infty)^{m-1}$, which implies:

$$\begin{aligned}
&\lim_{u\to\infty} \mathbb{P}\left(\sum_{i=1}^m X_i \leq -u,\ X_n \leq -\varepsilon u\ \forall n\middle| X_1 \leq -\varepsilon u\right) \\
&= \lim_{u\to\infty}\mu_u\left(\left\{(x_1,\ldots,x_m) \in (0,1/\varepsilon)^m : \sum_{i=1}^m 1/x_i \geq 1\right\}\right) \\
&= \nu_\varepsilon^{\lambda,\beta}\left(\left\{(x_1,\ldots,x_m) \in (0,1/\varepsilon)^m : \sum_{i=1}^m 1/x_i \geq 1\right\}\right) \\
&= \varepsilon^\beta H(\varepsilon)\ .
\end{aligned} \tag{2.7}$$

This finishes the proof of Lemma 2.6 . ■

Now we are ready to prove Theorem 1.4 a):

Proof of Theorem 1.4 a). From Lemma 2.6 the lower bound immediately follows:

$$\begin{aligned}
&\liminf_{u\to\infty} \frac{1}{F(-u)}\mathbb{P}\left(\sum_{i=1}^{m} X_i \le -u\right) \\
&\ge \liminf_{u\to\infty} \frac{F(-\varepsilon u)}{F(-u)}\mathbb{P}\left(\sum_{i=1}^{m} X_i \le -u,\ X_n \le -\varepsilon u\ \forall n \,\middle|\, X_1 \le -\varepsilon u\right) \\
&= \liminf_{u\to\infty} \frac{F(-\varepsilon u)}{F(-u)}\varepsilon^{\beta} H(\varepsilon) \ = \ H(\varepsilon). \qquad (2.8)
\end{aligned}$$

We can now use this inequality to give an upper bound for $H(\varepsilon)$; for all $\varepsilon > 0$

$$\begin{aligned}
H(\varepsilon) \ &\le\ \liminf_{u\to\infty} \frac{1}{F(-u)}\mathbb{P}\left(\sum_{i=1}^{m} X_i \le -u\right) \\
&\le\ \liminf_{u\to\infty} \frac{1}{F(-u)}\mathbb{P}\left(X_1 \le -u/m\right) = m^{\beta}, \qquad (2.9)
\end{aligned}$$

which, together with the fact that $H(\varepsilon)$ is decreasing, allows for the definition

$$q_g^F \overset{def.}{=} \lim_{\varepsilon\downarrow 0} H(\varepsilon) \le m^{\beta}. \qquad (2.10)$$

Note that this definition coincides with (1.9).

For the upper bound choose $\varepsilon < 1$ arbitrary. Then

$$\begin{aligned}
&\limsup_{u\to\infty} \frac{1}{F(-u)}\mathbb{P}\left(\sum_{i=1}^{m} X_i \le -u\right) \\
&\le \limsup_{u\to\infty} \frac{1}{F(-u)}\left(\mathbb{P}\left(\sum_{i=1}^{m} X_i \le -u, X_n \le -\varepsilon u\ \forall n\right)\right. \\
&\qquad \left. + m\mathbb{P}\left(\sum_{i=1}^{m} X_i \le -u, X_1 > -\varepsilon u\right)\right).
\end{aligned}$$

For the first term we have from Lemma 2.6:

$$\limsup_{u\to\infty} \frac{1}{F(-u)}\mathbb{P}\left(\sum_{i=1}^{m} X_i \le -u, X_n \le -\varepsilon u\ \forall n\right) = H(\varepsilon). \qquad (2.11)$$

For the second term we get:

$$\begin{aligned}
&\limsup_{u\to\infty} \frac{m}{F(-u)} \mathbb{P}\left(\sum_{i=1}^{m} X_i \leq -u, X_1 > -\varepsilon u\right) \\
&\leq \limsup_{u\to\infty} \frac{m(m-1)}{F(-u)} \mathbb{P}\left(X_m \leq -u\frac{1-\varepsilon}{m-1}, X_1 > -\varepsilon u\right) \qquad (2.12)\\
&= \limsup_{u\to\infty} \frac{m^2-m}{F(-u)} \left(\mathbb{P}\left(X_m \leq -u\frac{1-\varepsilon}{m-1}\right) - \mathbb{P}\left(X_m \leq -u\frac{1-\varepsilon}{m-1}, X_1 \leq -\varepsilon u\right)\right) .
\end{aligned}$$

Now, using symmetry of C and substituting κ for $(1-\varepsilon)/(m-1)$ we obtain

$$\begin{aligned}
&\limsup_{u\to\infty} \frac{m^2-m}{F(-u)} \left(\mathbb{P}\left(X_m \leq -u\kappa\right) - \mathbb{P}\left(X_m \leq -u\kappa, X_1 \leq -\varepsilon u\right)\right) \qquad (2.13)\\
&\leq (m^2-m)\left(\kappa^{-\beta} - \liminf_{u\to\infty} \frac{1}{F(-u)} \mathbb{P}\left(X_1 \leq -u\kappa, X_2 \leq -\varepsilon u, \ldots, X_m \leq -\varepsilon u\right)\right),
\end{aligned}$$

where we calculated the limit of the first probability and put extra restrictions on the events in the second probability. Using the copula to rewrite the second probability, we get

$$\begin{aligned}
&\liminf_{u\to\infty} \frac{1}{F(-u)} \mathbb{P}\left(X_1 \leq -u\kappa, X_2 \leq -\varepsilon u, \ldots, X_m \leq -\varepsilon u\right) \\
&= \liminf_{u\to\infty} \frac{\kappa^{-\beta}}{F(-\kappa u)} C\left(F(-\kappa u), \left(\frac{\varepsilon}{\kappa}\right)^{-\beta} F(-\kappa u), \ldots, \left(\frac{\varepsilon}{\kappa}\right)^{-\beta} F(-\kappa u)\right) \\
&= \liminf_{v\downarrow 0} \frac{\kappa^{-\beta}}{v} C\left(v, \left(\frac{\kappa}{\varepsilon}\right)^{\beta} v, \ldots, \left(\frac{\kappa}{\varepsilon}\right)^{\beta} v\right) \\
&\overset{(*)}{=} \kappa^{-\beta}\lambda g\left(1, \left(\frac{\kappa}{\varepsilon}\right)^{\beta}, \ldots, \left(\frac{\kappa}{\varepsilon}\right)^{\beta}\right).
\end{aligned}$$

In (*) we just use (1.3). Together with (2.13) we obtain

$$\begin{aligned}
&\limsup_{u\to\infty} \frac{m^2-m}{F(-u)} \left(\mathbb{P}\left(X_m \leq -u\frac{1-\varepsilon}{m-1}\right) - \mathbb{P}\left(X_m \leq -u\frac{1-\varepsilon}{m^2-m}, X_1 \leq -\varepsilon u\right)\right) \\
&\leq (m^2-m)\kappa^{-\beta}\left(1 - \lambda g\left(1, \left(\frac{\kappa}{\varepsilon}\right)^{\beta}, \ldots, \left(\frac{\kappa}{\varepsilon}\right)^{\beta}\right)\right), \qquad (2.14)
\end{aligned}$$

which, together with (1.4) shows that the second term (2.12) converges to 0 as $\varepsilon \to 0$. This finishes the proof of the Fréchet part of Theorem 1.4 and (1.9). ■

Proof of Theorem 1.5. Since only in the last step of the proof of Theorem 1.4 we have used (1.4), dropping (1.4) we still obtain the same lower bound, but a different upper bound. This means that for $\delta > 0$ there is a $u_0 \in \mathbb{R}^+$ such that for $u \geq u_0$:

$$\lim_{\varepsilon \downarrow 0} H(\varepsilon) - \delta \leq \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^{m} X_m \leq -u \right)$$
$$\leq \lim_{\varepsilon \downarrow 0} H(\varepsilon) + \lim_{x \to \infty} m(m-1)^{1-\beta} \left(1 - \lambda g\left(1, x, \ldots, x\right)\right) + \delta \ . \quad (2.15)$$

We shall show the use of this upper bound in section 3.4, about elliptical copulas. ■

2.2 The Weibull case

The Weibull case is only slightly different from the Fréchet case. The main difference is that the random variables have a lower bound. As a result the proof of the Fréchet case only needs minor adjustments to prove the Weibull case.

Lemma 2.7 *(Weibull) Assume $(X_1, \ldots, X_m)$ satisfies Assumption 1.2 with identical marginals F, and let there be an $c \in \mathbb{R}$ such that $s \to F(c-1/s)$ is regularly varying at $-\infty$ with parameter $-\beta < 0$. Furthermore, let $\varepsilon \in (0,1)$, $x_1 \in (0, 1/\varepsilon)$ and $x_2, \ldots, x_m \geq 0$. Then:*

$$\lim_{u \to \infty} \mathbb{P}\left(X_1 \leq c + x_1/u, \ldots, X_m \leq c + x_m/u \mid X_1 \leq c + 1/\varepsilon u\right) =$$
$$\varepsilon^{\beta} \lambda g\left(x_1^{\beta}, \ldots, x_m^{\beta}\right) \ . \quad (2.16)$$

Proof of Lemma 2.7 and Theorem 1.4 b).
The proof of Lemma 2.7 follows, mutatis mutandis, the lines of the proof of Lemma 2.2 in the Fréchet case. The only change for the proof of Theorem 1.4 is that now we take

$$\mu_u^*(A) \stackrel{def.}{=} \mathbb{P}\left(\left(u(X_1 - c), \ldots, u(X_m - c)\right) \in A \middle| X_1 \leq c + 1/\varepsilon u\right) \quad (2.17)$$

such that in this case

$$\lim_{u\to\infty} \mathbb{P}\left(\sum_{i=1}^{m} X_i \leq mc + 1/u, X_n \leq c + \frac{1}{\varepsilon u}\middle| X_1 \leq c + \frac{1}{\varepsilon u}\right)$$
$$= \nu_\varepsilon^{\lambda,\beta}\left(\left\{(x_1,\ldots,x_m) \in (0,1/\varepsilon]^m : \sum_{i=1}^{m} x_i \leq 1\right\}\right), \tag{2.18}$$

where $\nu_\varepsilon^{\lambda,\beta}$ again is as defined in (2.4). Thus in this case

$$q_g^W \stackrel{def.}{=} \lim_{\varepsilon\downarrow 0} H^*(\varepsilon), \tag{2.19}$$

where

$$H^*(\varepsilon) \stackrel{def.}{=} \nu_\varepsilon^{\lambda,\beta}\left(\left\{(x_1,\ldots,x_m) \in (0,1/\varepsilon]^m : \sum_{i=1}^{m} x_i \leq 1\right\}\right). \tag{2.20}$$

Note that this definition gives (1.10). In the Weibull case one always has $\mathbb{P}(X_i \leq c) = 0$, so, since $\varepsilon \in (0,1)$, we can also look at "$(x_1,\ldots,x_m) \in (0,\infty)^m$" instead. This finishes the proofs in the Weibull case. ■

2.3 The Gumbel case

Eventually, for the proof in the Gumbel case we need to adapt the Fréchet proof a little more.

Lemma 2.8 *(Gumbel) Assume $(X_1,\ldots,X_m)$ satisfies Assumption 1.2 with identical marginals F, and let there be an $c \geq -\infty$ and a positive function $s \mapsto a(s)$ such that $\lim_{u\downarrow c} F(u + ta(u))/F(u) = e^t$, for all $t \in \mathbb{R}$. Furthermore, let $\varepsilon \in (0,1)$, $x_1 \in (-\infty, 1/\varepsilon)$ and $x_2,\ldots,x_m \in \mathbb{R}$ Then:*

$$\lim_{u\downarrow c} \mathbb{P}\left(X_1 \leq u + x_1 a(u),\ldots,X_m \leq u + x_m a(u) \mid X_1 \leq u + a(u)/\varepsilon\right)$$
$$= e^{-1/\varepsilon}\lambda g\left(e^{x_1},\ldots,e^{x_m}\right). \tag{2.21}$$

Proof of Lemma 2.8 and Theorem 1.4 c). The proof of this Lemma is very similar to that of Lemma 2.2. Only this time instead of (2.4) we define:

$$\pi_\varepsilon^{\lambda,\beta}(A) \stackrel{def.}{=} e^{-1/\varepsilon}\lambda\nu\left(\{(e^{x_1},\ldots,e^{x_m}) : (x_1,\ldots,x_m) \in A\}\right)$$

be its density function and define:

$$L(\varepsilon) \stackrel{def.}{=} e^{1/\varepsilon}\nu_\varepsilon^{\lambda,\beta}\left(\left\{(x_1,\ldots,x_m)\in(0,1/\varepsilon]^m : \sum_{i=1}^m x_i \le 1\right\}\right) . \tag{2.22}$$

Since $L(\varepsilon)$ is increasing for $\varepsilon \downarrow 0$, one can define

$$q_g^F \stackrel{def.}{=} \lim_{\varepsilon\downarrow 0} L(\varepsilon) \le \infty . \tag{2.23}$$

This definition gives us (1.11) and thus completes Proposition 1.6. The lower bound follows immediately, similar to (2.8). For the upper bound we again split the term into two parts, the first part exactly the same as the lower bound (compare with (2.8) and (2.11)). The second part is somewhat different, however:

$$\begin{aligned}
&\limsup_{u\downarrow c}\frac{1}{F(u)}\mathbb{P}\left(\sum_{i=1}^m X_i \le mu + a(u), X_1 > u + \frac{a(u)}{\varepsilon}\right)\\
&\le \limsup_{u\downarrow c}\frac{m-1}{F(u)}\mathbb{P}\left(X_m \le u + \frac{1-1/\varepsilon}{m-1}a(u), X_1 > u + \frac{a(u)}{\varepsilon}\right)\\
&\le \limsup_{u\downarrow c}\frac{m-1}{F(u)}\mathbb{P}\left(X_m \le u + \frac{1-1/\varepsilon}{m-1}a(u)\right)\\
&= (m-1)e^{\frac{1-1/\varepsilon}{m-1}},
\end{aligned} \tag{2.24}$$

which goes to 0 as ε goes to 0, which in turn shows that the lower and upper bound coincide, which in its turn finishes the proof in the Gumbel case, which finally finishes the proof of Theorem 1.4. ■

Remark 2.9 In the Gumbel case (1.4) is not needed for the proof. So the fact that we have a very "thin" tail in the Gumbel case actually reduces the demands on the copula. We can see this in (2.24). Here we condition on X_1 being relatively large ($X_1 > u + a(u)/\varepsilon$). In the Fréchet case we find that if X_1 is relatively large ($X_1 > -\varepsilon u$), the other random variables will also be fairly large, so that the sum will be larger than $-u$, so there the estimate comes from the dependency (the copula) of the X_i's. But here the fact that X_1 is relatively large demands for the others to be extra small ($X_i \le u + \frac{1-1/\varepsilon}{m-1}a(u)$, for at least one $i \in \{2,\ldots,m\}$), which only occurs with very small probability, even if the X_i's were independent.

3 Examples

3.1 Archimedean copula

The Archimedean case is already solved in Chapter 1, Theorem 2.1. If we assume that the generator θ of the Archimedean copula C_α^θ is regularly varying at 0 with index $-\alpha < 0$ and the marginals are regularly varying at $-\infty$ with parameter $-\beta = -2$ (i.e. of the Fréchet type) then, in the two-dimensional case, the constant q_g^F is given as follows

$$q_g^F = 2 + 2\frac{\Gamma(1+1/2\alpha)^2}{\Gamma(1+1/\alpha)}, \tag{3.1}$$

and the lower-tail dependence coefficient is given by $\lambda = 2^{-1/\alpha}$ (cf. [20], Theorem 3.9). Now we can explain Remark 1.7:

Example 3.1 *If we define the following function $C_\diamondsuit : [0,1]^2 \to [0,1]$:*

$$C_\diamondsuit(x,y) \overset{def.}{=} \begin{cases} x & \textit{if } \min(3x, \frac{x+2}{3}) \le y; \\ \frac{x+y}{4} & \textit{if } \frac{x}{3} \le y < \min(3x, 1-x); \\ \frac{3(x+y)-2}{4} & \textit{if } \max(1-x, 3x-2) \le y < \frac{x+2}{3}; \\ y & \textit{if } y < \max(\frac{x}{3}, 3x-2), \end{cases}$$

(see figures), it is a copula. This can best be seen by the fact that it is the bivariate distribution function of a random vector (X_0, X_1) that takes its value uniformly on the edge of the diamond $((0,0),(1/4,3/4),(1,1),(3/4,1/4))$ with probability $3/4$ and uniformly on the line $((1/4,3/4),(3/4,1/4))$ otherwise. We see that this copula has the same lower-tail dependence coefficient

$$\lambda = \lim_{x\downarrow 0} \frac{C_\diamondsuit(x,x)}{x} = \lim_{x\downarrow 0} \frac{(x+x)/4}{x} = 1/2 \ , \tag{3.2}$$

as C_1^θ, but a different g and a different limiting constant q_g^F (as above we take $\beta = 2$):

$$q_g^F = \lim_{\varepsilon\downarrow 0} H(\varepsilon) = \lambda\nu\left(\left\{\left(x^2,y^2\right) : \frac{1}{x} + \frac{1}{y} \ge 1\right\}\right) \ . \tag{3.3}$$

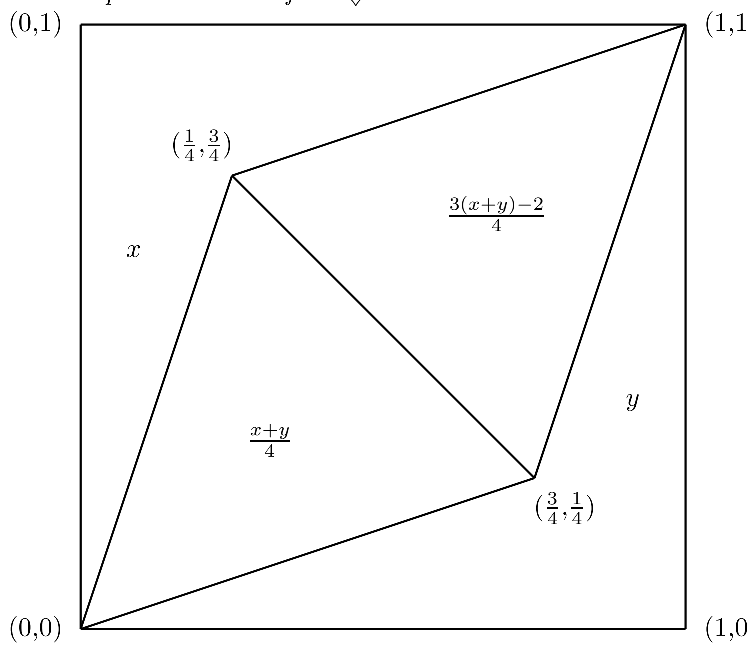
Now it is not difficult to see that

$$\nu(A) = \frac{1}{2}\left(\mu_{Lebesgue}\left(\left\{x : \left(x, \frac{x}{3}\right) \in A\right\}\right) + \mu_{Lebesgue}\left(\{x : (x, 3x) \in A\}\right)\right), \tag{3.4}$$

which leads to

$$q_g^F = \frac{1}{2} \cdot \frac{1}{2} \left(\left(4 + 2\sqrt{3}\right) + \left(4 + 2\sqrt{3}\right) \right) = 2 + \sqrt{3} \ . \tag{3.5}$$

Note here that Assumption 1.2 holds for $C_\diamond$.



3.2 Archimedean survival copula

If the upper-tail-behaviour of random variables is of interest, rather than the lower-tail-behaviour, we need to extend our theorem. To this end we define survival copulas. (Also see Nelsen [23].)

Definition 3.2 *The survival copula $\hat{C}$ of a bivariate copula C is defined as follows:*

$$\hat{C}(x, y) \stackrel{def.}{=} x + y - 1 + C(1 - x, 1 - y). \tag{3.6}$$

This $\hat{C}$ is again a copula.

This definition is used in the following way: If X and Y have copula C, then by definition that

$$\mathbb{P}(X \leq x, Y \leq y) = C(\mathbb{P}(X \leq x), \mathbb{P}(Y \leq y)),$$

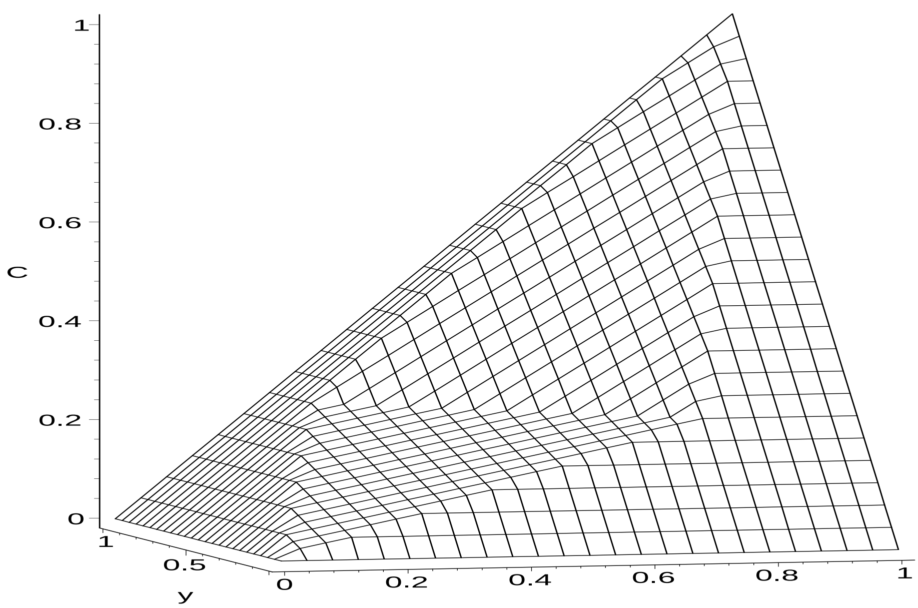


Figure 3.1: The copula $C_{\diamondsuit}$.

and thus

$$\mathbb{P}(X > x, Y > y) = \hat{C}(\mathbb{P}(X > x), \mathbb{P}(Y > y)).$$

We consider the case of a bivariate Gumbel copula. Note here that the Gumbel copula is something entirely different than Gumbel marginals. The Gumbel copula is an Archimedean copula with generator $t \mapsto (-\log(t))^\theta$, for some $\theta \geq 1$. Now let us assume (Z_1, Z_2) has Gumbel copula $C^{Gu,\alpha}$ $(\alpha > 1)$ and Pareto marginals (i.e. Fréchet type)

$$D(x) = 1 - (x/\theta)^{-\beta} \qquad \text{for} \quad x \geq \theta.$$

We investigate

$$\mathbb{P}[Z_1 + Z_2 > u], \qquad \text{for large } u.$$

First, to translate the problem to the setting of Theorem 1.4 , we define $(X_1, X_2) = (-Z_1, -Z_2)$. We have

$$F(-u) = \mathbb{P}[X_1 \leq -u] = \mathbb{P}[Z_1 \geq u] = 1 - D(u) = (u/\theta)^{-\beta}, \tag{3.7}$$

which is regularly varying with parameter $-\beta$. Furthermore the copula of (X_1, X_2) is given by

$$\begin{aligned} C(u,v) &= \mathbb{P}\left(X_1 \leq F^{-1}(u), X_2 \leq F^{-1}(v)\right) = \mathbb{P}\left(Z_1 \geq -F^{-1}(u), Z_2 \geq -F^{-1}(v)\right) \\ &= \mathbb{P}\left(Z_1 \geq D^{-1}(1-u), Z_2 \geq D^{-1}(1-v)\right) \\ &= u + v - 1 + C^{Gu,\alpha}(1-u, 1-v) = \hat{C}^{Gu,\alpha}(u,v). \end{aligned} \tag{3.8}$$

The generator of the Gumbel copula is regularly varying at 1 with parameter α. From Theorems 3.4 and 4.4 in [21] we have

$$\lambda g(1,x) = 1 + x - (1 + x^\alpha)^{1/\alpha} \ . \tag{3.9}$$

Note that for $\alpha > 1$ one has $\lim_{x\to\infty} x - (1+x^\alpha)^{1/\alpha} = 0$, so we get $\lim_{x\to\infty} \lambda g(1,x) = 1$ (i.e. condition (1.4)). It follows that $\lambda = 2 - 2^{1/\alpha}$ and

$$\begin{aligned} \nu_\varepsilon^{\lambda,\beta}(A) &= \varepsilon^\beta \lambda \nu\left(A^\beta\right) \\ &= \varepsilon^\beta \mu_{Lebesgue}\left(\left\{x^\beta + y^\beta - \left(x^{\alpha\beta} + y^{\alpha\beta}\right)^{1/\alpha} : (x,y) \in A\right\}\right) \end{aligned} \tag{3.10}$$

Now we can calculate the limiting constant

$$\begin{aligned} q_g^F &= \varepsilon^{-\beta} \int_{(x,y)\in[0,\infty)^2: \frac{1}{x}+\frac{1}{y}\geq 1} \mathbb{1} \, d\nu_\varepsilon^{\lambda,\beta} \\ &= \int_0^1 \int_0^\infty \frac{d^2}{dxdy} \lambda g(x^\beta, y^\beta) dydx + \int_1^\infty \int_1^{\frac{x}{x-1}} \frac{d^2}{dxdy} \lambda g(x^\beta, y^\beta) dydx \\ &= 1 + \beta \int_1^\infty x^{\beta-1} - \left(x^{\alpha\beta} + \left(\frac{x}{x-1} \right)^{\alpha\beta} \right)^{1/\alpha - 1} x^{\alpha\beta-1} dx \qquad (3.11) \\ &= 1 + \beta \int_1^\infty x^{\beta-1} \left(1 - \left(1 + (x-1)^{-\alpha\beta} \right)^{1/\alpha-1} \right) dx. \end{aligned}$$

Doing the transformation $z = (x-1)^\beta$ we find

$$q_g^F = 1 + \int_0^\infty \left(1 + z^{-1/\beta} \right)^{\beta-1} \left(1 - \left(1 + z^{-\alpha} \right)^{1/\alpha - 1} \right) dz. \qquad (3.12)$$

Consider for $y \geq 0$

$$f_\alpha(y) = 1 - \left(1 + y^{-\alpha} \right)^{1/\alpha-1}. \qquad (3.13)$$

Proposition 3.3 *f_α is a probability density on $[0, \infty)$. Choose $Z_\alpha \sim f_\alpha$, hence*

$$q_g^F = 1 + \mathbb{E}\left(\left(1 + Z_\alpha^{-1/\beta} \right)^{\beta-1} \right). \qquad (3.14)$$

Moreover we have:

- *For $\beta > 1$: q_g^F is strictly increasing in α.*
- *For $\beta = 1$: $q_g^F = 2$.*
- *For $\beta < 1$: q_g^F is strictly decreasing in α.*

Proof. f_α is positive on $[0, \infty)$. Choose $0 < c_1 < c_2$ and set $y = z^\alpha$

$$\begin{aligned} \int_{c_1}^{c_2} f_\alpha(z) dz &= \int_{c_1}^{c_2} 1 - \left(\frac{z^\alpha}{1+z^\alpha} \right)^{1-1/\alpha} dz \qquad (3.15) \\ &= \frac{1}{\alpha} \int_{c_1^\alpha}^{c_2^\alpha} y^{1/\alpha-1} - (1+y)^{1/\alpha-1} dy = \lambda \cdot \left(g(1, c_2) - g(1, c_1) \right). \end{aligned}$$

From $\lim_{x\to\infty} g(1,x) = 1/\lambda$ and $\lim_{x\to 0} g(1,x) = 0$ (see (3.9)) follows that we have a density.

Next we consider $Z_\alpha \sim f_\alpha$. Define

$$H(c;\alpha) = \mathbb{P}\left(Z_\alpha \geq c\right) = 1 - \lambda g(1-c). \tag{3.16}$$

Hence for $c > 0$

$$\frac{dH(c;\alpha)}{d\alpha} = -\frac{(1+c^\alpha)^{1/\alpha-1}}{\alpha^2}\left((1+c^\alpha)\log(1+c^\alpha) - c^\alpha \log c^\alpha\right) < 0. \tag{3.17}$$

Moreover $\lim_{c\to 0} \frac{dH(c;\alpha)}{d\alpha} = 0$. This proves that $\mathbb{P}\left(Z_\alpha \geq c\right)$ is strictly increasing in α, which immediately implies the properties of q_g^F (the proof is similar to the proof of Theorem 2.1 in Chapter 1). ■

3.3 A generator that is not regularly varying

Now we shall look at X, Y, random variables on $(-\infty, 0]$ with marginal distribution

$$F : x \mapsto (1 - \frac{x}{\theta})^{-\beta} ,$$

for some $\beta > 0$ and Archimedean copula

$$C_{19} : (x,y) \to \frac{1}{\log(e^{1/x} + e^{1/y} - e)}, \tag{3.18}$$

generated by

$$\phi : t \mapsto e^{1/t} - e .$$

(See also [23], (4.2.19).)

Note that this generator is not regularly varying in 0, so it was not covered by Chapter 1. But here

$$\lambda = \lim_{u\downarrow 0} \frac{C_{19}(u,u)}{u} = 1$$

and

$$\lambda g(x,y) = \lim_{u\downarrow 0} \frac{C_{19}(ux,uy)}{u} = \min(x,y),$$

so that (1.2) is fulfilled, with

$$\nu : A \mapsto \mu_{Lebesgue}(\{x : (x,x) \in A\}) .$$

So now we can apply Theorem 1.4 a) and find

$$\lim_{u\to\infty} \frac{\mathbb{P}(X+Y\le -u)}{\left(1+\frac{u}{\theta}\right)^{-\beta}} = q_g^F = \lim_{\varepsilon\downarrow 0} H(\varepsilon)$$
$$= \lambda\nu\left(\left\{\left(x^\beta, y^\beta\right) : (x,y)\in(0,\infty)^m, \frac{1}{x}+\frac{1}{y}\ge 1\right\}\right) = 2^\beta \ . \qquad (3.19)$$

3.4 Elliptical copulas

Even in the case where $m=2$, for general copulas the limiting constant cannot always be easily calculated. E.g. we choose the bivariate t-copula with γ degrees of freedom and correlation ρ:

$$C_{\gamma,\rho}(u,v) = \int_\infty^{t_\gamma^{-1}(u)} \int_\infty^{t_\gamma^{-1}(v)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \left(1+\frac{s^2-2\rho st+t^2}{\gamma(1-\rho)^2}\right)^{-(\gamma+2)/2} dsdt, \qquad (3.20)$$

where t_γ denotes the standard univariate t-distribution function with γ degrees of freedom, and t_γ^{-1} its inverse. We shall see that this copula does not fulfill (1.4), and we shall assume Fréchet marginals and use Theorem 1.5

Lemma 3.4 (t-copula) *We have for $x>0$*

$$\lim_{u\to 0}\frac{C(xu,yu)}{u} = \lambda g(x,y) = x\cdot t_{\gamma+1}\left(\left(\rho-\left(\frac{x}{y}\right)^{1/\gamma}\right)\left(\frac{\gamma+1}{1-\rho^2}\right)^{1/2}\right)$$
$$+y\cdot t_{\gamma+1}\left(\left(\rho-\left(\frac{y}{x}\right)^{1/\gamma}\right)\left(\frac{\gamma+1}{1-\rho^2}\right)^{1/2}\right), \qquad (3.21)$$

and the tail dependence coefficient is given by

$$\lambda = 2t_{\gamma+1}\left(-(\gamma+1)^{1/2}\left(\frac{1-\rho}{1+\rho}\right)^{1/2}\right). \qquad (3.22)$$

Proof. Our first observation is that

$$\lim_{u\to-\infty}\frac{t_\gamma(ux^{-1/\gamma})}{t_\gamma(u)} = 1. \qquad (3.23)$$

Assume $(U_1, U_2) \sim C_{\gamma,\rho}$ and $(X,Y) \overset{(d)}{=} (t_\gamma^{-1}(U_1), t_\gamma^{-1}(U_2))$, i.e. (X,Y) has a bivariate t-distribution. Applying de l'Hôpital's rule we find

$$\begin{aligned}
\lambda g(1,x) &= \lim_{u\to 0} x\mathbb{P}\left(U_2 \le u | U_1 = xu\right) + \mathbb{P}\left(U_1 \le ux | U_1 = u\right) && (3.24)\\
&= \lim_{u\to -\infty} x\mathbb{P}\left(U_2 \le t_\gamma(u) | U_1 = xt_\gamma(u)\right) + \mathbb{P}\left(U_1 \le t_\gamma(u)x | U_1 = t_\gamma(u)\right)\\
&= \lim_{u\to -\infty} x\mathbb{P}\left(U_2 \le t_\gamma(u) | U_1 = t_\gamma(ux^{-1/\gamma})\right)\\
&\qquad + \mathbb{P}\left(U_1 \le t_\gamma(ux^{-1/\gamma}) | U_1 = t_\gamma(u)\right) && (3.25)\\
&= \lim_{u\to -\infty} x\mathbb{P}\left(Y \le u | X = ux^{-1/\gamma}\right) + \mathbb{P}\left(X \le ux^{-1/\gamma} | Y = u\right).
\end{aligned}$$

Conditional on $X = x$

$$\left(\frac{\gamma+1}{\gamma+x^2}\right)^{1/2} \frac{Y-\rho x}{(1-\rho^2)^{1/2}} \sim t_{\gamma+1}. \tag{3.26}$$

Hence we have

$$\begin{aligned}
\lambda g(1,x) = \lim_{u\to -\infty} &x t_{\gamma+1}\left(\frac{u-\rho u x^{-1/\gamma}}{(1-\rho^2)^{1/2}}\left(\frac{\gamma+1}{\gamma+u^2x^{-2/\gamma}}\right)^{1/2}\right) && (3.27)\\
&+ t_{\gamma+1}\left(\frac{ux^{-1/\gamma}-\rho u}{(1-\rho^2)^{1/2}}\left(\frac{\gamma+1}{\gamma+u^2}\right)^{1/2}\right)\\
= x t_{\gamma+1}&\left(\frac{\rho x^{-1/\gamma}-1}{(1-\rho^2)^{1/2}}(\gamma+1)^{1/2}x^{1/\gamma}\right) + t_{\gamma+1}\left(\frac{\rho - x^{-1/\gamma}}{(1-\rho^2)^{1/2}}(\gamma+1)^{1/2}\right)\\
= x t_{\gamma+1}&\left(\left(\rho - x^{1/\gamma}\right)\left(\frac{\gamma+1}{1-\rho^2}\right)^{1/2}\right) + t_{\gamma+1}\left(\left(\rho - x^{-1/\gamma}\right)\left(\frac{\gamma+1}{1-\rho^2}\right)^{1/2}\right).
\end{aligned}$$

The identity for λ follows if we set $x = 1$, statement (3.21) follows from $\lambda g(x,y) = \lambda \cdot x \cdot g(1, y/x)$.

□

From Lemma 3.4 it follows:

$$\lim_{x\to\infty} \lambda g(1,x) = t_{\gamma+1}\left(\rho\left(\frac{\gamma+1}{1-\rho^2}\right)^{1/2}\right) < 1, \tag{3.28}$$

so (1.4) does not hold, but we shall use Theorem 1.5. First we calculate $\lim_{\varepsilon\downarrow 0} H(\varepsilon)$:

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} H(\varepsilon) &= \lim_{\varepsilon\downarrow 0}\int_{1/x+1/y\geq 1,\ x\leq 1/\varepsilon,\ y\leq 1/\varepsilon}\frac{d^2}{dxdy}\lambda g(x^\beta,y^\beta)dxdy\\ &= \lim_{\varepsilon\downarrow 0}\int_{x\leq 1,\ y\leq 1/\varepsilon}\frac{d^2}{dxdy}\lambda g(x^\beta,y^\beta)dxdy\\ &\quad+\lim_{\varepsilon\downarrow 0}\int_{1\leq x\leq 1/\varepsilon,\ y\leq \frac{1}{\varepsilon}\wedge\frac{x}{x-1}}\frac{d^2}{dxdy}\lambda g(x^\beta,y^\beta)dxdy\\ &= t_{\gamma+1}\left(\rho\left(\frac{\gamma+1}{1-\rho^2}\right)^{1/2}\right)+\int_1^\infty\left[\frac{d}{dx}\lambda g\left(x^\beta,y^\beta\right)\right]_{y=0}^{\frac{x}{x-1}}dx\ . \end{aligned}\tag{3.29}$$

To keep the terms short we write τ for $\left(\frac{\gamma+1}{1-\rho^2}\right)^{1/2}$. After a tedious calculation (calculate the derivative above, and use partial integration) we arrive at

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} H(\varepsilon) \quad &= \quad t_{\gamma+1}(\rho\tau)-t_{\gamma+1}(\rho\tau) \qquad (3.30)\\ &+\int_1^\infty\frac{\beta}{\gamma}\tau x^{\beta-1}\left(x-1\right)^{\beta/\gamma-1}t'_{\gamma+1}\left(\left(\rho-\left(x-1\right)^{\beta/\gamma}\right)\tau\right)dx\\ &+\int_1^\infty\frac{\beta}{\gamma}\tau x^{\beta-1}\left(x-1\right)^{-\beta/\gamma-\beta}t'_{\gamma+1}\left(\left(\rho-\left(x-1\right)^{-\beta/\gamma}\right)\tau\right)dx,\end{aligned}$$

where $t'_{\gamma+1}$ is the density of the one-dimensional t-distribution with $\gamma+1$ degrees of freedom. After two substitutions we find

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} H(\varepsilon) \quad &= \quad 2\int_{-\infty}^{\rho\tau}\left(1+\left(\rho-y\tau^{-1}\right)^{\gamma/\beta}\right)^{\beta-1}t'_{\gamma+1}\left(y\right)dy\\ &= \quad \frac{\mathbb{E}\left(\left(1+\left(\rho-T\tau^{-1}\right)^{\gamma/\beta}\right)^{\beta-1}\middle| T\leq\rho\tau\right)}{t_{\gamma+1}\left(\rho\tau\right)},\end{aligned}\tag{3.31}$$

where T is a $t_{\gamma+1}$-distributed variable. Now, in order to calculate the upper bound, we have to calculate the second term of the right-hand side of (2.15) :

$$\begin{aligned}&\lim_{x\to\infty} m(m-1)^{1-\beta}\left(1-\lambda g\left(1,x,\ldots,x\right)\right)\\ &=2-2\lim_{x\to\infty}\lambda g(1,x)=2-2t_{\gamma+1}\left(\rho\tau\right)\ .\end{aligned}\tag{3.32}$$

These calculations lead to the following:

Corollary 3.5 *When the dependence structure of random vector (X,Y) is given by a bivariate t-copula with γ degrees of freedom and correlation ρ and when this random vector (X,Y) has Fréchet marginals with index $-\beta$, then, for all $\delta > 0$, for large u:*

$$\begin{aligned} &\frac{\mathbb{E}\left(\left(1+\left(\rho - T\tau^{-1}\right)^{\gamma/\beta}\right)^{\beta-1} \middle| T \leq \rho\tau\right)}{t_{\gamma+1}\left(\rho\tau\right)} - \delta \\ \leq\ & \frac{1}{F(-u)}\mathbb{P}\left(X+Y \leq -u\right) \qquad (3.33) \\ \leq\ & \frac{\mathbb{E}\left(\left(1+\left(\rho - T\tau^{-1}\right)^{\gamma/\beta}\right)^{\beta-1} \middle| T \leq \rho\tau\right)}{t_{\gamma+1}\left(\rho\tau\right)} + 2 - 2t_{\gamma+1}\left(\rho\tau\right) + \delta\ , \end{aligned}$$

where T is a $t_{\gamma+1}$-distributed variable and where $\tau := \left(\frac{\gamma+1}{1-\rho^2}\right)^{1/2}$.

Chapter 4

Expected Shortfall

1 Introduction

In Chapter 1 and [27] solvency requirements were calculated using Value-at-Risk by showing that:

$$\mathbb{P}\left(\sum_{i=1}^{m} X_i \leq -u\right) \approx q_m \cdot \mathbb{P}\left(X_1 \leq -u\right), \qquad \text{as } u \to \infty, \tag{1.1}$$

where the constant q_m quantifies the diversification effect between the dependent risks. From this analysis of the asymptotic behaviour of quantiles of the aggregate risks we were able to deduce as a main result an asymptotic Value-at-Risk estimate.

However, even though being very popular, Value-at-Risk has some disadvantageous properties, e.g. it is not a coherent risk measure (Value-at-Risk generally lacks the subadditivity property, cf. Artzner-Delbaen-Eber-Heath [4],or Chapter 1, Theorem 2.5 for $\beta < 1$). Therefore various efforts are undertaken to look for more suitable, coherent risk measures. In many countries the regulators tend to use expected shortfall or worst conditional expectation, which in the case of continuous random variables are equivalent (see Acerbi-Tasche [1]). In this book we do not want to enter the discussion about "good" and "bad" risk measures, we simply choose expected shortfall as our risk measure for this chapter. It is coherent under the assumption that our random variables have continuous marginals (cf. Acerbi-Tasche [1]). We consider (for small p's) $\mathbb{E}(X|X \leq u_p)$, where u_p is the p-quantile of X. (To facilitate the analysis, we still assume losses to be negative, i.e. we study lower tails.)

It turned out that without much work the proof could be extended (and actually shortened), with respect to early results, towards moment estimates of the sum we are considering. Therefore, even though our main interest lies in the analysis of expected shortfall (for which we need to take $\kappa = 1$ in Theorem 3.1 below), the proof also covers, for instance, the conditional variance (for which we would consider the case $\kappa = 2$).

This chapter is organized as follows. In Section 2, we briefly describe our model. Section 3 contains the formulation of our main results, while Section 4 is devoted to examples. Finally in Section 5 we give the proofs, which use, or are at least inspired by the results in Chapter 1. We conclude this introduction with a quick review on the concept of copulas.

With expected shortfall as our risk measure, we concentrate on the case of aggregating dependent risks. The dependency of the risks is modelled by copulas. In this chapter we again focus on Archimedean copulas, as defined in the Introduction of this book.

The main results in this chapter can be described as follows. Assume the risks $X_1 \ldots X_d$ have the same continuous marginal distribution function F and $(X_1, \ldots, X_m)$ has an Archimedian copula. Then we are able to compute the asymptotic behaviour of the expected shortfall, i.e. we are able to compute the decay of

$$\mathbb{E}\left(\sum_{i=1}^{m} X_i \,\middle|\, \sum_{i=1}^{m} X_i \leq -u\right)$$

as u tends to infinity, or – more generally – moments of the form

$$\mathbb{E}\left(\left(\sum_{i=1}^{m} X_i\right)^{\kappa} \,\middle|\, \sum_{i=1}^{m} X_i \leq -u\right)$$

for sufficiently small $\kappa \in \mathbb{N}$.

In this article we define expected shortfall as

$$\mathbb{E}(X) - \mathbb{E}(X|X \leq u).$$

Just like in Chapter 1 we can distinguish between three different cases with respect to marginal distribution functions: the Fréchet case, the Gumbel case, and the Weibull case, of which only the two (most) interesting ones, the Fréchet and the Gumbel case will be considered here.

2 The model

As already mentioned in the introduction we study a multivariate model describing the diversification effect when aggregating d dependent risks. The dependence structure will be given by an Archimedian copula, and losses are assumed to be negative. More precisely our assumptions read as follows:

Assumption 2.1 *We assume that the random vector* $(X_1, \ldots, X_m)$ *satisfies:*
1) All coordinates X_i *are negative and have the same continuous marginal*

$$F(x) = \mathbb{P}(X_1 \leq x).$$

2) $(X_1, \ldots, X_m)$ *has an Archimedean copula with generator* ϕ*.*
3) This generator ϕ *is regularly varying at* 0^+ *with index* $-\alpha$*, where* $\alpha > 0$*.*

The first condition is nescessary for our proof and seems rather restrictive, but when one has different marginals, one could take the heavier tail and see our same-marginal result as an upper bound for the various-marginal case.

3 Results

In this section we formulate our central results. Depending on the extreme value behaviour of the underlying risks, we distinguish two cases: the Fréchet case and the Gumbel case.

3.1 Fréchet case

In the Fréchet case we look at (dependent) random variables that have a Fréchet-type distribution: their marginal distributions are regularly varying at $-\infty$ with paramater $-\beta$, for some $\beta > 0$. In our case we additionally assume that $\beta > 1$. The latter assumption is needed in order for the random variables to have a (finite) mean.

Theorem 3.1 (Fréchet case) *Let* $\kappa \in [0, \infty)$ *, assume Assumption 2.1 and assume that* F *is regularly varying at* $-\infty$ *with parameter* $-\beta$*,* $\beta > \kappa$*. We have*

$$\lim_{u \to \infty} \frac{-1}{u^\kappa} \mathbb{E}\left(\left(\sum_{i=1}^{m} X_i \right)^\kappa \middle| \sum_{i=1}^{m} X_i \leq -u \right) = c_m^F(\alpha, \beta), \tag{3.1}$$

where

$$c_m^F(\alpha,\beta) \quad = \quad \frac{\beta}{\beta-\kappa}. \tag{3.2}$$

Remark 3.2 *Note that $c_m^F(\alpha,\beta)$ is constant in α and m.*

Hence we find the following asymptotic behaviour for the conditional expectation ($\kappa = 1$): As $u \to \infty$ we have

$$\mathbb{E}\left(\sum_{i=1}^m X_i \middle| \sum_{i=1}^m X_i \le -u\right) \approx -\frac{\beta}{\beta-1}u, \tag{3.3}$$

which is essentially the asymptotic behaviour of the conditional expectation of the Pareto distribution (see Karamata's Theorem, [13] Theorem A3.6). The dependence strength now enters via the following observation: For the expected shortfall, conditioned on an event with probability p we obtain the following result: Denote by $-u_p$ the p-quantile of $\sum_{i=1}^m X_i$. From the above theorem and Theorem 2.1 in Chapter 1, we get, as $p \to 0$

$$\mathbb{E}\left(\sum_{i=1}^m X_i \middle| \sum_{i=1}^m X_i \le -u_p\right) \approx -\frac{\beta}{\beta-1}u_p \approx \frac{\beta}{\beta-1}F^{-1}\left(\frac{p}{q_m^F(\alpha,\beta)}\right), \tag{3.4}$$

where

$$q_m^F(\alpha,\beta) = \int\limits_{\substack{x_i\ge 0\ \forall i \\ \sum_{i=1}^m 1/x_i \ge 1}} \left(\frac{d^m}{dx_1\dots dx_m}\left(\sum_{i=1}^m x_i^{-\alpha\beta}\right)^{-1/\alpha}\right) dx_1\dots dx_m. \tag{3.5}$$

For $m = 2$, $q_m^F(\alpha,\beta)$ can be calculated explicitly, (see Theorem 2.5 in 1):
Let Y_α have density $f_\alpha = (1+x^\alpha)^{-1/\alpha-1}$, $\alpha > 0$ and $x > 0$, then

$$q_2^F(\alpha,\beta) = 1 + \mathbb{E}\left(\left(1+Y_\alpha^{-1/\beta}\right)^{\beta-1}\right). \tag{3.6}$$

For $\beta > 1$, $q_2^F(\alpha,\beta)$ is increasing in α (see also Theorem 2.5 in 1). Hence we have found:

Corollary 3.3 *Choose $m = 2$ and assume that (X_1, X_2) satisfies the assumptions of Theorem 3.1. For $p \to 0$ we have*

$$\mathbb{E}\left(X_1+X_2 \middle| X_1+X_2 \le -u_p\right) \approx \frac{\beta}{\beta-1}F^{-1}\left(\frac{p}{q_2^F(\alpha,\beta)}\right), \tag{3.7}$$

where the right-hand side of (3.7) *is strictly decreasing in α.*

This shows that the right-hand side of (3.7) is decreasing in α, i.e. the bigger α, the smaller the diversification effect. This is not surprising since α measures the dependence strength in the tails (see Juri-Wüthrich [20]). In the bivariate situation a coefficient for the dependence strength in the tails is the so-called tail dependence coefficient λ (see Embrechts-McNeil-Straumann [14]). For Archimedean copulas we have $\lambda = 2^{-1/\alpha}$ (see [20], Theorem 3.9), which is increasing in α.

3.2 Gumbel case

In the Gumbel case we look at (dependent) random variables that lie in the domain of attraction of the exponential limit law for exceedances: there is a $c \geq -\infty$ and a positive measurable function $s \mapsto a(s)$ such that for $t \in \mathbb{R}$ one has for marginals F that $\lim_{u \downarrow c} F(u + ta(u))/F(u) = e^t$.

Theorem 3.4 (Gumbel case) *Under Assumption 2.1 and F of Gumbel type we have that*

$$\lim_{u \to c} \frac{1}{a(u)} \mathbb{E}\left(\sum_{i=1}^m X_i \,\middle|\, \sum_{i=1}^m X_i \leq mu + a(u) \right) - \frac{mu}{a(u)} = c_m^G(\alpha), \tag{3.8}$$

where

$$c_m^G(\alpha) = \frac{1}{q_m^G(\alpha)} \int\limits_{\sum_{i=1}^m x_i \leq 1} \left(\sum_{i=1}^m x_i \right) \frac{d^m}{dx_1 \ldots dx_m} \left(\sum_{i=1}^m e^{-x_i \alpha} \right)^{-1/\alpha} dx_1 \ldots dx_m, \tag{3.9}$$

with q_m^G given by

$$q_m^G(\alpha) = \int\limits_{\sum_{i=1}^m x_i \leq 1} \frac{d^m}{dx_1 \ldots dx_m} \left(\sum_{i=1}^m e^{-x_i \alpha} \right)^{-1/\alpha} dx_1 \ldots dx_m. \tag{3.10}$$

In particular we get

$$c_2^G(\alpha) = 1 + \frac{\mathbb{E}\left(Y_\alpha^{-1/2} \log Y_\alpha \right)}{\mathbb{E}\left(Y_\alpha^{-1/2} \right)} = -1, \tag{3.11}$$

where Y_α has probability density $f_\alpha = (1 + x^\alpha)^{-1/\alpha - 1}$ on $x > 0$.

Remark 3.5 *Note that $c_2^G(\alpha)$ is constant in α.*

We can now do similar considerations as in the Fréchet case, assume that F is strictly increasing, then as $u \downarrow c$:

$$\begin{aligned}\mathbb{E}\left(\sum_{i=1}^m X_i \middle| \sum_{i=1}^m X_i \le mu + a(u)\right) &\sim mu + c_m^G(\alpha)a(u)\\ &= mF^{-1}(F(u + c_m^G(\alpha)a(u)/m)) \sim mF^{-1}\left(e^{c_m^G(\alpha)/m}F(u)\right)\\ &\sim mF^{-1}\left(\frac{e^{c_m^G(\alpha)/m}}{q_m^G(\alpha)}\mathbb{P}\left(\sum_{i=1}^m X_i \le mu + a(u)\right)\right), \qquad (3.12)\end{aligned}$$

where in the last step we have used formula (5.22) of [2].

Denote by u_p the p-quantile of $\sum_{i=1}^m X_i$. Then as $p \to 0$ we get

$$\mathbb{E}\left(\sum_{i=1}^m X_i \middle| \sum_{i=1}^m X_i \le u_p\right) \sim mF^{-1}\left(\frac{p\cdot\exp\left\{c_m^G(\alpha)/m\right\}}{q_m^G(\alpha)}\right), \qquad (3.13)$$

hence expected shortfall can be approximated asymptotically.

Using Theorem 3.9 of [2] we find:

Corollary 3.6 *Choose $m = 2$ and assume that (X_1, X_2) satisfies the assumptions of Theorem 3.4. For $p \to 0$ we have*

$$\begin{aligned}\mathbb{E}\left(X_1 + X_2 \middle| X_1 + X_2 \le u_p\right) &\sim 2F^{-1}\left(\frac{p\cdot\exp\{-1/2\}}{q_2^G(\alpha)}\right) \qquad (3.14)\\ &= 2F^{-1}\left(\frac{p\cdot\Gamma(1+1/\alpha)}{e\cdot\Gamma^2(1+1/(2\alpha))}\right),\end{aligned}$$

where we use Theorem 2.9 in Chapter 1 for the equality and see that the right-hand side of (3.14) *isstrictly decreasing in α.*

3.3 Conclusions

In Corollaries 3.3 and 3.6 we are able to study the asymptotic behaviour of expected shortfall, which gives upper and lower bounds for small p. Once again the estimate only depends on the marginals F and on the dependence strength α. So in the Archimedean situation we can avoid the difficulty of choosing an explicit copula for the dependence structure. All we need to estimate are the marginals and the (tail) dependence strength α (or the tail dependence coefficient $\lambda = 2^{-1/\alpha}$, resp.). As expected, the bounds are decreasing for increasing dependence strength α, i.e. the larger the dependence strength, the smaller the diversification effect.

4 Examples

The results from the previous section can be used to estimate the expected shortfall in cases where the assumptions of that section are met. In this section we shall do the calculations for one such case. We shall also show the accuracy of our estimate for another case.

4.1 How can we use this result?

First we revisit the example given in Chapter 1. There we took two dependent motor liability portfolios X_1 and X_2. As risk measure we considered Value-at-Risk at a certain probability level. Using Value-at-Risk we studied then the diversification effect when merging these two dependent portfolios into one big portfolio $X_1 + X_2$. Here we examine the same example, but this time we choose expected shortfall as our risk measure (which in our continuous setup is a coherent risk measure).

Assume X_1 and X_2 have Archimedean copula generated by a regularly varying function with index $-\alpha$ at 0^+ ($\alpha > 0$). Moreover assume that $-X_1$ and $-X_2$ have translated Pareto marginals with translation $V_1 = 880$ and $V_2 = 820$, i.e. $Y_i := -(X_i + V_i)$ is Pareto distributed with $\theta = 80$ and $\beta = 3$: for $i = 1, 2$.

$$\mathbb{P}(X_i \leq x) = \mathbb{P}(X_i+V_i \leq x+V_i) = \left(\frac{\theta}{-(x+V_i)}\right)^{\beta} \qquad \text{for } x \leq -(\theta+V_i). \tag{4.1}$$

We define expected shortfall for $p \in (0, 1)$:

$$\mathrm{ES}_{X_i}(p) = -\mathbb{E}\left(X_i \middle| X_i < u_p(X_i)\right), \tag{4.2}$$

where $u_p(X_i)$ is the p-quantile of X_i.

Hence we have for $p = 0.5\%$

	portfolio 1	portfolio 2
translation V_i	880	820
mean $\mathbb{E}(-X_i)$	$1'000$	940
$u_p(X_i)$	$-1'347.8$	$-1'287.8$
$\mathrm{ES}_{X_i}(p)$	1581.8	1521.8

Now we merge these two dependent portfolios to one big portfolio and we study expected shortfall as a function of the dependence strength α:

$$\mathrm{ES}_{X_1+X_2}(p;\alpha) \stackrel{def.}{=} -\mathbb{E}\left(X_1+X_2 \middle| X_1+X_2 < u_p^\alpha(X_1+X_2)\right), \tag{4.3}$$

where $u_p^\alpha(X_1+X_2)$ is the p-quantile of X_1+X_2. Using Corollary 3.3 on $(-Y_1,-Y_2)$ (note that this random vector has the same copula as (X_1,X_2), and furthermore identical marginals, which is nescessary in order to apply 3.3 , we see that we have the following approximation as $p \to 0$

$$\mathrm{ES}_{X_1+X_2}(p;\alpha) \approx \frac{\beta}{\beta-1}\theta\left(\frac{q_2^F(\alpha,\beta)}{p}\right)^{1/\beta} + V_1 + V_2 \stackrel{def.}{=} E_{X_1+X_2}(\alpha). \tag{4.4}$$

In order to quantify the benefits gained by merging the portfolios we introduce the diversification effect on expected shortfall.

Definition 4.1 *The diversification effect on expected shortfall, as a function of α is given by*

$$Div.eff.ES(\alpha) \stackrel{def.}{=} \frac{E_{X_1+X_2}(\infty) - E_{X_1+X_2}(\alpha)}{E_{X_1+X_2}(\infty) - \mathbb{E}(X_1+X_2)} \tag{4.5}$$

If we evaluate $E_{X_1+X_2}(\alpha)$ for different α's ($p = 0.5\%$) we obtain the following table (note that in the independent case we calculated the exact values, rather than the approximated values):

α	*indep.*	0.5	1.0	1.5	2.0	3.0	4.0	∞
$-\mathbb{E}(X_1+X_2)$	1′940	1′940	1′940	1′940	1′940	1′940	1′940	1′940
$E_{X_1+X_2}(\alpha)$	2711	2918	3′032	3′066	3′080	3′092	3′097	3′104
Div.eff.ES(α)	33.7%	16.0%	6.2%	3.2%	2.0%	1.0%	0.6%	0%
Div.eff.VaR(α)	31.6%	17.8%	6.9%	3.6%	2.2%	1.1%	0.6%	0%

$\alpha = \infty$ belongs to the comonotonic case (total positive dependence), and Div.eff.VaR(α) gives the comparison to the results obtained in Chapter 1 for Value-at-Risk.

Not surprisingly, we see that the diversification effect decreases for increasing dependence strength α. One also observes that the decrease is rather fast, i.e. already introducing slight dependencies in the tails reduces the diversification savings substantially.

For small α (i.e. close to the independent case), p should be even smaller than 0.5% in order for the approximation to be sharp. This is not a serious problem,

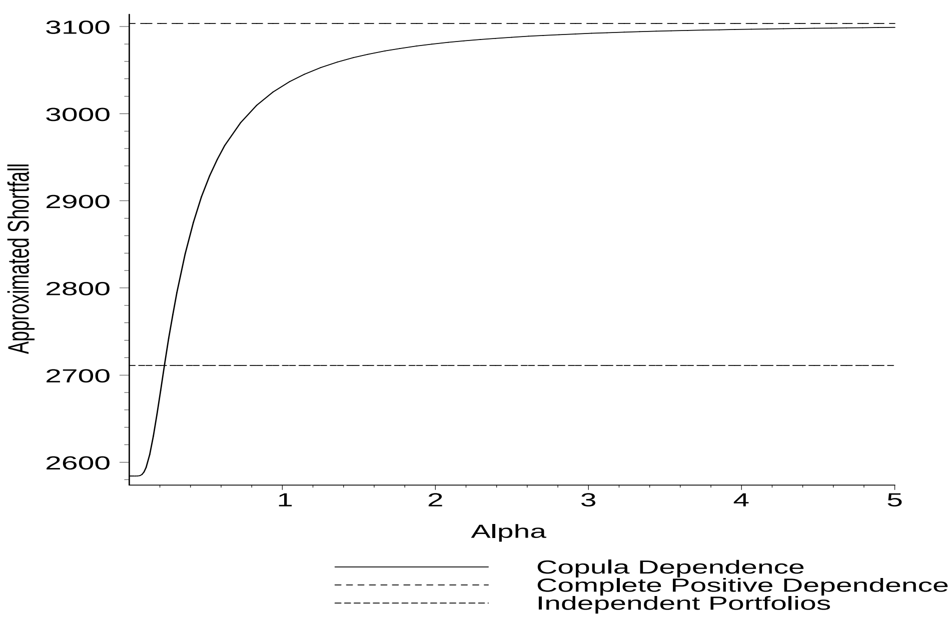


Figure 4.1: *The expected shortfall as a function of α.*

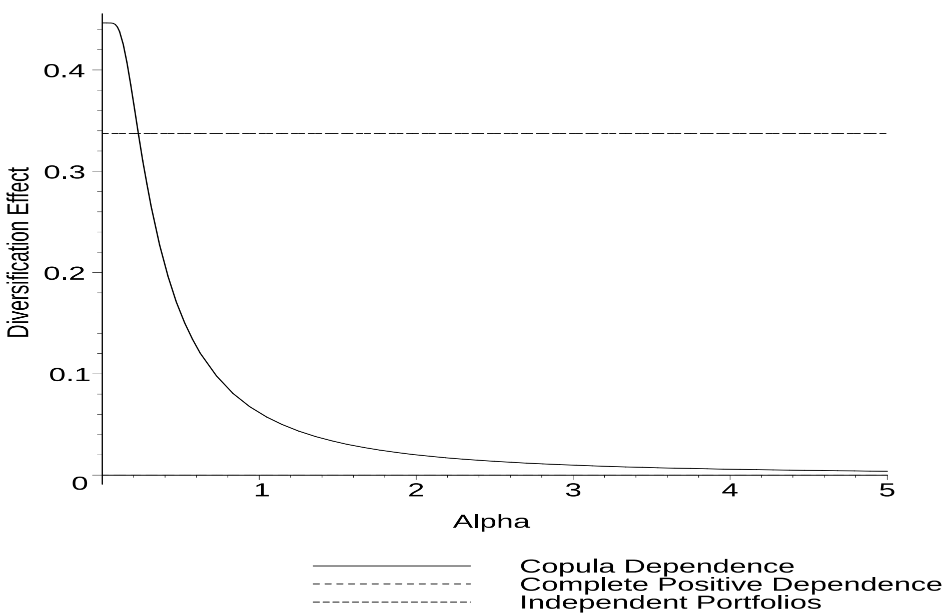


Figure 4.2: *The diversification effect as a function of α. The complete positive dependence coincides with the x-axis.*

however, since we can calculate the expected shortfall and the diversification effect directly in the independent case. A more detailed account of the accuracy of our approximation shall be given in the next subsection.

4.2 How accurate is the estimate?

Now we shall show the efficiency of our estimate for the case where X and Y are random variables with a dependency structure described by a Clayton copula and such that $-X$ and $-Y$ have Generalised Pareto distribution. The definitions of these can e.g. be found in [13], Definition 3.4.9 on page 162 and [23], (4.2.1) , page 94. We shall recall them here:

Definition 4.2 *$-X$ and $-Y$ have Gen. Pareto marginals F_β^{GP}; i.e:*

$$F_\beta^{GP}(t) := \mathbb{P}(X \le t) = \mathbb{P}(Y \le t) = \left(1 - \frac{1}{\beta}t\right)^{-\beta}, \ \forall t \le 0 \ . \tag{4.6}$$

Definition 4.3 *And X and Y have Clayton copula C_α^{Cl}, as given by:*

$$C_\alpha^{Cl} : (x, y) \to \left(x^{-\alpha} + y^{-\alpha} - 1\right)^{-1/\alpha} \ . \tag{4.7}$$

This means that X and Y have joint distribution function F on $(-\infty, 0]^2$ as follows:

$$\begin{aligned} F(x, y) &:= \mathbb{P}(X \le x, Y \le y) = C_\alpha^{Cl}\left(F_\beta^{GP}(x), F_\beta^{GP}(y)\right) \\ &= \left(\left(1 - \frac{1}{\beta}x\right)^{\alpha\beta} + \left(1 - \frac{1}{\beta}y\right)^{\alpha\beta} - 1\right)^{(-1/\alpha)} \ . \end{aligned} \tag{4.8}$$

According to Theorem 3.1 and especially (3.3) we have in the bivariate case:

$$\mathbb{E}(X + Y | X + Y \le -u) \approx -u\frac{\beta}{\beta - 1} \ , \tag{4.9}$$

for large u. In order to show the efficiency we calculate the value of $\mathbb{E}(X + Y | X + Y \le -u)$. With some straightforward calculations we see:

$$\mathbb{E}(X + Y | X + Y \le -u) = 2\frac{\int_{-\infty}^{-u} x\left(1 - \frac{1}{\beta}x\right)^{-\beta-1} dx + \int_{-u}^{0} x \cdot J_{\alpha,\beta}(x, u)dx}{\int_{-\infty}^{-u} \left(1 - \frac{1}{\beta}x\right)^{-\beta-1} dx + \int_{-u}^{0} J_{\alpha,\beta}(x, u)dx}, \tag{4.10}$$

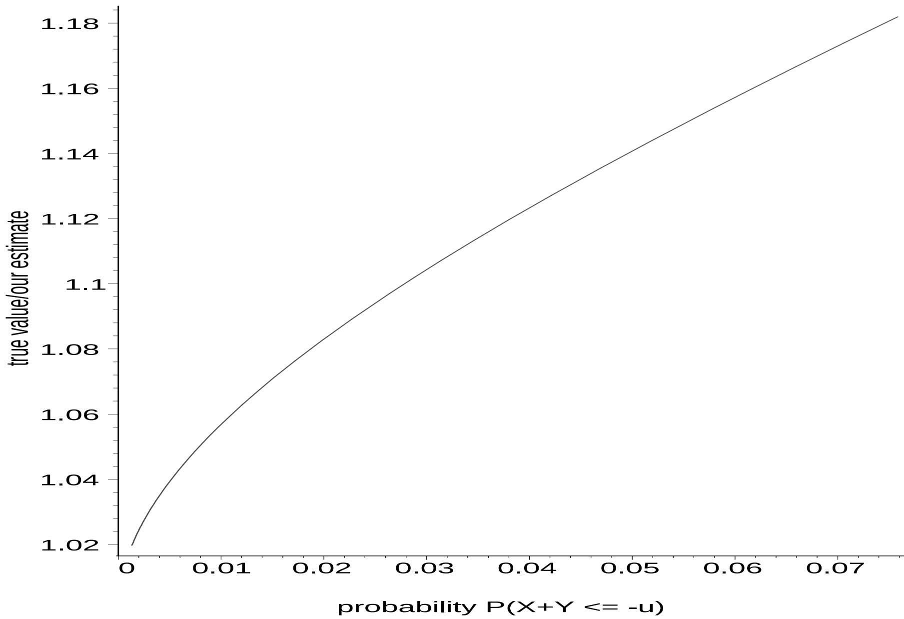


Figure 4.3: $\frac{\mathbb{E}(X+Y|X+Y\le -u)}{-u\beta/(\beta-1)}$ *as a function of* $\mathbb{P}(X+Y\le -u)$, *for* $\alpha = 1$ *and* $\beta = 2$.

where $J_{\alpha,\beta}$ is given by:

$$J_{\alpha,\beta}(x,u) := \left[\frac{d}{dx}F(x,y)\right]_{y=-\infty}^{-u-x}$$
$$= \left(1-\frac{x}{\beta}\right)^{\alpha\beta-1}\left(\left(1-\frac{x}{\beta}\right)^{\alpha\beta}+\left(1+\frac{u+x}{\beta}\right)^{\alpha\beta}-1\right)^{(-1/\alpha)-1} \qquad (4.11)$$

We fed this formula into the computer-algebra-package Maple to draw the following result (Figure 4.3) for the case where $\alpha = 1$ and $\beta = 2$. The figure shows the exact value of $E(X+Y|X+Y \le -u_p)$, divided by our estimate $-u_p\beta/(\beta-1)$, as a function of p $(= P(X+Y \le -u_p))$.

Remark 4.4 *It is clear that our estimate cannot be sharp. One only has to take* $u = 0$, *and one immediately sees that for negative random variables* $X_1, \dots, X_m$

typically

$$\mathbb{E}(X+Y|X+Y\leq -u)=\mathbb{E}(X+Y)\neq 0=-u\frac{\beta}{\beta-1}\ . \tag{4.12}$$

So our estimate is not exact for $u=0$ (or $p=1$, which is the same in this case), no matter what copula and marginal distribution one takes.

The fact that these estimates are not exact is not in contradiction with the results of [20] and [21]. Their results state that the Clayton copula $C^{Cl}(x,y)$ is the 'limiting copula' when one looks at the quotient $C(x\varepsilon,y\varepsilon)/C(\varepsilon,\varepsilon)$, and that the behaviour of the Clayton copula itself is invariant. But here we condition on $X+Y\leq -u$ rather than $X\leq -u\ \wedge\ Y\leq -u$. So our estimate is slightly smaller than the real value, since we not only condition on (and thus divide by) the probability that both X and Y are very small, but also the probability that only one of them is very small. But as we take $\alpha>0$ and thus positive dependency, this last probability is very small, but large enough to show up in Figure 4.3.

5 Proofs

Proof of Theorem 3.1.
The main idea here comes from Chapter 1. Theorem 2.1 of that chapter states that

$$\lim_{u\to\infty}\frac{1}{F(-u)}\mathbb{P}\left(\sum_{i=1}^{m}X_i\leq -u\right)=q_m^F(\alpha,\beta), \tag{5.1}$$

for a certain constant $q_m^F(\alpha,\beta)$.

For simplicity, let us write S for $\sum_{i=1}^{m}X_i$. Let F_S be its distribution function. By the simple substitution $s=-t$ we obtain

$$\begin{aligned}&\lim_{u\to\infty}\left(\frac{-1}{u}\right)^{\kappa}\mathbb{E}\left(\left(\sum_{i=1}^{m}X_i\right)^{\kappa}\middle|\sum_{i=1}^{m}X_i\leq -u\right)\\&=\lim_{u\to\infty}\frac{\int_{-\infty}^{-u}(-t)^{\kappa}mF_S(t)}{u^{\kappa}F_S(-u)}\overset{t\to -s}{=}\lim_{u\to\infty}\frac{\int_{u}^{\infty}(s)^{\kappa}mF_S(-s)}{u^{\kappa}F_S(-u)}\ .\end{aligned} \tag{5.2}$$

With (5.1) we can now see that the distribution function F_S is regular varying at $-\infty$ with parameter $-\beta$. We therefore may apply [6], Theorem 1.6.5 to the right hand side of the above equation to obtain

$$\lim_{u\to\infty}\left(\frac{-1}{u}\right)^{\kappa}\mathbb{E}\left(\left(\sum_{i=1}^{m}X_i\right)^{\kappa}\middle|\sum_{i=1}^{m}X_i\le -u\right)\overset{[6]}{=}\frac{\beta}{\beta-\kappa}\ . \tag{5.3}$$

■

Proof of Theorem 3.4. For the lower bound note that

$$\begin{aligned}1-\mathbb{E}&\left(\sum_{i=1}^{m}\frac{X_i-u}{a(u)}\middle|\sum_{i=1}^{m}X_i\le mu+a(u)\right)\\&=\mathbb{E}\left(1-\sum_{i=1}^{m}\frac{X_i-u}{a(u)}\middle|\sum_{i=1}^{m}\frac{X_i-u}{a(u)}\le 1\right)\end{aligned} \tag{5.4}$$

has a positive argument in the integral. We define $Y_i(u)=(X_i-u)/a(u)$. Hence for all $\varepsilon>0$

$$\begin{aligned}1-\mathbb{E}&\left(\sum_{i=1}^{m}\frac{X_i-u}{a(u)}\middle|\sum_{i=1}^{m}X_i\le mu+a(u)\right)\\&=\int_0^{\infty}\mathbb{P}\left(1-\sum_{i=1}^{m}Y_i(u)>z\middle|\sum_{i=1}^{m}Y_i(u)\le 1\right)dz\end{aligned} \tag{5.5}$$

$$\begin{aligned}&=\int_0^{\infty}\mathbb{P}\left(\sum_{i=1}^{m}Y_i(u)<1-z\middle|\sum_{i=1}^{m}Y_i(u)\le 1\right)dz\\&=\int_0^{\infty}\frac{\mathbb{P}\left(\sum_{i=1}^{m}Y_i(u)<1-z\right)}{\mathbb{P}\left(\sum_{i=1}^{m}Y_i(u)\le 1\right)}dz\\&=\frac{F(u+a(u)/\varepsilon)}{\mathbb{P}\left(\sum_{i=1}^{m}Y_i(u)\le 1\right)}\int_0^{\infty}\frac{\mathbb{P}\left(\sum_{i=1}^{m}Y_i(u)<1-z\right)}{F(u+a(u)/\varepsilon)}dz.\end{aligned}$$

From the Gumbel assumption on F and formula (4.21) in Chapter 1, we find that the first term on the right-hand side in (5.5) satisfies

$$\lim_{u\to c}\frac{F(u+a(u)/\varepsilon)}{\mathbb{P}\left(\sum_{i=1}^{m}Y_i(u)\le 1\right)}=\frac{e^{1/\varepsilon}}{q_m^G(\alpha)}. \tag{5.6}$$

It remains to study the integral. Choose $M > 1$ and $\varepsilon < m$ and divide the integral into two parts:

$$\begin{aligned}
&\int_0^\infty \frac{\mathbb{P}\left(\sum_{i=1}^m Y_i(u) < 1-z\right)}{F(u+a(u)/\varepsilon)} dz \qquad (5.7)\\
&\quad = \int_0^M \frac{\mathbb{P}\left(\sum_{i=1}^m Y_i(u) < 1-z\right)}{F(u+a(u)/\varepsilon)} dz + \int_M^\infty \frac{\mathbb{P}\left(\sum_{i=1}^m Y_i(u) < 1-z\right)}{F(u+a(u)/\varepsilon)} dz.
\end{aligned}$$

To the first term we apply the dominated convergence theorem, the second term becomes arbitrarily small for large M.

Term 1. For $z > 0$

$$\mathbb{P}\left(\sum_{i=1}^m Y_i(u) < 1-z\right) \leq m \cdot \mathbb{P}\left(Y_1(u) < (1-z)/m\right) \leq m \cdot F(u+a(u)/m). \quad (5.8)$$

Hence for all large u we have that

$$\frac{\mathbb{P}\left(\sum_{i=1}^m Y_i(u) < 1-z\right)}{F(u+a(u)/\varepsilon)} \leq m \cdot \frac{F(u+a(u)/m)}{F(u+a(u)/\varepsilon)} \leq (m+1)\exp\{1/m - 1/\varepsilon\}. \quad (5.9)$$

Henceforth we have found an uniform upper bound, which implies that our function is L^1 on $[0, M]$. There remains to prove pointwise convergence in z so that we can apply the dominated convergence theorem to the first term on the right-hand side of (5.7).

We introduce the events $\{Y_1(u) < 1/\varepsilon\}$.

$$\begin{aligned}
&\frac{\mathbb{P}\left(\sum_{i=1}^m Y_i(u) < 1-z\right)}{F(u+a(u)/\varepsilon)} \qquad (5.10)\\
&= \mathbb{P}\left(\sum_{i=1}^m Y_i(u) < 1-z \,\middle|\, Y_1(u) < 1/\varepsilon\right)\\
&\qquad + \frac{\mathbb{P}\left(\sum_{i=1}^m Y_i(u) < 1-z, Y_1(u) \geq 1/\varepsilon\right)}{F(u+a(u)/\varepsilon)}. \qquad (5.11)
\end{aligned}$$

Lemma 5.3 of [2] states:

$$\begin{aligned}
&\lim_{u\to c} \mathbb{P}\left(X_i \leq u + x_i a(u), i = 1, \ldots, m \mid X_1 \leq u + a(u)/\varepsilon\right)\\
&= e^{-1/\varepsilon}\left(\sum_{i=1}^m e^{-\alpha x_i}\right)^{-1/\alpha}. \qquad (5.12)
\end{aligned}$$

When we apply this to the first term on the right-hand side of (5.10), we find

$$
\begin{aligned}
e^{-1/\varepsilon} f_{1,\varepsilon}(z) &\stackrel{def.}{=} \lim_{u\to c} \mathbb{P}\left(\sum_{i=1}^{m} Y_i(u) < 1-z \,\middle|\, Y_1(u) < 1/\varepsilon\right) \qquad (5.13)\\
&= e^{-1/\varepsilon} \int\limits_{\substack{\sum_i x_i < 1-z \\ x_1 < 1/\varepsilon}} \left(\frac{d^m}{dx_1\cdots dx_m} \left(\sum_{i=1}^{m} e^{-\alpha x_i}\right)^{-1/\alpha} \right) dx_1 \ldots dx_m.
\end{aligned}
$$

To the second term on the right-hand side of (5.10) we give an estimate which is similar to (4.11) in Chapter 1.

$$
\begin{aligned}
&\limsup_{u\to c} \frac{\mathbb{P}\left(\sum_{i=1}^{m} Y_i(u) < 1-z, Y_1(u) \geq 1/\varepsilon\right)}{F(u+a(u)/\varepsilon)}\\
&\quad\leq \limsup_{u\to c} \frac{(m-1)\cdot \mathbb{P}\left(Y_2(u) < (1-z)/m, Y_1(u) \geq 1/\varepsilon\right)}{F(u+a(u)/\varepsilon)} \qquad (5.14)\\
&\quad\leq \limsup_{u\to c} \frac{(m-1)\cdot F(u+a(u)(1-z)/m)}{F(u+a(u)/\varepsilon)}\\
&\qquad\quad \times \left(1 - \frac{\phi^{-1}\left(\phi\left(F(u+a(u)(1-z)/m)\right) + \phi\left(F(u+a(u)/\varepsilon)\right)\right)}{F(u+a(u)(1-z)/m)}\right)\\
&\quad= (m-1)e^{-1/\varepsilon}\left(e^{(1-z)/m} - \left(e^{-\alpha(1-z)/m} + e^{-\alpha/\varepsilon}\right)^{-1/\alpha}\right)\\
&\quad\leq (m-1)e^{-1/\varepsilon}e^{(1-z)/m}\left(1 - \left(1 + e^{-\alpha/\varepsilon+\alpha/m}\right)^{-1/\alpha}\right) \stackrel{def.}{=} e^{-1/\varepsilon} f_{2,\varepsilon}(z).
\end{aligned}
$$

Now we come to the last term on the right-hand side of (5.7). For $M > 1$,

$$
\begin{aligned}
&\int_M^\infty \frac{\mathbb{P}\left(\sum_{i=1}^m Y_i(u) < 1 - z\right)}{F(u + a(u)/\varepsilon)} dz \\
&\le m \int_M^\infty \frac{F(u + (1-z)a(u)/m)}{F(u + a(u)/\varepsilon)} dz \\
&= m \frac{F\left(u - \frac{M-1}{m} a(u)\right)}{F(u + a(u)/\varepsilon)} \int_{(M-1)/m}^\infty \frac{F(u - xa(u))}{F\left(u - \frac{M-1}{m} a(u)\right)} dx \qquad (5.15) \\
&= m \frac{F\left(u - \frac{M-1}{m} a(u)\right)}{F(u + a(u)/\varepsilon)} \int_{(M-1)/m}^\infty \mathbb{P}\left(Y_1(u) < -x \,|\, Y_1(u) < -(M-1)/m\right) dx \\
&= m \frac{F\left(u - \frac{M-1}{m} a(u)\right)}{F(u + a(u)/\varepsilon)} \mathbb{E}\left(-Y_1(u) | -Y_1(u) > (M-1)/m\right).
\end{aligned}
$$

Next we consider the expectation in the expression above:

$$
\begin{aligned}
\mathbb{E}\left(-Y_1(u) \middle| -Y_1(u) > \frac{M-1}{m}\right) &= \mathbb{E}\left(-\frac{X_1 - u}{a(u)} \middle| -\frac{X_1 - u}{a(u)} > \frac{M-1}{m}\right) \\
&= \frac{M-1}{m} + \frac{1}{a(u)} \mathbb{E}\left(-X_1 - v_M(u) | -X_1 > v_M(u)\right), \qquad (5.16)
\end{aligned}
$$

where $v_M(u) = (M-1)a(u)/m - u$. Now we may use the that we are working with marginals which have Gumbel type, henceforth (see [13], formula (3.3.34))

$$
\begin{aligned}
\limsup_{u \to c} \frac{1}{a(u)} \mathbb{E}\left(-X_1 - v_M(u) | -X_1 > v_M(u)\right) &= \limsup_{u \to c} \frac{a(-v_M(u))}{a(u)} \\
&= \limsup_{u \to c} \frac{a\left(-\frac{M-1}{m} a(u) + u\right)}{a(u)} = 1, \qquad (5.17)
\end{aligned}
$$

where in the last step we have used that $\lim_{u \to c} a'(u) = 0$ (see [13], Theorem 3.3.26 and formula (3.3.31)).

Hence we find for all $\varepsilon < m$ and all $M > 1$ (see (5.13), (5.14), (5.15), (5.17))

$$
\begin{aligned}
&\limsup_{u \to c} \int_0^\infty \frac{\mathbb{P}\left(\sum_{i=1}^m Y_i(u) < 1 - z\right)}{F(u + a(u)/\varepsilon)} dz \qquad (5.18) \\
&\quad \le e^{-1/\varepsilon} \left(\int_0^M f_{1,\varepsilon}(x) + f_{2,\varepsilon}(x) dx + m e^{-(M-1)/m} (M - 1/m + 1) \right).
\end{aligned}
$$

The function $f_{1,\varepsilon}$ is increasing in ε. Moreover

$$\int_0^M f_{2,\varepsilon}(x)dx = (m-1)m\left(e^{1/m} - e^{(1-M)/m}\right)\left(1-\left(1+e^{-\alpha/\varepsilon+\alpha/m}\right)^{-1/\alpha}\right), \tag{5.19}$$

which converges to 0 for $\varepsilon \to 0$. Hence we find (see (5.5), (5.6), (5.18))

$$\limsup_{M\to\infty}\limsup_{\varepsilon\to 0}\left(1-\mathbb{E}\left(\sum_{i=1}^m Y_i(u)\middle|\sum_{i=1}^m Y_i(u)\le 1\right)\right) \tag{5.20}$$
$$\le \frac{1}{q_m^G(\alpha)}\int_0^\infty\int_{\sum_i x_i<1-z}\left(\frac{d^m}{dx_1\cdots dx_m}\left(\sum_{i=1}^m e^{-\alpha x_i}\right)^{-1/\alpha}\right)dx_1\ldots dx_m\;dz$$
$$= \frac{1}{q_m^G(\alpha)}\int_{\sum_{i=1}^m x_i\le 1}\left(1-\sum_{i=1}^m x_i\right)\left(\frac{d^m}{dx_1\ldots dx_m}\left(\sum_{i=1}^m e^{-x_i\alpha}\right)^{-1/\alpha}\right)dx_1\ldots dx_m$$
$$= 1-\frac{1}{q_m^G(\alpha)}\int_{\sum_{i=1}^m x_i\le 1}\left(\sum_{i=1}^m x_i\right)\left(\frac{d^m}{dx_1\ldots dx_m}\left(\sum_{i=1}^m e^{-x_i\alpha}\right)^{-1/\alpha}\right)dx_1\ldots dx_m.$$

Exchanging the two integration finishes to proof of the upper bound. The same lower bound is found only considering the term coming from $f_{1,\varepsilon}$. This finishes the proof of (3.8).
Now, for the case $m=2$ we find

$$\begin{aligned}
c_2^G(\alpha) &= \frac{2}{q_2^G(\alpha)}\int_{x_1+x_2\le 1} x_1\left[\frac{d^2}{dx_1dx_2}\left(\sum_{i=1}^2 e^{-x_i\alpha}\right)^{-1/\alpha}\right]dx_1dx_2\\
&= \frac{2}{q_2^G(\alpha)}\int_{-\infty}^\infty xe^{-\alpha x}\left(e^{-\alpha x}+e^{-\alpha(1-x)}\right)^{-1/\alpha-1}dx\\
&= \frac{2}{q_2^G(\alpha)}\int_{-\infty}^\infty xe^{x}\left(1+e^{-\alpha(1-2x)}\right)^{-1/\alpha-1}dx \qquad (5.21)\\
&\overset{y=e^{-(1-2x)}}{=} \frac{2}{q_2^G(\alpha)}\left(\int_{-\infty}^\infty \frac{e^{1/2}}{4}(1+\log(y))y^{-1/2}\left(1+y^\alpha\right)^{-1/\alpha-1}dy\right)\\
&= \frac{2}{q_2^G(\alpha)}\left(\frac{e^{1/2}}{4}\mathbb{E}\left(Y_\alpha^{-1/2}\left(1+\log Y_\alpha\right)\right)\right).
\end{aligned}$$

Recall (5.39) from [2]:

$$q_2^G(\alpha) = \frac{e^{1/2}}{2}\mathbb{E}\left(Y_\alpha^{-1/2}\right), \tag{5.22}$$

and find:

$$c_2^G(\alpha) = 1 + \frac{\mathbb{E}\left(Y_\alpha^{-1/2}\log Y_\alpha\right)}{\mathbb{E}\left(Y_\alpha^{-1/2}\right)}. \tag{5.23}$$

This proves the left equality of (3.11); for a proof of the right equality we introduce $\gamma < 0$ and generalize:

$$\begin{aligned}
\frac{\mathbb{E}\left(Y_\alpha^\gamma \log(Y)\right)}{\mathbb{E}\left(Y_\alpha^\gamma\right)} &= \frac{\int_0^\infty y^\gamma \log(y)(1+y^\alpha)^{-\frac{1}{\alpha}-1}dy}{\int_0^\infty y^\gamma (1+y^\alpha)^{-\frac{1}{\alpha}-1}dy} \\
&= \frac{d}{d\gamma}\log\left(\int_0^\infty y^\gamma(1+y^\alpha)^{-\frac{1}{\alpha}-1}dy\right) \\
&\overset{z=y^\alpha}{=} \frac{d}{d\gamma}\log\left(\frac{1}{\alpha}\int_0^\infty z^{\frac{\gamma+1}{\alpha}-1}(1+z)^{-\frac{1}{\alpha}-1}dz\right) \\
&\overset{s=\frac{z}{1+z}}{=} \frac{d}{d\gamma}\log\left(\frac{1}{\alpha}\int_0^1 s^{\frac{\gamma+1}{\alpha}-1}(1-s)^{\frac{\alpha-\gamma}{\alpha}-1}ds\right) \\
&= \frac{d}{d\gamma}\log\left(\frac{1}{\alpha}B(\frac{\gamma+1}{\alpha},\frac{\alpha-\gamma}{\alpha})\right) = \frac{d}{d\gamma}\log\left(\frac{\Gamma\left(\frac{\gamma+1}{\alpha}\right)\Gamma\left(1-\frac{\gamma}{\alpha}\right)}{\alpha\Gamma\left(\frac{\alpha+1}{\alpha}\right)}\right) \\
&= \frac{d}{d\gamma}\left(\log\Gamma\left(\frac{\gamma+1}{\alpha}\right) + \log(-\gamma) + \log\Gamma\left(-\frac{\gamma}{\alpha}\right)\right) \\
&= \frac{1}{\alpha}\left(\log\Gamma\right)'\left(\frac{\gamma+1}{\alpha}\right) + \frac{1}{\gamma} - \frac{1}{\alpha}\left(\log\Gamma\right)'\left(-\frac{\gamma}{\alpha}\right).
\end{aligned}$$

Now we take $\gamma = -1/2$ and find:

$$\frac{\mathbb{E}\left(Y_\alpha^{-1/2}\log(Y_\alpha)\right)}{\mathbb{E}\left(Y_\alpha^{-1/2}\right)} = -2\ , \tag{5.24}$$

which, together with (5.23) finishes proof of Theorem 3.4. ■

Chapter 5

Esscher Premium

1 Introduction

In this chapter we shall take a look at the Esscher Premium. In general the Esscher Premium is defined as

$$\frac{\mathbb{E}\left(Xe^{T}\right)}{\mathbb{E}\left(e^{T}\right)}, \tag{1.1}$$

where X and T are random variables (see e.g. [26]). Note that this is equal to $\mathbb{E}(X)$ if X and T are independent. It is typically bigger than $\mathbb{E}(X)$ if X and T have positive dependence and smaller in the case of negative dependence, since in these cases either the high or the low values of X are weighed extra. In this manner it is sort of a measure for co-dependence. The Esscher Premium is based on the Esscher transform (Esscher [15]) and it was first used in the form

$$\frac{\mathbb{E}\left(Xe^{\lambda\sum_{i=1}^{m}X_i}\right)}{\mathbb{E}\left(e^{\lambda\sum_{i=1}^{m}X_i}\right)}, \tag{1.2}$$

for some original risk (random variable) X, some tradeable risks $X_1, \ldots, X_m$ and some constant λ (see [7]). Because in Bühlmann's article he mentions that for independent $X_1, \ldots, X_m$ and $X = X_1$:

$$\frac{\mathbb{E}\left(X_1e^{\lambda\sum_{i=1}^{m}X_i}\right)}{\mathbb{E}\left(e^{\lambda\sum_{i=1}^{m}X_i}\right)} = \frac{\mathbb{E}\left(X_1e^{\lambda X_1}\right)}{\mathbb{E}\left(e^{\lambda X_1}\right)}\frac{\mathbb{E}\left(e^{\lambda\sum_{i=2}^{m}X_i}\right)}{\mathbb{E}\left(e^{\lambda\sum_{i=2}^{m}X_i}\right)} = \frac{\mathbb{E}\left(X_1e^{\lambda X_1}\right)}{\mathbb{E}\left(e^{\lambda X_1}\right)}, \tag{1.3}$$

the Esscher Premium is also often described as

$$\frac{\mathbb{E}\left(Xe^{\lambda X}\right)}{\mathbb{E}\left(e^{\lambda X}\right)}, \tag{1.4}$$

for some random variable X and some constant λ. Although these premiums are used in a variety of ways (e.g. option pricing, [18]), we think of them in the way they were originally used; as ways to calculate the risk premium of an insurance.

Since we are only interested in the extreme values (as opposed to the premium that has to be paid to compensate for the expected value of the risk), we make use of a conditional Esscher Premium, conditioned on a large aggregate value of the risks:

$$\frac{\mathbb{E}\left(X_1 e^{\sum_{i=1}^m X_i} \middle| \sum_{i=1}^m X_i \le mu\right)}{\mathbb{E}\left(e^{\sum_{i=1}^m X_i} \middle| \sum_{i=1}^m X_i \le mu\right)}. \tag{1.5}$$

Because of this conditioning (and simply because in our case the X_i are not independent) we cannot use (1.3) to simplify this. We rather use the results of Chapter 1 and the fact that we have identical marginals to calculate (1.5).

2 The model

We study a multivariate model describing the Esscher Premium of one of m dependent risks. The dependence structure will be given by an Archimedian copula, and losses are assumed to be negative. More precisely our assumptions read as follows:

Assumption 2.1 *We assume that the random vector $(X_1, \ldots, X_m)$ satisfies:*
1) All coordinates X_i are negative and have the same continuous marginal

$$F(x) = \mathbb{P}(X_1 \le x).$$

2) $(X_1, \ldots, X_m)$ has an Archimedean copula with generator ϕ.
3) This generator ϕ is regularly varying at 0^+ with index $-\alpha$, where $\alpha > 0$.

These conditions are the same as in Chapter 1 for the exact reason of using the results of that chapter. Probably the first condition could be loosened a bit by using the results of Chapter 2 instead, but then we would run into problems since then we cannot easily connect

$$\frac{\mathbb{E}\left(X_1 e^{S} \middle| S \le mu\right)}{\mathbb{E}\left(e^{S} \middle| S \le mu\right)} \tag{2.1}$$

to

$$\frac{\mathbb{E}\left(Se^S \middle| S \leq mu\right)}{\mathbb{E}\left(e^S \middle| S \leq mu\right)} \tag{2.2}$$

anymore.

3 Results

In this section we formulate our results. Depending on the extreme value behaviour of the underlying risks, we distinguish two cases: the Fréchet case and the Gumbel case.

3.1 Fréchet case

In the Fréchet case we look at (dependent) random variables that have a Fréchet-type distribution: their marginal distributions are regularly varying at $-\infty$ with paramater $-\beta$, for some $\beta > 0$. In our case we additionally assume that $\beta > 1$. The latter assumption is needed in order for the random variables to have a (finite) mean.

Theorem 3.1 (Fréchet case) *Assume Assumption 2.1 and assume that F is regularly varying at $-\infty$ with parameter $-\beta$, $\beta > 1$. We have*

$$\lim_{u\to\infty} -\frac{1}{u}\frac{\mathbb{E}\left(X_1 e^S \middle| S \leq -u\right)}{\mathbb{E}\left(e^S \middle| S \leq -u\right)} = \frac{1}{m} \tag{3.1}$$

3.2 Gumbel case

In the Gumbel case we look at (dependent) random variables that lie in the domain of attraction of the exponential limit law for exceedances: there is a $c \geq -\infty$ and a positive measurable function $s \mapsto a(s)$ such that for $t \in \mathbb{R}$ one has for marginals F that $\lim_{u\downarrow c} F(u + ta(u))/F(u) = e^t$. In this case we take $c = -\infty$.

Theorem 3.2 (Gumbel case) *Assume 2.1 and F of Gumbel type with $c = -\infty$ and such that $a := \lim_{u\to-\infty} a(u)$ exists, then*

$$\lim_{u\to-\infty} \frac{1}{u} \cdot \frac{\mathbb{E}\left(X_1 e^S \middle| S \leq mu\right)}{\mathbb{E}\left(e^S \middle| S \leq mu\right)} = \frac{1}{1+ma} \ . \tag{3.2}$$

Remark 3.3 *It is remarkable that, even though the Esscher Premium can be seen as a measure of co-dependence, the dependence strength α seems to play no role in our results.*

4 Proofs

Proof of Theorem 3.1.
Again, we use the results of Chapter 1. The Fréchet part of Theorem 2.1 of that chapter states that

$$\lim_{u\to\infty} \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^{m} X_i \leq -u\right) = q_m^F(\alpha,\beta), \tag{4.1}$$

for a certain constant $q_m^F(\alpha,\beta)$. This tells us that the distribution function of the sum is also regularly varying in $-\infty$ with factor $-\beta$. For simplicity, let us write S for $\sum_{i=1}^{m} X_i$. Let F_S be its distribution function. Because of the exchangeability of the random variables we see that:

$$\begin{aligned}
&\mathbb{E}\left(X_1 e^S \middle| S \leq -u\right) \\
&= \frac{1}{m}\sum_{i=1}^{m} \mathbb{E}\left(X_i e^S \middle| S \leq -u\right) \\
&= \frac{1}{m}\mathbb{E}\left(S e^S \middle| S \leq -u\right)
\end{aligned}$$

We obtain, since $X_i \leq 0$ and so also $S \leq 0$

$$\begin{aligned}
&\mathbb{E}\left(S e^S \middle| S \leq -u\right) \\
&= -\int_{-\infty}^{0} \mathbb{P}\left(S e^S \leq x \middle| S \leq -u\right) dx
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&\mathbb{E}\left(e^S \middle| S \leq -u\right) \\
&= \int_{0}^{\infty} \mathbb{P}\left(e^S \geq x \middle| S \leq -u\right) dx
\end{aligned} \tag{4.3}$$

To prepare (4.2) for a substitution, we split it:

$$\begin{aligned}
&-\int_{-\infty}^{0} \mathbb{P}\left(Se^S \leq x \middle| S \leq -u\right) dx \\
&= -\int_{-\infty}^{-e^{-1}} \mathbb{P}\left(Se^S \leq x \middle| S \leq -u\right) dx \\
&\quad - \int_{-e^{-1}}^{0} \mathbb{P}\left(Se^S \leq x \middle| S \leq -u\right) dx \\
&= -\int_{-e^{-1}}^{0} \mathbb{P}\left(Se^S \leq x \middle| S \leq -u\right) dx, \qquad (4.4)
\end{aligned}$$

where we use that $s \mapsto se^s$ is larger than $-e^{-1}$ on $(-\infty, 0]$. Now we can substitute $x := se^s$, and assume $u > 1$ to get:

$$\begin{aligned}
&-\int_{-e^{-1}}^{0} \mathbb{P}\left(Se^S \leq x \middle| S \leq -u\right) dx \\
&= \int_{-\infty}^{-1} \mathbb{P}\left(Se^S \leq se^s \middle| S \leq -u\right) (1+s)e^s ds \\
&= \int_{-\infty}^{-1} \mathbb{P}\left(S \geq s \middle| S \leq -u\right) (1+s)e^s ds, \qquad (4.5)
\end{aligned}$$

where we use that $s \mapsto se^s$ is decreasing on $(-\infty, -1]$. In order to get a formula with distribution functions, that we can work with, we get out the conditional probability and substitute $s := -ut$:

$$\begin{aligned}
&\int_{-\infty}^{-1} \mathbb{P}\left(S \geq s \middle| S \leq -u\right) (1+s)e^s ds \\
&= \frac{1}{F_S(-u)} \int_{-\infty}^{-u} \mathbb{P}\left(S \geq s,\ S \leq -u\right) (1+s)e^s ds \\
&= \frac{1}{F_S(-u)} \int_{-\infty}^{-u} \left(\mathbb{P}\left(S \leq -u\right) - \mathbb{P}\left(S \leq s\right)\right) (1+s)e^s ds \\
&= \int_{-\infty}^{-u} \left(1 - \frac{F_S(s)}{F_S(-u)}\right) (1+s)e^s ds \\
&= u \int_{1}^{\infty} \left(1 - \frac{F_S(-ut)}{F_S(-u)}\right) (1-ut)e^{-ut} dt \qquad (4.6)
\end{aligned}$$

We use the fact that F_S is regularly varying at $-\infty$ with factor $-\beta$ to get the following: Let $\varepsilon > 0$, then there is an $u_0 \in \mathbb{R}$ such that $\forall u > u_0$:

$$\left| \frac{F_S(-ut)}{F_S(-u)} - t^{-\beta} \right| < \varepsilon \ , \tag{4.7}$$

for all $t \in [1, 2]$. We need this estimate for integration purposes. With this in the back of our mind we rewrite (4.6) into:

$$\begin{aligned}
& u \int_1^\infty \left(1 - \frac{F_S(-ut)}{F_S(-u)}\right)(1-ut)e^{-ut}dt \\
& = u \int_1^\infty (1-ut)e^{-ut}dt \\
& \qquad - u \int_1^\infty \left(\frac{F_S(-ut)}{F_S(-u)}\right)(1-ut)e^{-ut}dt \\
& = -ue^{-u} - u \int_1^2 \left(\frac{F_S(-ut)}{F_S(-u)} - t^{-\beta}\right)(1-ut)e^{-ut}dt \\
& \qquad - u \int_1^2 t^{-\beta}(1-ut)e^{-ut}dt \\
& \qquad - u \int_2^\infty \left(\frac{F_S(-ut)}{F_S(-u)}\right)(1-ut)e^{-ut}dt
\end{aligned} \tag{4.8}$$

To analyze the second term we can use (4.7):

$$\begin{aligned}
& \left| u \int_1^2 \left(\frac{F_S(-ut)}{F_S(-u)} - t^{-\beta}\right)(1-ut)e^{-ut}dt \right| \\
& \leq -u \int_1^2 \left| \left(\frac{F_S(-ut)}{F_S(-u)} - t^{-\beta}\right) \right| (1-ut)e^{-ut}dt \\
& \leq -u \int_1^2 \varepsilon(1-ut)e^{-ut}dt \\
& = \varepsilon u(e^{-u} - 2e^{-2u}) \ ,
\end{aligned} \tag{4.9}$$

whereas the last term of (4.8) gives us:

$$\begin{aligned}
&\left| u \int_2^\infty \left(\frac{F_S(-ut)}{F_S(-u)} \right) (1-ut) e^{-ut} dt \right| \\
&\leq u \int_2^\infty \left| \left(\frac{F_S(-ut)}{F_S(-u)} \right) (1-ut) e^{-ut} \right| dt \\
&\leq -u \int_2^\infty (1-ut) e^{-ut} dt \\
&= u 2 e^{-2u} \ , \qquad (4.10)
\end{aligned}$$

where we used the obvious fact that F_S is increasing. With (4.9) and (4.10) we can now rewrite (4.8) into:

$$\begin{aligned}
& u \int_1^\infty \left(1 - \frac{F_S(-ut)}{F_S(-u)} \right) (1-ut) e^{-ut} dt \\
&= -ue^{-u} + O(\varepsilon u e^{-u}) + o(ue^{-u}) \\
&\qquad -u \int_1^2 t^{-\beta} (1-ut) e^{-ut} dt. \qquad (4.11)
\end{aligned}$$

To calculate this last integral, we note that:

$$\begin{aligned}
&\left| u \int_2^\infty t^{-\beta} (1-ut) e^{-ut} dt \right| \\
&\leq -u \int_2^\infty (1-ut) e^{-ut} dt \\
&= u 2 e^{-2u} = o(ue^{-u}) \qquad (4.12)
\end{aligned}$$

This tells us that for the behaviour of the last integral in (4.11), we can take a look at the limit behaviour of the following fraction:

$$\begin{aligned}
&\lim_{u \to \infty} \frac{u \int_1^\infty t^{-\beta} (1-ut) e^{-ut} dt}{ue^{-u}} \\
&\overset{t \to s/u}{=} \lim_{u \to \infty} \frac{\int_u^\infty s^{-\beta} (1-s) e^{-s} ds}{u^{1-\beta} e^{-u}},
\end{aligned}$$

so we can use de l'Hôpital:

$$\begin{aligned}
&\lim_{u\to\infty} \frac{\int_u^\infty s^{-\beta}(1-s)e^{-s}ds}{u^{1-\beta}e^{-u}} \\
&= \lim_{u\to\infty} \frac{u^{-\beta}(1-u)e^{-u}}{-u^{1-\beta}e^{-u}+(1-\beta)u^{-\beta}e^{-u}} \\
&= \lim_{u\to\infty} \frac{1-u}{-u+(1-\beta)} = 1 \qquad (4.13)
\end{aligned}$$

Here we momentarily let our analysis of (4.2) rest, and start working on (4.3):

$$\begin{aligned}
&\int_0^\infty \mathbb{P}\left(e^S \geq x \middle| S \leq -u\right) dx \\
&= \int_1^\infty \mathbb{P}\left(e^S \geq x \middle| S \leq -u\right) dx \\
&+ \int_0^1 \mathbb{P}\left(e^S \geq x \middle| S \leq -u\right) dx \\
&= \int_0^1 \mathbb{P}\left(e^S \geq x \middle| S \leq -u\right) dx, \qquad (4.14)
\end{aligned}$$

where we used that $e^S \leq 1$. We substitute $x := e^s$ and get:

$$\begin{aligned}
&\int_0^1 \mathbb{P}\left(e^S \geq x \middle| S \leq -u\right) dx \\
&= \int_{-\infty}^0 \mathbb{P}\left(S \geq s \middle| S \leq -u\right) e^s ds \qquad (4.15)
\end{aligned}$$

We get out the conditional probability and substitute $s := -ut$:

$$\begin{aligned}
&\int_{-\infty}^0 \mathbb{P}\left(S > s \middle| S \leq -u\right) e^s ds \\
&= \frac{1}{F_S(-u)} \int_{-\infty}^{-u} \mathbb{P}\left(S > s,\ S \leq -u\right) e^s ds \\
&= \frac{1}{F_S(-u)} \int_{-\infty}^{-u} \left(\mathbb{P}\left(S \leq -u\right) - \mathbb{P}\left(S \leq s\right)\right) e^s ds \\
&= \int_{-\infty}^{-u} \left(1 - \frac{F_S(s)}{F_S(-u)}\right) e^s ds \\
&= u \int_1^\infty \left(1 - \frac{F_S(-ut)}{F_S(-u)}\right) e^{-ut} dt \qquad (4.16)
\end{aligned}$$

Now, using the same ε and u_0 as in (4.7), we rewrite (4.16):

$$\begin{aligned}
& u\int_1^\infty \left(1-\frac{F_S(-ut)}{F_S(-u)}\right)e^{-ut}dt \\
&= u\int_1^\infty e^{-ut}dt \\
&\quad -u\int_1^\infty \left(\frac{F_S(-ut)}{F_S(-u)}\right)e^{-ut}dt \\
&= e^{-u} - u\int_1^2 \left(\frac{F_S(-ut)}{F_S(-u)} - t^{-\beta}\right)e^{-ut}dt \\
&\quad -u\int_1^2 t^{-\beta}e^{-ut}dt \\
&\quad -u\int_2^\infty \left(\frac{F_S(-ut)}{F_S(-u)}\right)e^{-ut}dt
\end{aligned} \tag{4.17}$$

About the second term we can say:

$$\begin{aligned}
& \left|u\int_1^2 \left(\frac{F_S(-ut)}{F_S(-u)} - t^{-\beta}\right)e^{-ut}dt\right| \\
&\leq u\int_1^2 \left|\left(\frac{F_S(-ut)}{F_S(-u)} - t^{-\beta}\right)\right|e^{-ut}dt \\
&\leq u\int_1^2 \varepsilon e^{-ut}dt \\
&= \varepsilon(e^{-u} - e^{-2u}) \ ,
\end{aligned} \tag{4.18}$$

whereas the last term of (4.17) gives us:

$$\begin{aligned}
\left|u\int_2^\infty \left(\frac{F_S(-ut)}{F_S(-u)}\right)e^{-ut}dt\right| &\leq u\int_2^\infty \left|\left(\frac{F_S(-ut)}{F_S(-u)}\right)e^{-ut}\right|dt \\
&\leq u\int_2^\infty e^{-ut}dt \\
&= e^{-2u} \ ,
\end{aligned} \tag{4.19}$$

where we again used that F_S is increasing. With (4.18) and (4.19) we can rewrite (4.17):

$$
\begin{aligned}
& u\int_1^\infty \left(1-\frac{F_S(-ut)}{F_S(-u)}\right)e^{-ut}dt \\
& = e^{-u}+O(\varepsilon e^{-u})+o(e^{-u})-u\int_1^2 t^{-\beta}e^{-ut}dt. \qquad (4.20)
\end{aligned}
$$

To calculate the last integral we note that:

$$
\begin{aligned}
\left|u\int_2^\infty t^{-\beta}e^{-ut}dt\right| &\leq -u\int_2^\infty e^{-ut}dt \\
&= e^{-2u}=o(e^{-u})\ . \qquad (4.21)
\end{aligned}
$$

So, for the calculation of the last integral in (4.20), we can look at the limit behaviour of the following fraction:

$$
\begin{aligned}
\lim_{u\to\infty}\frac{u\int_1^\infty t^{-\beta}e^{-ut}dt}{e^{-u}} &\overset{t\to s/u}{=} \lim_{u\to\infty}\frac{u\int_u^\infty \left(\frac{s}{u}\right)^{-\beta}e^{-s}d\frac{s}{u}}{e^{-u}} \\
&= \lim_{u\to\infty}\frac{\int_u^\infty s^{-\beta}e^{-s}ds}{u^{-\beta}e^{-u}}, \qquad (4.22)
\end{aligned}
$$

and again we use de l'Hôpital:

$$
\begin{aligned}
\lim_{u\to\infty}\frac{\int_u^\infty s^{-\beta}e^{-s}ds}{u^{-\beta}e^{-u}} &= \lim_{u\to\infty}\frac{u^{-\beta}e^{-u}}{-u^{-\beta}e^{-u}-\beta u^{-\beta-1}e^{-u}} \\
&= \lim_{u\to\infty}\frac{1}{-1-\beta u^{-1}}=-1 \qquad (4.23)
\end{aligned}
$$

After these preparations, we combine (4.2), (4.11), (4.12), (4.13) ,(4.20), (4.21) and (4.23) to get:

$$
\begin{aligned}
&\lim_{u\to\infty} -\frac{1}{u}\frac{\mathbb{E}\left(X_1 e^S \middle| S \le -u\right)}{\mathbb{E}\left(e^S \middle| S \le -u\right)} \\
&\lim_{u\to\infty} -\frac{1}{mu}\frac{\mathbb{E}\left(S e^S \middle| S \le -u\right)}{\mathbb{E}\left(e^S \middle| S \le -u\right)} \\
&= \frac{1}{m}\lim_{u\to\infty}\frac{-ue^{-u} + O(\varepsilon u e^{-u}) + o(ue^{-u}) - u\int_1^\infty t^{-\beta}(1-ut)e^{-ut}dt}{-u\left(e^{-u} + O(\varepsilon e^{-u}) + o(e^{-u}) - u\int_1^\infty t^{-\beta}e^{-ut}dt\right)} \\
&= \frac{1}{m}\lim_{u\to\infty}\frac{1 + O(\varepsilon) + \frac{u\int_1^\infty t^{-\beta}(1-ut)e^{-ut}dt}{ue^{-u}}}{\left(-1 + O(\varepsilon) + \frac{u\int_1^\infty t^{-\beta}e^{-ut}dt}{e^{-u}}\right)} \\
&= \frac{1}{m}\frac{1 + O(\varepsilon) + 1}{-1 + O(\varepsilon) - 1}. \qquad (4.24)
\end{aligned}
$$

This completes the proof as we let ε go to 0.

■

Proof of Theorem 3.2. We shall use the same equations as in the Fréchet case for the first part and the same techniques for the second part. (We once again write S for $\sum_{i=1}^m X_i$.) We note that, since $c = -\infty$, we can safely assume $u \le 0$ and use the representations (4.5) and (4.15) for the denominator and the numerator. But first we use the Gumbel result of Theorem (2.1) in Chapter 1 (twice) with $c = -\infty$ to see that for every $t \in \mathbb{R}$:

$$
\begin{aligned}
&\lim_{u\to-\infty}\frac{F_S(m(u+ta(u)))}{F_S(mu)} \\
&= \lim_{u\to-\infty}\frac{F_S(m(u+ta(u)))}{F(u+ta(u))}\frac{F(u)}{F_S(mu)}\frac{F(u+ta(u))}{F(u)} \\
&= \lim_{u\to-\infty}\frac{F(u+ta(u))}{F(u)} = e^t \qquad (4.25)
\end{aligned}
$$

Here we use and rewrite (4.5), where we use mu instead of $-u$ and substitute $s := m(u + ta(u))$:

$$
\begin{aligned}
&\int_{-\infty}^{-1} \mathbb{P}\left(S \geq s \middle| S \leq mu\right)(1+s)e^s ds \\
&= \frac{1}{F_S(mu)} \int_{-\infty}^{mu} \mathbb{P}\left(S \geq s,\ S \leq mu\right)(1+s)e^s ds \\
&= \frac{1}{F_S(mu)} \int_{-\infty}^{mu} \left(\mathbb{P}\left(S \leq mu\right) - \mathbb{P}\left(S \leq s\right)\right)(1+s)e^s ds \\
&= \int_{-\infty}^{mu} \left(1 - \frac{F_S(s)}{F_S(mu)}\right)(1+s)e^s ds \\
&\overset{s=m(u+ta(u))}{=} ma(u) \int_{-\infty}^{0} \left(1 - \frac{F_S(m(u+ta(u)))}{F_S(mu)}\right) \cdot \\
&\qquad (1+m(u+ta(u)))e^{m(u+ta(u))} dt\ . \qquad (4.26)
\end{aligned}
$$

We shall also use (4.25) in a way similar to the Fréchet case. First, however; let $M > 0$, let $\varepsilon > 0$, then there is an $u_0 \in \mathbb{R}$ such that $\forall u > u_0$:

$$
\left| \frac{F_S(m(u+ta(u)))}{F_S(mu)} - e^t \right| < \varepsilon\ , \qquad (4.27)
$$

for all $t \in [-M, 0]$. To use this inequality, we rewrite (4.26), where we shall use $s(u,t) := m(u+ta(u))$, to keep it readable:

$$
\begin{aligned}
&ma(u) \int_{-\infty}^{0} \left(1 - \frac{F_S(s(u,t))}{F_S(mu)}\right)(1+s(u,t))e^{s(u,t)} dt \\
&= ma(u) \int_{-\infty}^{0} (1+s(u,t))e^{s(u,t)} dt \\
&-ma(u) \int_{-\infty}^{0} \left(\frac{F_S(s(u,t))}{F_S(mu)}\right)(1+s(u,t))e^{s(u,t)} dt \\
&= mue^{mu} \\
&-ma(u) \int_{-M}^{0} \left(\frac{F_S(s(u,t))}{F_S(mu)} - e^t\right)(1+s(u,t))e^{s(u,t)} dt \\
&-ma(u) \int_{-M}^{0} e^t (1+s(u,t))e^{s(u,t)} dt \\
&-ma(u) \int_{-\infty}^{-M} \left(\frac{F_S(s(u,t))}{F_S(mu)}\right)(1+s(u,t))e^{s(u,t)} dt. \qquad (4.28)
\end{aligned}
$$

When we assume $u \leq -1/m$, the second term becomes:

$$\begin{aligned}
&\left| ma(u) \int_{-M}^{0} \left(\frac{F_S(s(u,t))}{F_S(mu)} - e^t \right) (1+s(u,t))e^{s(u,t)} dt \right| \\
&\leq ma(u) \int_{-M}^{0} \left| \frac{F_S(s(u,t))}{F_S(mu)} - e^t \right| (1+s(u,t))e^{s(u,t)} dt \\
&\leq ma(u) \int_{-M}^{0} \varepsilon (1+s(u,t))e^{s(u,t)} dt \\
&= \varepsilon \left(mue^{mu} - m(u - Ma(u))e^{m(u-Ma(u))} \right) \ , \qquad (4.29)
\end{aligned}$$

whereas the last term of (4.28) gives us (again with $u \leq -1/m$):

$$\begin{aligned}
&\left| ma(u) \int_{-\infty}^{-M} \left(\frac{F_S(s(u,t))}{F_S(mu)} \right) (1+s(u,t))e^{s(u,t)} dt \right| \\
&\leq ma(u) \int_{-\infty}^{-M} \left| \left(\frac{F_S(s(u,t))}{F_S(mu)} \right) (1+s(u,t))e^{s(u,t)} \right| dt \\
&\leq ma(u) \int_{-\infty}^{-M} |1+m(u+ta(u))| e^{m(u+ta(u))} dt \\
&= m(Ma(u) - u)e^{m(u-Ma(u))} \ , \qquad (4.30)
\end{aligned}$$

where we again used the obvious fact that F_S is increasing. With (4.29) and (4.30) we can rewrite (4.28) as follows:

$$\begin{aligned}
&ma(u) \int_{-\infty}^{0} \left(1 - \frac{F_S(m(u+ta(u)))}{F_S(mu)} \right) (1+m(u+ta(u)))e^{m(u+ta(u))} dt \\
&= mue^{mu} + O(\varepsilon mue^{mu}) + O(m(u-Ma(u))e^{m(u-Ma(u))}) \\
&\quad + ma(u) \int_{-M}^{0} e^t (1+m(u+ta(u)))e^{m(u+ta(u))} dt. \qquad (4.31)
\end{aligned}$$

In order to estimaate this last term, we note that:

$$\begin{aligned}
&\left| ma(u)\int_{-\infty}^{-M} e^t(1+m(u+ta(u)))e^{m(u+ta(u))}dt\right| \\
&\leq e^{-M}\left| ma(u)\int_{-\infty}^{-M} (1+m(u+ta(u)))e^{m(u+ta(u))}dt\right| \\
&= e^{-M}m(u-Ma(u))e^{m(u-Ma(u))}, \qquad (4.32)
\end{aligned}$$

so we can look at:

$$\begin{aligned}
&ma(u)\int_{-\infty}^{0} e^t(1+m(u+ta(u)))e^{m(u+ta(u))}dt \\
&= ma(u)e^{mu}\int_{-\infty}^{0} (1+mu)e^{t(1+ma(u))} + tma(u)e^{t(1+ma(u))}dt \\
&= ma(u)e^{mu}\left[\frac{1+mu}{1+ma(u)}e^{t(1+ma(u))}\right]_{-\infty}^{0} \\
&\quad + ma(u)e^{mu}\int_{-\infty}^{0} tma(u)e^{t(1+ma(u))}dt \\
&= ma(u)e^{mu}\frac{1+mu}{1+ma(u)} + ma(u)e^{mu}\left[\frac{ma(u)}{1+ma(u)}te^{t(1+ma(u))}\right]_{-\infty}^{0} \\
&\quad - ma(u)e^{mu}\left[\frac{ma(u)}{(1+ma(u))^2}e^{t(1+ma(u))}\right]_{-\infty}^{0} \\
&= ma(u)e^{mu}\left(\frac{1+mu}{1+ma(u)} - \frac{ma(u)}{(1+ma(u))^2}\right) \qquad (4.33)
\end{aligned}$$

For now, we have worked enough on the denominator and we shall first rewrite the numerator (4.15), where again we use mu instead of $-u$ and substitute $s := m(u + ta(u))$:

$$
\begin{aligned}
&\int_{-\infty}^{-1} \mathbb{P}\left(S \geq s \middle| S \leq mu \right) e^s ds \\
&= \frac{1}{F_S(mu)} \int_{-\infty}^{mu} \mathbb{P}\left(S \geq s,\ S \leq mu \right) e^s ds \\
&= \frac{1}{F_S(mu)} \int_{-\infty}^{mu} \left(\mathbb{P}\left(S \leq mu \right) - \mathbb{P}\left(S \leq s \right) \right) e^s ds \\
&= \int_{-\infty}^{mu} \left(1 - \frac{F_S(s)}{F_S(mu)} \right) e^s ds \\
&= ma(u) \int_{-\infty}^{0} \left(1 - \frac{F_S(m(u+ta(u)))}{F_S(mu)} \right) e^{m(u+ta(u))} dt
\end{aligned}
\tag{4.34}
$$

We take the same M, ε and u_0 as in (4.27). Then for $u > u_0$:

$$
\begin{aligned}
&ma(u) \int_{-\infty}^{0} \left(1 - \frac{F_S(m(u+ta(u)))}{F_S(mu)} \right) e^{m(u+ta(u))} dt \\
&= ma(u) \int_{-\infty}^{0} e^{m(u+ta(u))} dt \\
&-ma(u) \int_{-\infty}^{0} \left(\frac{F_S(m(u+ta(u)))}{F_S(mu)} \right) e^{m(u+ta(u))} dt \\
&= e^{mu} - ma(u) \int_{-M}^{0} \left(\frac{F_S(m(u+ta(u)))}{F_S(mu)} - e^t \right) e^{m(u+ta(u))} dt \\
&-ma(u) \int_{-M}^{0} e^t e^{m(u+ta(u))} dt \\
&-ma(u) \int_{-\infty}^{-M} \left(\frac{F_S(m(u+ta(u)))}{F_S(mu)} \right) e^{m(u+ta(u))} dt
\end{aligned}
\tag{4.35}
$$

About that second term we can say:

$$\begin{aligned}
&\left| ma(u) \int_{-M}^{0} \left(\frac{F_S(m(u+ta(u)))}{F_S(mu)} - e^t \right) e^{m(u+ta(u))} dt \right| \\
&\leq ma(u) \int_{-M}^{0} \left| \frac{F_S(m(u+ta(u)))}{F_S(mu)} - e^t \right| e^{m(u+ta(u))} dt \\
&\leq ma(u) \int_{-M}^{0} \varepsilon e^{m(u+ta(u))} dt \\
&= \varepsilon \left(e^{mu} - e^{m(u-Ma(u))} \right) \ , \qquad (4.36)
\end{aligned}$$

and about the last term of (4.35) we conclude:

$$\begin{aligned}
&\left| ma(u) \int_{-\infty}^{-M} \left(\frac{F_S(m(u+ta(u)))}{F_S(mu)} \right) e^{m(u+ta(u))} dt \right| \\
&\leq ma(u) \int_{-\infty}^{-M} \left| \left(\frac{F_S(m(u+ta(u)))}{F_S(mu)} \right) e^{m(u+ta(u))} \right| dt \\
&\leq ma(u) \int_{-\infty}^{-M} e^{m(u+ta(u))} dt \\
&= e^{m(u-Ma(u))} \ , \qquad (4.37)
\end{aligned}$$

where we, for the last time, used the obvious fact that F_S is increasing. With (4.36) and (4.37) we can now rewrite (4.35) into:

$$\begin{aligned}
&ma(u) \int_{-\infty}^{0} \left(1 - \frac{F_S(m(u+ta(u)))}{F_S(mu)} \right) e^{m(u+ta(u))} dt \\
&= e^{mu} + O(\varepsilon e^{mu}) + O(e^{m(u-Ma(u))}) \\
&\qquad + ma(u) \int_{-M}^{0} e^t e^{m(u+ta(u))} dt. \qquad (4.38)
\end{aligned}$$

To calculate this last term, we note that:

$$\begin{aligned}
&\left| ma(u) \int_{-\infty}^{-M} e^t e^{m(u+ta(u))} dt \right| \\
&\leq e^{-M} \left| ma(u) \int_{-\infty}^{-M} e^{m(u+ta(u))} dt \right| \\
&= e^{-M} e^{m(u-Ma(u))}, \qquad (4.39)
\end{aligned}$$

so we can look at:

$$\begin{aligned}
&ma(u) \int_{-\infty}^{0} e^t e^{m(u+ta(u))} dt \\
&= ma(u) e^{mu} \int_{-\infty}^{0} e^{t(1+ma(u))} dt \\
&= ma(u) e^{mu} \left[\frac{1}{1+ma(u)} e^{t(1+ma(u))} \right]_{-\infty}^{0} \\
&= ma(u) e^{mu} \frac{1}{1+ma(u)} \qquad (4.40)
\end{aligned}$$

And finally we combine (4.2) (4.31), (4.32), (4.33), (4.38), (4.39) and (4.40) to get to:

$$\lim_{u \to -\infty} \frac{1}{u} \frac{\mathbb{E}\left(X_1 e^S \middle| S \leq -u \right)}{\mathbb{E}\left(e^S \middle| S \leq -u \right)} = \lim_{u \to -\infty} \frac{1}{mu} \frac{\mathbb{E}\left(S e^S \middle| S \leq -u \right)}{\mathbb{E}\left(e^S \middle| S \leq -u \right)}, \qquad (4.41)$$

where the numerator becomes:

$$\begin{aligned}
&mue^{mu} + O(\varepsilon mue^{mu}) + O(m(u - Ma(u))e^{m(u-Ma(u))}) \\
&+O(e^{-M} m(u - Ma(u))e^{m(u-Ma(u))}) \\
&+ma(u) \int_{-M}^{0} e^t (1 + m(u + ta(u)))e^{m(u+ta(u))} dt \ , \qquad (4.42)
\end{aligned}$$

and the denominator of (4.41) becomes:

$$\begin{aligned}
&e^{mu} + O(e^{-M} e^{m(u-Ma(u))}) + O(\varepsilon e^{mu}) + O(e^{m(u-Ma(u))}) \\
&-ma(u) \int_{-M}^{0} e^t e^{m(u+ta(u))} dt \ . \qquad (4.43)
\end{aligned}$$

When we take M very large and ε very small, (4.41) becomes:

$$\begin{aligned}
&\lim_{u\to-\infty}\frac{1}{mu}\frac{mue^{mu}-ma(u)\int_{-\infty}^{0}e^{t}(1+m(u+ta(u)))e^{m(u+ta(u))}dt}{e^{mu}-ma(u)\int_{-\infty}^{0}e^{t}e^{m(u+ta(u))}dt}\\
&=\lim_{u\to-\infty}\frac{1}{mu}\frac{mue^{mu}-ma(u)e^{mu}\left(\frac{1+mu}{1+ma(u)}-\frac{ma(u)}{(1+ma(u))^2}\right)}{e^{mu}-e^{mu}\frac{ma(u)}{1+ma(u)}}\\
&=\lim_{u\to-\infty}\frac{1-\frac{a(u)}{u}\left(\frac{1+mu}{1+ma(u)}-\frac{ma(u)}{(1+ma(u))^2}\right)}{1-\frac{ma(u)}{u(1+ma(u))}}\\
&=\lim_{u\to-\infty}1-\frac{ma(u)}{1+ma(u)}=\lim_{u\to-\infty}\frac{1}{1+ma(u)}=\frac{1}{1+ma}.
\end{aligned} \tag{4.44}$$

This finishes the proof.

■

5 An example

In this section we shall compare the Esscher Premium of an example to its Expected Shortfall (the corresponding Expected Value).
As mentioned before, by comparing the Esscher Premium to the expected value we can say something about the dependence. In this chapter the dependence between X_1 and S. Since we look at a conditional Esscher Premium, conditioned on a large S, we have to compare it to the expected value with the same condition. For this we can use our results in Chapter 4.
For our model we shall look at a two dependent random variables X_1 and X_2 and their sum S. They have marginal distribution $F(x) := e^x$ on $(-\infty, 0]$ and Clayton copula with index α. So their joint distribution F_{X_1,X_2} function looks like:

$$F_{X_1,X_2} : (-\infty,0]^2 \to [0,1] : (x_1,x_2) \to \left(e^{-\alpha x_1}+e^{-\alpha x_2}-1\right)^{-1/\alpha} \quad . \tag{5.1}$$

These random variables fulfill the requirements for Theorem 3.2 with
$a(s) = 1$, $\forall s$ so we can state that

$$\lim_{u\to-\infty}\frac{1}{u}\cdot\frac{\mathbb{E}\left(X_1e^S\middle|\,S\le 2u\right)}{\mathbb{E}\left(e^S\middle|\,S\le 2u\right)}=\frac{1}{1+2\cdot 1}=\frac{1}{3}\ . \tag{5.2}$$

We shall see if this means that the (conditioned) Esscher Premium is larger than the (conditioned) expected value, which is what we expect, since X_1 and S are obviously co-dependent. The way we do this is by calculating the ratio between the Esscher Premium and the corresponding expected value. Note here that both the Esscher Premium and the expected value are negative, so that this ratio should be smaller than 1.
We have to determine how $\mathbb{E}(X_1|S < 2u)$ behaves for large, negative u. For this we notice that this model also fulfills the requirements of Theorem 3.4 in Chapter 4 and look at the result we derive from that:

$$\lim_{u\to-\infty} \mathbb{E}\left(S|\, S \le 2u+1\right) - 2u = c_2^G(\alpha) = -1 \ . \tag{5.3}$$

We can rewrite this:

$$\lim_{v\to-\infty} \mathbb{E}\left(X_1|\, S \le 2v\right) - v + \frac{1}{2} = \frac{-1}{2} \ . \tag{5.4}$$

This leads us to the ratio:

$$\begin{aligned}
&\lim_{u\to-\infty} \frac{\text{conditional Esscher Premium}}{\text{conditional expected value}} \\
&= \lim_{u\to-\infty} \frac{\frac{\mathbb{E}\left(X_1 e^S \middle| S \le 2u\right)}{\mathbb{E}\left(e^S | S \le 2u\right)}}{\mathbb{E}\left(X_1|\, S \le 2u\right)} \\
&= \lim_{u\to-\infty} \frac{1}{u} \frac{\mathbb{E}\left(X_1 e^S \middle|\, S \le 2u\right)}{\mathbb{E}\left(e^S|\, S \le 2u\right)} \cdot \frac{u}{\mathbb{E}\left(X_1|\, S \le 2u\right)} \\
&= \lim_{u\to-\infty} \frac{1}{3} \frac{u}{u-1} \\
&= \frac{1}{3},
\end{aligned} \tag{5.5}$$

which is indeed smaller than 1.

It is not strange that this ratio is significantly smaller than 1, even for small α, since it gives a measure for the dependence between X_1 and S, and not between X_1 and X_2, like α does. It is however remarkable that α has no effect whatsoever on this ratio, since for large α, X_1 and S are stronger dependent than for small α. For this conditioned Esscher Premium it seems to be more important that there is a dependence of this form, than the strenght of this dependence. This could serve as a warning to be careful when using our results for small α.

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Summary in Dutch

Copulas en Extreme Waarden

Voor een verzekering is het nodig dat er voldoende reserves zijn om uit te kunnen betalen wanneer dat nodig is. Om te bepalen hoe groot die reserves moeten zijn, is het nodig om te weten hoeveel schade er kan zijn binnen een bepaalde periode, bijvoorbeeld één jaar. Hierbij dient bij voorkeur niet te worden gekeken naar een gemiddeld jaar, maar naar een (uitzonderlijk) slecht jaar. Als de reserve namelijk voldoende is om dat op te vangen, dan is het uiteraard ook voldoende voor de minder slechte jaren. Nu kan zo'n uitzonderlijk slecht jaar voorkomen doordat één verzekerde een buitengewoon groot bedrag claimt, maar ook doordat vele verzekerden tegelijkertijd een bedrag claimen. Dit eerste wordt vaak afgedekt door in de voorwaarden een maximum dekking op te nemen. Het tweede kan echter tot grote problemen leiden, vooral ook omdat een ongeluk zelden alleen komt. Denk hierbij aan een kettingbotsing (meerdere autoschades) of een grote brand (meerdere woningschades), maar ook aan hagelschade (potentieel heel veel autoschades) of een aardbeving (potentieel enorme woningschades).

Om de zinsnede "een ongeluk komt zelden alleen" te modelleren, moet gekeken worden naar de afhankelijkheden tussen de individuele uitkeringen. Een algemene manier om afhankelijkheid te modelleren is via het concept van copulas.

Copulas zijn een manier om een multivariate kansverdeling te beschrijven. Ze zijn geïntroduceerd door Sklar [25], die aantoonde dat elke kansverdeling van eindige dimensie kan worden geschreven als de samenstelling van de individuele marginale verdelingen en een copula, die de afhankelijkheid beschrijft. Dit gebeurt volgens de volgende formule:

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_m(x_m)),$$

waar F een m-dimensionale verdelingsfunctie is met marginale verdelingsfuncties $F_1, \ldots, F_m$ en waar C de copula is.

Voorbeelden van copulas zijn:
Definitie [De product copula]

$$\Pi : [0,1]^m \to [0,1] \ : \ (x_1, \ldots, x_m) \to \prod_{i=1}^{m} x_i \ ,$$

en:
Definitie [De comonotone copula]

$$M : [0,1]^m \to [0,1] \ : \ (x_1, \ldots, x_m) \to \min(\{x_1, \ldots, x_m\}) \ ,$$

die respectievelijk onafhankelijkheid en totale afhankelijkheid vastleggen.

In dit boek wordt voornamelijk gekeken naar zogenaamde Archimedische copulas, dit zijn copulas van de vorm:

$$C^\phi(x_1, \ldots, x_m) \stackrel{def.}{=} \phi^{-1}\left(\sum_{i=1}^{m} \phi(x_i)\right),$$

waar $m \geq 2$ en $\phi : [0,1] \to [0,\infty]$ strikt dalend en convex met $\phi(0) = \infty$ en $\phi(1) = 0$. Deze functie ϕ wordt de *generator* van de copula C^ϕ genoemd.

Het grote voordeel van het gebruik van Archimedische copulas om de afhankelijkheid te modelleren is dat slechts de generator geschat dient te worden in plaats van de gehele afhankelijkheidsstructuur. Om statistici verder van dienst te zijn, beperken we ons in de meeste hoofdstukken van dit boek zelfs tot Archimedische copulas waarvan de generator regulier variërend (zie Bingham-Goldie-Teugels [6]) is in 0^+ met index $-\alpha$, waardoor slechts deze α geschat hoeft te worden.

Al lijken deze beperkingen misschien op het eerste gezicht zwaar, enkele zeer interessante copulas, die daadwerkelijk door verzekeringsmaatschappijen gebruikt worden, vallen hier binnen.

Voor de marginale verdelingen, die de kansverdeling van de schadepost van een individuele deelnemer beschrijven, halen we onze inspiratie uit de extreme waarden theorie. Deze theorie is een soort tegenhanger van de Centrale Limiet Stelling. Waar de Centrale Limiet Stelling zegt dat het gewogen gemiddelde van een groot aantal stochasten zich ongeveer zoals een normaal verdeelde stochast gedraagt, zo zegt de hoofdstelling van de extreme waarden theorie, de Fisher-Tippett Stelling (zie [13]), iets over het gewogen maximum van een groot aantal stochasten. Hier is er echter geen universele limietverdeling, zoals de normale verdeling bij de Centrale Limiet Stelling, maar valt de stelling uiteen in drie delen, die, afhankelijk van de verdelingen van de individuele stochasten, drie mogelijke limietverdelingen geven voor hun gewogen maximum.

Volgens exact deze lijnen zijn ook de stellingen in dit boek verdeeld. Het Fréchet-type zijn machtsfuncties, het Weibull-type zijn begrensde functies en het Gumbel-type zijn exponentiele functies. Omdat we willen berekenen of de reserves een erg slecht jaar (maximum van de jaarschades) kunnen overleven, is het een prettig feit dat onze stellingen bij deze theorie aansluiting vinden.

Structuur

De structuur is als volgt: In Hoofdstuk 1 bewijzen we dat

$$\lim_{u\to\infty} \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^{m} X_i \leq -u\right)$$

een constante is die slechts van de α (die de generator en daarmee de Archimedische copula kenmerkt) en van eigenschappen van de marginale verdelingen afhangt. Het grote voordeel hiervan is dat nu met behulp van Value-at-Risk eenvoudig de benodigde reserve bij een samengestelde portefeuille te berekenen is. Dit doen we dan ook in een voorbeeld. In dit hoofdstuk, dat het uitgangspunt is van het boek, gaan we uit van een mooie Archimedische copula en identieke marginale verdelingen. Dat wil zeggen dat elke individuele deelnemer dezelfde schadekansverdeling heeft.

In Hoofdstuk 2 bewijzen we dat we de eis van identieke marginale verdelingen kunnen laten varen. Dat is erg prettig aangezien zelden deelnemers exact dezelfde schadekansverdeling hebben, en ook in al die overige gevallen de reserves bepaald dienen te worden.

In Hoofdstuk 3 laten we de eis vallen dat de copula Archimedisch moet zijn. Dit lukt slechts tot op zekere hoogte. De eis die we krijgen in plaats van de eis dat de copula Archimedisch is, is dermate ondoorzichtig dat we enkele voorbeelden hebben toegevoegd, alleen om te laten zien dat het echt een uitbreiding is van hetgeen in Hoofdstuk 1 bewezen is. De stille hoop hier was om een unieke parameter of set parameters te vinden die de afhankelijkheid bepaalt met betrekking tot het staartgedrag van de samengestelde kansverdeling. Een poging hiertoe met de zogenaamde "lower-tail dependence coefficient" blijkt geen succes (dit laten we zien aan de hand van een tegenvoorbeeld), maar levert wel een afschatting op, die we in weer een ander voorbeeld gebruiken.

In Hoofdstuk 4 gooien we het over een andere boeg. Hier berekenen we het Expected Shortfall. Dit is de omvang van de vergoeding die niet uitgekeerd kan worden doordat de reserve niet groot genoeg is. Voor herverzekeraars is het interessant om dit te weten (zij moeten namelijk die klap opvangen), maar ook voor overheden, aangezien zij dan kunnen inschatten wat de schade voor de economie is als een verzekeraar niet meer kan uitkeren wegens gebrek aan reserves. Oorspronkelijk was het bewijs van het Fréchet-geval in dit hoofdstuk veel langer, maar het lukte om dit bewijs erg te verkorten en tegelijkertijd de stelling sterker te maken.

Tenslotte, in Hoofdstuk 5, kijken we naar de Esscher Premie. Dit is een variant op het Expected Shortfall, waar gekeken wordt wat een individuele deelnemer aan schade kan kosten, waarbij alvast rekening wordt gehouden met de omvang van de totale schade.

Curriculum Vitae

Op 20 september 1976 ben ik, Stan Henk Frederik Alink, geboren te Sambeek als eerste kind van Herman Alink en Henra Alink-Van Kemenade. In 1994 behaalde ik het Gymnasiumdiploma aan het Elzendaalcollege te Boxmeer. Aansluitend volgde de studie wiskunde aan de Katholieke Universiteit Nijmegen en in 2000 trouwde ik met mijn jeugdliefde Friederieke Steinberg. In 2001 studeerde ik cum laude af in de functionaalanalyse bij Prof. Dr. Van Rooij. Hierna werkte ik als promovendus in de kanstheorie onder prof. dr. habil. Löwe en prof. dr. dr. h.c. Van Zuijlen aan de Radboud Universiteit Nijmegen en gedeeltelijk aan de Westfälische Wilhelmsuniversität Münster. In deze periode werd ik vader van Afke (2003), Simon (2005) en Matthijs (2005). Sinds medio 2006 ben ik werkzaam als kwantitatief analist bij de afdeling Credit Risk Management van Van Lanschot Bankiers te 's-Hertogenbosch.

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