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A CONSTRUCTIVE CONVERSE OF THE MEAN VALUE THEOREM

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ABSTRACT. Consider the following converse of the Mean Value Theorem.
Let f be a continuously differentiable function on $[a, b]$. If $c \in (a, b)$, then there are α and β in $[a, b]$ such that $(f(\beta) - f(\alpha))/(\beta - \alpha) = f'(c)$. Assuming some weak conditions to be mentioned in Section 3, Tong and Braza [3] were able to prove this statement. Unfortunately their proof does not provide a method to compute α and β. We give a constructive proof.

1. INTRODUCTION

Constructive mathematics tries to determine the constructive or computational content of mathematics. One sometimes distinguishes several varieties of constructive mathematics [2]. Our result is acceptable to all of them as it belongs to so-called Bishop-style mathematics [1] (E. Bishop 1928–1983). We avoid non-constructive steps, but do not assume axioms that are classically false.

In constructive mathematics 'there exists an x' is interpreted as 'there is an effective construction for x'. A constructive proof of 'A or B' is a proof of A or a proof of B. In order to prove 'A or not A' we have to prove or refute A. As there will always be unsolved problems, we do not recognize the scheme $A \vee \neg A$, *Tertium non datur*, as a valid principle.

We will first introduce the real numbers and some relations between them in the usual way, keeping in mind the constructive interpretation of 'there exists' and 'or'. A *real number* x is a sequence of rational numbers $x(0), x(1), \ldots$, such that for all k there exists(!) an N satisfying $|x(N) - x(n)| < 1/k$, for all $n > N$. Let x and y be real numbers. x and y are *equal* ($x = y$) if for all k there exists an N such that $|x(n) - y(n)| < 1/k$, for all $n > N$.

x is *greater than* y ($x > y$) if there are k and N such that $|x(N) - y(N + n)| > 1/k$, for all n. Notice that if $x < y$ then $x < z$ or $z < y$ for all z.

x is *not-greater-than* y ($x \leq y$) if not $x > y$. Finally, x is *apart from* y ($x \# y$) if $x > y$ or(!) $x < y$. Now $x = y$ if and only if not $x \# y$, but conversely it is not true in general that if not $x = y$ then $x \# y$. Addition, subtraction, multiplication, etc. are defined in the usual way.

We only consider *strongly extensional* functions, i.e. functions f satisfying, for all x and y, if $f(x) \# f(y)$ then $x \# y$.

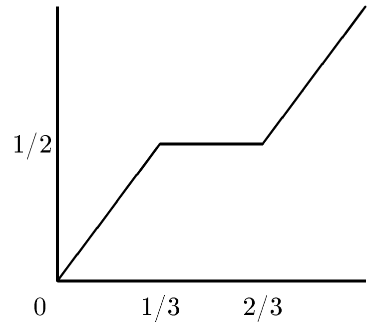

FIGURE 2.1. The function f

2. A Constructive Mean Value Theorem

We start by giving a so-called Brouwerian counterexample (L.E.J. Brouwer 1881–1966) for the Intermediate Value Theorem. Let f be the function (Figure 2.1) from [0,1] to [0,1] given by:

$$f(x) := \inf(3x/2, 1/2) + \sup(3x/2 - 1, 0).$$

Define a function k_{99} from $\mathbb{N}$ to $\mathbb{N}$ by:

$$k_{99}(n) := \begin{cases} k & \text{if the first block of 99 nines starts at position } k \text{ in the} \\ & \text{decimal expansion of } \pi \text{ and } k < n, \\ n & \text{if such } k \text{ does not exist.} \end{cases}$$

Define a function t from $\mathbb{N}$ to $\mathbb{N}$ by: $t(n) := \frac{1}{2} + (-1/2)^{k_{99}(n)}$. Observe that t is a real number. Suppose we find x such that $f(x) = t$; then we are able to decide either $x < \frac{2}{3}$ or $x > \frac{1}{3}$. If $x < \frac{2}{3}$, then, if there exists a block of 99 nines in the decimal expansion of π the first one will start at an odd position. Similarly if $x > \frac{1}{3}$, then, if there exists a block of 99 nines in the decimal expansion of π, the first one will start at an even position. Both conclusions are unjustified.

Observe that this difficulty arises as soon as a function is constant on an interval.

Definition 2.1. Let f be a function on $[a, b]$ and let y be a real number. f is called *densely apart from* y if in every interval there exists a real number x such that $f(x) \# y$.

If p is a polynomial function of degree at least one, then p is densely apart from y for all y in $\mathbb{R}$. The function f in Figure 2.1 is not densely apart from 1/2.

Lemma 2.2. *If f is continuous on $[a, b]$, then there is a countable $T \subset \mathbb{R}$, such that if $s \# t$ for all $t \in T$, then f is densely apart from s.*

We express this fact as follows: f is densely apart from *all but countably many* real numbers.

Proof. Take $T := \{f(x) : x \in \mathbb{Q} \cap [a,b]\}$. □

Lemma 2.3. [Intermediate Value Lemma] *Let f be continuous on $[a,b]$. If $f(a) < t < f(b)$ and f is densely apart from t, then there exists c in $[a,b]$ such that $f(c) = t$.*

Proof. We use successive bisection. Choose $x \in (a + \frac{b-a}{4}, a + \frac{3(b-a)}{4})$ for which $f(x) \# t$. This means that either $f(x) < t$ or $f(x) > t$. If $f(x) < t$ let $a_1 := x$ and $b_1 := b$, otherwise let $a_1 := a$ and $b_1 := x$. Now $f(a_1) < t < f(b_1)$ and $b_1 - a_1 < 3/4$. This process, applied recursively, produces sequences $a_0 < a_1 < \ldots$ and $b_0 > b_1 > \ldots$, such that for each i, $0 < b_i - a_i < (\frac{3}{4})^i(b-a)$ and $f(a_i) < t < f(b_i)$.
Therefore $c := \lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i$ satisfies $f(c) = t$. □

A direct consequence of Lemma 2.2 and Lemma 2.3 is the following constructive version of the Intermediate Value Theorem.

Theorem 2.4. [Intermediate Value Theorem] *Let f be continuous on $[a,b]$. For all but countably many t: if $f(a) < t < f(b)$ then there is c in $[a,b]$ satisfying $f(c) = t$.*

A countable set of exceptions may indeed occur. Consider Cantor's function (Figure 2.2). This is the unique continuous and nondecreasing function f, which is constant on every interval outside Cantor's discontinuum and satisfies $f(x) = 1/2$ for x in $[1/3, 2/3]$, $f(x) = 1/4$ for x in $[1/9, 2/9]$, $f(x) = 3/4$ for x in $[7/9, 8/9]$, etc.
By a proof similar to that of the Intermediate Value Theorem one may establish:

Theorem 2.5. [Mean Value Theorem] *Let f be continuously differentiable on $[a,b]$. There is a countable set T, such that for all α and β, if $a < \alpha < \beta < b$ and $(f(\alpha) - f(\beta))/(\alpha - \beta)$ is apart from every t in T, then there is c in (α, β) such that $f'(c) = (f(\alpha) - f(\beta))/(\alpha - \beta)$.*

3. A constructive converse of the Mean Value Theorem

We will obtain a converse of the Mean Value Theorem in which we do not have to make exceptions as in the Theorems of Section 2. We need a few preparations.

For x and y in $\mathbb{R}$ and $x \# y$, define the difference quotient

$$\Delta(x,y) := \frac{f(x) - f(y)}{x - y}.$$

Then for each z in $\mathbb{R}$, such that $z \# x$ and $z \# y$

$$\Delta(x,y) = \frac{(x-z)\Delta(x,z) + f(z) - f(y)}{x - y}$$

$$= (\frac{x-z}{x-y})\Delta(x,z) + (1 - \frac{x-z}{x-y})\Delta(z,y). \quad (3.1)$$

Lemma 3.1. *Let f be continuously differentiable on $[a,b]$. If $t < f'(x)$ and $\delta > 0$, then for all $z < x$ there exists w in $(x - \delta, x)$ apart from z, such that $t < f'(w)$ and $\Delta(z,w) \# t$.*

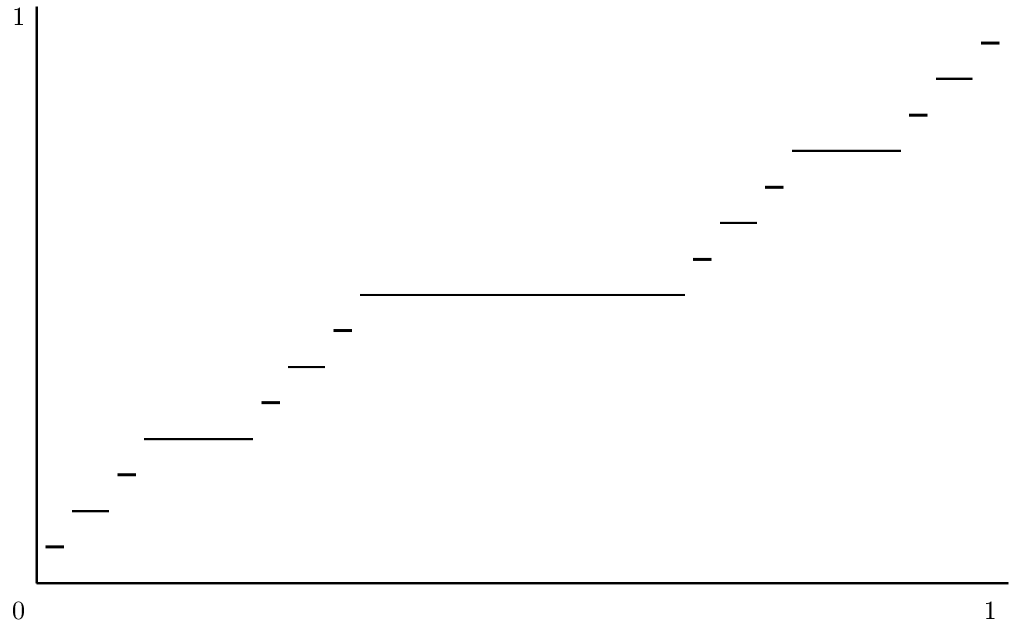

FIGURE 2.2. Cantor's function.

Proof. Choose y in $(x-\delta, x)$, such that $z < y$ and $\Delta(x,y) > (f'(x)+t)/2$. Let $r := \frac{z-x}{z-y}$ and choose $\epsilon < \frac{(f'(x)-t)}{2}\frac{1-r}{r}$. Now either $|\Delta(z,x) - t| > \epsilon/2$ or $|\Delta(z,x) - t| < \epsilon$. In the former case take w close enough to x. In the latter case:

$$\begin{aligned}
\Delta(z,y) &= r\Delta(z,x) + (1-r)\Delta(x,y) && \text{by Formula 3.1} \\
&\geq r(t-\epsilon) + (1-r)(f'(x)+t)/2 \\
&\geq t + (1-r)(f'(x)-t)/2 - \epsilon r \\
&> t && \text{by choice of } \epsilon
\end{aligned}$$

So in this case let $w := y$. □

Theorem 3.2. *Let f be continuously differentiable on $[a,b]$ and $\epsilon > 0$. If $f'(c_1) < t < f'(c_2)$ and $a < c_1 < c_2 < b$, then there exist α and β such that $\alpha < \beta$ and for which $\Delta(\beta, \alpha) = t$ and $\alpha \in (c_1, c_1 + \epsilon)$ or $\beta \in (c_2 - \epsilon, c_2)$.*

The condition $f'(c_1) < t < f'(c_2)$ is necessary: consider the function $f : [-1,1] \to \mathbb{R}$ given by $f(x) = x^3$ and let $t := 0$.

Proof. Because $f'(c_1) = \lim_{y \to c_1} \Delta(c_1, y)$, there exists y_0 such that $\Delta(c_1, y_0) < t$ and $c_1 < y_0 < c_2$. Lemma 3.1 provides z_0 in (y_0, c_2) such that $t < f'(z_0)$ and $\Delta(c_1, z_0) \# t$. By taking z_0 close enough to c_2 we ensure $\Delta(c_2, z_0) > t$. Now there are two possibilities: 1. $\Delta(c_1, z_0) > t$ or 2. $\Delta(c_1, z_0) < t$. We first consider case 1.

In classical mathematics one could simply define $\alpha := c_1$ and then use successive bisection in order to find β such that $\Delta(\alpha, \beta) = t$. In constructive mathematics we have to construct both α and β, but we may ensure that α is not too far away from c_1.

Let $\alpha_0 := c_1$. Now $\Delta(\alpha_0, z_0) > t$ and $\Delta(\alpha_0, y_0) < t$. Since Δ is continuous, there exists an open interval I containing α_0 such that for all x in I: $\Delta(x, z_0) > t$ and $\Delta(x, y_0) < t$. Let $y := \frac{z_0+y_0}{2}$. Lemma 3.1 applied to $-f$ provides $\alpha_1 < y_0$ in $I \cap (\alpha_0 - \epsilon/2, \alpha_0 + \epsilon/2)$ satisfying $f'(\alpha_1) < t$ and $\Delta(y, \alpha_1) \# t$. If $\Delta(\alpha_1, y) > t$ let $y_1 := y_0$ and $z_1 := y$, if $\Delta(\alpha_1, y) < t$ let $y_1 := y$ and $z_1 := z_0$. Thus $\alpha_1 \in (\alpha_0 - \epsilon/2, \alpha_0 + \epsilon/2)$ and $|z_1 - y_1| < \frac{|z_0 - y_0|}{2}$.

By repeating the above construction, we obtain sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ such that for all n: $\Delta(\alpha_n, y_n) < t$ and $\Delta(\alpha_n, z_n) > t$. Let $\alpha := \lim_{n\to\infty} \alpha_n$ and $\beta := \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n$, and observe $\Delta(\alpha, \beta) = t$ and $|\alpha - c_1| < \epsilon$.
In case 2 we follow a similar construction and obtain the conclusion $|\beta - c_2| < \epsilon$. □

From the previous theorem one may derive a constructive weak converse of the Mean Value Theorem, in which α and β are found such that $f'(c) = \Delta(\beta, \alpha)$, possibly not satisfying $\alpha < c < \beta$. For the stronger conclusion we need one more lemma.

Lemma 3.3. *Let f be continuously differentiable on $[a, b]$. If $f'(x) \leq t$ for all x in $[a, b]$, then $\Delta(a, b) \leq t$. If furthermore $f'(z) < t$ for some z in $[a, b]$, then $\Delta(a, b) < t$.*

Proof. Suppose $\Delta(a, b) > t$, say $\Delta(a, b) = t + \epsilon$. Because $f'(a) \leq t$ there is an a' such that $\Delta(a, a') < t + \frac{1}{2}\epsilon$. Now $(t + \frac{1}{2}\epsilon, t + \epsilon)$ is uncountable, so by the Intermediate Value Theorem (2.4) we construct x for which we can apply the Mean Value Theorem (2.5) in order to find y in (a, x) satisfying $f'(y) = K(a, x) > t + \frac{1}{2}\epsilon$. But this is a contradiction, so $\Delta(a, b) \leq t$.

Assume $z \in (a, b)$ and $f'(z) < t$. Because $f'(z) = \lim_{y\to z} \Delta(z, y)$ there is $y_0 > z$ such that $\Delta(z, y_0) < t$. Now apply the argument above to $[a, z]$ and $[y_0, b]$ and conclude that $\Delta(a, z) \leq t$ and $\Delta(y_0, b) \leq t$. Applying Formula 3.1 twice we find first $\Delta(a, y_0) < t$, and then $\Delta(a, b) < t$. □

We now prove the promised constructive strong converse of the Mean Value Theorem. The conditions in this theorem are classically equivalent to: f' does not have a local extremum in c and c is not an accumulation point of $A_c := \{x \in (a, b) : f'(x) = f'(c)\}$. Tong and Braza [3] proved the necessity of these conditions: it suffices to consider the continuous function $g : [-1/2, 1/2] \to \mathbb{R}$ satisfying $g(x) := x^3 \sin(1/x) + x|x|/2$ for $x \# 0$, and $g(0) := 0$.

Theorem 3.4. *Let f be continuously differentiable on $[a, b]$ and $\delta > 0$ such that for all x in $(c - \delta, c + \delta)$ apart from c: $f'(c) \# f'(x)$. If for all $\epsilon > 0$ there exist c_1 and c_2 in $(c - \epsilon, c + \epsilon)$, satisfying $f'(c_1) < f'(c) < f'(c_2)$, then there are α and β in (a, b) such that $\alpha < c < \beta$ and $\Delta(\alpha, \beta) = f'(c)$.*

Proof. Take c_1 and c_2 in $(c-\delta, c+\delta)$ satisfying $f'(c_1) < f'(c) < f'(c_2)$. Now $c_1 \# c$, say $c_1 < c$. Suppose that $c_2 < c$, then the Intermediate Value Lemma 2.3 provides y, satisfying $c_1 < y < c$ and $f(y) = f(c)$, which contradicts the assumptions. Hence $c_2 \geq c$, moreover because $f(c_2) \# f(c)$ it follows that $c_2 > c$.
Theorem 3.2 provides α and β, such that $\alpha < \beta$ and $\Delta(\alpha, \beta) = f'(c)$. We may decide $\alpha < c$ or $c < \beta$. We only consider the case $\alpha < c$. By Lemma 3.3 we have $\Delta(\alpha, x) < f'(c)$ for $x \in [\alpha, c]$. So $\alpha < c < \beta$. □

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