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The vanishing relaxation time behavior of multi-term nonlocal Jordan–Moore–Gibson–Thompson equations

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ABSTRACT

The family of Jordan–Moore–Gibson–Thompson (JMGT) equations arises in nonlinear acoustics when a relaxed version of the heat flux law is employed within the system of governing equations of sound motion. Motivated by the propagation of sound waves in complex media with anomalous diffusion, we consider here a generalized class of such equations involving two (weakly) singular memory kernels in the principal and non-leading terms. To relate them to the second-order wave equations, we investigate their vanishing relaxation time behavior. The key component of this singular limit analysis are the uniform bounds for the solutions of these nonlinear equations of fractional type with respect to the relaxation time. Their availability turns out to depend not only on the regularity and coercivity properties of the two kernels, but also on their behavior relative to each other and the type of nonlinearity present in the equations.

1. Introduction

The family of Jordan–Moore–Gibson–Thompson (JMGT) equations [1] arises in nonlinear acoustics when the classical Fourier heat flux law is replaced by the Maxwell–Cattaneo law [2] within the system of governing equations of sound propagation. The latter introduces thermal relaxation with the parameter $\tau > 0$ thereby avoiding the so-called paradox of infinite speed of propagation. The resulting acoustic equations are then third-order in time:

$$\tau u_{ttt} + \mathfrak{a}(u, u_t)u_{tt} - c^2\mathfrak{b}(u, u_t)\Delta u - \tau c^2\Delta u - \delta\Delta u_t + \mathcal{N}(u_t, \nabla u, \nabla u_t) = 0. \tag{JMGT}$$

The function u stands for either the acoustic pressure or acoustic velocity potential. The functions $\mathfrak{a}$, $\mathfrak{b}$, and $\mathcal{N}$ dictate the type of nonlinearity present in the equation; we will discuss them in depth together with

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modeling in Section 2. The JMGT equations and their linearizations (known as Moore–Gibson–Thompson (MGT) equation) have been extensively studied in the recent mathematical literature; we refer to, e.g., [3–7] for a selection of the results on their well-posedness, regularity of solutions, and long-term behavior. Also the stability of (J)MGT equations with additional memory terms has been extensively researched; see, e.g., [8–11] and the references provided therein.

Recently nonlocal generalizations of these equations of higher order have been put forward in [12] based on using the Compte–Metzler fractional interpolations [13] of the Fourier and Maxwell–Cattaneo flux laws valid in media with anomalous diffusion, such as biological tissues. This type of modeling has significantly gained in importance with the rise of ultrasound imaging applications [14]. Motivated by such sound propagation, we consider here the following family of nonlocal generalizations of the (JMGT) equation, given by

$$\tau^a \mathfrak{K}_1 * u_{ttt} + \mathfrak{a} u_{tt} - c^2 \mathfrak{b} \Delta u - \tau^a c^2 \mathfrak{K}_1 * \Delta u_t - \delta \mathfrak{K}_2 * \Delta u_{tt} + \mathcal{N} = 0, \tag{1.1}$$

where $*$ denotes the Laplace convolution in time. The power a is dependent on the kernel $\mathfrak{K}_1$ and included to ensure dimensional homogeneity. When $\mathfrak{K}_1$ is the Dirac delta distribution δ_0 (with $a = 1$) and $\mathfrak{K}_2 = 1$, (1.1) formally reduces to the (JMGT) equation, up to modifying the right-hand side. However, the presence of the two kernels allows us to treat a much richer family of equations here than the (JMGT) equation. For example, (1.1) with suitable Abel kernels covers the time-fractional equations introduced and analyzed in [12] under the name fractional Jordan–Moore–Gibson–Thompson (fJMGT) equations that correspond to the four fractional flux laws of Compte and Metzler [13]; we refer to upcoming Section 2 for details.

As the relaxation parameter $\tau > 0$ is typically small, it is of high interest to determine the behavior of solutions to (1.1) as it vanishes. This is the main goal of the present work. Such analysis will, first of all, result in sufficient conditions for stability of solutions with respect to τ, which might be needed, for example, in numerical simulations. Secondly, it will provide an additional (mathematical) justification of these models as it will relate them to their limiting (established) counterparts. While we verify the conditions needed in our analysis for the kernels from [13], since this work has strongly motivated our studies, we also point to the wide range of possibilities opened up by the general framework (1.1). In particular, our mathematical analysis can serve as a theoretical foundation for developing – by a proper choice of $\mathfrak{K}_1$, $\mathfrak{K}_2$ – models with desirable properties such as finite speed of propagation.

By formally setting the relaxation parameter to zero, one arrives at equations with the leading term of second order:

$$\mathfrak{a} u_{tt} - c^2 \mathfrak{b} \Delta u - \delta \mathfrak{K}_2 * \Delta u_{tt} + \mathcal{N} = 0.$$

Our singular limit analysis is based on proving well-posedness of (1.1), which we consider with homogeneous Dirichlet data on bounded domains and three initial conditions, uniformly in τ. As one might expect, whether one can obtain the uniform bounds on the solutions is heavily influenced by the properties of the two kernels and their interplay with the nonlinearities present in the equations (that is, the properties of the functions $\mathfrak{a}$, $\mathfrak{b}$, and $\mathcal{N}$). This will necessarily lead to delicate case distinctions. More precisely, we will consider two sets of assumptions on the kernels and then develop the corresponding theories; the details can be found in Sections 3 and 4, respectively. The first set of assumptions will allow for a more standard testing strategy, using $(-\Delta)^\nu u_{tt}$ with $\nu \in \{0, 1, 2\}$ as test functions; the second set of assumptions will involve $\mathfrak{K}_1 * (-\Delta)^\nu u_{tt}$ as a test function.

The analysis will cover the following fractional JMGT equations introduced in [12] as special cases:

$$\tau^\alpha \mathrm{D}_t^\alpha u_{tt} + \mathfrak{a} u_{tt} - c^2 \mathfrak{b}(u, u_t) \Delta u - \tau^\alpha c^2 \mathrm{D}_t^\alpha \Delta u - \delta \mathrm{D}_t^{1-\alpha} \Delta u_t + \mathcal{N} = 0, \tag{fJMGT I}$$

$$\tau u_{ttt} + \mathfrak{a} u_{tt} - c^2 \mathfrak{b} \Delta u - \tau c^2 \Delta u - \delta \mathrm{D}_t^{1-\alpha} \Delta u_t + \mathcal{N} = 0, \tag{fJMGT III}$$

$$\tau^\alpha \mathrm{D}_t^\alpha u_{tt} + \mathfrak{a} u_{tt} - c^2 \mathfrak{b} \Delta u - \tau^\alpha c^2 \mathrm{D}_t^\alpha \Delta u - \delta \Delta u_t + \mathcal{N} = 0, \tag{fJMGT}$$

Table 1
Main results of this work for the nonlinear JMGT and fJMGT equations.

Equation	τ-uniform well-posedness or existence	τ weak limits
(JMGT)	Theorems 3.1 and 3.3	Theorems 3.2 and 3.4
(fJMGT I)	with $\alpha < 1/2$, $u_{tt}\|_{t=0} = 0$, Theorem 4.1 with $\alpha \geq 1/2$, $\mathfrak{a} \equiv 1$, $u_{tt}\|_{t=0} = 0$, Theorem 4.2 (existence, Westervelt–Blackstock nonlinearities)	Theorem 4.3
(fJMGT III)	with $\alpha > 1/2$, Theorems 3.1 and 3.3	Theorems 3.2 and 3.4
(fJMGT)	with $\alpha \geq 1/2$, $\mathfrak{a} \equiv 1$, $u_{tt}\|_{t=0} = 0$, Theorem 4.2 (existence, Westervelt–Blackstock nonlinearities)	Theorem 4.3

under various (different) restrictions in terms of the order of differentiation α and the involved nonlinearities (where we distinguish the so-called Westervelt–Blackstock and Kuznetsov–Blackstock type). Here numbers I and III indicate that the equations stem from the Compte–Metzler fractional flux laws introduced under the same numbers in [13]; the last one is unnumbered in [13].

In terms of closely related works, we point out the singular limit analysis of third-order Eqs. (JMGT) on bounded domains in [15–17]. In particular, our analysis follows in the spirit of [16] by employing an energy method on a linearized problem in combination with a fixed-point strategy. We also point out two works which consider (1.1) in simplified settings that allow for optimizing the theory. The first one is [18] with a tailored treatment of (1.1) in the case $\mathfrak{K}_2 = 1$ (leading to the (fJMGT) equation) which allows for a different testing strategy compared to ours here and less restrictive assumptions on $\mathfrak{K}_1$. The second is [19] which investigates linear versions of (1.1) allowing for a broader family of kernels and the treatment of equations based on the second Compte–Metzler law, among others.

We organize the rest of the exposition as follows. Section 2 first gives the necessary background details on the modeling. We then split the analysis into two parts, corresponding to two sets of assumptions on the kernels. Section 3 considers kernels and equations amenable to testing with $(-\Delta)^\nu u_{tt}$ with $\nu \in \{1, 2\}$; this is, for example, the fJMGT III equation. Section 4 considers kernels and equations amenable to testing with u_{tt} but also $-\Delta \mathfrak{K}_1 * u_{tt}$; this will be the case for the fJMGT I equation, for instance. We summarize in Table 1 the uniform well-posedness and existence results of this work for the fractional JMGT equations as particular cases.

2. Acoustic equations based on nonlocal flux laws

In this section, we discuss acoustic modeling to motivate the wave equations considered in this work in the context of nonlinear sound propagation in tissue-like media. For a deeper insight into nonlinear acoustic modeling, we refer the interested readers to [20–23] and the references contained therein.

2.1. A generalized nonlocal heat flux law of the Maxwell–Cattaneo type

Classical acoustic equations that describe nonlinear sound motion in thermoviscous fluids are derived as approximations of the Navier–Stokes–Fourier system based on the following Fourier heat flux law:

$$\boldsymbol{q} = -\kappa \nabla \theta; \tag{2.1}$$

see, for example [21,22,24]. Here $\boldsymbol{q}$ is the heat flux, θ the absolute temperature, and $\kappa > 0$ denotes the thermal conductivity.

The acoustic equations studied in this work are based on employing a nonlocal generalization of (2.1) of the Maxwell–Cattaneo type which incorporates thermal relaxation as follows:

$$\boldsymbol{q}(t) + \tau^a \int_0^t \mathfrak{K}_1(t-s)\boldsymbol{q}_t(s)\,\mathrm{d}s = -\tau_\theta^b \kappa \int_0^t \mathfrak{K}_2(t-s)\nabla\theta_t(s)\,\mathrm{d}s. \tag{2.2}$$

Table 2
Kernels for the Compte–Metzler fractional laws.

Flux law	$\mathfrak{K}_1$	$\mathfrak{K}_2$	a	b
GFE I	$g_{1-\alpha}$	g_α	α	$1-\alpha$
GFE III	δ_0	g_α	1	$1-\alpha$
GFE	$g_{1-\alpha}$	1	α	–

Here $\tau > 0$ stands for the thermal relaxation time and the constant τ_θ as well as the powers a and b are there to ensure the dimensional homogeneity of the equation. The introduced memory kernels $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are assumed to be independent of τ.

This relation generalizes many flux laws in the literature. Fourier law (2.1) follows by setting $\tau = 0$, $\mathfrak{K}_2 = 1$, and $\tau_\theta^b = 1$, where we assume that $\boldsymbol{q}(0) = 0$ and $\nabla\theta(0) = 0$. The well-known Maxwell–Cattaneo law [2]:

$$\boldsymbol{q} + \tau \boldsymbol{q}_t = -\kappa \nabla\theta$$

follows by setting $a = 1$, $\mathfrak{K}_1 = \delta_0$ (the Dirac delta distribution), $\tau_\theta^b = 1$, and $\mathfrak{K}_2 = 1$. Importantly, (2.2) unifies (and generalizes) the Compte–Metzler fractional laws [13]:

$$\begin{aligned}
&\text{(GFE I)} && (1+\tau^\alpha \mathrm{D}_t^\alpha)\boldsymbol{q}(t) = -\kappa\tau_\theta^{1-\alpha}\mathrm{D}_t^{1-\alpha}\nabla\theta;\\
&\text{(GFE II)} && (1+\tau^\alpha \mathrm{D}_t^\alpha)\boldsymbol{q}(t) = -\kappa\tau_\theta^{\alpha-1}\mathrm{D}_t^{\alpha-1}\nabla\theta;\\
&\text{(GFE III)} && (1+\tau \partial_t)\boldsymbol{q}(t) = -\kappa\tau_\theta^{1-\alpha}\mathrm{D}_t^{1-\alpha}\nabla\theta;\\
&\text{(GFE)} && (1+\tau^\alpha \mathrm{D}_t^\alpha)\boldsymbol{q}(t) = -\kappa\nabla\theta.
\end{aligned}$$

Although in [13] the fractional derivative is understood in the Riemann–Liouville sense, in this work D_t^η denotes the Caputo–Djrbashian fractional derivative:

$$\mathrm{D}_t^\eta w(t) = \frac{1}{\Gamma(1-\eta)}\int_0^t (t-s)^{-\eta}\mathrm{D}_t^{\lceil\eta\rceil} w(s)\,\mathrm{d}s, \quad -1 < \eta < 1;$$

see, for example, [25, §1] and [26, §2.4.1] for the definition. Here $\lceil\eta\rceil$ is the integer obtained by rounding up η and $\mathrm{D}_t^{\lceil\eta\rceil}$ is the zeroth or first derivative operator. We may do this exchange at this point since it is assumed that $\boldsymbol{q}(0) = 0$ and $\nabla\theta(0) = 0$.

Looking at the Compte–Metzler laws, we can see that GFE II has a different form compared to others since $\mathrm{D}_t^{\alpha-1}$ is an integral operator rather than a derivative for $\alpha \in (0,1)$. It thus does not fit properly into the framework of (2.2) and we do not consider it going forward. We refer to [19] for the treatment of acoustic waves based on this law in a linear setting.

Before proceeding we collect in Table 2 the kernels $\mathfrak{K}_1$ and $\mathfrak{K}_2$ for the three Compte–Metzler laws relevant for this work (and the powers a and b) which allow writing them in form of (2.2). Here we use the short-hand notation for the Abel kernel:

$$g_\alpha(t) := \frac{1}{\Gamma(\alpha)} t^{\alpha-1}.$$

Note that the GFE laws have possible non-physical shortcomings, which are discussed in detail in [13]. The fractional heat equations resulting from the Compte–Metzler laws have been numerically compared in [27]. It was determined that they can all lead to negative absolute temperatures, however the heat equation based on using law GFE I avoids this non-physical behavior in numerical experiments for $\alpha \in (1/2, 1)$ close enough to 1/2.

Remark 1 (*On the Gurtin–Pipkin Approach*)**.** An alternative approach of generalizing the heat flux laws is given by the Gurtin–Pipkin flux law [28]:

$$\boldsymbol{q}(t) = -\kappa \int_0^t \mathcal{K}_\tau(t-s)\nabla\theta(s)\,\mathrm{d}s.$$

The resulting acoustic equations are then second order in time with damping of fractional type but the kernel $\mathcal{K}_\tau$ may depend on τ; we refer to [29,30] for their derivation and limiting analysis.

2.2. Acoustic modeling with the generalized heat flux law

By closely following the steps of the derivation in [12] only now with generalized heat flux law instead of the fractional ones, the following nonlinear acoustic wave equation can be derived:

$$\begin{aligned}\tau^a \mathfrak{K}_1 * \psi_{ttt} + \mathfrak{a}(\psi_t)\psi_{tt} - c^2\mathfrak{b}(\psi_t)\Delta\psi - \tau^a c^2 \Delta \mathfrak{K}_1 * \psi_t - \tau_\theta^b \delta \mathfrak{K}_2 * \Delta\psi_{tt} \\ +\mathcal{N}(\nabla\psi, \nabla\psi_t) = 0,\end{aligned} \tag{2.3}$$

where either

$$\mathfrak{a} = 1, \quad \mathfrak{b}(\psi_t) = 1 - 2\tilde{k}\psi_t, \quad \mathcal{N}(\nabla\psi, \nabla\psi_t) = \tilde{\ell}\partial_t(|\nabla\psi|)^2 = 2\tilde{\ell}\nabla\psi\cdot\nabla\psi_t \tag{2.4}$$

or

$$\mathfrak{a}(\psi_t) = 1 + 2\tilde{k}\psi_t, \quad \mathfrak{b} = 1, \quad \mathcal{N}(\nabla\psi, \nabla\psi_t) = \tilde{\ell}\partial_t(|\nabla\psi|)^2 = 2\tilde{\ell}\nabla\psi\cdot\nabla\psi_t. \tag{2.5}$$

In (2.3), $\psi = \psi(x,t)$ denotes the acoustic velocity potential. It is connected to the acoustic pressure p via the relation

$$p = \varrho\psi_t, \tag{2.6}$$

where ϱ is the medium mass density. The damping coefficient δ is usually referred to as sound diffusivity [31]. In (2.4), $\tilde{k}$ and ℓ are nonlinearity coefficients, which are assumed to be real numbers. Eq. (2.3) can be understood as a generalization of the time-fractional models considered in [12], with kernels $\mathfrak{K}_1$ and $\mathfrak{K}_2$ generalizing those in Table 2.

In the limiting case $\tau = \delta = 0$, having nonlinearities (2.4) corresponds to the classical Blackstock equation [24] in nonlinear acoustics:

$$\psi_{tt} - c^2(1 - 2\tilde{k}\psi_t)\Delta\psi + 2\tilde{\ell}\nabla\psi\cdot\nabla\psi_t = 0,$$

and (2.5) to the Kuznetsov equation [32]:

$$(1 + 2\tilde{k}\psi_t)\psi_{tt} - c^2\Delta\psi + 2\tilde{\ell}\nabla\psi\cdot\nabla\psi_t = 0.$$

For the Kuznetsov equation above, it is usual to employ the approximation

$$|\nabla\psi|^2 \approx c^{-2}\psi_t^2,$$

when cumulative nonlinear effects dominate the local ones, and in this manner simplify it by the Westervelt equation [33]. Using this approximation in (2.3) results in

$$\tau^a \mathfrak{K}_1 * \psi_{ttt} + \mathfrak{a}(\psi_t)\psi_{tt} - c^2\mathfrak{b}(\psi_t)\Delta\psi - \tau^a c^2 \mathfrak{K}_1 * \Delta\psi_t - \tau_\theta^b \delta \mathfrak{K}_2 * \Delta\psi_{tt} = 0 \tag{2.7}$$

where now either

$$\mathfrak{a} = 1, \quad \mathfrak{b}(\psi_t) = 1 - 2k\psi_t, \quad \mathcal{N} \equiv 0$$

or

$$\mathfrak{a}(\psi_t) = 1 + 2k\psi_t, \quad \mathfrak{b} = 1, \quad \mathcal{N} \equiv 0$$

with $k = \tilde{k} + c^{-2}\tilde{\ell}$.

It is also common to express the Westervelt equation in terms of the acoustic pressure p. Formally taking the time derivative of (2.7) and employing the pressure-potential relation (2.6) as well as the approximation $\Delta\psi \approx c^{-2}\psi_{tt}$ justified by Blackstock's scheme [20,31] that allows to use a lower order approximation within a higher order term (here it is a first linear approximation used in a quadratic term) leads to the pressure form

$$\tau^a \mathfrak{K}_1 * (p_{tt} - c^2 \Delta p)_t + \mathfrak{a}(\tfrac{p}{\rho}) p_{tt} - c^2 \mathfrak{b}(\tfrac{p}{\rho}) \Delta p - \tau_\theta^b \delta \mathfrak{K}_2 * \Delta p_{tt} + \tfrac{1}{\varrho}(\mathfrak{a}'(\tfrac{p}{\varrho}) - \mathfrak{b}'(\tfrac{p}{\varrho})) p_t^2 = f \tag{2.8}$$

with the right-hand side

$$f(t) = -\tau^a \mathfrak{K}_1(t) p_{tt}(0) + \tau^a c^2 \mathfrak{K}_1(t) \Delta p(0) + \tau_\theta^b \delta \mathfrak{K}_2(t) \Delta p_t(0).$$

To be able to treat all these different acoustic equations, we unify them into one model.

Equations considered in this work. We assume throughout that $\Omega \subset \mathbb{R}^d$ with $d \in \{1, 2, 3\}$ is a bounded and sufficiently smooth domain. $T > 0$ denotes the final propagation time. Motivated by the modeling discussed above, we study the following general equation:

$$\tau^a \mathfrak{K}_1 * u_{ttt} + \mathfrak{a}\psi_{tt} - c^2 \mathfrak{b} \Delta u - \tau^a c^2 \mathfrak{K}_1 * \Delta u_t - \delta \mathfrak{K}_2 * \Delta u_{tt} + \mathcal{N} = f, \tag{2.9}$$

coupled with initial data

$$(u, u_t, u_{tt})|_{t=0} = (u_0, u_1, u_2), \tag{2.10}$$

and homogeneous Dirichlet boundary conditions

$$u|_{\partial\Omega} = 0. \tag{2.11}$$

We distinguish the following two cases that require different regularity assumptions on the initial data:

- Equations of Westervelt–Blackstock-type with

$$\mathfrak{a} = \mathfrak{a}(u) = 1 + 2k_1 u, \quad \mathfrak{b} = 1 - 2k_2 u, \quad \mathcal{N} = \mathcal{N}(u_t) = 2k_3 u_t^2;$$

- Equations of Kuznetsov–Blackstock type with

$$\mathfrak{a} = \mathfrak{a}(u_t) = 1 + 2k_1 u_t, \;\; \mathfrak{b} = \mathfrak{b}(u_t) = 1 - 2k_2 u_t, \;\; \mathcal{N} = \mathcal{N}(\nabla u, \nabla u_t) = 2k_3 \nabla u \cdot \nabla u_t,$$

where we assume $k_{1,2,3} \in \mathbb{R}$. Note that to relax the notation we have merged the constant $\tau_\theta^b \delta$ into the coefficient $\delta > 0$ in (2.9), which therefore no longer has the usual dimension of sound diffusivity. To be able to take limits as $\tau \searrow 0$ we will rely on the damping term containing $\mathfrak{K}_2$ and therefore assume $\delta > 0$ to be fixed. As we are interested in the vanishing behavior $\tau \searrow 0$, we assume throughout that $\tau \in (0, \bar{\tau}]$ for some fixed $\bar{\tau} > 0$.

The Westervelt–Blackstock-type equation incorporates the nonlinearities that arise in pressure form (2.8). We point out that this pressure form is more general than the equation considered in [12,18] where $\mathfrak{b} \equiv 1$ and which was termed of Westervelt-type. Here we also allow for $\mathfrak{b} = \mathfrak{b}(u)$ and thus call it of Westervelt–Blackstock type.

The Kuznetsov–Blackstock-type equation encompasses (2.4), (2.5), and (2.7). It involves u_t instead of u in the nonlinearities tied to the leading and second terms in the limiting equation ($\tau = 0$) and also contains a quadratic gradient term. For this reason, it requires stronger regularity assumptions on the data in the well-posedness analysis compared to the Westervelt–Blackstock case.

2.3. General strategy in the singular limit analysis

Our strategy in the singular limit analysis is based on first deriving τ-uniform bounds for a linearization of (2.9) with a source term, given by

$$\tau^a \mathfrak{K}_1 * u_{ttt} + \mathfrak{a}(x,t) u_{tt} - c^2 \mathfrak{b}(x,t) \Delta u - \tau^a c^2 \mathfrak{K}_1 * \Delta u_t - \delta \mathfrak{K}_2 * \Delta u_{tt} = f(x,t), \tag{2.12}$$

and later using a suitable fixed-point theorem on the mapping

$$\mathcal{T} : u^* \mapsto u$$

to transfer the result to the nonlinear problem. To this end, the two types of nonlinearities will necessitate different assumptions on the smoothness of data and coefficients $\mathfrak{a}$ and $\mathfrak{b}$ (and in some cases prevent certain arguments):

- Westervelt–Blackstock type with

$$\mathfrak{a} = 1 + 2k_1 u^*, \quad \mathfrak{b} = 1 - 2k_2 u^*, \qquad f = -2k_3 (u_t^*)^2;$$

- Kuznetsov–Blackstock-type with

$$\mathfrak{a} = 1 + 2k_1 u_t^*, \quad \mathfrak{b} = 1 - 2k_2 u_t^*, \quad f = -2k_3 \nabla u^* \cdot \nabla u_t^*,$$

The fixed-point of the mapping $u^* = u$ will give us the solution of the nonlinear problem. The key component of this analysis are the uniform bounds for the solutions of (2.12) with respect to τ. To obtain them, we will employ the following two testing strategies:

- Section 3: Testing with $-\Delta u_{tt}$ and $\Delta^2 u_{tt}$;
- Section 4: Testing with u_{tt} and $-\Delta \mathfrak{K}_1 * u_{tt}$,

which can be rigorously justified through a Faedo–Galerkin procedure; cf. Appendix A. To make the testing procedure work we will introduce two sets of regularity and coercivity assumptions on the kernels in the corresponding sections. As a consequence, among the equations obtained from the Compte–Metzler laws, the first testing strategy will work for the (fJMGT III) equation. The second testing strategy will turn out to work for proving uniform solvability of the (fJMGT) and (fJMGT I) equations but if $\alpha > 1/2$, we will have the restriction of Westervelt–Blackstock type on the allowed nonlinearities.

2.4. Notation

Below we often use the notation $x \lesssim y$ for $x \leq C\, y$ with a constant $C > 0$ that does not depend on the thermal relaxation time τ. We use $\lesssim_T$ to emphasize that $C = C(T)$ tends to ∞ as $T \to \infty$.

We use $(\cdot, \cdot)_{L^2}$ to denote the scalar product on $L^2(\Omega)$ and $\sim$ to denote equivalence of norms. We often omit the spatial and temporal domain when writing norms; for example, $\| \cdot \|_{L^p(L^q)}$ denotes the norm on the Bochner space $L^p(0, T; L^q(\Omega))$. We use $\| \cdot \|_{L^p_t(L^q)}$ to denote the norm on $L^p(0, t; L^q(\Omega))$ for $t \in (0, T)$.

2.5. Regularity of the kernels

As already mentioned above, several coercivity assumptions will have to be made on the kernels below. However, concerning their regularity, throughout this paper we will only assume that

$$\mathfrak{K}_1,\, \mathfrak{K}_2 \in \{\delta_0\} \cup L^1(0, T). \tag{2.13}$$

In fact, the analysis below would also apply to measures $\mathfrak{K}_1 *$, $\mathfrak{K}_2 * \in \mathcal{M}(0, t) = C_b([0, T])^*$, however, at the cost of somewhat increased technicality. We will be reminded of this possibility when using the following norm below:

$$\|\mathfrak{K}_1\|_{\mathcal{M}(0,T)} = \begin{cases} 1 & \text{if } \mathfrak{K}_1 = \delta_0, \\ \|\mathfrak{K}_1\|_{L^1(0,T)} & \text{if } \mathfrak{K}_1 \in L^1(0, T). \end{cases}$$

2.6. Auxiliary theoretical results

We recall from [19] a result that will allow us to extract appropriately converging subsequences later on in the existence proofs; see, for example, Proposition 3.1.

Lemma 2.1 (*Sequential Compactness with the Caputo–Djrbashian Derivative, See [19]*)**.** *Let $1 \leq p \leq \infty$, and let $\mathfrak{K} \in L^1(0,T)$ be such that there exists $\tilde{\mathfrak{K}} \in L^{p'}(0,T)$ for which $\tilde{\mathfrak{K}} * \mathfrak{K} = 1$ with $p' = \frac{p}{p-1}$. Consider the space*

$$X^p_{\mathfrak{K}} = \{u \in L^p(0,T) \mid \mathfrak{K} * u_t \in L^p(0,T)\}, \tag{2.14}$$

with the norm

$$\|\cdot\|_{X^p_{\mathfrak{K}}} = \left(\|u\|^p_{L^p} + \|(\mathfrak{K} * u_t)\|^p_{L^p}\right)^{1/p},$$

and the usual modification for $p = \infty$. The following statements hold true:

- *The space $X^p_{\mathfrak{K}}$ is reflexive for $1 < p < \infty$ and separable for $1 \leq p < \infty$.*
- *The unit ball B^p_X of $X^p_{\mathfrak{K}}$ is weakly sequentially compact for $1 < p < \infty$, and B^∞_X is weak-$*$ sequentially compact.*
- *The space $X^p_{\mathfrak{K}}$ continuously embeds into $C[0,T]$.*

We also state and prove a result which will be helpful in verifying a coercivity assumption on the two kernels in Section 3.

Lemma 2.2. *Given $t > 0$, let $y \in W^{1,1}(0,t)$ with $y' \in H^{-\alpha/2}(0,t)$ for $\alpha \in (0,1)$. Then*

$$\|y - y(0)\|_{L^\infty(0,t)} \lesssim \|y - y(0)\|_{H^{1-\alpha/2}(0,t)} \sim \|I^{\alpha/2}_t y'\|_{L^2(0,t)} \sim \|y'\|_{H^{-\alpha/2}(0,t)}.$$

Proof. We introduce the time-flip operator as

$$\overline{w}^t(s) = w(t-s).$$

By [25, Theorem 2.2, Corollary 2.1], we have

$$\|\overline{w}^t\|_{H^{\alpha/2}(0,t)} \sim \|(\mathrm{I}^{\alpha/2}_t)^{-1}\overline{w}^t\|_{L^2(0,t)},$$

where

$$\mathrm{I}^\eta_t y(t) = \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} y(s)\,\mathrm{d}s, \ \ \eta > 0.$$

Additionally employing the identity $\langle \overline{a}^t, b\rangle_{L^2(0,t)} = (a * b)(t) = (b * a)(t)$, leads to

$$\begin{aligned}
\|y'\|_{H^{-\alpha/2}(0,t)} &= \sup_{w \in C^\infty_0(0,t)} \frac{|\langle y', w\rangle_{L^2(0,t)}|}{\|w\|_{H^{\alpha/2}(0,t)}} = \sup_{w \in C^\infty_0(0,t)} \frac{|\langle \overline{y'}^t, \overline{w}^t\rangle_{L^2(0,t)}|}{\|\overline{w}^t\|_{H^{\alpha/2}(0,t)}} \\
&\sim \sup_{\tilde{w} \in C^\infty_0(0,t)} \frac{|\langle \overline{y'}^t, \mathrm{I}^{\alpha/2}_t \tilde{w}\rangle_{L^2(0,t)}|}{\|\tilde{w}\|_{L^2(0,t)}} = \sup_{\tilde{w} \in C^\infty_0(0,t)} \frac{|(y' * g_{\alpha/2} * \tilde{w})(t)|}{\|\tilde{w}\|_{H^{\alpha/2}(0,t)}} \\
&= \sup_{\tilde{w} \in C^\infty_0(0,t)} \frac{|\langle \mathrm{I}^{\alpha/2}_t y', \overline{\tilde{w}}^t\rangle_{L^2(0,t)}|}{\|\tilde{w}\|_{L^2(0,t)}} = \|I^{\alpha/2} y'\|_{L^2(0,t)},
\end{aligned}$$

where we have set $\tilde{w} = (\mathrm{I}^{\alpha/2}_t)^{-1}\overline{w}^t$ and used $\|\overline{\tilde{w}}^t\|_{L^2(0,t)} = \|\tilde{w}\|_{L^2(0,t)}$. Now due to Sobolev's embedding and $1 - \frac{\alpha}{2} > \frac{1}{2}$, as well as [25, Theorems 2.4, 2.5, p. 22], we have

$$\|y - y(0)\|_{L^\infty(0,t)} \lesssim \|y - y(0)\|_{H^{1-\alpha/2}(0,t)} \sim \|I^{\alpha/2}_t y'\|_{L^2(0,t)} \sim \|y'\|_{H^{-\alpha/2}(0,t)},$$

as claimed. □

3. Testing with canonical test functions

In this section, we perform the analysis of Eqs. (2.9) amenable to testing with $(-\Delta)^\nu u_{tt}$ where $\nu \geq 0$. These are the canonical test functions for the third-order (Jordan–)Moore–Gibson–Thompson equations and thus make an obvious choice for an attempt at a uniform analysis here. Unsurprisingly, among the equations based on the Compte–Metzler laws, this testing will suffice for the acoustic Eq. (fJMGT III) with the integer-order leading term, obtained using the third Compte–Metzler law.

Assumptions on the two kernels in this section. Recall that throughout the paper, we assume the regularity of the two kernels given in (2.13). In this section we make the following additional assumption on the resolvent of the leading kernel:

$$\text{there exists } \tilde{\mathfrak{K}}_1 \in L^2(0,T), \text{ such that } \mathfrak{K}_1 * \tilde{\mathfrak{K}}_1 = 1. \tag{$\mathcal{A}_1$}$$

In case of the Abel kernel, this is equivalent to asking that the fractional order of differentiation α is larger than 1/2. We next need the coercivity assumptions; one on the leading kernel and one on the two kernels combined. We assume that there exist constants C, $\underline{c}$, $\overline{C} > 0$, independent of τ, such that

$$\int_0^{t'} (\mathfrak{K}_1 * y')(t)\, y(t)\, \mathrm{d}t \geq -C|y(0)|^2, \quad y \in X^2_{\mathfrak{K}_1}(0,t'), \tag{$\mathcal{A}_2$}$$

and for all $\tau \geq 0$

$$\int_0^{t'} \left(\tau^a c^2 \mathfrak{K}_1 * y + \delta \mathfrak{K}_2 * y'\right)(t) y'(t)\, \mathrm{d}t \geq \quad \underline{c}\|y\|^2_{L^\infty(0,t')} - \overline{C}|y(0)|^2, \ y \in W^{1,1}(0,t'); \tag{$\mathcal{A}_3$}$$

recall that the space $X^2_{\mathfrak{K}_1}(0,t')$ is defined in (2.14) with $p = 2$.

These assumptions are suited to the analysis of Eq. (1.1) that relies on an energy method based on testing with $(-\Delta)^\nu u_{tt}$ for $\nu \geq 0$. In this case, the leading term $\tau^a \mathfrak{K}_1 * u_{ttt}$ will invoke assumption $(\mathcal{A}_2)$, whereas the combination of the other two nonlocal terms, $-\tau^a c^2 \mathfrak{K}_1 * \Delta u_t - \delta \mathfrak{K}_2 * \Delta u_{tt}$ will invoke $(\mathcal{A}_3)$. Note that we assume $(\mathcal{A}_3)$ to hold also for $\tau = 0$, thereby having the coercivity of the kernel $\mathfrak{K}_2$ alone as well.

How to verify the coercivity assumptions

Assumption $(\mathcal{A}_2)$ clearly holds for $\mathfrak{K}_1 = \delta_0$. For kernels in $L^p(0,T)$, sufficient conditions under which $(\mathcal{A}_2)$ holds for $y \in W^{1,1}(0,t')$ can be found, for example, in Lemma 5.1 in [29]; see also [34, Lemma B.1] and [35, Lemma 3.1]. These are as follows:

$$\begin{aligned} &\mathfrak{K}_1 \in L^1(0,T), \quad (\forall \varepsilon > 0) \quad \mathfrak{K}_1 \in W^{1,1}(\varepsilon,T), \\ &\mathfrak{K}_1 \geq 0 \ \text{ on } (0,T), \quad \mathfrak{K}_1'|_{[\varepsilon,T]} \leq 0 \ \text{ a.e.} \end{aligned} \tag{3.1}$$

By a density argument, $(\mathcal{A}_2)$ then also holds for any $y \in X^2_{\mathfrak{K}_1}(0,t')$ under assumptions (3.1).

Assumption $(\mathcal{A}_3)$ is fulfilled for $\mathfrak{K}_1 = \delta_0$ and $\mathfrak{K}_2 = 1$; that is, for the third-order (JMGT) equation.

For other kernels, one can verify $(\mathcal{A}_3)$ by applying the Fourier analysis technique from [36, Lemma 2.3] to the combined kernel

$$\tilde{\mathcal{K}} := \tau^a c^2 1 * \mathfrak{K}_1 + \delta \mathfrak{K}_2,$$

see Table 3, where $\mathcal{F}$ denotes the Fourier transform, $\Re$ the real part, and $\overline{f}^t$ the extension of a function f by zero outside $\mathbb{R} \setminus (0,t)$.

Among the fractional Compte–Metzler laws, the combined uniform lower bound $(\mathcal{A}_3)$ only holds for the kernels in the GFE III law. Indeed, in case of the equation obtained using the third Compte–Metzler law:

$$\tau u_{ttt} + \mathfrak{a} u_{tt} - c^2 \mathfrak{b} \Delta u - \tau c^2 \Delta u_t - \delta \mathfrak{K}_2 * \Delta u_{tt} + \mathcal{N} = 0$$

Table 3
Real part of Fourier transforms of the Compte–Metzler laws.

Flux law	$(\Re\mathcal{F}\overline{(\tau^a c^2 1 * \mathfrak{K}_1 + \delta\mathfrak{K}_2)}^{\infty})(\imath\omega)$
GFE I	$\tau^a c^2 \cos((2-\alpha)\pi/2)\omega^{\alpha-2} + \delta\cos(\alpha\pi/2)\omega^{-\alpha}$
GFE III	$\delta\cos(\alpha\pi/2)\omega^{-\alpha}$
GFE	$\tau^a c^2 \cos((2-\alpha)\pi/2)\omega^{\alpha-2}$

we have $\mathfrak{K}_1 = \delta_0$ and $a = 1$. Then for $\delta > 0$, condition $(\mathcal{A}_3)$ holds (uniformly with respect to $\alpha \in [0,1]$) due to the estimate

$$\begin{aligned}\int_0^{t'} (\tau^a c^2 y + \delta g_\alpha * y')(t) y'(t)\,\mathrm{d}t &\geq \frac{1}{2}\tau^a c^2 (y(t')^2 - y(0)^2) + \delta\cos(\alpha\pi/2)\|y'\|^2_{H^{-\alpha/2}(0,t')} \\ &\geq \delta\cos(\alpha\pi/2)\|y'\|^2_{H^{-\alpha/2}(0,t')}\end{aligned}$$

and Lemma 2.2, which provides a lower bound for $\|y'\|^2_{H^{-\alpha/2}(0,t')}$.

3.1. Well-posedness of a linearized Westervelt–Blackstock problem

Under assumptions (2.13) and $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ on the two kernels, we next discuss the well-posedness of (2.9) with Westervelt–Blackstock nonlinearities, uniformly in τ. Recall that this means we consider equation

$$\tau^a \mathfrak{K}_1 * u_{ttt} + \mathfrak{a} u_{tt} - c^2 \mathfrak{b}\Delta u - \tau^a c^2 \mathfrak{K}_1 * \Delta u_t - \delta\mathfrak{K}_2 * \Delta u_{tt} = f, \tag{2.9}$$

where we have in mind

- $\mathfrak{a} = 1 + 2k_1 u$, $\mathfrak{b} = 1 - 2k_2 u$, and $f = -2k_3 u_t^2$

and that, among the discussed equations, the assumptions on the kernels in this section are verified by the third-order (JMGT) and (fJMGT III) equations.

As announced, we first analyze a linearization where $\mathfrak{a} = \mathfrak{a}(x,t)$ and $\mathfrak{b} = \mathfrak{b}(x,t)$. We assume that these coefficients are sufficiently regular in the following sense:

$$\begin{aligned}&\mathfrak{a} \in \left\{\mathfrak{a} \in C([0,T]; L^\infty(\Omega)) : \nabla\mathfrak{a} \in L^\infty(0,T; L^4(\Omega))\right\}, \\ &\mathfrak{b} \in W^{1,1}(0,T; L^\infty(\Omega)) \hookrightarrow C([0,T]; L^\infty(\Omega)).\end{aligned} \tag{3.2}$$

Furthermore, the coefficient $\mathfrak{a}$ should not degenerate: we assume that there exist $\underline{\mathfrak{a}}$, $\overline{\mathfrak{a}} > 0$, independent of τ, such that

$$\underline{\mathfrak{a}} < \mathfrak{a}(x,t) < \overline{\mathfrak{a}} \quad \text{a.e. in } \Omega \times (0,T). \tag{3.3}$$

To state the well-posedness result, we introduce the solution space

$$\begin{aligned}\mathcal{X}^{\mathrm{WB}} = \{u \in L^\infty(0,T; H^2(\Omega) \cap H_0^1(\Omega)) : u_t &\in L^\infty(0,T; H^2(\Omega) \cap H_0^1(\Omega)), \\ u_{tt} &\in L^2(0,T; H_0^1(\Omega))\}.\end{aligned}$$

Note that if $u \in \mathcal{X}^{\mathrm{WB}}$, we have the following (weak) continuity in time

$$u \in \mathcal{X}^{\mathrm{WB}} \implies u \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)),\ u_t \in C_w(0,T; H^2(\Omega) \cap H_0^1(\Omega));$$

see [37, Ch. 2, Lemma 3.3].

Proposition 3.1. *Let $T > 0$ and $\tau \in (0, \bar{\tau}]$. Let assumptions (2.13), $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ on the kernels hold. Let the coefficients $\mathfrak{a}$ and $\mathfrak{b}$ satisfy regularity assumption (3.2). Assume that $\mathfrak{a}$ does not degenerate so that (3.3) holds. Let $f \in W^{1,1}(0,T; L^2(\Omega))$ and*

$$(u, u_t, u_{tt})|_{t=0} = (u_0, u_1, u_2) \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega).$$

Then there exists $m>0$, independent of τ, such that if

$$\|\nabla \mathfrak{a}\|_{L^\infty(L^4)} \leq m,$$

then there is a solution

$$u \in \mathcal{X}^{\text{WB}}, \quad \tau^a \mathfrak{K}_1 * u_{ttt} \in L^2(0,T;H^{-1}(\Omega))$$

of the initial boundary-value problem

$$\begin{cases} \tau^a \mathfrak{K}_1 * u_{ttt} + \mathfrak{a}(x,t)u_{tt} - c^2\mathfrak{b}(x,t)\Delta u - \tau^a c^2 \mathfrak{K}_1 * \Delta u_t \\ \qquad\qquad\qquad\qquad - \delta \mathfrak{K}_2 * \Delta u_{tt} = f(x,t) \ \text{ in } \Omega \times (0,T), \\ u|_{\partial\Omega} = 0, \\ (u, u_t, u_{tt})|_{t=0} = (u_0, u_1, u_2). \end{cases} \tag{3.4}$$

The solution satisfies the following estimate:

$$\|u\|^2_{\mathcal{X}^{\text{WB}}} \lesssim_T \|u_0\|^2_{H^2} + \|u_1\|^2_{H^2} + \tau^a\|u_2\|^2_{H^1} + \|f\|^2_{W^{1,1}(L^2)},$$

where the hidden constant does not depend on τ. If additionally $\mathfrak{b} \in W^{1,1}(0,T;W^{1,4}(\Omega))$, the solution is unique.

Proof. We conduct the proof by using a standard Faedo–Galerkin procedure where we first construct an approximate solution $u^{(n)} \in W^{2,\infty}(0,T;V_n)$ with V_n being a finite-dimensional subspace of $H^2(\Omega) \cap H^1_0(\Omega)$; the details can be found in Appendix A.

The next step in the proof is to obtain a bound that is uniform in n (and also τ, having in mind the later singular limit analysis). By testing the semi-discrete problem with $-\Delta u^{(n)}_{tt}$ and using the assumptions we have made on the kernels, we find that

$$\begin{aligned} &\int_0^t \|\sqrt{\mathfrak{a}}\nabla u^{(n)}_{tt}\|^2_{L^2}\,\mathrm{d}s + \underline{c}\|\Delta u^{(n)}_t\|^2_{L^\infty(L^2)} \\ \lesssim &\int_0^t \left\{-(u^{(n)}_{tt}\nabla\mathfrak{a}, \nabla u^{(n)}_{tt})_{L^2} - c^2(\mathfrak{b}\Delta u, \Delta u^{(n)}_{tt})_{L^2} + (f, -\Delta u^{(n)}_{tt})_{L^2}\right\}\mathrm{d}s + \tau^a\|\nabla u^{(n)}_2\|^2_{L^2} \\ &+ \|\Delta u^{(n)}_1\|^2_{L^2}. \end{aligned}$$

Integrating by parts in time in the c^2 and f terms on the right-hand side yields

$$\begin{aligned} &\int_0^t \|\sqrt{\mathfrak{a}}\nabla u^{(n)}_{tt}\|^2_{L^2}\,\mathrm{d}s + \underline{c}\|\Delta u^{(n)}_t\|^2_{L^\infty(L^2)} \\ \lesssim &-\int_0^t (u^{(n)}_{tt}\nabla\mathfrak{a}, \nabla u^{(n)}_{tt})_{L^2}\,\mathrm{d}s + c^2\left(-\mathfrak{b}\Delta u^{(n)}(s), \Delta u^{(n)}_t(s)\right)_{L^2}\Big|_0^t + (f(s), -\Delta u^{(n)}_t(s))_{L^2}\Big|_0^t \\ &+ \int_0^t \left\{c^2(\mathfrak{b}\Delta u^{(n)}_t + \mathfrak{b}_t\Delta u^{(n)}, \Delta u^{(n)}_t)_{L^2} + (f_t, \Delta u^{(n)}_t)_{L^2}\right\}\mathrm{d}s \\ &+ \tau^a\|\nabla u^{(n)}_2\|^2_{L^2} + \|\Delta u^{(n)}_1\|^2_{L^2}. \end{aligned}$$

From here using Hölder's inequality and the embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$, we have

$$\begin{aligned} &\underline{\mathfrak{a}}\int_0^t \|\nabla u^{(n)}_{tt}\|^2_{L^2}\,\mathrm{d}s + \underline{c}\|\Delta u^{(n)}_t\|^2_{L^\infty(L^2)} \\ \lesssim &\|\nabla\mathfrak{a}\|_{L^\infty(L^4)}\|\nabla u^{(n)}_{tt}\|^2_{L^2(L^2)} + \overline{\mathfrak{b}}^2\|\Delta u^{(n)}\|^2_{L^\infty(L^2)} + \varepsilon\|\Delta u^{(n)}_t\|^2_{L^\infty(L^2)} \\ &+ \|\mathfrak{b}(0)\|^2_{L^\infty}\|\Delta u^{(n)}_0\|^2_{L^2} + \|\Delta u^{(n)}_1\|^2_{L^2} + \|f\|^2_{W^{1,1}(L^2)} \\ &+ \overline{\mathfrak{b}}\|\Delta u^{(n)}_t\|^2_{L^2(L^2)} + \|\mathfrak{b}_t\|^2_{L^1(L^\infty)}\|\Delta u^{(n)}\|^2_{L^\infty(L^2)} + \tau^a\|\nabla u^{(n)}_2\|^2_{L^2}. \end{aligned} \tag{3.5}$$

We can further bound $\|\Delta u^{(n)}\|_{L^\infty(L^2)}$ by using the estimate

$$\|\Delta u^{(n)}\|_{L^\infty(L^2)} \leq \sqrt{T}\|\Delta u_t^{(n)}\|_{L^2(L^2)} + \|\Delta u_0^{(n)}\|_{L^2}.$$

If C is the hidden constant in (3.5), then provided we take $\mathfrak{a}$ small enough so that

$$\|\nabla \mathfrak{a}\|_{L^\infty(L^4)} \leq \frac{1}{2} C \underline{\mathfrak{a}},$$

and also choose $\varepsilon > 0$ small enough, we have via Grönwall's inequality

$$\begin{aligned} &\int_0^t \|\nabla u_{tt}^{(n)}\|_{L^2}^2 \,\mathrm{d}s + \|\Delta u_t^{(n)}\|_{L^\infty(L^2)}^2 + \|\Delta u^{(n)}\|_{L^\infty(L^2)}^2 \\ &\lesssim_T \|\Delta u_0^{(n)}\|_{L^2}^2 + \|\Delta u_1^{(n)}\|_{L^2}^2 + \tau^a \|\nabla u_2^{(n)}\|_{L^2}^2 + \|f\|_{W^{1,1}(L^2)}^2, \end{aligned} \tag{3.6}$$

where the hidden constant has the form

$$C = C_1 \exp\left(C_2 T (1 + \|\mathfrak{b}_t\|_{L^1(L^\infty)}^2)\right).$$

Additionally, from the PDE and Young's convolution inequality, we have

$$\begin{aligned} &\tau^a \|\mathfrak{K}_1 * u_{ttt}^{(n)}\|_{L^2(H^{-1})} \\ &\lesssim \overline{\mathfrak{a}} \|u_{tt}^{(n)}\|_{L^2(L^2)} + \overline{\mathfrak{b}} \|\Delta u^{(n)}\|_{L^2(L^2)} + \tau^a \|\mathfrak{K}_1\|_{\mathcal{M}} \|\Delta u_t^{(n)}\|_{L^2(L^2)} \\ &\quad + \|\mathfrak{K}_2\|_{\mathcal{M}} \|\nabla u_{tt}^{(n)}\|_{L^2(L^2)} + \|f\|_{L^2(H^{-1})}, \end{aligned} \tag{3.7}$$

which, taken together with (3.6), provides us with a uniform bound on $\tau^a \|\mathfrak{K}_1 * u_{ttt}^{(n)}\|_{L^2(H^{-1})}$ as well. Since we have assumed that $\tilde{\mathfrak{K}}_1 \in L^2(0,T)$, then from the bound on

$$\tau^a \mathfrak{K}_1 * u_{ttt}^{(n)} := \tilde{f}^{(n)}$$

in $L^2(0,T;H^{-1}(\Omega))$, we also obtain a uniform bound on

$$\begin{aligned} \tau^a \|u_{tt}^{(n)}\|_{L^\infty(H^{-1})} &= \tau^a \|u_2^{(n)} + \tilde{\mathfrak{K}}_1 * \tilde{f}^{(n)}\|_{L^\infty(H^{-1})} \\ &\lesssim \|u_2^{(n)}\|_{H^{-1}} + \|\tilde{\mathfrak{K}}_1\|_{L^2} \|\tilde{f}^{(n)}\|_{L^2(H^{-1})}. \end{aligned} \tag{3.8}$$

From the above analysis, we conclude that there is a subsequence (not relabeled), such that

$$\begin{aligned} u^{(n)} &\rightharpoonup u && \text{weakly-}\star \text{ in } L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \\ u_t^{(n)} &\rightharpoonup u_t && \text{weakly-}\star \text{ in } L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \\ u_{tt}^{(n)} &\rightharpoonup u_{tt} && \text{weakly} \quad \text{in } L^2(0,T;H_0^1(\Omega)), \end{aligned} \tag{3.9}$$

as $n \to \infty$. From (3.9), by [38, Theorem 3.1.1], there is a subsequence (again not relabeled), such that

$$\begin{aligned} u^{(n)} &\longrightarrow u && \text{strongly in } L^2(0,T;H_0^1(\Omega)), \\ u_t^{(n)} &\longrightarrow u_t && \text{strongly in } L^2(0,T;H_0^1(\Omega)). \end{aligned} \tag{3.10}$$

Furthermore, by Young's convolution inequality

$$\begin{aligned} \|\mathfrak{K}_1 * \Delta u_t^{(n)}\|_{L^\infty(L^2)} &\leq \|\mathfrak{K}_1\|_{\mathcal{M}} \|\Delta u_t^{(n)}\|_{L^\infty(L^2)}, \\ \|\mathfrak{K}_2 * \Delta u_{tt}^{(n)}\|_{L^2(H^{-1})} &\leq \|\mathfrak{K}_2\|_{\mathcal{M}} \|\Delta u_{tt}^{(n)}\|_{L^2(H^{-1})}, \end{aligned}$$

so we conclude that (up to a subsequence)

$$\begin{aligned} \mathfrak{K}_1 * \Delta u_t^{(n)} &\rightharpoonup \mathfrak{K}_1 * \Delta u_t && \text{weakly-}\star \text{ in } L^\infty(0,T;L^2(\Omega)), \\ \mathfrak{K}_2 * \Delta u_{tt}^{(n)} &\rightharpoonup \mathfrak{K}_2 * \Delta u_{tt} && \text{weakly} \quad \text{in } L^2(0,T;H^{-1}(\Omega)). \end{aligned}$$

By (3.7) and Lemma 2.1, we also have

$$\tau^a \mathfrak{K}_1 * u_{ttt}^{(n)} \rightharpoonup \tau^a \mathfrak{K}_1 * u_{ttt} \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)). \tag{3.11}$$

These weak convergence results allow us to pass to the limit in the semi-discrete equation. From (3.10), and uniqueness of limits, we conclude that $(u,u_t)|_{t=0}=(u_0,u_1)$. It remains to interpret how u_2 is attained. Following [39, Ch. 7], let $v \in C^1([0,T];H_0^1(\Omega))$ with $v(T)=v_t(T)=0$. We have

$$\begin{aligned}
&-\tau^a \int_0^T \int_\Omega \mathfrak{K}_1 * u_{tt} v_t \,\mathrm{d}x\mathrm{d}s - \tau^a \int_0^T (\mathfrak{K}_1 u_{tt}(0), v)_{L^2}\,\mathrm{d}s \\
&+\int_0^T (\mathfrak{a} u_{tt} - c^2 \mathfrak{b}\Delta u - \tau^a c^2 \mathfrak{K}_1 * \Delta u_t, v)_{L^2}\,\mathrm{d}s + \int_0^T \delta(\mathfrak{K}_2 * \nabla u_{tt}, \nabla v)_{L^2}\,\mathrm{d}s = \int_0^T (f,v)\,\mathrm{d}s.
\end{aligned}$$

For the Galerkin approximation, we similarly have

$$\begin{aligned}
&-\tau^a \int_0^T \int_\Omega \mathfrak{K}_1 * u_{tt}^{(n)} v_t \,\mathrm{d}x\mathrm{d}s - \tau^a \int_0^T (\mathfrak{K}_1 u_{tt}^{(n)}(0), v)_{L^2}\,\mathrm{d}s \\
&+\int_0^T (\mathfrak{a} u_{tt}^{(n)} - c^2 \mathfrak{b}\Delta u^{(n)} - \tau^a c^2 \mathfrak{K}_1 * \Delta u_t^{(n)}, v)_{L^2}\,\mathrm{d}s + \int_0^T \delta(\mathfrak{K}_2 * \nabla u_{tt}^{(n)}, \nabla v)_{L^2}\,\mathrm{d}s \\
&= \int_0^T (f,v)\,\mathrm{d}s.
\end{aligned} \tag{3.12}$$

Note that with L^∞ regularity in time for $u_{tt}^{(n)}$ obtained in (3.8), $\tau^a(\mathfrak{K}_1 * u_{tt}^{(n)})(0)=0$. We also infer that $\tau^a(\mathfrak{K}_1 * u_{tt})(0)=0$ since due to (3.11) and (3.8)

$$\tau^a \|u_{tt}\|_{L^\infty(H^{-1})} \leq \tau^a \liminf_{n\to\infty} \|u_{tt}^{(n)}\|_{L^\infty(H^{-1})} \leq C.$$

The uniform bound on $u_{tt}^{(n)}$ taken together with (3.9)–(3.11), allows us to pass to the limit (in possibly a subsequence) in (3.12). Comparing the resulting identities gives

$$\tau^a \int_0^T (\mathfrak{K}_1 u_{tt}(0), v)_{L^2}\,\mathrm{d}s = \tau^a \int_0^T (\mathfrak{K}_1 u_2, v)_{L^2}\,\mathrm{d}s,$$

from which we conclude (since $\mathfrak{K}_1 \neq 0$) that $u_{tt}(0)=u_2$.

Uniqueness. Uniqueness of solutions in $\mathcal{X}^{\mathrm{WB}}$ follows by proving that the only solution of the homogeneous problem with $f=0$ and zero data is $u=0$. Since $-\Delta u_{tt}$ is not a valid test function in this setting, we can test this homogeneous problem with $u_{tt} \in L^2(0,T;H_0^1(\Omega))$ instead. Similarly to before, we have

$$\begin{aligned}
&\int_0^t \|\sqrt{\mathfrak{a}} u_{tt}\|_{L^2}^2\,\mathrm{d}s + \underline{c}\|\nabla u_t\|_{L^\infty(L^2)}^2 \\
\leq &\int_0^t \left\{-c^2(\mathfrak{b}\nabla u, \nabla u_{tt})_{L^2} - c^2(u\nabla\mathfrak{b}, \nabla u_{tt})_{L^2}\right\}\,\mathrm{d}s \\
= &-c^2(\mathfrak{b}(t)\nabla u(t), \nabla u_t(t))_{L^2} - c^2(u(t)\nabla \mathfrak{b}(t), \nabla u_t(t))_{L^2} \\
&-\int_0^t \left\{-c^2(\mathfrak{b}_t\nabla u + \mathfrak{b}\nabla u_t, \nabla u_t)_{L^2} - c^2(u_t\nabla\mathfrak{b} + u\nabla\mathfrak{b}_t, \nabla u_t)_{L^2}\right\}\,\mathrm{d}s.
\end{aligned} \tag{3.13}$$

We can further bound the right-hand side using Hölder's inequality and the additional smoothness assumption on $\mathfrak{b}$:

$$\begin{aligned}
&\left|-c^2(\mathfrak{b}(t)\nabla u(t), \nabla u_t(t))_{L^2} - c^2(u(t)\nabla\mathfrak{b}(t), \nabla u_t(t))_{L^2}\right| \\
\lesssim\, &\overline{\mathfrak{b}}\|\nabla u(t)\|_{L^2}^2 + \|u(t)\|_{L^4}\|\nabla\mathfrak{b}(t)\|_{L^4}\|\nabla u_t(t)\|_{L^2}
\end{aligned}$$

a.e. in time. Similarly,

$$\begin{aligned}&\left|-\int_0^t \left\{-c^2(\mathfrak{b}_t\nabla u+\mathfrak{b}\nabla u_t,\nabla u_t)_{L^2}-c^2(u_t\nabla\mathfrak{b}+u\nabla\mathfrak{b}_t,\nabla u_t)_{L^2}\right\}\,\mathrm{d}s\right|\\ &\lesssim \|\mathfrak{b}_t\|_{L^1(L^\infty)}\|\nabla u\|_{L^\infty(L^2)}\|\nabla u_t\|_{L^\infty(L^2)}+\overline{\mathfrak{b}}\|\nabla u_t\|^2_{L^2(L^2)}\\ &\quad+\|\nabla\mathfrak{b}\|_{L^2(L^4)}\|u_t\|_{L^2(L^4)}\|\nabla u_t\|_{L^\infty(L^2)}+\|\nabla\mathfrak{b}_t\|_{L^1(L^4)}\|u\|_{L^\infty(L^4)}\|\nabla u_t\|_{L^\infty(L^2)}.\end{aligned} \tag{3.14}$$

Employing these bounds together with Young's inequality and the embedding $H^1(\Omega)\hookrightarrow L^4(\Omega)$ in (3.13) leads at first to

$$\begin{aligned}&\int_0^t \|\sqrt{\mathfrak{a}}u_{tt}\|^2_{L^2}\,\mathrm{d}s+\underline{c}\|\nabla u_t\|^2_{L^\infty_t(L^2)}\\ &\lesssim \|\nabla u(t)\|^2_{L^2}+\varepsilon\|\nabla\mathfrak{b}(t)\|^2_{L^4}\|\nabla u_t(t)\|^2_{L^2}+\|\nabla u\|^2_{L^\infty(L^2)}\\ &\quad+\varepsilon\|\mathfrak{b}_t\|^2_{L^1(L^\infty)}\|\nabla u_t\|^2_{L^\infty_t(L^2)}\\ &\quad+\|\nabla u_t\|^2_{L^2_t(L^2)}+\varepsilon\|\nabla\mathfrak{b}\|^2_{L^2(L^4)}\|\nabla u_t\|^2_{L^\infty_t(L^2)}\\ &\quad+\varepsilon\|\nabla\mathfrak{b}_t\|^2_{L^1(L^4)}\|\nabla u_t\|^2_{L^\infty_t(L^2)}\end{aligned}$$

for any $\varepsilon>0$. Note that

$$\|\nabla u(t)\|_{L^2}\leq\sqrt{T}\|\nabla u_t\|_{L^2_t(L^2)}.$$

Therefore from (3.14), for small enough ε, by using Grönwall's inequality and assumed smoothness of $\mathfrak{a}$ and $\mathfrak{b}$, we obtain

$$\int_0^t \|u_{tt}\|^2_{L^2}\,\mathrm{d}s+\|\nabla u_t\|^2_{L^\infty_t(L^2)}\leq 0,$$

from which (combined with the zero initial data) it follows that $u=0$. This concludes the proof. □

3.2. Uniform well-posedness with Westervelt–Blackstock nonlinearities

To treat Eqs. (2.9) with Westervelt–Blackstock nonlinearities under assumptions (2.13) and $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ on the two kernels, we next set up a fixed-point mapping $\mathcal{T}:\mathcal{B}^{\mathrm{WB}}\ni u^*\mapsto u$, where u solves (3.4) with

$$\mathfrak{a}(u^*)=1+2k_1u^*,\quad \mathfrak{b}(u^*)=1-2k_2u^*,\quad f=-\mathcal{N}(u_t^*)=-2k_3(u_t^*)^2,$$

and u^* is taken from the ball

$$\mathcal{B}^{\mathrm{WB}}=\left\{u^*\in\mathcal{X}^{\mathrm{WB}}:\ \|u^*\|_{\mathcal{X}^{\mathrm{WB}}}\leq R,\quad (u^*,u^*_t,u^*_{tt})|_{t=0}=(u_0,u_1,u_2)\right\}.$$

The radius $R>0$ is independent of τ and will be determined by the upcoming proof.

Theorem 3.1. *Let $T>0$ and $\tau\in(0,\bar{\tau}]$. Let $\delta>0$ and $k_{1,2,3}\in\mathbb{R}$. Let assumptions (2.13), $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ on the kernels hold. There exists a size of data $r=r(T)>0$, independent of τ, such that if*

$$\|u_0\|^2_{H^2}+\|u_1\|^2_{H^2}+\bar{\tau}^a\|u_2\|^2_{H^1}\leq r^2,$$

then there is a unique solution $u\in\mathcal{B}^{\mathrm{WB}}$ of

$$\begin{aligned}&\tau^a\mathfrak{K}_1*u_{ttt}+(1+2k_1u)u_{tt}-c^2(1-2k_2u)\Delta u-\tau^ac^2\mathfrak{K}_1*\Delta u_t-\delta\mathfrak{K}_2*\Delta u_{tt}\\ &\quad+2k_3u_t^2=0,\end{aligned} \tag{3.15}$$

with initial (2.10) and boundary (2.11) conditions. The solution satisfies the following estimate:

$$\|u\|^2_{\mathcal{X}^{\mathrm{WB}}}\lesssim_T \|u_0\|^2_{H^2}+\|u_1\|^2_{H^2}+\tau^a\|u_2\|^2_{H^1},$$

where the hidden constant does not depend on τ.

Proof. We check that the conditions of the Banach fixed-point theorem are satisfied for the introduced mapping $\mathcal{T}$. We note that the set $\mathcal{B}^{\mathrm{WB}}$ is non-empty, as the solution of the linear problem with $k_1 = k_2 = k_3 = 0$ belongs to it for small enough (with respect to R) initial data.

To show that $\mathcal{T}(\mathcal{B}^{\mathrm{WB}}) \subset \mathcal{B}^{\mathrm{WB}}$, take $u^* \in \mathcal{B}^{\mathrm{WB}} \subset \mathcal{X}^{\mathrm{WB}}$. Then the smoothness assumptions on $\mathfrak{a}$ and $\mathfrak{b}$ in Proposition 3.1 are fulfilled and the smallness assumption on $\mathfrak{a}$ is fulfilled by reducing $R > 0$. The non-degeneracy condition on $\mathfrak{a}$ is fulfilled for small enough R. To see this, note that due to the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we have

$$\|2k_1 u^*\|_{L^\infty(L^\infty)} \leq C(\Omega, T)|k_1|\|u^*\|_{L^\infty(H^2)} \leq C(\Omega, T)|k_1|R.$$

Thus $R > 0$ should be chosen so that

$$1 - C(\Omega, T)|k_1|R \geq \underline{\mathfrak{a}} > 0.$$

Furthermore, we have

$$\|\mathcal{N}(u_t^*)\|_{W^{1,1}(L^2)} \lesssim \|u_t^*\|^2_{L^2(L^4)} + \|u_t^*\|_{L^\infty(L^4)}\|u_{tt}^*\|_{L^1(L^4)} \leq C(\Omega, T)R^2,$$

where we have relied on the embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$. By employing Proposition 3.1, we have

$$\begin{aligned}\|u\|_{\mathcal{X}^{\mathrm{WB}}} \leq C_1 \exp\left(C_2 T(1 + \|\mathfrak{b}_t\|^2_{L^1(L^\infty)})\right) \Big(&\|u_0\|^2_{H^2} + \|u_1\|^2_{H^2} + \tau^a\|\nabla u_2\|^2_{L^2} \\ &+ \|\mathcal{N}(u_t^*)\|^2_{W^{1,1}(L^2)}\Big).\end{aligned}$$

Since

$$\|\mathfrak{b}_t\|_{L^1(L^\infty)} \lesssim \|u_t^*\|_{L^1(L^\infty)} \lesssim T\|u_t^*\|_{L^\infty(H^2)} \lesssim TR,$$

we conclude that

$$\|u\|_{\mathcal{X}^{\mathrm{WB}}} \leq C_1 \exp\left(C_2 T(1 + T^2R^2)\right)(r^2 + CR^4).$$

Therefore, $u \in \mathcal{B}^{\mathrm{WB}}$ for sufficiently small radius R and data size r.

To prove strict contractivity, let $\mathcal{T}u^* = u$ and $\mathcal{T}v^* = v$. Denote $\phi = u - v$ and $\phi^* = u^* - v^*$. Then ϕ solves

$$\begin{aligned}&\tau^a \mathfrak{K}_1 * \phi_{ttt} + \mathfrak{a}(u^*)\phi_{tt} - c^2\mathfrak{b}(u^*)\Delta\phi - \tau^a c^2 \mathfrak{K}_1 * \Delta\phi_t - \delta\mathfrak{K}_2 * \Delta\phi_{tt} \\ =& -2k_1\phi^* v_{tt} - 2k_2\phi^*\Delta v - 2k_3\phi_t^*(u_t^* + v_t^*) := f.\end{aligned}$$

Note that we cannot prove contractivity with respect to the $\|\cdot\|_{\mathcal{X}^{\mathrm{WB}}}$ norm by exploiting the linear bound in Proposition 3.1, as the right-hand side of this equation does not belong to $W^{1,1}(0, T; L^2(\Omega))$ due to the first term on the right-hand side in the last line (we do not have control over the time derivative of v_{tt}). Similarly to the proof of uniqueness for the linear problem (see (3.13)), we can test this equation with ϕ_{tt} to obtain

$$\begin{aligned}&\int_0^t \|\phi_{tt}\|^2_{L^2}\,\mathrm{d}s + \|\nabla\phi_t\|^2_{L^\infty(L^2)} \\ \lesssim& \|f\|^2_{L^2(L^2)} + \left|-c^2(\mathfrak{b}(u^*)\nabla\phi(t), \nabla\phi_t(t))_{L^2} + 2k_2c^2(\phi(t)\nabla u^*, \nabla\phi_t(t))_{L^2}\right| \\ &+ \left|-\int_0^t \left\{2k_2c^2(u_t^*\nabla\phi + \mathfrak{b}(u^*)\nabla\phi_t, \nabla\phi_t)_{L^2} + 2k_2c^2(u_t\nabla u^* + \phi\nabla u_t^*, \nabla\phi_t)_{L^2}\right\}\mathrm{d}s\right|.\end{aligned}$$

The terms on the right-hand side not containing f can be further estimated as in the proof of uniqueness for the linear problem. We additionally have

$$\begin{aligned}\|f\|_{L^2(L^2)} \lesssim& \|\phi^*\|_{L^\infty(L^4)}\|v_{tt}\|_{L^2(L^4)} + \|\phi^*\|_{L^\infty(L^4)}\|\Delta v\|_{L^\infty(L^2)} \\ &+ \|\phi_t^*\|_{L^2(L^4)}\|u_t^* + v_t^*\|_{L^\infty(L^4)}.\end{aligned}$$

Since $v \in \mathcal{B}^{\mathrm{WB}}$, we have $\|v_{tt}\|_{L^2(H^1(\Omega))} \lesssim \|v\|_{\mathcal{X}^{\mathrm{WB}}} \lesssim R$. We can reason similarly for the Δv and $u_t^* + v_t^*$ terms above. Altogether, it follows that

$$\int_0^t \|\phi_{tt}\|^2_{L^2}\,\mathrm{d}s + \|\nabla\phi\|^2_{W^{1,\infty}(L^2)} \lesssim R^2 \left(\int_0^t \|\phi^*_{tt}\|^2_{L^2}\,\mathrm{d}s + \|\nabla\phi^*\|^2_{W^{1,\infty}(L^2)}\right).$$

Therefore, we can obtain strict contractivity of the mapping $\mathcal{T}$ with respect to the norm of $W^{1,\infty}(0,T; H_0^1(\Omega)) \cap H^2(0,T;L^2(\Omega))$ by reducing R. The closedness of $\mathcal{B}^{\mathrm{WB}}$ with respect to this norm can be argued similarly to, e.g., [40, Theorem 4.1] to conclude the proof. □

We note that this uniform well-posedness result generalizes [16, Theorem 4.1], where the (fJMGT III) equation with Westervelt nonlinearities is considered, to Eq. (3.15) with Westervelt–Blackstock nonlinearities and general kernels satisfying the assumptions of this section.

3.3. Weak singular limit with Westervelt–Blackstock nonlinearities

Equipped with the previous uniform analysis, we are now ready to discuss the limiting behavior of these equations as $\tau \searrow 0$. Let $\tau \in (0, \bar{\tau}]$. Under the assumptions of Theorem 3.1 with the uniform smallness condition

$$\|u_0^\tau\|^2_{H^2} + \|u_1^\tau\|^2_{H^2} + \bar{\tau}^a \|u_2^\tau\|^2_{H^1} \le r^2,$$

let u^τ be the solution of

$$\begin{cases} \tau^a \mathfrak{K}_1 * u^\tau_{ttt} + \mathfrak{a}(u^\tau) u^\tau_{tt} - c^2 \mathfrak{b}(u^\tau)\Delta u^\tau - \tau^a c^2 \mathfrak{K}_1 * \Delta u^\tau_t \\ \qquad\qquad - \delta \mathfrak{K}_2 * \Delta u^\tau_{tt} + \mathcal{N}(u^\tau_t) = 0 \quad \text{in } \Omega \times (0,T), \\ u^\tau|_{\partial\Omega} = 0, \\ (u^\tau, u^\tau_t, u^\tau_{tt})|_{t=0} = (u_0^\tau, u_1^\tau, u_2^\tau). \end{cases} \tag{3.16}$$

Based on the previous analysis and the obtained uniform bounds with respect to the thermal relaxation time τ, there exists a subsequence, not relabeled, such that

$$\begin{aligned} u^\tau &\rightharpoonup u && \text{weakly-}\star \text{ in } L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \\ u^\tau_t &\rightharpoonup u_t && \text{weakly-}\star \text{ in } L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \\ u^\tau_{tt} &\rightharpoonup u_{tt} && \text{weakly} \quad \text{in } L^2(0,T;H_0^1(\Omega)) \end{aligned} \tag{3.17}$$

as $\tau \searrow 0$. By the Aubin–Lions–Simon lemma (see [41, Corollary 4]), this further implies that

$$\begin{aligned} u^\tau &\longrightarrow u && \text{strongly in } C([0,T];H_0^1(\Omega)), \\ u^\tau_t &\longrightarrow u_t && \text{strongly in } C([0,T];H_0^1(\Omega)). \end{aligned} \tag{3.18}$$

Therefore, we know that

$$\begin{aligned} u_0^\tau = u^\tau(0) &\longrightarrow u(0) := u_0 && \text{strongly in } H_0^1(\Omega), \\ u_1^\tau = u^\tau_t(0) &\longrightarrow u_t(0) := u_1 && \text{strongly in } H_0^1(\Omega). \end{aligned} \tag{3.19}$$

On top of this, by (3.17), we have

$$\mathfrak{K}_2 * \nabla u^\tau_{tt} \rightharpoonup \mathfrak{K}_2 * \nabla u_{tt} \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \tag{3.20}$$

We next prove that u satisfies the limiting problem. Let $v \in C_0^\infty([0,T];C_0^\infty(\Omega))$. We have with $\bar{u} = u - u^\tau$:

$$\begin{aligned} &\int_0^T \int_\Omega \mathfrak{a}(u) u_{tt} v \,\mathrm{d}x\mathrm{d}s - c^2 \int_0^T \int_\Omega \mathfrak{b}(u)\Delta u v \,\mathrm{d}x\mathrm{d}s + \delta \int_0^T \int_\Omega \mathfrak{K}_2 * \nabla u_{tt} \cdot \nabla v \,\mathrm{d}x\mathrm{d}s \\ &+ \int_0^T \int_\Omega \mathcal{N}(u_t) v \,\mathrm{d}x\mathrm{d}s = \mathrm{rhs} \end{aligned}$$

with the right-hand side

$$
\begin{aligned}
\text{rhs} := & \int_0^T \int_\Omega \mathfrak{a}(u)\bar{u}_{tt} v \,\mathrm{d}x\mathrm{d}s - c^2 \int_0^T \int_\Omega \mathfrak{b}(u)\Delta \bar{u} v \,\mathrm{d}x\mathrm{d}s + \delta \int_0^T \int_\Omega \mathfrak{K}_2 * \nabla \bar{u}_{tt} \cdot \nabla v \,\mathrm{d}x\mathrm{d}s \\
& - \int_0^T \int_\Omega \tau^a \mathfrak{K}_1 * u^\tau_{ttt} v \,\mathrm{d}x\mathrm{d}s + \tau^a c^2 \int_0^T \int_\Omega \mathfrak{K}_1 * \Delta u^\tau_t v \,\mathrm{d}x\mathrm{d}s \\
& - \int_0^T \int_\Omega (\mathfrak{a}(u^\tau) - \mathfrak{a}(u)) u^\tau_{tt} v \,\mathrm{d}x\mathrm{d}s + c^2 \int_0^T \int_\Omega (\mathfrak{b}(u^\tau) - \mathfrak{b}(u)) \Delta u^\tau v \,\mathrm{d}x\mathrm{d}s \\
& - \int_0^T \int_\Omega (\mathcal{N}(u^\tau_t) - \mathcal{N}(u_t)) v \,\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

We wish to prove that rhs tends to zero as $\tau \searrow 0$. To this end, we rely on the established weak convergence. We first discuss the terms involving the kernels. Note that

$$
\begin{aligned}
& \int_0^T \int_\Omega \tau^a \mathfrak{K}_1 * u^\tau_{ttt} v \,\mathrm{d}x\mathrm{d}s \\
= & -\tau^a \int_0^T \int_\Omega \mathfrak{K}_1 * u^\tau_{tt}\, v_t \,\mathrm{d}x\mathrm{d}s - \tau^a \int_0^T \int_\Omega \mathfrak{K}_1(s) u_2\, v \,\mathrm{d}x\mathrm{d}s \to 0 \quad \text{as } \tau \searrow 0.
\end{aligned}
$$

Above we have relied on the uniform bound on

$$
\left| \int_0^T \int_\Omega \mathfrak{K}_1 * u^\tau_{tt}\, v_t \,\mathrm{d}x\mathrm{d}s \right| \leq \|\mathfrak{K}_1\|_{\mathcal{M}} \|u^\tau_{tt}\|_{L^2(L^2)} \|v_t\|_{L^2(L^2)}.
$$

Similarly, we have

$$
\tau^a c^2 \int_0^T \int_\Omega \mathfrak{K}_1 * \Delta u^\tau_t v \,\mathrm{d}x\mathrm{d}s \to 0 \quad \text{as } \tau \searrow 0.
$$

By the limit in (3.20), it also follows that

$$
\delta \int_0^T \int_\Omega \mathfrak{K}_2 * \nabla \bar{u}_{tt} \cdot \nabla v \,\mathrm{d}x\mathrm{d}s \to 0 \quad \text{as } \tau \searrow 0.
$$

By relying on (3.17) and the equivalence of the norms $\|\mathfrak{a}(u)v\|_{L^2}$, $\|v\|_{L^2}$, and $\|\mathfrak{b}(u)v\|_{L^2}$ (under the assumptions of Theorem 3.1), we can conclude that

$$
\begin{aligned}
\mathfrak{a}(u)u^\tau_{tt} & \rightharpoonup \mathfrak{a}(u)u_{tt} && \text{weakly in } L^2(0,T;L^2(\Omega)), \\
\mathfrak{b}(u)\Delta u^\tau & \rightharpoonup \mathfrak{b}(u)\Delta u && \text{weakly in } L^2(0,T;L^2(\Omega)),
\end{aligned}
$$

and thus

$$
\int_0^T \int_\Omega \mathfrak{a}(u)\bar{u}_{tt} v \,\mathrm{d}x\mathrm{d}s - c^2 \int_0^T \int_\Omega \mathfrak{b}(u)\Delta \bar{u} v \,\mathrm{d}x\mathrm{d}s \to 0 \quad \text{as } \tau \searrow 0.
$$

Additionally, we have

$$
\begin{aligned}
& - \int_0^T \int_\Omega (\mathfrak{a}(u^\tau) - \mathfrak{a}(u)) u^\tau_{tt} v \,\mathrm{d}x\mathrm{d}s + c^2 \int_0^T \int_\Omega (\mathfrak{b}(u^\tau) - \mathfrak{b}(u)) \Delta u^\tau v \,\mathrm{d}x\mathrm{d}s \\
= & -2 \int_0^T \int_\Omega k_1 (u^\tau - u) u^\tau_{tt} v \,\mathrm{d}x\mathrm{d}s - 2c^2 k_2 \int_0^T \int_\Omega (u^\tau - u) \Delta u^\tau v \,\mathrm{d}x\mathrm{d}s,
\end{aligned}
$$

which tends to zero as well thanks to (3.17) and (3.18). Finally,

$$
\int_0^T \int_\Omega (\mathcal{N}(u^\tau_t) - \mathcal{N}(u_t)) v \,\mathrm{d}x\mathrm{d}s = 2k_3 \int_0^T \int_\Omega \bar{u}_t (u^\tau_t + u_t) v \,\mathrm{d}x\mathrm{d}s,
$$

which also tends to zero on account of again (3.17) and the uniform bounds in (3.17).

We have thus proven that there is a subsequence of $\{u^\tau\}_{\tau\in(0,\bar{\tau}]}$ that converges to a solution $u \in \mathcal{X}^{\mathrm{WB}}$ of the following problem:

$$\begin{cases}(1+2k_1u)u_{tt} - c^2(1-2k_2u)\Delta u - \delta\mathfrak{K}_2 * \Delta u_{tt} + 2k_3u_t^2 = 0,\\ u|_{\partial\Omega} = 0,\\ (u,u_t)|_{t=0} = (u_0,u_1).\end{cases} \tag{3.21}$$

The initial conditions (u_0, u_1) are obtained in the limit of (u_0^τ, u_1^τ) as $\tau \searrow 0$ in the sense of (3.19). The uniqueness of this solution in $\mathcal{X}^{\mathrm{WB}}$ can be shown by noting that the difference $\bar{u} = u^{(1)} - u^{(2)}$ of two solutions would have to satisfy

$$\begin{aligned}&(1+2k_1u^{(1)})\bar{u}_{tt} - c^2(1-2k_2u^{(1)})\Delta\bar{u} - \delta\mathfrak{K}_2 * \Delta\bar{u}_{tt}\\ =& -2k_1\bar{u}u_{tt}^{(2)} - 2k_2c^2\bar{u}\Delta u - 2k_3\bar{u}_t(u_t^{(1)} + u_t^{(2)})\end{aligned}$$

with zero initial data and then testing this problem by $\bar{u}_{tt}$, similarly to the proof of contractivity in Theorem 3.1. Thus by a subsequence-subsequence argument, the whole sequence $\{u^\tau\}_{\tau\in(0,\bar{\tau}]}$ converges to u. Altogether, we arrive at the following result.

Theorem 3.2. *Let the assumptions of Theorem 3.1 hold for (3.16). Then the family $\{u^\tau\}_{\tau\in(0,\bar{\tau}]}$ of solutions to (3.16) converges as $\tau \searrow 0$ in the sense of (3.17), (3.18) to the solution $u \in \mathcal{X}^{\mathrm{WB}}$ of (3.21).*

Note that as a by-product of this analysis, we obtain the unique solvability of the limiting problem given in (3.21) for small data in $\big(H^2(\Omega)\cap H_0^1(\Omega)\big) \times \big(H^2(\Omega)\cap H_0^1(\Omega)\big)$.

3.4. Well-posedness of a linearized Kuznetsov–Blackstock problem

We next show how the previous arguments can be adapted to the equations with Kuznetsov–Blackstock nonlinearities. That is, we consider again the equation

$$\tau^a\mathfrak{K}_1 * u_{ttt} + \mathfrak{a}u_{tt} - c^2\mathfrak{b}\Delta u - \tau^a c^2\mathfrak{K}_1 * \Delta u_t - \delta\mathfrak{K}_2 * \Delta u_{tt} = f, \tag{2.12}$$

but now we have in mind that

- $\mathfrak{a} = 1 + 2k_1u_t,\ \ \mathfrak{b} = 1 - 2k_2u_t$, and $f = -2k_3\nabla u\cdot\nabla u_t$.

As mentioned before, we need more regularity of the variable coefficients and data in this setting. In particular, we need

$$\mathfrak{a} \in C([0,T]; H^2(\Omega)\cap H_0^1(\Omega)), \quad \mathfrak{b} \in H^1(0,T; H^2(\Omega)\cap H_0^1(\Omega)). \tag{3.22}$$

We still assume that $\mathfrak{a}$ does not degenerate so that there exist $\underline{\mathfrak{a}}$, $\overline{\mathfrak{a}} > 0$, independent of τ, such that (3.3) holds. Furthermore, we need smallness in the following sense: there exists as small enough $m > 0$, independent of τ, such that

$$\|\Delta\mathfrak{a}\|_{L^\infty(L^2)} \leq m. \tag{3.23}$$

Let also $f \in W^{1,1}(0,T; H^1(\Omega))$. Now the τ-independent solution space is

$$\mathcal{X}^{\mathrm{KB}} = \{u \in W^{1,\infty}(0,T; H^3_\diamondsuit(\Omega)) :\ u_{tt} \in L^2(0,T; H^2(\Omega)\cap H_0^1(\Omega))\}$$

with

$$H^3_\diamondsuit(\Omega) = \{u \in H^3(\Omega) : u_{|\partial\Omega} = \Delta u_{|\partial\Omega} = 0\}.$$

The analysis of this linearized problem can be conducted as before through a Faedo–Galerkin procedure based on the smooth eigenfunctions of the Dirichlet–Laplace operator; cf. Appendix A. In this way, we can rely on $\Delta u^{(n)} = \Delta u_t^{(n)} = \Delta u_{tt}^{(n)} =$ on $\partial\Omega$. In what follows, we only discuss the testing procedure as the other details follow analogously to the Westervelt–Blackstock case. For ease of notation, we drop the superscript (n). By testing the (semi-discretized) problem with $\Delta^2 u_{tt}$ and employing coercivity assumptions $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$ on the kernels, we obtain

$$\begin{aligned}
&\int_0^t \|\sqrt{\mathfrak{a}}\Delta u_{tt}\|^2_{L^2} + \underline{c}\|\nabla\Delta u_t\|^2_{L^\infty(L^2)}\\
\leq &-c^2(\nabla[\mathfrak{b}\Delta u], \nabla\Delta u_t)_{L^2}\Big|_0^t + \overline{C}\|\nabla\Delta u_1\|^2_{L^2} + C\tau^a\|\Delta u_2\|^2_{L^2}\\
&+\int_0^t \Big(-(u_{tt}\,\Delta\mathfrak{a} + 2\nabla u_{tt}\cdot\nabla\mathfrak{a}, \Delta u_{tt})_{L^2} + c^2(\nabla[\mathfrak{b}\Delta u]_t, \nabla\Delta u_t)_{L^2}\Big)\,\mathrm{d}s\\
&+\int_0^t (f, \Delta^2 u_{tt})_{L^2}\,\mathrm{d}s,
\end{aligned} \tag{3.24}$$

where we have used the identity $\mathfrak{a}\Delta u_{tt} - \Delta[\mathfrak{a}u_{tt}] = -u_{tt}\,\Delta\mathfrak{a} - 2\nabla u_{tt}\cdot\nabla\mathfrak{a}$, and integrated by parts to obtain

$$\int_0^t (\mathfrak{b}\Delta u, \Delta^2 u_{tt})_{L^2}\,\mathrm{d}s = \int_0^t (\nabla[\mathfrak{b}\Delta u]_t, \nabla\Delta u_t)_{L^2}\,\mathrm{d}s - (\nabla[\mathfrak{b}\Delta u], \nabla\Delta u_t)_{L^2}\Big|_0^t.$$

The terms on the right hand side of (3.24) can be further estimated as follows. First, using the embeddings $H^1(\Omega)\hookrightarrow L^4(\Omega)$ and $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$, we have

$$\begin{aligned}
-(\nabla[\mathfrak{b}(t)\Delta u(t)], \nabla\Delta u_t(t))_{L^2} &\leq (\|\nabla\mathfrak{b}\Delta u\|_{L^\infty(L^2)} + \|\mathfrak{b}\nabla\Delta u\|_{L^\infty(L^2)})\|\nabla\Delta u_t\|_{L^\infty(L^2)}\\
&\lesssim \Big(\|\Delta\mathfrak{b}\|_{L^\infty(L^2)} + 1\Big)\|\nabla\Delta u\|_{L^\infty(L^2)}\|\nabla\Delta u_t\|_{L^\infty(L^2)},
\end{aligned}$$

where we have also used the fact that $\mathfrak{b}-1$ vanishes on the boundary and that $\Delta(\mathfrak{b}-1) = \Delta\mathfrak{b}$. We can employ

$$\|\nabla\Delta u\|_{L^\infty(L^2)} \leq \sqrt{T}\|\nabla\Delta u_t\|_{L^2(L^2)} + \|\nabla\Delta u_0\|_{L^2}$$

to further bound the u term. We also have

$$\begin{aligned}
\int_0^t -(u_{tt}\,\Delta\mathfrak{a} + \nabla u_{tt}\cdot\nabla\mathfrak{a}, \Delta u_{tt})_{L^2}\,\mathrm{d}s &\leq \|u_{tt}\,\Delta\mathfrak{a} + \nabla u_{tt}\cdot\nabla\mathfrak{a}\|_{L^2(L^2)}\|\Delta u_{tt}\|_{L^2(L^2)}\\
&\lesssim \|\Delta\mathfrak{a}\|_{L^\infty(L^2)}\|\Delta u_{tt}\|^2_{L^2(L^2)} \lesssim m\|\Delta u_{tt}\|^2_{L^2(L^2)}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\int_0^t (\nabla[\mathfrak{b}\Delta u]_t, \nabla\Delta u_t)_{L^2}\,\mathrm{d}s\\
\lesssim &\|\Delta\mathfrak{b}_t\|_{L^2(L^2)}\|\nabla\Delta u\|_{L^\infty(L^2)}\|\nabla\Delta u_t\|_{L^2(L^2)} + \Big(\|\Delta\mathfrak{b}\|_{L^\infty(L^2)} + 1\Big)\|\nabla\Delta u_t\|^2_{L^2(L^2)}\\
\lesssim &\|\Delta\mathfrak{b}_t\|_{L^2(L^2)}(\sqrt{T}\|\nabla\Delta u_t\|_{L^2(L^2)} + \|\nabla\Delta u_0\|_{L^2})\|\nabla\Delta u_t\|_{L^2(L^2)}\\
&+\Big(\|\Delta\mathfrak{b}\|_{L^\infty(L^2)} + 1\Big)\|\nabla\Delta u_t\|^2_{L^2(L^2)}.
\end{aligned}$$

We can estimate the term containing f using integration by parts as follows:

$$\begin{aligned}
\int_0^t (f, \Delta^2 u_{tt})_{L^2}\,\mathrm{d}s &= (f, \Delta^2 u_t)_{L^2}\Big|_0^t - \int_0^t (f_t, \Delta^2 u_t)_{L^2}\,\mathrm{d}s\\
&= -(\nabla f, \nabla\Delta u_t)_{L^2}\Big|_0^t + \int_{\partial\Omega} f\nabla\Delta u_t\cdot n\,\mathrm{d}\Gamma\Big|_0^t - \int_0^t (\nabla f_t, \nabla\Delta u_t)_{L^2}\,\mathrm{d}s\\
&\quad + \int_0^t\int_{\partial\Omega} f_t\nabla\Delta u_t\cdot n\,\mathrm{d}\Gamma\,\mathrm{d}s\\
&\lesssim \|f\|^2_{W^{1,1}(H^1)} + \varepsilon\|\Delta u_t\|^2_{L^\infty(H^1)} + \|\Delta u_1\|^2_{H^1}
\end{aligned}$$

for any $\varepsilon > 0$. Employing the above bounds to further estimate the right-hand side terms in (3.24) and assuming that ε and $\|\Delta\mathfrak{a}\|_{L^\infty(L^2)}$ are small enough (independently of τ) thus leads to

$$\int_0^t \|\Delta u_{tt}\|^2_{L^2} + \|\nabla\Delta u_t\|^2_{L^\infty(L^2)} \lesssim_T \|\nabla\Delta u_0\|^2_{L^2} + \|\Delta u_1\|^2_{H^1} + \tau^a\|\Delta u_2\|^2_{L^2} + \|f\|^2_{W^{1,1}(H^1)}.$$

We can further use

$$\|\nabla\Delta u\|_{L^\infty(L^2)} \leq T\|\nabla\Delta u_t\|_{L^\infty(L^2)} + \|\nabla\Delta u_0\|_{L^2}$$

and

$$\|\Delta u\|_{L^\infty(L^2)} \lesssim_T \|\Delta u_{tt}\|_{L^2(L^2)} + \|\Delta u_0\|_{L^2} + \|\Delta u_1\|_{L^2}$$

to obtain a bound on $\|u\|_{\mathcal{X}^{\mathrm{KB}}}$:

$$\|u\|_{\mathcal{X}^{\mathrm{KB}}} \lesssim_T \|u_0\|^2_{H^3} + \|u_1\|^2_{H^3} + \tau^a\|u_2\|^2_{H^2} + \|f\|^2_{W^{1,1}(H^1)}, \tag{3.25}$$

where the hidden constant has the form

$$C = C_1 \exp\left((1 + \|\Delta\mathfrak{b}_t\|_{L^2(L^2)} + \|\Delta\mathfrak{b}\|_{L^\infty(L^2)})C_2 T\right).$$

This bound for the linearized problem forms the foundation for the well-posedness and limiting analysis in the Kuznetsov–Blackstock case.

3.5. Uniform well-posedness with Kuznetsov–Blackstock-type nonlinearities

The next step is as before to set up a fixed-point mapping $\mathcal{T} : \mathcal{B}^{\mathrm{KB}} \ni u^* \mapsto u$, where u solves the linear equation in (3.4) with

$$\mathfrak{a}(u_t^*) = 1 + 2k_1 u_t^*, \quad \mathfrak{b}(u_t^*) = 1 - 2k_2 u_t^*, \;\; f = -\mathcal{N}(\nabla u^*, u_t^*) = -2k_3 \nabla u^* \cdot \nabla u_t^*,$$

and u^* is taken from the ball

$$\mathcal{B}^{\mathrm{KB}} = \left\{u^* \in \mathcal{X}^{\mathrm{KB}} : \|u^*\|_{\mathcal{X}^{\mathrm{KB}}} \leq R, \; (u^*, u_t^*, u_{tt}^*)|_{t=0} = (u_0, u_1, u_2)\right\}. \tag{3.26}$$

Theorem 3.3. *Let $T > 0$ and $\tau \in (0, \bar{\tau}]$. Let assumptions (2.13) and $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ on the kernels $\mathfrak{K}_1$ and $\mathfrak{K}_2$ hold. Let*

$$(u_0, u_1, u_2) \in H^3_\diamondsuit(\Omega) \times H^3_\diamondsuit(\Omega) \times \left(H^2(\Omega) \cap H^1_0(\Omega)\right).$$

There exists $r = r(T) > 0$, independent of τ, such that if

$$\|u_0\|^2_{H^3} + \|u_1\|^2_{H^3} + \bar{\tau}^a\|u_2\|^2_{H^2} \leq r^2,$$

then there is a unique solution $u \in \mathcal{B}^{\mathrm{KB}}$, such that $\tau^a\mathfrak{K}_1 * u_{ttt} \in L^2(0, T; L^2(\Omega))$, of

$$\begin{aligned}\tau^a\mathfrak{K}_1 * u_{ttt} + (1 + 2k_1 u_t)u_{tt} - c^2(1 - 2k_2 u_t)\Delta u - \tau^a c^2\mathfrak{K}_1 * \Delta u_t - \delta\mathfrak{K}_2 * \Delta u_{tt} \\ +2k_3\nabla u \cdot \nabla u_t = 0\end{aligned} \tag{3.27}$$

with initial (2.10) and boundary (2.11) data. The solution satisfies

$$\|u\|^2_{\mathcal{X}^{\mathrm{KB}}} \lesssim_T \|u_0\|^2_{H^3} + \|u_1\|^2_{H^3} + \tau^a\|u_2\|^2_{H^2},$$

where the hidden constant does not depend on τ.

Proof. The proof follows by employing the Banach fixed-point theorem on $\mathcal{T}$, analogously to the proof of Theorem 3.1. We provide the details in Appendix B. □

This uniform well-posedness result generalizes [16, Theorem 6.1], where the (fJMGT III) equation with Kuznetsov nonlinearities is considered, to Eq. (3.27) with Kuznetsov–Blackstock nonlinearities and kernels satisfying the assumptions of this section.

3.6. Weak singular limit with Kuznetsov–Blackstock type nonlinearities

We next discuss the limiting behavior of the equations with Kuznetsov–Blackstock nonlinearities. Let $\tau \in (0, \bar{\tau}]$. Under the assumptions of Theorem 3.3, with

$$\|u_0^\tau\|^2_{H^3} + \|u_1^\tau\|^2_{H^3} + \bar{\tau}^a \|u_2^\tau\|^2_{H^2} \leq r^2,$$

consider

$$\begin{cases} \tau^a \mathfrak{K}_1 * u_{ttt}^\tau + (1 + 2k_1 u_t^\tau) u_{tt}^\tau - c^2(1 - 2k_2 u_t^\tau)\Delta u^\tau - \tau^a c^2 \mathfrak{K}_1 * \Delta u_t^\tau \\ \quad - \delta \mathfrak{K}_2 * \Delta u_{tt}^\tau + 2k_3 \nabla u^\tau \cdot \nabla u_t^\tau = 0, \\ u^\tau|_{\partial\Omega} = 0, \\ (u^\tau, u_t^\tau, u_{tt}^\tau)|_{t=0} = (u_0^\tau, u_1^\tau, u_2^\tau). \end{cases} \tag{3.28}$$

From the previous analysis we knowc that a unique solution of this problem exists in $\mathcal{B}^{\mathrm{KB}}$. Therefore, there exists a subsequence, not relabeled, such that

$$\begin{aligned} u^\tau &\rightharpoonup u && \text{weakly-}\star \text{ in } L^\infty(0,T; H^3_\diamondsuit(\Omega)), \\ u_t^\tau &\rightharpoonup u_t && \text{weakly-}\star \text{ in } L^\infty(0,T; H^3_\diamondsuit(\Omega)), \\ u_{tt}^\tau &\rightharpoonup u_{tt} && \text{weakly} \quad \text{in } L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)). \end{aligned} \tag{3.29}$$

Similarly to before, using the Aubin–Lions–Simon lemma, this further implies

$$\begin{aligned} u^\tau &\longrightarrow u && \text{strongly in } C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)), \\ u_t^\tau &\longrightarrow u_t && \text{strongly in } C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)), \end{aligned} \tag{3.30}$$

and therefore,

$$\begin{aligned} u_0^\tau = u^\tau(0) &\longrightarrow u(0) := u_0 && \text{strongly in } H^2(\Omega) \cap H_0^1(\Omega), \\ u_1^\tau = u_t^\tau(0) &\longrightarrow u_t(0) := u_1 && \text{strongly in } H^2(\Omega) \cap H_0^1(\Omega). \end{aligned} \tag{3.31}$$

It remains to prove that u solves the limiting problem. Let $v \in C_0^\infty([0,T]; C_0^\infty(\Omega))$. We have

$$\begin{aligned} &\int_0^T \int_\Omega \mathfrak{a}(u_t) u_{tt} v \,\mathrm{d}x\mathrm{d}s - c^2 \int_0^T \int_\Omega \mathfrak{b}(u_t) \Delta u v \,\mathrm{d}x\mathrm{d}s + \delta \int_0^T \int_\Omega \mathfrak{K}_2 * \nabla u_{tt} \cdot \nabla v \,\mathrm{d}x\mathrm{d}s \\ &+ \int_0^T \int_\Omega \mathcal{N}(\nabla u, \nabla u_t) v \,\mathrm{d}x\mathrm{d}s = \mathrm{rhs}, \end{aligned}$$

where, with $\bar{u} = u - u^\tau$, the right-hand side is

$$\begin{aligned} \mathrm{rhs} := &\int_0^T \int_\Omega \mathfrak{a}(u_t) \bar{u}_{tt} v \,\mathrm{d}x\mathrm{d}s - c^2 \int_0^T \int_\Omega \mathfrak{b}(u_t) \Delta \bar{u} v \,\mathrm{d}x\mathrm{d}s + \delta \int_0^T \int_\Omega \mathfrak{K}_2 * \nabla \bar{u}_{tt} \cdot \nabla v \,\mathrm{d}x\mathrm{d}s \\ &- \int_0^T \int_\Omega \tau^a \mathfrak{K}_1 * u_{ttt}^\tau v \,\mathrm{d}x\mathrm{d}s + \tau^a c^2 \int_0^T \int_\Omega \mathfrak{K}_1 * \Delta u_t^\tau v \,\mathrm{d}x\mathrm{d}s \\ &- \int_0^T \int_\Omega (\mathfrak{a}(u_t^\tau) - \mathfrak{a}(u_t)) u_{tt}^\tau v \,\mathrm{d}x\mathrm{d}s + c^2 \int_0^T \int_\Omega (\mathfrak{b}(u_t^\tau) - \mathfrak{b}(u_t)) \Delta u^\tau v \,\mathrm{d}x\mathrm{d}s - \\ &- \int_0^T \int_\Omega (\mathcal{N}(\nabla u^\tau, \nabla u_t^\tau) - \mathcal{N}(\nabla u, \nabla u_t)) v \,\mathrm{d}x\mathrm{d}s. \end{aligned}$$

We should prove that rhs tends to zero as $\tau \searrow 0$. We only comment here on how to tackle the $\mathfrak{a}$, $\mathfrak{b}$ and $\mathcal{N}$ terms; the other terms can be treated as in Section 3.3. By relying on the equivalence of the norms $\|\mathfrak{a}(u_t)v\|_{L^2}$, $\|v\|_{L^2}$, and $\|\mathfrak{b}(u_t)v\|_{L^2}$ (under the assumptions of Theorem 3.3), we can conclude that

$$\int_0^T \int_\Omega \mathfrak{a}(u_t) \bar{u}_{tt} v \,\mathrm{d}x\mathrm{d}s + c^2 \int_0^T \int_\Omega \mathfrak{b}(u_t) \Delta \bar{u} v \,\mathrm{d}x\mathrm{d}s \to 0 \quad \text{as } \tau \searrow 0.$$

We also have

$$
\begin{aligned}
&-\int_0^T\int_\Omega(\mathfrak{a}(u_t^\tau)-\mathfrak{a}(u_t))u_{tt}^\tau v\,\mathrm{d}x\mathrm{d}s+c^2\int_0^T\int_\Omega(\mathfrak{b}(u_t^\tau)-\mathfrak{b}(u_t))\Delta u^\tau v\,\mathrm{d}x\mathrm{d}s\\
=&-2k_1\int_0^T\int_\Omega \bar{u}_t u_{tt}^\tau v\,\mathrm{d}x\mathrm{d}s-2k_2c^2\int_0^T\int_\Omega \bar{u}_t\Delta u^\tau v\,\mathrm{d}x\mathrm{d}s,
\end{aligned}
$$

which tends to zero as well thanks to (3.29) and (3.30). Finally,

$$
\int_0^T\int_\Omega(\mathcal{N}(\nabla u^\tau,u_t^\tau)-\mathcal{N}(\nabla u,\nabla u_t))v\,\mathrm{d}x\mathrm{d}s=\,2k_3\int_0^T\int_\Omega(\nabla\bar{u}\cdot\nabla u_t^\tau+\nabla u\cdot\nabla\bar{u}_t)v\,\mathrm{d}x\mathrm{d}s,
$$

which tends to zero on account of again (3.30) and the uniform bounds obtained in Theorem 3.3. Uniqueness of solutions for the limiting problem follows by testing the equation solved by the difference $\bar{u}$ of two solutions by $\bar{u}_{tt}$, similarly to the proof of contractivity in Theorem 3.3 given in Appendix B. This allows us to employ a subsequence-subsequence argument on $\{u^\tau\}_{\tau\in(0,\bar{\tau}]}$. Altogether, we arrive at the following result.

Theorem 3.4. *Let the assumptions of Theorem 3.3 hold for problem (3.28). Then the family $\{u^\tau\}_{\tau\in(0,\bar{\tau}]}$ of solutions to (3.28) converges in the sense of (3.29), (3.31) as $\tau\searrow 0$ to the solution $u\in\mathcal{X}^{\mathrm{WB}}$ of*

$$
\begin{cases}
(1+2k_1u_t)u_{tt}-c^2(1-2k_2u_t)\Delta u-\delta\mathfrak{K}_2*\Delta u_{tt}\\
\qquad\qquad\qquad\qquad\qquad\qquad+2k_3\nabla u\cdot\nabla u_t=0\quad\text{in }\Omega\times(0,T),\\
u|_{\partial\Omega}=0,\\
(u,u_t)|_{t=0}=(u_0,u_1).
\end{cases}
\tag{3.32}
$$

As a by-product of the previous analysis, we obtain unique solvability of (3.32) for small data in $H^3_\diamondsuit(\Omega)\times H^3_\diamondsuit(\Omega)$.

4. Testing with u_{tt} and $-\Delta\mathfrak{K}_1*u_{tt}$

We have seen in Section 3 that among the Compte–Metzler laws, testing with $(-\Delta)^\nu u_{tt}$ leads to a τ-uniform bound only in the (fJMGT III) case. Therefore in this section, we will investigate an alternative way of testing by somewhat weakening the time derivative in the multiplier: instead of $-\Delta u_{tt}$ we will use $-\mathfrak{K}_1*\Delta u_{tt}$. Complementary to this, we will also test with u_{tt}.

This testing strategy turns out to be applicable to all three Compte–Metzler laws of interest, with caveats. The price to pay is that we have to restrict ourselves to Westervelt–Blackstock nonlinearities, unless dealing with the (fJMGT I) equation with $\alpha\le 1/2$ (Case I below). In this case we can prove existence of solutions but not uniqueness.

Besides, when $\mathfrak{K}_1\in L^1(0,T)$, for the analysis in this section to go through we need to rewrite the leading term in (1.1) and consider

$$
\tau^a(\mathfrak{K}_1*u_{tt})_t+\mathfrak{a}u_{tt}-c^2\mathfrak{b}\Delta u-\tau^ac^2\mathfrak{K}_1*\Delta u_t-\delta\mathfrak{K}_2*\Delta u_{tt}+\mathcal{N}=\tau^a\mathfrak{K}_1(t)u_{tt}(0),
$$

which will force us to assume $u_2=0$ (and thus have that the right-hand side above is zero).

Assumptions on the two kernels in this section. In this section, we assume that

$$
\text{there exists }\tilde{\mathfrak{K}}_1\in L^1(0,T),\ \text{such that }\mathfrak{K}_1*\tilde{\mathfrak{K}}_1=1.
\tag{$\mathcal{H}_1$}
$$

Note that unlike in Section 3, here the resolvent is only required to be L^1 regular. Furthermore, we need the following coercivity property of the kernel in the leading term:

$$
\int_0^t(\mathfrak{K}_1*y)'(s)\,y(s)\,\mathrm{d}s\ge -C_{\mathfrak{K}_1}|y(0)|^2,\quad y\in X^2_{\mathfrak{K}_1}(0,t),
\tag{$\mathcal{H}_2$}
$$

for some (possibly large, but possibly also vanishing) constant $C_{\mathfrak{K}_1}$. We also assume that

$$\int_0^t (\mathfrak{K}_1 * y)(s)\, y(s)\, \mathrm{d}s \geq \Psi[y](t), \quad y \in L^2(0,t), \tag{$\mathcal{H}_3$}$$

for some functional Ψ. More precisely, we will assume the energy contribution due to Ψ to be quantified by an estimate in some Sobolev norm

$$\int_0^t \Psi[\nabla u_{tt}](s)\, \mathrm{d}s \sim \|\nabla u\|^2_{W^{\sigma,\rho}(0,t;L^2(\Omega))} \tag{$\mathcal{H}_3$ Ψ}$$

for some $\sigma \in [0,2]$ and $\rho \in [2,\infty]$; see Table 4 for the Compte–Metzler laws.

Furthermore, we impose a coercivity condition on $\mathfrak{K}_2$:

$$\int_0^t (\mathfrak{K}_2 * y)(s) y(s)\, \mathrm{d}s \geq 0, \quad y \in L^2(0,t). \tag{$\mathcal{H}_4$}$$

How the two kernels behave relative to each other will also be important. More precisely, besides the above assumptions, the key coercivity property that we assume in this section is the following:

$$\int_0^t (\mathfrak{K}_2 * y)(s)(\mathfrak{K}_1 * y)(s)\, \mathrm{d}s \geq \Phi[y](t), \quad y \in L^2(0,t), \tag{$\mathcal{H}_5$}$$

where Φ is a nonnegative functional satisfying

$$\sup_{t'\in(0,t)} \Phi[y](t') \geq \underline{c}_2 \max\Big\{\|\mathfrak{K}_1 * 1 * y\|^2_{L^2(0,t)}, \|1 * 1 * y\|^2_{L^2(0,t)}\Big\}, \quad y \in L^2(0,t) \tag{4.1}$$

for some (possibly small) $\underline{c}_2 > 0$. Later on, we will assume the energy contribution due to Φ to be quantified by an estimate in some Sobolev norm

$$\sup_{t'\in(0,t)} \Phi[\Delta u_{tt}](t') \sim \|\Delta u\|^2_{W^{s,\mathtt{r}}(0,t;L^2(\Omega))}, \tag{$\mathcal{H}_5$ Φ}$$

for some $s \in [0,2]$ and $\mathtt{r} \in [2,\infty]$; see Table 4. Concerning the further behavior of Φ, we distinguish between two cases.

Case I : "Kernel $\mathfrak{K}_2$ is more singular than $\mathfrak{K}_1$"

This condition constitutes the following: We assume that there exist constants $\underline{c}_3 > 0$ and a sufficiently large $\overline{C}_3$, such that

$$\sup_{t'\in(0,t)} \Phi[y](t') \geq \max\Big\{\underline{c}_3\|\mathfrak{K}_1 * y\|^2_{L^2(0,t)}, \overline{C}_3\|1 * 1 * y\|^2_{L^2(0,t)}\Big\}, \quad y \in L^2(0,t). \tag{$\mathcal{H}_5$ I}$$

Case II : "Kernel $\mathfrak{K}_2$ is less singular than $\mathfrak{K}_1$"

This condition should be understood as follows: We assume that

$$\begin{aligned}&\sup_{t'\in(0,t)} \Phi[y](t')\\ &\geq \overline{C}_3 \max\Big\{\|1 * y\|^2_{L^p(0,t)}, \|1 * 1 * y\|^2_{L^\infty(0,t)}, \|\mathfrak{K}_1 * 1 * y\|^2_{L^\infty(0,t)},\Big\}, \quad y \in L^2(0,t)\end{aligned} \tag{$\mathcal{H}_5$ II}$$

for some $p \in [1,\infty]$ and $\overline{C}_3 > 0$ sufficiently large. Further,

$$\nabla\mathfrak{a} = 0, \quad \mathfrak{a}_t = 0, \quad \mathfrak{b}_t \in L^{\tilde{q}}(0,T;L^\infty(\Omega)), \tag{$\mathcal{H}_5$ II $\mathfrak{a},\mathfrak{b}$}$$

Table 4
Kernels $\mathfrak{K}_1(t) = g_{\alpha_1}(t)$ and $\mathfrak{K}_2 = g_{\alpha_2}$ for different examples of generalized flux laws; values of relevant quantities in the assumptions on kernels; here $\tilde{\iota}$, $\tilde{\tilde{\iota}} > 0$ can be arbitrarily small; in particular, they can be chosen such that $q = p'$ and thus $\tilde{q} = 1$ in all cases.

Flux law	α_1	α_2	p	p'	q	$\tilde{q}$	s	$\mathtt{r}$	σ	ρ
GFE I	$1-\alpha$	α	$< \infty$	$1+\tilde{\iota}$	$\frac{1}{\alpha} - \tilde{\tilde{\iota}}$	1	$\frac{3}{2}$	2	$\frac{3+\alpha}{2}$	2
GFE III	0	α	∞	1	1	1	$2-\frac{\alpha}{2}$	2	2	2
GFE	$1-\alpha$	1	$\frac{2}{1-\alpha}$	$\frac{2}{1+\alpha}$	$\frac{1}{\alpha} - \tilde{\tilde{\iota}}$	1	$1+\frac{\alpha}{2}$	2	$\frac{3+\alpha}{2}$	2

where

$$\begin{aligned} &\text{either } \mathfrak{K}_1 = \delta_0 \text{ and } \tilde{q} = p' = \frac{p}{p-1} \\ &\text{or} \quad \mathfrak{K}_1 \in L^q(0,T), \quad q \le p', \quad \tilde{q} = \frac{p'q}{p'q + q - p'} = \frac{pq}{2pq - q - p} \end{aligned} \tag{$\mathcal{H}_5$ II $\mathfrak{K}_1$}$$

with p as in ($\mathcal{H}_5$ II).

Assumptions on the variable coefficients. We will need the following smoothness and non-degeneracy assumptions:

$$\mathfrak{a},\ \frac{1}{\mathfrak{a}},\ \mathfrak{b} \in L^\infty(0,T;L^\infty(\Omega)), \quad \|\mathfrak{a} - \mathfrak{a}_0\|_{L^\infty(L^\infty)} \le \overline{c}_4 \tag{4.2}$$

for some constants $\mathfrak{a}_0 > 0$ and $\overline{c}_4$, where the latter will be assumed to be small enough (independently of τ); cf. (4.7) for Case I. In Case II we will even need $\mathfrak{a} - \mathfrak{a}_0 = 0$.

4.1. How to verify assumptions on the kernels

As noted before, assumption ($\mathcal{H}_2$) is satisfied for all Compte–Metzler laws considered due to [29, Lemma 5.1]. Conditions ($\mathcal{H}_3$) and ($\mathcal{H}_4$) hold for the three laws considered in this work due to [36, Lemma 2.3] with the choice

$$\mathfrak{K}_1 = g_{\alpha_1} = \frac{1}{\Gamma(\alpha_1)} t^{\alpha_1 - 1}, \qquad \Psi[y] = \cos(\alpha_1\pi/2)\|y\|_{H^{-\alpha_1/2}(0,t)}, \qquad \mathfrak{K}_2 = g_{\alpha_2}$$

with $\alpha_1,\ \alpha_2 \in [0,1]$. Also assumption ($\mathcal{H}_5$) which relates the two kernels is satisfied for all three Compte–Metzler laws of interest. More precisely, we have

$$\Phi[y] = \tilde{c}(\alpha_1,\alpha_2)\ \cos((\alpha_2 - \alpha_1)\pi/2)\|y\|_{H^{-(\alpha_2+\alpha_1)/2}(0,T)}.$$

Condition ($\mathcal{H}_5$ I) means that the damping by $\mathfrak{K}_2$ is at least as strong (in the sense of higher order of differentiation) as the one by $\mathfrak{K}_1$. However, among the Compte–Metzler laws it is only satisfied for the GFE I kernel with $\alpha \le \frac{1}{2}$.

Condition ($\mathcal{H}_5$ II) is satisfied for the GFE I kernel with $\alpha \ge \frac{1}{2}$, as well as for the GFE III and GFE kernels. Largeness of the constant $\overline{C}_3$ in assumptions ($\mathcal{H}_5$ I) and ($\mathcal{H}_5$ II) can be achieved by using Hölder's inequality and making T small enough.

Assumption ($\mathcal{H}_5$ II $\mathfrak{K}_1$) holds for the GFE III kernel and, with any $q \in [1, \frac{1}{1-\alpha})$ for the GFE I and GFE kernels.

The three Compte–Metzler laws and relevant further information are summarized in Table 4, where we have also listed s, $\mathtt{r}$, σ, and ρ, such that ($\mathcal{H}_5$ Φ) holds.

4.2. Well-posedness of the linearized problem

Again, we first present the main ideas of establishing well-posedness of the linearized equation

$$\tau^a(\mathfrak{K}_1 * u_{tt})_t + \mathfrak{a}u_{tt} - c^2\mathfrak{b}\Delta u - \tau^a c^2 \mathfrak{K}_1 * \Delta u_t - \delta\mathfrak{K}_2 * \Delta u_{tt} = f, \tag{4.3}$$

in particular, the energy estimates that will be needed for this purpose in each of the above-mentioned Cases I and II.

Testing with u_{tt}. We start with the low-order energy estimate that is the same for both cases, obtained by testing with u_{tt}. To this end, we interpret Eq. (4.3) as

$$\tau^a(\mathfrak{K}_1 * u_{tt})_t + \mathfrak{a}u_{tt} - \delta\mathfrak{K}_2 * \Delta u_{tt} = \bar{r}$$

with the right-hand side

$$\bar{r} = f + c^2\mathfrak{b}\Delta u + \tau^a c^2\mathfrak{K}_1 * \Delta u_t$$

and test it with u_{tt}. By assumptions $(\mathcal{H}_2)$ and $(\mathcal{H}_4)$, we obtain

$$\int_0^t |\sqrt{\mathfrak{a}}u_{tt}|^2_{L^2}\,\mathrm{d}s \leq \left\|\frac{1}{\sqrt{\mathfrak{a}}}\bar{r}\right\|_{L^2_t(L^2)} \|\sqrt{\mathfrak{a}}u_{tt}\|_{L^2_t(L^2)} + C_{\mathfrak{K}_1}\|u_{tt}(0)\|^2_{L^2}.$$

The r term can be estimated as follows:

$$\left\|\frac{1}{\sqrt{\mathfrak{a}}}\bar{r}\right\|_{L^2_t(L^2)} \leq \left\|\frac{1}{\sqrt{\mathfrak{a}}}\right\|_{L^\infty(L^\infty)} \|\bar{r}\|_{L^2_t(L^2)}$$

with

$$\begin{aligned}\|\bar{r}\|_{L^2_t(L^2)} &\leq \|f\|_{L^2_t(L^2)} + c^2\|\mathfrak{b}\|_{L^\infty(L^\infty)}\|\Delta u\|_{L^2_t(L^2)} + \tau^a c^2\|\mathfrak{K}_1 * \Delta u_t\|_{L^2_t(L^2)}\\ &\leq \|f\|_{L^2_t(L^2)} + c^2\|\mathfrak{b}\|_{L^\infty(L^\infty)}(t^{1/2}\|\Delta u(0)\|_{L^2} + t^{3/2}\|\Delta u_t(0)\|_{L^2})\\ &\quad + c^2\tau^a t^{1/2}\|\mathfrak{K}_1\|_{\mathcal{M}(0,t)}\|\Delta u_t(0)\|_{L^2} + c^2(\|\mathfrak{b}\|_{L^\infty(L^\infty)} + \tau^a)/\sqrt{\underline{c}_2}\,\sqrt{\Phi[\Delta u_{tt}]},\end{aligned}$$

where we have relied on (4.1). Altogether,

$$\begin{aligned}\|u_{tt}\|^2_{L^2_t(L^2)} \lesssim\, &\|f\|^2_{L^2(L^2)} + \Phi[\Delta u_{tt}] + C_{\mathfrak{K}_1}\|u_{tt}(0)\|^2_{L^2} + t\|\Delta u(0)\|^2_{L^2}\\ &+ t^3\|\Delta u_t(0)\|^2_{L^2}.\end{aligned} \tag{4.4}$$

We keep track of $C_{\mathfrak{K}_1}$ here because it comes with high-order initial data but is nonzero only in the exceptional case $\mathfrak{K}_1 = \delta_0$. Later on we will see that for a different reason we will anyway have to assume $u_{tt}(0) = 0$. The hidden constant in (4.4) is independent of both T and τ.

Testing with $-\mathfrak{K}_1 * \Delta u_{tt}$. The higher-order estimate is obtained by rearranging Eq. (4.3) as

$$\tau^a(\mathfrak{K}_1 * u_{tt})_t + \mathfrak{a}_0 u_{tt} - \tau^a c^2\mathfrak{K}_1 * \Delta u_t - \delta\mathfrak{K}_2 * \Delta u_{tt} = \bar{r}$$

for $\mathfrak{a}_0 > 0$ with now

$$\bar{r} = f - (\mathfrak{a} - \mathfrak{a}_0)u_{tt} + c^2\mathfrak{b}\Delta u$$

and testing it with

$$-\mathfrak{K}_1 * \Delta u_{tt} = -(\mathfrak{K}_1 * \Delta u_t)_t + \mathfrak{K}_1 \cdot \Delta u_t(0).$$

Using assumption $(\mathcal{H}_3)$ with $y = \nabla u_{tt}(x)$ as well as $(\mathcal{H}_5)$ with $y = -\Delta u_{tt}(x)$ yields

$$\begin{aligned}&\frac{1}{2}\tau^a\Big[\|\mathfrak{K}_1 * \nabla u_{tt}\|^2_{L^2} + c^2\|\mathfrak{K}_1 * \Delta u_t\|^2_{L^2}\Big]_0^t + \mathfrak{a}_0\int_0^t \Psi[\nabla u_{tt}](s)\,\mathrm{d}s + \delta\,\Phi[\Delta u_{tt}](t)\\ &\leq \mathrm{rhs} + \tau^a c^2\int_0^t\int_\Omega \mathfrak{K}_1\,\Delta u_t(0)\,\mathfrak{K}_1 * \Delta u_t\,\mathrm{d}x\mathrm{d}s,\end{aligned} \tag{4.5}$$

where

$$\mathrm{rhs} := \int_0^t\int_\Omega (f - (\mathfrak{a} - \mathfrak{a}_0)u_{tt} + c^2\mathfrak{b}\Delta u)(-\mathfrak{K}_1 * \Delta u_{tt})\,\mathrm{d}x\mathrm{d}s. \tag{4.6}$$

We can estimate the second term on the right in (4.5) using

$$\Big|\int_0^t\int_\Omega \mathfrak{K}_1\,\Delta u_t(0)\,\mathfrak{K}_1*\Delta u_t\,\mathrm{d}x\mathrm{d}s\Big| \leq \|\mathfrak{K}_1\|_{\mathcal{M}(0,t)}\|\Delta u_t(0)\|_{L^2}\|\mathfrak{K}_1*\Delta u_t\|_{L_t^\infty(L^2)}.$$

We distinguish between two cases in the further treatment of the right-hand side.

Case I: Due to ($\mathcal{H}_5$ I), assumptions (4.2) on the variable coefficients, and Young's inequality, as well as the elementary estimate

$$\|\mathfrak{K}_1*\Delta u_t\|_{L_t^\infty(L^2)} \leq \|(\mathfrak{K}_1*\Delta u_t)(0)\|_{L^2} + \sqrt{t}\|(\mathfrak{K}_1*\Delta u_t)_t\|_{L_t^2(L^2)},$$

all terms containing u on the right-hand side of (4.5) can be estimated by means of $\Phi(\Delta u_{tt})$ and the already obtained $L^2(0,t;L^2(\Omega))$ estimate of u_{tt}. Indeed,

$$\begin{aligned}
&\text{rhs} + \tau^a c^2\int_0^t\int_\Omega \mathfrak{K}_1\,\Delta u_t(0)\,\mathfrak{K}_1*\Delta u_t\,\mathrm{d}x\mathrm{d}s\\
&\leq \epsilon\|\mathfrak{K}_1*\Delta u_{tt}\|^2_{L_t^2(L^2)} + \frac{1}{\epsilon}\|f\|^2_{L_t^2(L^2)} + \frac{1}{\epsilon}\|\mathfrak{a}-\mathfrak{a}_0\|^2_{L_t^\infty(L^\infty)}\|u_{tt}\|^2_{L_t^2(L^2)}\\
&\quad + \frac{1}{\epsilon}c^4\|\mathfrak{b}\|^2_{L_t^\infty(L^\infty)}\|\Delta u\|^2_{L_t^2(L^2)} + \mu\|\mathfrak{K}_1*\Delta u_t\|^2_{L_t^\infty(L^2)} + \frac{1}{4\mu}\tau^{2a}c^4\|\mathfrak{K}_1\|^2_{\mathcal{M}(0,t)}\|\Delta u_t(0)\|^2_{L^2}\\
&\leq \Big(\frac{\epsilon+2t\mu}{\underline{c}_3} + \frac{c^4\|\mathfrak{b}\|^2_{L^\infty(L^\infty)}}{\epsilon\overline{C}_3}\Big)\Phi(\Delta u_{tt}) + \frac{1}{\epsilon}\|f\|^2_{L_t^2(L^2)} + \frac{1}{\epsilon}\|\mathfrak{a}-\mathfrak{a}_0\|^2_{L_t^\infty(L^\infty)}\|u_{tt}\|^2_{L_t^2(L^2)}\\
&\quad + 2\mu\|(\mathfrak{K}_1*\Delta u_t)(0)\|^2_{L^2} + \frac{1}{4\mu}\tau^{2a}c^4\|\mathfrak{K}_1\|^2_{\mathcal{M}(0,t)}\|\Delta u_t(0)\|^2_{L^2}
\end{aligned}$$

for any μ, $\epsilon>0$. By first choosing ϵ and μ sufficiently small and then assuming $\overline{C}_3$ to be sufficiently large (which might necessitate a decrease of T, see the comment on largeness of $\overline{C}_3$ in Section 4.1), we can achieve

$$\frac{\epsilon+2t\mu}{\underline{c}_3} + \frac{c^4\|\mathfrak{b}\|^2_{L^\infty(L^\infty)}}{\epsilon\overline{C}_3} \leq \frac{\delta}{2}$$

and therefore end up with

$$\begin{aligned}
&\frac{1}{2}\tau^a\Big[\|\mathfrak{K}_1*\nabla u_{tt}\|^2_{L^2} + c^2\|\mathfrak{K}_1*\Delta u_t\|^2_{L^2}\Big]_0^t + \mathfrak{a}_0\int_0^t \Psi[\nabla u_{tt}](s)\,\mathrm{d}s + \delta\,\Phi[\Delta u_{tt}](t)\\
&\leq C(t)\,\Big(\|f\|^2_{L^2(L^2)} + \|\mathfrak{a}-\mathfrak{a}_0\|^2_{L_t^\infty(L^\infty)}\|u_{tt}\|^2_{L^2(L^2)} + \|(\mathfrak{K}_1*\Delta u_t)(0)\|^2_{L^2}\\
&\quad + \tau^{2a}\|\mathfrak{K}_1\|^2_{\mathcal{M}(0,T)}\|\Delta u_t(0)\|^2_{L^2}\Big),
\end{aligned}$$

where $C(t)>0$ might depend on t but not on τ. Combining this bound with estimate (4.4) where $C>0$ is the hidden constant and assuming

$$C\,C(t)\|\mathfrak{a}-\mathfrak{a}_0\|^2_{L_t^\infty(L^\infty)} < \frac{\delta}{8}, \tag{4.7}$$

we arrive at the τ-uniform estimate

$$\begin{aligned}
&\tau^a\Big[\|\mathfrak{K}_1*\nabla u_{tt}\|^2_{L^2} + c^2\|\mathfrak{K}_1*\Delta u_t\|^2_{L^2}\Big]_0^t + \|u_{tt}\|^2_{L_t^2(L^2)} + \int_0^t \Psi[\nabla u_{tt}](s)\,\mathrm{d}s\\
&\quad + \sup_{t'\in(0,t)} \Phi[\Delta u_{tt}](t')\\
&\lesssim_T \|f\|^2_{L^2(L^2)} + \|(\mathfrak{K}_1*\Delta u_t)(0)\|^2_{L^2} + \tau^{2a}\|\mathfrak{K}_1\|^2_{\mathcal{M}(0,t)}\|\Delta u_t(0)\|^2_{L^2} + C_{\mathfrak{K}_1}\|u_{tt}(0)\|^2_{L^2}\\
&\quad + \|\Delta u(0)\|^2_{L^2} + \|\Delta u_t(0)\|^2_{L^2}.
\end{aligned} \tag{4.8}$$

Recall that we assume δ to be fixed while $\tau\in(0,\overline{\tau}]$ might be arbitrarily small.

Case II: Here the $\mathfrak{a}-\mathfrak{a}_0$ term in (4.6) cannot be estimated by means of $\Phi[\Delta u_{tt}]$ any more due to the factor $\mathfrak{K}_1*\Delta u_{tt}$. We therefore have to assume $\mathfrak{a}-\mathfrak{a}_0$ to vanish, that is, $\mathfrak{a}$ to be constant; cf. ($\mathcal{H}_5$ II $\mathfrak{a},\mathfrak{b}$).

The further estimate of

$$\text{rhs} = \int_0^t \int_\Omega (f + c^2\mathfrak{b}\Delta u)(-\mathfrak{K}_1 * \Delta u_{tt})\,\mathrm{d}x\mathrm{d}s$$

becomes somewhat complicated since now $\Phi[\Delta u_{tt}]$ cannot dominate the $L^2(0,t;L^2(\Omega))$ norm of the multiplier $\mathfrak{K}_1 * \Delta u_{tt}$. We estimate $h := f + c^2\mathfrak{b}\Delta u$ using the integration by parts and transposition identities involving kernels

$$\begin{aligned}&\int_0^T y_t(T-t)w(t)\,\mathrm{d}t = \int_0^T y(T-t)w_t(t)\,\mathrm{d}t - w(T)y(0) + w(0)y(T), \ \ y,w \in W^{1,1}(0,T)\\ &\int_0^T (\mathfrak{K}_1 * y)(T-t)w(t)\,\mathrm{d}t = \int_0^T (\mathfrak{K}_1 * w)(t)y(T-t)\,\mathrm{d}t, \ \ y,w \in L^1(0,T),\end{aligned} \tag{4.9}$$

as well as the timeflip operator defined by $\overline{h}^t(s) = h(t-s)$, and the identity

$$(\overline{a}^t, b)_{L^2(0,t)} = (a*b)(t) = (b*a)(t).$$

In this manner, we obtain the following identities:

$$\begin{aligned}&\int_0^t \int_\Omega h(s)\,(\mathfrak{K}_1 * \Delta u_{tt})(s)\,\mathrm{d}x\mathrm{d}s\\ &= \int_0^t \int_\Omega \overline{h}^t(t-s)\,\left((\mathfrak{K}_1 * \Delta u_t)_t(s) - \mathfrak{K}_1(s)\Delta u_t(0)\right)\,\mathrm{d}x\mathrm{d}s\\ &= \int_\Omega \Big(\Delta u_t * \mathfrak{K}_1 * (\overline{h}^t)_t\Big)(t) + \Big[\overline{h}^t(t-s)\,(\mathfrak{K}_1 * \Delta u_t)(s)\Big]_{s=0}^t - \int_0^t \overline{h}^t(t-s)\,\mathfrak{K}_1(s)\Delta u_t(0)\,\mathrm{d}x\mathrm{d}s\end{aligned}$$

and thus

$$\begin{aligned}&\int_0^t \int_\Omega h(s)\,(\mathfrak{K}_1 * \Delta u_{tt})(s)\,\mathrm{d}x\mathrm{d}s\\ &= \int_0^t \int_\Omega \Delta u_t(s)\,(\mathfrak{K}_1 * (\overline{h}^t)_t)(t-s)\,\mathrm{d}x\mathrm{d}s\\ &\quad + \int_\Omega \Big(h(t)(\mathfrak{K}_1 * \Delta u_t)(t) - h(0)(\mathfrak{K}_1 * \Delta u_t)(0) - \Delta u_t(0)\int_0^t \mathfrak{K}_1(s)h(s)\,\mathrm{d}s\Big)\,\mathrm{d}x.\end{aligned}$$

From here we have

$$\begin{aligned}&\Big|\int_0^t \int_\Omega h(s)\,(\mathfrak{K}_1 * \Delta u_{tt})(s)\,\mathrm{d}x\mathrm{d}s\Big|\\ &\leq \|\Delta u_t\|_{L^p_t(L^2)}\|\mathfrak{K}_1 * (\overline{h}^t)_t\|_{L^{p'}_t(L^2)} + \|h\|_{L^\infty_t(L^2)}\|\mathfrak{K}_1 * \Delta u_t\|_{L^\infty_t(L^2)}\\ &\quad + \|h(0)\|_{L^2}\|(\mathfrak{K}_1 * \Delta u_t)(0)\|_{L^2} + \|\Delta u_t(0)\|_{L^2}\|h\|_{L^\infty_t(L^2)}\|\mathfrak{K}_1\|_{\mathcal{M}(0,t)}\end{aligned}$$

for $p \in [1,\infty]$ as in ($\mathcal{H}_5$ II) and $p' = \frac{p}{p-1}$. Further, by Young's convolution inequality

$$\begin{aligned}&\|\mathfrak{K}_1 * (\overline{\mathfrak{b}\Delta u}^t)_t\|_{L^{p'}_t(L^2)}\\ &\leq \|\mathfrak{K}_1\|_{L^q(0,t)}\|\mathfrak{b}_t\Delta u + \mathfrak{b}\Delta u_t\|_{L^{\tilde{q}}_t(L^2)}\\ &\leq \|\mathfrak{K}_1\|_{L^q(0,t)}\left(\|\mathfrak{b}_t\|_{L^{\tilde{q}}_t(L^\infty)}\|\Delta u\|_{L^\infty_t(L^2)} + \|\mathfrak{b}\|_{L^\infty_t(L^\infty)}\|\Delta u_t\|_{L^{\tilde{q}}_t(L^2)}\right),\end{aligned} \tag{4.10}$$

where $\tilde{q} = \frac{p'q}{p'q+q-p'}$ so that $1 + \frac{1}{p'} = \frac{1}{q} + \frac{1}{\tilde{q}}$ and $q \leq p'$ as in ($\mathcal{H}_3$). Note that estimate (4.10) is the general kernel substitute for the Kato–Ponce inequality used in [12] to analyze the fJMGT equations. In case $\mathfrak{K}_1 = \delta_0$, (4.10) remains valid with $\|\mathfrak{K}_1\|_{L^q(0,t)}$ replaced by one and $\tilde{q} = p'$.

Estimating further by means of assumption ($\mathcal{H}_5$ II) (which again might require a decrease of T to achieve $\overline{C}_3$ to be large enough) yields the τ-uniform estimate:

$$\begin{aligned}
&\tau^a\Big[\|\mathfrak{K}_1 * \nabla u_{tt}\|^2_{L^2} + c^2\|\mathfrak{K}_1 * \Delta u_t\|^2_{L^2}\Big]_0^t + \|u_{tt}\|^2_{L^2_t(L^2)} + \int_0^t \Psi[\nabla u_{tt}](s)\,\mathrm{d}s \\
&+ \sup_{t'\in(0,t)} \Phi[\Delta u_{tt}](t') \\
&\lesssim_T \|f\|^2_{L^\infty(L^2)} + \|f_t\|^2_{L^{\tilde q}_t(L^2)} + \|\Delta u_t(0)\|^2_{L^2} + \|\Delta u(0)\|^2_{L^2} + C_{\mathfrak{K}_1}\|u_{tt}(0)\|^2_{L^2},
\end{aligned} \tag{4.11}$$

again assuming δ to be fixed while $\tau \in (0, \overline{\tau}]$ might be arbitrarily small. As a consequence, the natural solution space here is

$$\begin{aligned}
\mathcal{X}^\tau = \{\, u \in & W^{s,\mathfrak{r}}(0,T; H^2(\Omega)\cap H^1_0(\Omega)) \cap W^{\sigma,\rho}(0,T; H^1_0(\Omega)) : \\
& u_{tt} \in L^2(0,T; L^2(\Omega)),\ u_t,\ \Delta u \in X^\infty_{\mathfrak{K}_1}(0,T; L^2(\Omega))\,\}
\end{aligned} \tag{4.12}$$

and the τ-uniform solution space is

$$\begin{aligned}
\mathcal{X} = \{u \in W^{s,\mathfrak{r}}(0,T; H^2(\Omega)\cap H^1_0(\Omega)) \cap W^{\sigma,\rho}(0,T; H^1_0(\Omega)) : \\
u_{tt} \in L^2(0,T; L^2(\Omega))\},
\end{aligned}$$

where the space $X^\infty_{\mathfrak{K}_1}(0,t')$ is defined in (2.14).

As previously mentioned, since we have used $(\mathfrak{K}_1 * u_{tt})_t$ in place of $\mathfrak{K}_1 * u_{ttt}$ in the derivation of both energy estimates (4.8) and (4.11) above, we have to add $\mathfrak{K}_1(s)\cdot u_{tt}(0)$ to the right-hand side. Consequently we would have to assume $\mathfrak{K}_1 \in L^2(0,T)$ in Case I and even $\mathfrak{K}_1 \in L^\infty(0,T)\cap W^{1,\tilde q}(0,T)$ in Case II. To avoid this, we impose the condition $u_2 = 0$ in the upcoming analysis.

Proposition 4.1. *Let $T>0$ and $\tau \in (0,\bar\tau]$ for some fixed $\bar\tau > 0$. Let assumptions (2.13) and ($\mathcal{H}_1$)–($\mathcal{H}_5$ Φ) on the kernels hold, as well as*

$$u_0,\ u_1 \in H^2(\Omega)\cap H^1_0(\Omega), \quad u_2 = 0$$

and the additional following assumptions on the kernels, coefficients $\mathfrak{a}$, $\mathfrak{b}$, and the right-hand side:

(I) ($\mathcal{H}_5$ I), (4.7), $f \in L^2(0,T;L^2(\Omega))$, or
(II) ($\mathcal{H}_5$ II), ($\mathcal{H}_5$ II $\mathfrak{a},\mathfrak{b}$), ($\mathcal{H}_5$ II $\mathfrak{K}_1$), $f \in L^\infty(0,T;L^2(\Omega))\cap W^{1,\tilde q}(0,T;L^2(\Omega))$

with $\tilde q$ as in ($\mathcal{H}_5$ II $\mathfrak{K}_1$). Then there exists a unique solution $u \in \mathcal{X}^\tau \subseteq \mathcal{X}$ of the initial boundary-value problem

$$\begin{cases}
\tau^a(\mathfrak{K}_1 * u_{tt})_t + \mathfrak{a}(x,t)u_{tt} - c^2\mathfrak{b}(x,t)\Delta u - \tau^a c^2 \mathfrak{K}_1 * \Delta u_t - \delta\mathfrak{K}_2 * \Delta u_{tt} = f(x,t), \\
u|_{\partial\Omega} = 0, \\
(u, u_t, u_{tt})|_{t=0} = (u_0, u_1, u_2).
\end{cases} \tag{4.13}$$

The solution satisfies the following τ-uniform estimate:

$$\|u\|^2_{\mathcal{X}} \lesssim_T \begin{cases}
\|f\|^2_{L^2(L^2)} + \|\Delta u(0)\|^2_{L^2} + \|\Delta u_t(0)\|^2_{L^2} + \tau^{2a}\|\mathfrak{K}_1\|^2_{\mathcal{M}(0,T)}\|\Delta u_t(0)\|^2_{L^2} & \text{(I)}, \\
\|f\|^2_{L^\infty(L^2)} + \|f_t\|^2_{L^{\tilde q}(L^2)} + \|\Delta u(0)\|^2_{L^2} + \|\Delta u_t(0)\|^2_{L^2} & \text{(II)}.
\end{cases}$$

Proof. The proof follows by a Faedo–Galerkin procedure, where Appendix A can be used to establish well-posedness of the ODEs resulting from Galerkin semidiscretization. The testing strategies shown above and resulting in (4.8), (4.11) are then applied to the Galerkin solutions in place of u.

Stating the energy estimates in terms of Sobolev norms $\|u_{tt}\|$, $\|\nabla u\|_{W^{s,r}(0,t)}$, and $\|\Delta u\|_{W^{\sigma,\rho}(0,t)}$ (cf. ($\mathcal{H}_5$ Φ)), we can rely on weak−∗ lower continuity of these norms when taking weak limits. Moreover, we use

Lemma 2.1 with $p = \infty$ to also take limits in the τ^a terms of (4.8) and (4.11). This is useful for enabling the use of Δu_t as a multiplier in the uniqueness proof.

Uniqueness. Since $u \in \mathcal{X}^\tau$, with $\mathcal{X}^\tau$ defined in (4.12), we can test the time integrated homogeneous PDE with vanishing f and initial data (not only its Galerkin semidiscretization) in the way described above (that is, testing with the time integrated versions $-\mathfrak{K}_1 * \Delta u_t$, u_t of $-\mathfrak{K}_1 * \Delta u_{tt}$, u_{tt}) and analogously obtain from (4.8), (4.11) that its solution needs to vanish. □

4.3. Uniform well-posedeness with Kuznetsov–Blackstock and Westervelt–Blackstock nonlinearities in Case I

To relate the previous analysis to the nonlinear equation, we again consider the fixed-point mapping $\mathcal{T} : \mathcal{B} \ni u^* \mapsto u$, on the ball

$$\mathcal{B} = \{u^* \in \mathcal{X} : \|u^*\|_{\mathcal{X}} \le R,\ (u^*, u^*_t, u^*_{tt})|_{t=0} = (u_0, u_1, 0)\}.$$

Here $u = \mathcal{T}u^*$ solves (4.3) with

$$\mathfrak{a}(u^*, u^*_t) = \begin{cases} 1 + 2k_1 u^* & \text{(WB)}, \\ 1 + 2k_1 u^*_t & \text{(KB)}, \end{cases} \qquad \mathfrak{b}(u^*, u^*_t) = \begin{cases} 1 - 2k_2 u^* & \text{(WB)}, \\ 1 - 2k_2 u^*_t & \text{(KB)}, \end{cases} \tag{4.14}$$

and

$$f(x,t) = -\mathcal{N}(u^*_t, \nabla u^*, \nabla u^*_t) = \begin{cases} -2k_3 (u^*_t)^2 & \text{(WB)}, \\ -2k_3 \nabla u^* \cdot \nabla u^*_t & \text{(KB)}, \end{cases} \tag{4.15}$$

where (WB) stands for the Westervelt–Blackstock and (KB) for Kuznetsov–Blackstock nonlinearities.

Theorem 4.1. *Let $T > 0$ and $\tau \in (0, \bar{\tau}]$. Let assumptions (2.13) and ($\mathcal{H}_1$)–($\mathcal{H}_5$ Φ), ($\mathcal{H}_5$ I) on the kernels $\mathfrak{K}_1$ and $\mathfrak{K}_2$ hold with s, $\mathtt{r}$, σ, and ρ so that the following continuous embedding holds:*

$$\begin{aligned} & W^{s,\mathtt{r}}(0,T; H^2(\Omega) \cap H^1_0(\Omega)) \cap W^{\sigma,\rho}(0,T; H^1(\Omega)) \\ \hookrightarrow & \begin{cases} L^\infty(0,T; L^\infty(\Omega)) \cap W^{1,4}(0,T; L^4(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H^1_0(\Omega)) & \text{(WB)}, \\ W^{1,\infty}(0,T; L^\infty(\Omega)) \cap H^1(0,T; W^{1,4}(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H^1_0(\Omega)) & \text{(KB)}. \end{cases} \end{aligned} \tag{4.16}$$

There exists a data size $r = r(T) > 0$, independent of τ, such that if

$$\|\Delta u(0)\|^2_{L^2} + \|\Delta u_t(0)\|^2_{L^2} + \tau^{2a} \|\mathfrak{K}_1\|^2_{\mathcal{M}(0,T)} \|\Delta u_t(0)\|^2_{L^2} \le r^2, \tag{4.17}$$

then there is a unique solution $u \in \mathcal{B}$ of the nonlinear initial boundary-value problem

$$\begin{cases} \tau^a \mathfrak{K}_1 * u_{ttt} + \mathfrak{a}(u, u_t) u_{tt} - c^2 \mathfrak{b}(u, u_t) \Delta u - \tau^a c^2 \mathfrak{K}_1 * \Delta u_t - \delta \mathfrak{K}_2 * \Delta u_{tt} \\ \quad = -\mathcal{N}(u_t, \nabla u, \nabla u_t), \\ u|_{\partial\Omega} = 0, \\ (u, u_t, u_{tt})|_{t=0} = (u_0, u_1, 0). \end{cases} \tag{4.18}$$

Proof. The proof goes analogously to the one of Theorem 3.3, using the fact that $u = \mathcal{T}u^*$ solves (4.13) with $\mathfrak{a}$, $\mathfrak{b}$, and f as in (4.14) and (4.15) to establish self-mapping and the fact that $\phi = u - v := \mathcal{T}(u^*) - \mathcal{T}(v^*)$ solves (4.13) with

$$\mathfrak{a}(u^*, u^*_t) = \begin{cases} 1 + 2k_1 u^* & \text{(WB)}, \\ 1 + 2k_1 u^*_t & \text{(KB)}, \end{cases} \qquad \mathfrak{b}(u^*, u^*_t) = \begin{cases} 1 - 2k_2 u^* & \text{(WB)}, \\ 1 - 2k_2 u^*_t & \text{(KB)}, \end{cases} \tag{4.19}$$

and, with $\phi^* = u^* - v^*$,

$$= \begin{cases} -2\phi^*(k_1 v_{tt} + k_2 \Delta v) - 2k_3\phi^*_t(u^*_t + v^*_t) & \text{(WB)}, \\ -2\phi^*_t(k_1 v_{tt} + k_2 \Delta v) - 2k_3 \nabla\phi^* \cdot \nabla u^*_t - 2k_3 \nabla v^* \cdot \nabla \phi^*_t & \text{(KB)}, \end{cases} \tag{4.20}$$

for establishing contractivity. Indeed, the continuous embedding assumption (4.16) together with smallness of data size r as well as R ensures the required bounds. □

The fact that a Kuznetsov-type nonlinearity is enabled here under only H^2 regularity of the initial data (cf. [42]) is due to the relative strength of the damping term with factor δ in Case I, where kernel $\mathfrak{K}_2$ is more singular than $\mathfrak{K}_1$.

4.4. Uniform existence with Westervelt–Blackstock nonlinearities in Case II

To treat Case II, we restrict our considerations to the Westervelt–Blackstock nonlinearities with $k_1 = 0$; that is, we analyze the following equation:

$$\tau^a \mathfrak{K}_1 * u_{ttt} + u_{tt} - c^2(1 - 2k_2 u)\Delta u - \tau^a c^2 \mathfrak{K}_1 * \Delta u_t - \delta \mathfrak{K}_2 * \Delta u_{tt} + 2k_3 u_t^2 = 0.$$

Theorem 4.2. *Let $T > 0$ and $\tau \in (0, \bar{\tau}]$. Let assumptions (2.13) and ($\mathcal{H}_1$)–($\mathcal{H}_5$ Φ), ($\mathcal{H}_5$ II), ($\mathcal{H}_5$ II $\mathfrak{K}_1$) on the kernels $\mathfrak{K}_1$ and $\mathfrak{K}_2$ hold and let*

$$u_0,\, u_1 \in H^2(\Omega) \cap H_0^1(\Omega),\ u_2 = 0,$$

and

$$W^{s,\mathfrak{r}}(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \cap W^{\sigma,\rho}(0,T; H^1(\Omega)) \hookrightarrow L^\infty(0,T; L^\infty(\Omega)). \tag{4.21}$$

There exists a data size $r = r(T) > 0$, independent of τ, such that if (4.17) holds, then there is a solution $u \in \mathcal{B}$ of the nonlinear initial boundary-value problem (4.18) in case (WB) *with $k_1 = 0$ in (4.14).*

Proof. Establishing a self-mapping property of $\mathcal{T}$ on a sufficiently small ball works as in the proof of Theorems 3.3 and 4.1, based on assumption of having the continuous embedding in (4.21), which implies continuity of the embedding

$$\begin{aligned} &W^{s,\mathfrak{r}}(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \cap W^{\sigma,\rho}(0,T; H^1(\Omega)) \\ &\hookrightarrow L^\infty(0,T; L^\infty(\Omega)) \cap W^{1,\tilde{q}}(0,T; L^\infty(\Omega)) \cap W^{1,\infty}(0,T; L^4(\Omega)) \cap W^{1,\frac{2\tilde{q}}{2-\tilde{q}}}(0,T; L^\infty(\Omega)). \end{aligned}$$

However, a corresponding assumption leading to contractivity based on (4.19), (4.20) would be unrealistically strong; see Remark 2. We therefore (similarly to [16,43]) only prove existence of solutions based on a general version of Schauder's fixed-point theorem in locally convex topological spaces (see [44]) and for this purpose establish weak−∗ continuity of $\mathcal{T}$ as follows.

For any sequence $(u^*_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}$ that weakly−∗ converges to $u^* \in \mathcal{B}$ in $\mathcal{X}$, we also have

$$(\mathcal{T}(u^*_n))_{n\in\mathbb{N}} \subseteq \mathcal{B}.$$

Thus, by compactness of the embedding $\mathcal{X} \to W^{1,\infty}(0,T; L^\infty(\Omega))$, there exists a subsequence $(u^*_{n_\ell})_{\ell\in\mathbb{N}}$ such that $1 \pm k_{1/2} u^*_{n_\ell\, t}$ converges to $1 \pm k_{1/2} u^*_t$ strongly in $L^\infty(0,T; L^\infty(\Omega))$ and $\mathcal{T}(u^*_{n_\ell})$ converges weakly* in $\mathcal{X}$ to some $u \in \mathcal{B}$, which by definition of $\mathcal{B}$ satisfies the initial and homogeneous Dirichlet boundary conditions. It is readily checked that u also solves the PDE defining $\mathcal{T}(u^*)$, which, by uniqueness in Proposition 4.1, implies $u = \mathcal{T}(u^*)$. A subsequence-subsequence argument yields weak−∗ convergence in $\mathcal{X}$ of the whole sequence $(\mathcal{T}(u^*_n))_{n\in\mathbb{N}}$ to $\mathcal{T}(u^*)$. □

As can be read off from Table 4, embedding (4.21) is satisfied for all Compte–Metzler laws. To establish embedding (4.16) for GFE I, we use interpolation

$$\|w\|_{H^{\frac{3}{2}+\frac{\alpha(1-\epsilon)}{4}}(0,T;H^{\frac{3+\epsilon}{2}}(\Omega))} \lesssim \|w\|^{\frac{1+\epsilon}{2}}_{H^{\frac{3}{2}}(0,T;H^2(\Omega)\cap H_0^1(\Omega))} \|w\|^{\frac{1-\epsilon}{2}}_{H^{\frac{3+\alpha}{2}}(0,T;H^1(\Omega))}$$

with $\alpha > 0$ and $\epsilon \in (0,1)$.

Remark 2 (*Uniqueness of Solutions Of (4.18) in Case II*)**.** To prove contractivity of $\mathcal{T}$ in Case II with Westervelt–Blackstock nonlinearity when $k_1 \neq 0$ (that is, with $\mathfrak{a} = 1 + 2k_1 u$), we would need

$$2\phi^*(k_1 v_{tt} + k_2 \Delta v) - 2(k_1+k_2)(u_t^* + v_t^*)\phi_t^* \in L^\infty(0,T;L^2(\Omega)) \cap W^{1,\tilde{q}}(0,T;L^2(\Omega)),$$

thus requiring an estimate on v_{ttt} (and thus ψ_{ttt} in the existence proof). This is clearly beyond the scope of the available energy estimates. Time differentiation of the PDE and further testing might enable this at the cost of stronger conditions on the initial data.

Likewise, Kuznetsov-type nonlinearities ($\mathfrak{a} = 1 + 2k_1 u_t$) in Case II would require $u_{tt} \in L^{\tilde{q}}(0,T;L^\infty(\Omega))$ (as needed for $\mathfrak{b}_t \in L^{\tilde{q}}(0,T;L^\infty(\Omega))$ in estimate (4.10)), which seems to be out of reach for most Compte–Metzler laws. The (fJMGT) equation based on the GFE law, where $\mathfrak{K}_2 = 1$, allows for an alternative testing strategy that makes it possible to incorporate also Kuznetsov-type nonlinearities under stronger assumptions on the regularity of data; we refer to [18] for details and the corresponding analysis.

Note that the results obtained for the (fJMGT III) equation here are weaker than in the previous Section 3.

4.5. Weak singular limits

We next discuss the weak limiting behavior of these nonlocal equations in both cases. Let $\tau \in (0,\bar{\tau}]$. Under the assumptions of Theorems 4.1 and 4.2 with uniformly bounded data

$$\|\Delta u^\tau(0)\|^2_{L^2} + \|\Delta u^\tau_t(0)\|^2_{L^2} + \tau^{2a}\|\mathfrak{K}_1\|^2_{\mathcal{M}(0,T)}\|\Delta u^\tau_t(0)\|^2_{L^2} \leq r^2$$

and r independent of τ as in Theorems 4.1 and 4.2, consider

$$\begin{cases} \tau^a \mathfrak{K}_1 * u^\tau_{ttt} + \mathfrak{a}(u^\tau, u^\tau_t)u^\tau_{tt} - c^2\mathfrak{b}(u^\tau,u^\tau_t)\Delta u^\tau - \tau^a c^2 \mathfrak{K}_1 * \Delta u^\tau_t \\ \qquad\qquad - \delta\mathfrak{K}_2 * \Delta u^\tau_{tt} + \mathcal{N}(u_t, \nabla u, \nabla u_t) = 0 \quad \text{in } \Omega\times(0,T), \\ u^\tau|_{\partial\Omega} = 0, \\ (u^\tau, u^\tau_t, u^\tau_{tt})|_{t=0} = (u^\tau_0, u^\tau_1, 0). \end{cases}$$

According to Theorems 4.1, 4.2 a solution of this problem exists in $\mathcal{B}$ (although it might not be unique in Case II). Therefore, we have the following uniform bounds with respect to the relaxation time:

$$\begin{cases} u^\tau \text{ is bounded in } W^{s,\mathfrak{r}}(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \cap W^{\sigma,\rho}(0,T;H^1(\Omega)), \\ u^\tau_{tt} \text{ is bounded in } L^2(0,T;L^2(\Omega)). \end{cases} \tag{4.22}$$

hence existence of a subsequence, not relabeled, such that

$$\begin{aligned} u^\tau &\rightharpoonup u \quad && \text{weakly-}\star \text{ in } W^{s,\mathfrak{r}}(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \cap W^{\sigma,\rho}(0,T;H^1(\Omega)), \\ u^\tau_{tt} &\rightharpoonup u_{tt} && \text{weakly} \quad \text{in } L^2(0,T;L^2(\Omega)). \end{aligned} \tag{4.23}$$

Assuming compactness of the embedding

$$\mathcal{X} \hookrightarrow\hookrightarrow C([0,T];H^{\beta_0}(\Omega)) \cap C^1([0,T];H^{\beta_1}(\Omega)), \quad \beta_0,\, \beta_1 \geq 0 \tag{4.24}$$

we have additionally

$$\begin{aligned} u^\tau &\longrightarrow u \qquad \text{strongly in } C([0,T];H^{\beta_0}(\Omega)),\\ u_t^\tau &\longrightarrow u_t \qquad \text{strongly in } C([0,T];H^{\beta_1}(\Omega)). \end{aligned} \tag{4.25}$$

We should prove that u solves the limiting problem. Let $v \in C_0^\infty([0,T];C_0^\infty(\Omega))$. We have with $\bar{u} = u - u^\tau$:

$$\begin{aligned} &\int_0^T \int_\Omega \mathfrak{a}(u,u_t)u_{tt}v \,\mathrm{d}x\mathrm{d}s - c^2\int_0^T \int_\Omega \mathfrak{b}(u,u_t)\Delta u v \,\mathrm{d}x\mathrm{d}s + \delta\int_0^T \int_\Omega \mathfrak{K}_2 * \nabla u_{tt}\cdot\nabla v \,\mathrm{d}x\mathrm{d}s\\ &+\int_0^T \int_\Omega \mathcal{N}(u_t,\nabla u,\nabla u_t)v \,\mathrm{d}x\mathrm{d}s\\ =&\int_0^T \int_\Omega \mathfrak{a}(u,u_t)\bar{u}_{tt}v \,\mathrm{d}x\mathrm{d}s - c^2\int_0^T \int_\Omega \mathfrak{b}(u,u_t)\Delta \bar{u} v \,\mathrm{d}x\mathrm{d}s + \delta\int_0^T \int_\Omega \mathfrak{K}_2 * \nabla \bar{u}_{tt}\cdot\nabla v \,\mathrm{d}x\mathrm{d}s\\ &-\int_0^T \int_\Omega \tau^a\mathfrak{K}_1 * (u^\tau_{ttt} - c^2\Delta u^\tau_t)v \,\mathrm{d}x\mathrm{d}s\\ &-\int_0^T \int_\Omega (\mathfrak{a}(u^\tau,u^\tau_t)-\mathfrak{a}(u,u_t))u^\tau_{tt}v \,\mathrm{d}x\mathrm{d}s + c^2\int_0^T \int_\Omega (\mathfrak{b}(u^\tau,u^\tau_t)-\mathfrak{b}(u,u_t))\Delta u^\tau v \,\mathrm{d}x\mathrm{d}s\\ &-\int_0^T \int_\Omega (\mathcal{N}(u^\tau_t,\nabla u^\tau,\nabla u^\tau_t)-\mathcal{N}(u_t,\nabla u,\nabla u_t))v \,\mathrm{d}x\mathrm{d}s. \end{aligned}$$

We wish to prove that the right-hand side tends to zero as $\tau \searrow 0$. To this end, we rely on the established weak and strong convergence in (4.23) and (4.25), respectively. We first discuss the terms involving the kernels and treat them by means of transposition and integration by parts (see (4.9)), which as compared to the proof of Theorem 3.2 is required due to the limited regularity of $\bar{u}_{tt}$:

$$\begin{aligned} &\int_0^T \int_\Omega (\mathfrak{K}_2 * \nabla\bar{u}_{tt})(s)\cdot\nabla v(s) \,\mathrm{d}x\mathrm{d}s = -\int_0^T \int_\Omega (\mathfrak{K}_2 * \bar{u}_{tt})(s)\cdot\overline{\Delta v}^T(T-s) \,\mathrm{d}x\mathrm{d}s\\ &= -\int_0^T \int_\Omega \bar{u}_{tt}(s)\cdot(\mathfrak{K}_2 * \overline{\Delta v}^T)(T-s) \,\mathrm{d}x\mathrm{d}s \ \to 0 \quad \text{as } \tau\searrow 0, \end{aligned}$$

and for $w^\tau := u^\tau_{tt} - c^2\Delta u^\tau$

$$\begin{aligned} &\tau^a\int_0^T \int_\Omega (\mathfrak{K}_1 * w^\tau_t)(s)v(s) \,\mathrm{d}x\mathrm{d}s = \tau^a\int_0^T \int_\Omega (\mathfrak{K}_1 * w^\tau_t)(s)\overline{v}^T(T-s) \,\mathrm{d}x\mathrm{d}s\\ &= \tau^a\int_0^T \int_\Omega w^\tau_t(s)(\mathfrak{K}_1 * \overline{v}^T)(T-s) \,\mathrm{d}x\mathrm{d}s\\ &= \tau^a\Big(\int_0^T \int_\Omega w^\tau(s)(\mathfrak{K}_1 * \overline{v}^T)_t(T-s) \,\mathrm{d}x\mathrm{d}s - \int_\Omega w^\tau(0)(\mathfrak{K}_1 * \overline{v}^T)_t(T) \,\mathrm{d}x\Big) \ \to 0 \quad \text{as } \tau\searrow 0. \end{aligned}$$

Here we have used the fact that due to $v \in C_0^\infty([0,T];C_0^\infty(\Omega))$ we have

$$(\mathfrak{K}_1 * \overline{v}^T)_t(t) = (\mathfrak{K}_1 * \overline{v}^T_t)(t) + \mathfrak{K}_1(t)\overline{v}^T(0) = -\int_0^t \mathfrak{K}_1(s)v_t(T-s)\,\mathrm{d}s$$

which vanishes for $t = 0$, even in case $\mathfrak{K}_1 = \delta_0$.

By relying on the equivalence of the norms $\|\mathfrak{a}(u,u_t)v\|_{L^2}$, $\|v\|_{L^2}$, and $\|\mathfrak{b}(u,u_t)v\|_{L^2}$, we can conclude that

$$\int_0^T \int_\Omega \mathfrak{a}(u,u_t)\bar{u}_{tt}v \,\mathrm{d}x\mathrm{d}s + c^2\int_0^T \int_\Omega \mathfrak{b}(u,u_t)\Delta\bar{u}v \,\mathrm{d}x\mathrm{d}s \to 0 \quad \text{as } \tau\searrow 0.$$

We have

$$\begin{aligned} &-\int_0^T \int_\Omega (\mathfrak{a}(u^\tau,u^\tau_t)-\mathfrak{a}(u,u_t))u^\tau_{tt}v \,\mathrm{d}x\mathrm{d}s + c^2\int_0^T \int_\Omega (\mathfrak{b}(u^\tau,u^\tau_t)-\mathfrak{b}(u,u_t))\Delta u^\tau v \,\mathrm{d}x\mathrm{d}s\\ =&-2\int_0^T \int_\Omega \bar{w}(k_1u^\tau_{tt} + k_2\Delta u^\tau)v \,\mathrm{d}x\mathrm{d}s \end{aligned}$$

with $\bar{w} = \bar{u}$ (WB) or $\bar{w} = \bar{u}_t$ (KB), which tends to zero as well thanks to (4.22) and (4.25). Finally, with the Westervelt–Blackstock type nonlinearity (cf. (4.19), (4.20)):

$$\begin{aligned}\int_0^T \int_\Omega (\mathcal{N}(u_t^\tau) - \mathcal{N}(u_t))v \,\mathrm{d}x\mathrm{d}s &= 2k_3 \int_0^T \int_\Omega \bar{u}_t(u_t + u_t^\tau)v \,\mathrm{d}x\mathrm{d}s \\ &= -2k_3 \int_0^T \int_\Omega \bar{u}((u_t + u_t^\tau)v)_t \,\mathrm{d}x\mathrm{d}s\end{aligned}$$

and with the Kuznetsov–Blackstock nonlinearity:

$$\int_0^T \int_\Omega (\mathcal{N}(\nabla u^\tau, u_t^\tau) - \mathcal{N}(\nabla u, \nabla u_t))v \,\mathrm{d}x\mathrm{d}s = 2k_3 \int_0^T \int_\Omega \Big(\nabla \bar{u} \cdot \nabla u_t^\tau + \nabla u \cdot \nabla \bar{u}_t\Big) v \,\mathrm{d}x\mathrm{d}s.$$

We have

$$\nabla \bar{u} \to 0 \text{ in } L^2(0,T; L^2(\Omega)), \qquad \|v \nabla u_t^\tau\|_{L^2(L^2)} \leq \|v\|_{L^\infty(L^\infty)} \|\nabla u_t^\tau\|_{L^2(L^2)}$$

and

$$\begin{aligned}\bar{u}_t \rightharpoonup 0 \text{ weakly in } L^2(0,T; L^2(\Omega)), \quad \|\nabla \cdot (v \nabla u)\|_{L^2(L^2)} &\leq \|v\|_{L^\infty(L^\infty)} \|\Delta u\|_{L^2(L^2)} \\ &+ \|\nabla v\|_{L^\infty(L^4)} \|\nabla u\|_{L^2(L^4)}.\end{aligned}$$

In both cases these terms tend to zero on account of again (4.22), (4.23), and (4.25). The attainment of initial conditions (u_1, u_2) follows by (4.25), analogously to (3.19). With a subsequence-subsequence argument and using uniqueness for the limiting equation, similarly to Theorem 3.2, this leads to the following result.

Theorem 4.3. *Let the assumptions of Theorems 4.1 or 4.2 with*

$$\|\Delta u^\tau(0)\|_{L^2}^2 + \|\Delta u_t^\tau(0)\|_{L^2}^2 + \tau^{2a} \|\mathfrak{K}_1\|_{\mathcal{M}(0,T)}^2 \|\Delta u_t^\tau(0)\|_{L^2}^2 \leq r^2,$$

for $\tau \in (0, \bar{\tau}]$, as well as embedding (4.24) hold. Then any family $\{u^\tau\}_{\tau \in (0,\bar{\tau}]}$ of solutions to (4.18) converges weakly in the sense of (4.23) to the solution $u \in \mathcal{X}$ of

$$\begin{cases} \mathfrak{a}(u, u_t)u_{tt} - c^2 \mathfrak{b}(u, u_t)\Delta u - \delta \mathfrak{K}_2 * \Delta u_{tt} + \mathcal{N}(u_t, \nabla u, \nabla u_t) = 0 \\ u|_{\partial\Omega} = 0, \\ (u, u_t)|_{t=0} = (u_0, u_1) \end{cases}$$

with $\mathfrak{a}$ and $\mathfrak{b}$ as in (4.14) and $\mathcal{N}$ as in (4.15) (and restricted to the Westervelt–Blackstock case with $k_1 = 0$ under the conditions of Theorem 4.2).

Remark 3 (*Weak Limits for Solutions of The fJMGT Equations*)**.** The condition on the compactness of the embedding (4.24) that is left to be verified, for equations based on the Compte–Metzler fractional laws holds by interpolation

$$\begin{aligned}H^\sigma(0,T; H^1(\Omega)) \cap H^s(0,T; H^2(\Omega)) &\subseteq H^{\theta_i \sigma + (1-\theta_i)s}(0,T; H^{2-\theta_i}(\Omega)) \\ &\hookrightarrow C^i([0,T]; H^{\beta_i}(\Omega)), \quad i \in \{0,1\}\end{aligned}$$

for $\theta_i > \frac{i+1/2-s}{\sigma-s}$. This yields $\beta_0 \in (0,2)$ and $\beta_1 \in (0, 2 - \frac{3-2s}{2(\sigma-s)})$; more precisely, $\beta_1 \in (0,2)$ for the laws GFE I, GFE III and $\beta_1 \in (0, 1+\alpha)$ for GFE.

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Appendix A. Analysis of the semi-discrete problems

We present in this appendix the proof of the unique solvability of the semi-discrete problem considered in Propositions 3.1 and 4.1. Let $\{\phi_i\}_{i\geq 1}$ be a basis of $V = H^2(\Omega)\cap H_0^1(\Omega)$ consisting of the eigenfunctions of the Dirichlet–Laplace operator. Let $V_n = \text{span}\{\phi_1,\dots,\phi_n\}\subset V$ and set

$$u^{(n)}(t) = \sum_{i=1}^{n} \xi_i^{(n)}(t)\phi_i.$$

We choose the approximate initial data as

$$u_0^{(n)} = \sum_{i=1}^{n}\xi_i^{(0,n)}\phi_i, \quad u_1^{(n)} = \sum_{i=1}^{n}\xi_i^{(1,n)}\phi_i, \quad u_2^{(n)} \sum_{i=1}^{n}\xi_i^{(2,n)}\phi_i \in V_n,$$

(with $u_2^{(n)} = 0$ in Proposition 4.1) such that

$$u_0^{(n)} \to u_0 \text{ in } H^2(\Omega)\cap H_0^1(\Omega),\ u_1^{(n)} \to u_1 \text{ in } H^2(\Omega)\cap H_0^1(\Omega),\ \text{and } u_2^{(n)} \to u_2 \text{ in } H_0^1(\Omega),\ n\to\infty.$$

For each $n\in\mathbb{N}$, the system of Galerkin equations is given by

$$\begin{aligned}
&\tau^a\sum_{i=1}^{n}(\mathfrak{K}_1 * \xi_{i,ttt}^{(n)})(t)(\phi_i,\phi_j)_{L^2} + \sum_{i=1}^{n}\xi_{i,tt}^{(n)}(\mathfrak{a}(t)\phi_i,\phi_j)_{L^2} + c^2\sum_{i=1}^{n}\xi_i^{(n)}(\mathfrak{b}(t)\Delta\phi_i,\phi_j)_{L^2}\\
&\quad + \tau^a c^2\sum_{i=1}^{n}(\mathfrak{K}_1 * \xi_{i,t}^{(n)})(t)(\nabla\phi_i,\nabla\phi_j)_{L^2} + \delta\sum_{i=1}^{n}(\mathfrak{K}_2 * \xi_{i,tt}^{(n)})(t)(\nabla\phi_i,\nabla\phi_j)_{L^2}\\
&= (f(t),\phi_j)_{L^2}
\end{aligned}$$

for a.e. $t\in(0,T)$ and all $j\in\{1,\dots,n\}$. With $\boldsymbol{\xi} = [\xi_1^{(n)}\ \dots\ \xi_n^{(n)}]^T$, we can write this system in matrix form

$$\begin{cases}\tau^a M\mathfrak{K}_1 * \boldsymbol{\xi_{ttt}} + M_{\mathfrak{a}}\boldsymbol{\xi_{tt}} + K_{\mathfrak{b}}\boldsymbol{\xi} + \tau^a c^2 K\mathfrak{K}_1 * \boldsymbol{\xi_t} + \delta K\mathfrak{K}_2 * \boldsymbol{\xi_{tt}} = \boldsymbol{f},\\ (\boldsymbol{\xi},\boldsymbol{\xi_t},\boldsymbol{\xi_{tt}})|_{t=0} = (\boldsymbol{\xi_0},\boldsymbol{\xi_1},\boldsymbol{\xi_2}),\end{cases}$$

where $(\boldsymbol{\xi_0},\boldsymbol{\xi_1},\boldsymbol{\xi_2}) = ([\xi_1^{(0,n)}\dots\xi_n^{(0,n)}]^T, [\xi_1^{(1,n)}\dots\xi_n^{(1,n)}]^T, [\xi_1^{(2,n)}\dots\xi_n^{(2,n)}]^T)$. Above, M and K are the standard mass and stiffness matrices, respectively. $M_{\mathfrak{a}}$ and $K_{\mathfrak{b}}$ are matrices with the following weighted entries:

$$M_{\mathfrak{a},ij} = (\mathfrak{a}\phi_i,\phi_j)_{L^2}, \quad K_{\mathfrak{b},ij} = -(\mathfrak{b}\Delta\phi_i,\phi_j)_{L^2}.$$

Let $\boldsymbol{\mu} = \mathfrak{K}_1 * \boldsymbol{\xi_{ttt}}$ be the new unknown. Then with $\tilde{\mathfrak{K}}_1$ defined by $(\mathcal{A}_1)$ we have

$$\begin{aligned}
\boldsymbol{\xi_{tt}} &= \tilde{\mathfrak{K}}_1 * \boldsymbol{\mu} + \boldsymbol{\xi_2},\\
\boldsymbol{\xi_t} &= 1 * \tilde{\mathfrak{K}}_1 * \boldsymbol{\mu} + \boldsymbol{\xi_2}t + \boldsymbol{\xi_1},\\
\boldsymbol{\xi} &= 1 * 1 * \tilde{\mathfrak{K}}_1 * \boldsymbol{\mu} + \boldsymbol{\xi_2}\frac{t^2}{2} + \boldsymbol{\xi_1}t + \boldsymbol{\xi_0}.
\end{aligned}$$

The system can then be equivalently rewritten as a system of Volterra equations:

$$\begin{aligned}
&\tau^a\boldsymbol{\mu} + M^{-1}M_{\mathfrak{a}}(\tilde{\mathfrak{K}}_1 * \boldsymbol{\mu} + \boldsymbol{\xi_2}) + M^{-1}K_{\mathfrak{b}}(1*1*\tilde{\mathfrak{K}}_1 * \boldsymbol{\mu} + \boldsymbol{\xi_2}\frac{t^2}{2} + \boldsymbol{\xi_1}t + \boldsymbol{\xi_0})\\
&+ \tau^a c^2 M^{-1}K\mathfrak{K}_1 * (1 * \tilde{\mathfrak{K}}_1 * \boldsymbol{\mu} + \boldsymbol{\xi_2}t + \boldsymbol{\xi_1}) + \delta M^{-1}K\mathfrak{K}_2 * (\tilde{\mathfrak{K}}_1 * \boldsymbol{\mu} + \boldsymbol{\xi_2}) = M^{-1}\boldsymbol{f}
\end{aligned}$$

or equivalently

$$\begin{aligned}
&\tau^a\boldsymbol{\mu} + M^{-1}M_{\mathfrak{a}}\tilde{\mathfrak{K}}_1 * \boldsymbol{\mu} + M^{-1}K_{\mathfrak{b}}1*1*\tilde{\mathfrak{K}}_1 * \boldsymbol{\mu} + \tau^a c^2 M^{-1}K\mathfrak{K}_1 * 1 * \tilde{\mathfrak{K}}_1 * \boldsymbol{\mu}\\
&+ \delta M^{-1}K\mathfrak{K}_2 * \tilde{\mathfrak{K}}_1 * \boldsymbol{\mu} = \tilde{\boldsymbol{f}}
\end{aligned}$$

with the right-hand side

$$\begin{aligned}\tilde{\boldsymbol{f}} =& M^{-1}\boldsymbol{f} - M^{-1}M_{\mathfrak{a}}\boldsymbol{\xi_2} - M^{-1}K_{\mathfrak{b}}(\boldsymbol{\xi_2}\frac{t^2}{2} + \boldsymbol{\xi_1}t + \boldsymbol{\xi_0}) - \tau^a c^2 M^{-1}K\mathfrak{K}_1 * (\boldsymbol{\xi_2}t + \boldsymbol{\xi_1}) \\ &- \delta M^{-1}K\mathfrak{K}_2 * \boldsymbol{\xi_2} \\ &\in L^\infty(0,T).\end{aligned}$$

By [45, Ch. 2, Theorem 3.5], the system has a unique solution $\boldsymbol{\mu} \in L^\infty(0,T)$. We then consider the problem

$$\begin{cases}\mathfrak{K}_1 * \boldsymbol{\xi_{ttt}} = \boldsymbol{\mu}, \\ (\boldsymbol{\xi}, \boldsymbol{\xi_t}, \boldsymbol{\xi_{tt}})|_{t=0} = (\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)\end{cases}$$

by rewriting it equivalently as

$$\begin{cases}\boldsymbol{\xi_{tt}} = \tilde{\mathfrak{K}}_1 * \boldsymbol{\mu} + \boldsymbol{\xi}_2 \in L^\infty(0,T), \\ (\boldsymbol{\xi}, \boldsymbol{\xi_t})|_{t=0} = (\boldsymbol{\xi}_0, \boldsymbol{\xi}_1),\end{cases}$$

which has a unique solution $\boldsymbol{\xi} \in W^{2,\infty}(0,T)$. In this way we obtain existence of a unique approximate solution $u^{(n)} \in W^{2,\infty}(0,T;V_n)$.

Appendix B. Proof of Theorem 3.3

We present here the proof of the τ-uniform well-posedness stated in Theorem 3.3.

Proof. Let $u^* \in \mathcal{B}^{\mathrm{KB}}$ with $\mathcal{B}^{\mathrm{KB}}$ defined in (3.26). Since this implies that $u^* \in \mathcal{X}^{\mathrm{KB}}$, the smoothness assumptions on $\mathfrak{a}$ and $\mathfrak{b}$ in (3.22) are fulfilled and the smallness assumption on $\mathfrak{a}$ given in (3.23) follows by reducing $R > 0$ (independently of τ). The non-degeneracy condition on $\mathfrak{a}$ is fulfilled for small enough R as well. Furthermore, we have

$$\begin{aligned}&\|\mathcal{N}(\nabla u^*, \nabla u_t^*)\|_{W^{1,1}(L^2)} \\ \lesssim& \|\nabla u^*\|_{L^\infty(L^4)}\|\nabla u_t^*\|_{L^1(L^4)} + \|\nabla u_t^*\|_{L^1(L^4)}\|\nabla u_t^*\|_{L^\infty(L^4)} + \|\nabla u^*\|_{L^\infty(L^4)}\|\nabla u_{tt}^*\|_{L^1(L^4)} \\ \leq& C(\Omega,T)R^2,\end{aligned}$$

where in the last line we have relied on the embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$. By employing bound (3.25) for the linear problem, we obtain

$$\begin{aligned}\|u\|_{\mathcal{X}^{\mathrm{KB}}} \leq& C_1 \exp\left(C_2T(1 + \|\Delta\mathfrak{b}\|_{H^1(L^2)})\right)\Big(\|u_0\|_{H^3}^2 + \|u_1\|_{H^3}^2 + \tau^a\|u_2\|_{H^2}^2 \\ &+ \|\mathcal{N}(\nabla u^*, \nabla u_t^*)\|_{W^{1,1}(L^2)}^2\Big).\end{aligned}$$

Since

$$\|\Delta\mathfrak{b}\|_{H^1(L^2)} \lesssim \|u_t^*\|_{L^2(H^2)} + \|u_{tt}^*\|_{L^2(H^2)} \lesssim_T R,$$

we have

$$\|u\|_{\mathcal{X}^{\mathrm{KB}}} \leq C_1 \exp\left(C_2T(1 + TR)\right)(r^2 + CR^4).$$

Therefore, $u \in \mathcal{B}^{\mathrm{KB}}$ for sufficiently small data size r and radius R.

Let $\mathcal{T}u^* = u$ and $\mathcal{T}v^* = v$. Set $\phi = u - v$ and $\phi^* = u^* - v^*$. Then the difference ϕ solves

$$\begin{aligned}&\tau^a\mathfrak{K}_1 * \phi_{ttt} + \mathfrak{a}(u_t^*)\phi_{tt} - c^2\mathfrak{b}(u_t^*)\Delta\phi - \tau^a c^2\mathfrak{K}_1 * \Delta\phi_t - \delta\mathfrak{K}_2 * \Delta\phi_{tt} \\ =& -2k_1\phi_t^* v_{tt} - 2k_2\phi_t^*\Delta v - 2k_3\nabla\phi^* \cdot \nabla u_t^* - 2k_3\nabla v^* \cdot \nabla\phi_t^* := f\end{aligned}$$

with homogeneous boundary and initial data. Note that as before we cannot prove contractivity with respect to the $\|\cdot\|_{\mathcal{X}^{\mathrm{KB}}}$ norm by exploiting the linear bound in $\mathcal{X}^{\mathrm{KB}}$ (see (3.25)), as the right-hand side of this equation does not belong to $W^{1,1}(0,T;H^1(\Omega))$. Instead, we test this equation with ϕ_{tt} and prove contractivity with respect to a lower-order norm. To this end, we rely on the following identity:

$$\begin{aligned}
&c^2\int_0^t\int_\Omega \mathfrak{b}(u_t^*)\Delta\phi\phi_{tt}\,\mathrm{d}x\mathrm{d}s\\
=&\,c^2\int_\Omega(1-2k_2u_t^*(t))\Delta\phi(t)\phi_t(t)\,\mathrm{d}x-c^2\int_0^t\int_\Omega(1-2k_2u_t^*)\Delta\phi_t\phi_t\,\mathrm{d}x\mathrm{d}s\\
&+2k_2c^2\int_0^t\int_\Omega u_{tt}^*\Delta\phi\phi_t\,\mathrm{d}x\mathrm{d}s,
\end{aligned}$$

from which we have

$$\begin{aligned}
&c^2\int_0^t\int_\Omega \mathfrak{b}(u_t^*)\Delta\phi\phi_{tt}\,\mathrm{d}x\mathrm{d}s\\
=&-c^2\int_\Omega(1-2k_2u_t^*(t))\nabla\phi(t)\cdot\nabla\phi_t(t)\,\mathrm{d}x+2k_2c^2\int_\Omega\nabla u_t^*(t)\cdot\nabla\phi(t)\phi_t(t)\,\mathrm{d}x\\
&+c^2\int_0^t\int_\Omega(1-2k_2u_t^*)\nabla\phi_t\cdot\nabla\phi_t\,\mathrm{d}x\mathrm{d}s-2k_2c^2\int_0^t\int_\Omega\nabla u_t^*\cdot\nabla\phi_t\phi_t\,\mathrm{d}x\mathrm{d}s\\
&-2k_2c^2\int_0^t\int_\Omega u_{tt}^*\nabla\phi\cdot\phi_t\,\mathrm{d}x\mathrm{d}s-2k_2c^2\int_0^t\int_\Omega\nabla u_{tt}^*\cdot\nabla\phi\phi_t\,\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Similarly to the proof of uniqueness for the linear problem in Section 3, we then have

$$\begin{aligned}
&\int_0^t\|\sqrt{\mathfrak{a}(u_t^*)}\phi_{tt}\|_{L^2}^2\,\mathrm{d}s+\|\nabla\phi_t\|_{L^\infty(L^2)}^2\\
\lesssim&\,\|f\|_{L^2(L^2)}^2+(1+\|u_t^*\|_{L^\infty(L^\infty)})\|\nabla\phi\|_{L^\infty(L^2)}\|\nabla\phi_t\|_{L^\infty(L^2)}\\
&+\|\nabla u_t^*\|_{L^\infty(L^\infty)}\|\nabla\phi_t\|_{L^\infty(L^2)}\|\phi_t\|_{L^\infty(L^2)}\\
&+\|u_t^*\|_{L^\infty(L^\infty)}\|\nabla\phi_t\|_{L^2(L^2)}^2+\|\nabla u_t^*\|_{L^\infty(L^\infty)}\|\nabla\phi_t\|_{L^2(L^2)}\|\phi_t\|_{L^2(L^2)}\\
&+\|u_{tt}^*\|_{L^2(L^\infty)}\|\nabla\phi\|_{L^2(L^2)}\|\phi_t\|_{L^\infty(L^2)}+\|\nabla u_{tt}^*\|_{L^2(L^4)}\|\nabla\phi\|_{L^\infty(L^2)}\|\phi_t\|_{L^2(L^4)}.
\end{aligned}\tag{B.1}$$

We note that

$$\begin{aligned}
\|f\|_{L^2(L^2)}\lesssim&\,\|\phi_t^*\|_{L^\infty(L^4)}\|v_{tt}\|_{L^2(L^4)}+\|\phi_t^*\|_{L^\infty(L^2)}\|\Delta v\|_{L^\infty(L^\infty)}\\
&+\|\nabla\phi^*\|_{L^\infty(L^2)}\|\nabla u_t^*\|_{L^2(L^\infty)}+\|\nabla v^*\|_{L^\infty(L^\infty)}\|\nabla\phi_t^*\|_{L^2(L^2)}.
\end{aligned}$$

Therefore, from (B.1) by employing Young's and Gronwall's inequalities, and the fact that u^*, v^*, $v\in\mathcal{B}^{\mathrm{KB}}$, we conclude that

$$\int_0^t\|\phi_{tt}\|_{L^2}^2\,\mathrm{d}s+\|\nabla\phi_t\|_{L^\infty(L^2)}^2\lesssim R^2\left(\int_0^t\|\phi_{tt}\|_{L^2}^2\,\mathrm{d}s+\|\nabla\phi_t\|_{L^\infty(L^2)}^2\right).$$

Thus, one can guarantee strict contractivity of the mapping $\mathcal{T}$ in the norm of the space $W^{1,\infty}(0,T;H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega))$ by reducing the radius R. □

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