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A REDUCTION OF THE JACOBIAN CONJECTURE TO THE SYMMETRIC CASE

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Abstract. The main result of this paper asserts that it suffices to prove the Jacobian Conjecture for all polynomial maps of the form $x + H$, where H is homogeneous (of degree 3) and JH is nilpotent and symmetric. Also a 6-dimensional counterexample is given to a dependence problem posed by de Bondt and van den Essen (2003).

Introduction

Let $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map, i.e. each F_i is a polynomial in n variables over $\mathbb{C}$, and denote by $JF := (\frac{\partial F_i}{\partial x_j})_{1 \leq i,j \leq n}$ the Jacobian matrix of F. Then the Jacobian Conjecture asserts that if $\det JF \in \mathbb{C}^*$, then F is invertible. It was shown in the classical papers [1] and [13] by Bass-Connell-Wright and Yagzhev, respectively, that it suffices to prove the Jacobian Conjecture for all $n \geq 2$ and all polynomial maps of the form $F = x + H$, where H is homogeneous (of degree 3) and JH is nilpotent.

In [12] and [7] the cubic homogeneous cases in dimension 3 (resp. 4) were treated by Wright (resp. Hubbers).

Recently, in [6] Washburn and the second author treated one more special case, namely they showed that if $n \leq 4$, then the Jacobian Conjecture holds for all polynomial maps of the form $F = x + H$, where JH is homogeneous, nilpotent and symmetric.

At first glance the condition that JH is symmetric seems rather special. However the main result of this paper, Theorem 1.1, asserts that it suffices to prove the Jacobian Conjecture for all $n \geq 2$ and all polynomial maps of the form $F = x + H$, where JH is homogeneous, nilpotent and symmetric!

The technique to obtain this result is used in section 2 to give a negative answer in dimension 6 to a dependence problem posed in [2] (which, if true, would have implied the Jacobian Conjecture). We refer to section 2 for more details. Finally we would like to mention that in [3] the authors have obtained the following extensions of the results from [6]: the Jacobian Conjecture holds for all F of the form $x + H$, where JH is nilpotent and symmetric in the case $n \leq 4$ (H need not be homogeneous) and in the case $n = 5$ when H is homogeneous.

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1. Reduction to symmetric matrices

Throughout this paper we use the following notation:

$\mathbb{C}[x] := \mathbb{C}[x_1, \dots, x_n]$ is the polynomial ring in n variables over $\mathbb{C}$ and $H := (H_1, \dots, H_n) : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map. Its Jacobian matrix is denoted by JH. It follows from the Poincaré lemma (see for example [5], 1.3.53) that JH is symmetric iff there exists $f \in \mathbb{C}[x]$ such that $H = (f_{x_1}, \dots, f_{x_n})$ or equivalently such that $JH = (\frac{\partial^2 f}{\partial x_i \partial x_j})$, the **Hessian** matrix of f. We denote this matrix by $h(f)$.

Observe that

(1) $$h(f) = J(f_{x_1}, \dots, f_{x_n}).$$

For $A \in M_n(\mathbb{C})$ we put $f \circ A := f(Ax)$. It is well known that

(2) $$h(f \circ A) = A^t h(f)_{|Ax} A.$$

Now we introduce n new variables $y_1, \dots, y_n$ and to H as above we associate the polynomial $f_H \in \mathbb{C}[x, y]$ defined by

(3) $$f_H := (-i)H_1(x_1 + iy_1, \dots, x_n + iy_n)y_1 + \dots + (-i)H_n(x_1 + iy_1, \dots, x_n + iy_n)y_n.$$

So if S is the (invertible) linear map given by

$$S := (x_1 - iy_1, \dots, x_n - iy_n, y_1, \dots, y_n),$$

then $g_H := f_H \circ S = (-i)H_1(x)y_1 + \dots + (-i)H_n(x)y_n$.

One readily verifies that $h(g_H)$ is of the form

(4) $$h(g_H) = \begin{pmatrix} * & (-i)(JH)^t \\ (-i)JH & 0 \end{pmatrix}.$$

In order to formulate the main result of this paper we introduce

Hessian Conjecture HC(n). Let $f \in \mathbb{C}[x]$. If $h(f)$ is nilpotent, then $F := (x_1 + f_{x_1}, \dots, x_n + f_{x_n})$ is invertible.

It follows from (1) that if the n-dimensional Jacobian Conjecture is true, then HC(n) is true as well. The surprising point is now

Theorem 1.1. *The Jacobian Conjecture is equivalent to the Hessian Conjecture. More precisely, if HC(2n) holds, then $x + H$ is invertible for every $H : \mathbb{C}^n \to \mathbb{C}^n$ with JH nilpotent.*

The proof of this result is based on the following lemma.

Lemma 1.2. *Let $H = (H_1, \dots, H_n) : \mathbb{C}^n \to \mathbb{C}^n$ and let $f_H \in \mathbb{C}[x, y]$ be as defined in* (3)*. Then JH is nilpotent iff $h(f_H)$ is nilpotent.*

Proof. Introduce an extra variable z and write f (resp. g) instead of f_H (resp. g_H). Then $h(f)$ is nilpotent iff $\det(zI_{2n} - h(f)) = z^{2n}$. Put $q := (1/2)\sum_{j=1}^n (x_j^2 + y_j^2)$. Then $h(zq) = zI_{2n}$, so

(5) $$h(zq - f) = zI_{2n} - h(f).$$

Since $\det S = 1$, it follows from (2) and (5) that

(6) $$\det h(zq \circ S - g) = \det h(zq - f)_{|S(x,y)}.$$

Since $q \circ S = \frac{1}{2}\sum_{j=1}^n x_j^2 - \sum_{j=1}^n i x_j y_j$, it follows from (4) that

$$h(zq \circ S - g) = \begin{pmatrix} * & -izI_n + i(JH)^t \\ -izI_n + iJH & 0 \end{pmatrix}.$$

Consequently

$$\det h(zq \circ S - g) = \det(zI_n - JH)\det(zI_n - (JH)^t). \tag{7}$$

So by (6) and (7) we obtain

$$\det(zI_{2n} - h(f))_{|S(x,y)} = \det(zI_n - JH)\det(zI_n - (JH)^t).$$

Hence $h(f)$ is nilpotent iff $\det(zI_{2n} - h(f)) = z^{2n}$ iff $\det(zI_n - JH) = z^n$ iff JH is nilpotent. □

Proof of Theorem 1.1. Let $H = (H_1, \ldots, H_n)$ be such that JH is nilpotent and let f_H be as in (3). Then by Lemma 1.2 $h(f)$ is nilpotent. So the assumption HC(2n) implies that $F = (x_1 + f_{x_1}, \ldots, x_n + f_{x_n}, y_1 + f_{y_1}, \ldots, y_n + f_{y_n})$ is invertible. Consequently $F \circ S$ is invertible. An easy calculation shows that

$$\begin{aligned} F \circ S \quad = \quad & \Bigg(x_1 - iy_1 - i\sum_j H_{jx_1}(x)y_j, \ldots, x_n - iy_n - i\sum_j H_{jx_n}(x)y_j, \\ & \quad y_1 + \sum_j H_{jx_1}(x)y_j - iH_1, \ldots, y_n + \sum_j H_{jx_n}(x)y_j - iH_n \Bigg). \end{aligned}$$

Hence $S^{-1} \circ F \circ S = (x_1 + H_1(x), \ldots, x_n + H_n(x), *, \ldots, *)$ is invertible, which in turn implies that $x + H$ is invertible. □

Corollary 1.3. *It suffices to prove the Jacobian Conjecture for all* $n \geq 2$ *and all* F *of the form* $F = (x_1 + f_{x_1}, \ldots, x_n + f_{x_n})$*, where* $h(f)$ *is nilpotent and* f *is homogeneous of degree* 4 *(or equivalently for all* $n \geq 2$ *and all* F *of the form* $F = x + H$ *with* JH *nilpotent and symmetric and* H *homogeneous of degree* 3*).*

Proof. Follows immediately from Theorem 1.1 and Corollary 2.2 of [1]. □

2. Dependence problems

In the search for the Jacobian Conjecture the following problems were formulated by several authors (see [8], Conjecture 1, p. 80, [10], Conjecture B, p. 135, [11], Conjecture 11.3, [4] and [5], 7.1.7)

(Homogeneous) Dependence Problem (H)DP(n). Let $H := (H_1, \ldots, H_n)$ with $H(0) = 0$ be (homogeneous of degree $d \geq 1$) such that JH is nilpotent. Are the H_i linearly dependent over $\mathbb{C}$?

One easily verifies that the linear dependence of the H_i is equivalent to the linear dependence of the rows of JH over $\mathbb{C}$. It is shown in [5], Theorem 7.1.7, that DP(2) has an affirmative answer and that for each $n \geq 3$ there are counterexamples. The easiest such example is the following:

$$H_1 = x_2 - x_1^2, \quad H_2 = x_3 + 2x_1(x_2 - x_1^2), \quad H_3 = -(x_2 - x_1^2)^2. \tag{8}$$

The homogeneous dependence problem is still open; but in the cases $n = 3, d = 3$ and $n = 4, d = 3$, affirmative answers were obtained by Wright in [12] and Hubbers in [7]. Recently in [2] the corresponding dependence problems were formulated for Hessian matrices, i.e.

(Homogeneous) Symmetric Dependence Problem (H)SDP(n). Let H with $H(0) = 0$ be (homogeneous of degree $d \geq 1$) such that JH is nilpotent and symmetric. Are the H_i linearly dependent over $\mathbb{C}$?

The importance of these problems becomes clear if one combines Theorem 1.1 with the following result of [2].

Theorem 2.1 ([2, Theorem 2.1])**.** i) *If SDP(p) has an affirmative answer for all $p \leq n$, then HC(n) holds.*

ii) *If SDP(p) has an affirmative answer for all $p \leq n-2$ and HSDP(p) for $p = n-1$ and $p = n$, then HC(n) holds for all homogeneous $f \in \mathbb{C}[x]$.*

The aim of this section is to relate the dependence problems stated before with the symmetric dependence problems. As a consequence we obtain a negative answer to SDP(6). More precisely

Example. Let $H = (H_1, H_2, H_3)$ be as in (8). Then JH is nilpotent and H_1, H_2, H_3 are linearly independent over $\mathbb{C}$. Now let f_H be as in (3). Then it follows from the next result and the fact that DP(2) holds, that $h(f_H)$ is a counterexample to SDP(6).

Proposition 2.2. *If n is minimal such that (H)DP(n) does not hold, then (H)SDP(2n) does not hold either.*

Proof. i) Suppose (H)DP(n) does not hold and n is minimal with this property. Then there exists $H : \mathbb{C}^n \to \mathbb{C}^n$ with $H(0) = 0$ such that JH is nilpotent and the rows of JH are independent over $\mathbb{C}$.

Claim. The columns of JH are also independent over $\mathbb{C}$.

Namely, if the columns of JH are dependent over $\mathbb{C}$, then there exists $0 \neq v \in \mathbb{C}^n$ with $JH \cdot v = 0$. Let $T \in Gl_n(\mathbb{C})$ be such that its last column equals v. Then the last column of $JH \cdot T$ equals zero. So if we put $\tilde{H} := T^{-1} \circ H \circ T$, then $J\tilde{H} = T^{-1}JH(Tx)T$ is nilpotent and also its last column equals zero. In particular $\tilde{H}_1, \ldots, \tilde{H_{n-1}} \in \mathbb{C}[x_1, \ldots, x_{n-1}]$. Finally put $H_* := (\tilde{H}_1, \ldots, \tilde{H_{n-1}})$. Since the last column of $J\tilde{H}$ is zero, it follows readily that JH_* is nilpotent and that the rows of JH_* are linearly independent over $\mathbb{C}$ (since the rows of $J\tilde{H}$ are because those of JH are by hypothesis). So H_* contradicts the minimality of n.

ii) Therefore, the colums of JH are independent over $\mathbb{C}$. Let g_H and f_H be as above. Then $h(g_H)$ has the form (4).

Claim. The rows R_j of $h(g_H)$ are independent over $\mathbb{C}$: namely suppose that $\sum_{j=1}^{2n} c_j R_j = 0$ for some $c_j \in \mathbb{C}$. Since the rows of $(-i)(JH)^t$ are independent over $\mathbb{C}$ (since the columns of JH are by i)), the zero matrix in the right corner of $h(g_H)$ in (4) implies that $c_1 = \ldots = c_n = 0$. So $\sum_{j=n+1}^{2n} c_j R_j = 0$. However the rows of $(-i)JH$ are also independent over $\mathbb{C}$ (by hypothesis), so also $c_j = 0$ if $j > n$, which proves the claim.

iii) Finally, since $f_H = g_H \circ T$ $(T := S^{-1})$ it follows from (2) that $h(f_H) = T^t h(g_H)_{|T(x,y)} T$. Therefore, the rows of $h(f_H)$ are also independent over $\mathbb{C}$, which concludes the proof. □

3. Final remarks

Almost three months after this paper was submitted, the authors were notified by David Wright that the paper [9] by Guowu Meng had appeared on the internet, in which he obtained a result similar to ours. He also formulates a Hessian Conjecture and shows that the Jacobian Conjecture is equivalent to his Hessian Conjecture. Meng's Hessian Conjecture states that the Jacobian Conjecture holds for all gradient maps $\nabla f := (f_{x_1}, \ldots, f_{x_n})$. The difference between our Hessian Conjecture and the one formulated by Meng is that he considers all polynomial maps of the form ∇f with $\det h(f) \in \mathbb{C}^*$, where we only need to consider all polynomial maps of the form $x + \nabla f$, with $h(f)$ nilpotent. So our reduction is more refined in the sense that it preserves the nilpotency as formulated in the classical reduction theorems of [1] and [13].

Added in proof

In a recent paper the authors gave an affirmative answer to HDP(3). Also, the first author found counterexamples to HDP(n) for all $n \geq 5$.

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