Yang's system of particles and Hecke algebras

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Summary

The graded Hecke algebra has a simple realization as a certain algebra of operators acting on a space of smooth functions. This operator algebra arises from the study of the root system analogue of Yang's system of n particles on the real line with delta function potential. It turns out that the spectral problem for this generalization of Yang's system is related to the problem of finding the spherical tempered representations of the graded Hecke algebra. This observation turns out to be very useful for both these problems. Application of our technique to affine Hecke algebras yields a simple formula for the formal degree of the generic Iwahori spherical discrete series representations.

1. Introduction

Consider a finite dimensional real vector space V equipped with an inner product $(\cdot,\cdot)$. For $\alpha \in V$ a nonzero vector we denote by

$$r_\alpha(\xi) = \xi - (\xi, \alpha^\vee)\alpha \quad \forall \xi \in V \tag{1.1}$$

the orthogonal reflection in the mirror $V_\alpha = \{\xi \in V \mid (\xi,\alpha) = 0\}$. Here $\alpha^\vee = 2(\alpha,\alpha)^{-1}\alpha$ is the covector of α. A root system R in V will be a finite set of nonzero vectors (called roots) such that $\mathbf{R}\alpha \cap R = \{\pm\alpha\}$ and $r_\alpha(\beta) \in R \quad \forall \alpha,\beta \in R$. The reflections r_α for $\alpha \in R$ generate a real finite reflection group $W = W(R) \subset O(V)$. It can be shown that each reflection in W is of the form r_α for some $\alpha \in R$, and therefore each mirror of the finite reflection group $W(R)$ is perpendicular to two opposite roots in R. Conversely, given a finite reflection group W in $O(V)$ we can find root systems R such that $W(R) = W$. For example the set of unit normals of the mirrors of W is such a root system.

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The root systems occurring in semisimple Lie theory satisfy the additional requirement

$$(\beta, \alpha^\vee) \in \mathbf{Z} \quad \forall \alpha, \beta \in R, \tag{1.2}$$

and we refer to such R as integral root systems. However, for the purpose of this paper the integrality condition is unnecessary. Sometimes we shall use the normalization

$$(\alpha, \alpha) = 2 (\Leftrightarrow \alpha = \alpha^\vee) \quad \forall \alpha \in R, \tag{1.3}$$

in which case we speak of R as a normalized root system.

The symmetric algebra SV and the algebra PV of polynomial functions on V can be identified by means of the inner product on V. For $p \in PV$ we write $\partial(p) \in SV$, and think of $\partial(p)$ as a constant coefficient differential operator on V. For example $\Delta = \partial(\xi \to (\xi, \xi))$ is the Laplace operator on V, and $\partial(\alpha) = \partial(\xi \to (\xi, \alpha))$ is the derivative with respect to the root $\alpha \in R$. We denote SV^W and PV^W for the algebras of invariants for W.

Definition 1.1. *A coupling parameter $k = (k_\alpha)_{\alpha \in R}$ for R is a collection of real numbers k_α for $\alpha \in R$ with $k_{w\alpha} = k_\alpha \quad \forall \alpha \in R, w \in W$. Let K denote the $\mathbf{R}$-vector space of coupling parameters for R. The Yang system for R with coupling parameter $k \in K$ and spectral parameter $\lambda \in V_c = \mathbf{C} \otimes_{\mathbf{R}} V$ is the boundary value problem on V given by the differential equations*

$$\partial(p)\phi(\xi) = p(\lambda)\phi(\xi) \quad \forall p \in PV^W, \xi \in V \setminus \cup V_\alpha \tag{1.4}$$

and the boundary conditions

$$\phi(\xi + 0\alpha) = \phi(\xi - 0\alpha) \quad \forall \xi \in V_\alpha \tag{1.5}$$

$$\partial(\alpha)\phi(\xi + 0\alpha) - \partial(\alpha)\phi(\xi - 0\alpha) = 2k_\alpha \phi(\xi) \quad \forall \xi \in V_\alpha \tag{1.6}$$

along the arrangement of mirrors $\cup V_\alpha$.

The Yang system is the completely integrable quantum system associated with a particle moving in V according to the Schrödinger operator

$$-\Delta + \sum_{\alpha \in R} k_\alpha \delta((\alpha, \cdot)) \tag{1.7}$$

In the case of the symmetric group S_n acting on $\mathbf{R}^n$ by permutations of the coordinates one recovers the n-particle problem in one dimension with a delta-function potential as was originally studied by Yang [35], [36]. Likewise the case of the hyperoctahedral group $C_2^n \rtimes S_n$ acting on $\mathbf{R}^n$ by permutations and sign changes of the coordinates corresponds to the (2n+1)-particle problem in one dimension with a delta-function potential, and being constrained by the symmetry $x \to -x$ of $\mathbf{R}$. Now the coupling between the middle particle (located at the origin by the constraint) and one of the remaining 2n particles is allowed to be different from the coupling between two of the 2n remaining particles. For the exceptional root systems no such interpretation is available. Nevertheless from a mathematical point of view root systems are the natural framework for dealing with these kind of problems.

The connection with analogous problems in harmonic analysis on homogeneous spaces of semisimple groups will become clear in Section 2. In fact one might think of the Yang system as the infinitesimal version of the problem of decomposing $L^2(G/K)$ as a representation space of G with G a semisimple group over a nonarchimedean local field F and K the compact open subgroup of the elements that are defined over the ring of integers in F.

Let V_+ be a connected component of $V \setminus \cup V_\alpha$, and let $R_+ = \{\alpha \in R \mid (\xi, \alpha) > 0 \quad \forall \xi \in V_+\}$ be the corresponding set of positive roots. The choice of the chamber V_+ is fixed once and for all.

THEOREM 1.2. *Introduce the $\tilde{c}$-function for the Yang system as the rational function on the parameter space $V \times K$ (or its complexification) given by the formula*

$$\tilde{c}(\lambda, k) = \prod_{\alpha \in R_+} \frac{(\lambda, \alpha) + k_\alpha}{(\lambda, \alpha)}. \tag{1.8}$$

Let $V_{c,\mathrm{reg}} = V_c \setminus \cup V_{\alpha,c}$ denote the complement in V_c of the complexified mirrors. For $(\lambda, k) \in V_{c,\mathrm{reg}} \times K_c$ let the function $\phi(\lambda, k; \cdot)$ on V be given by

$$\phi(\lambda, k; \xi) = |W|^{-1} \sum_{w \in W} \tilde{c}(w\lambda, k) \mathrm{e}^{(w\lambda, \xi)} \tag{1.9}$$

for ξ in the closure of V_+, and extended to all of V as a W-invariant function. Then the function ϕ has an entire extension in the parameters $(\lambda, k) \in V_c \times K_c$, which is again denoted by ϕ. This function $\phi(\lambda, k; \cdot)$ is a solution of (1.4), (1.5) and (1.6), and is normalized by $\phi(\lambda, k; 0) = 1$. Moreover, each W-invariant solution of (1.4), (1.5), and (1.6) is a multiple of $\phi(\lambda, k; \cdot)$.

The proof of this theorem is straightforward and will be given in Section 2. The explicit formula (1.9) is analogous to Macdonald's explicit formula for the elementary spherical function on a p-adic semisimple group [25]. In Section 2 we also explain the role of the graded Hecke algebra for the Yang system. Once this role is clear it follows that the solution of the spectral problem for the Yang system for general wave functions is equivalent to the same problem for W-invariant wave functions together with some knowledge of the representation theory of graded Hecke algebras. The results of this section were inspired by work of Drinfeld [9]. It follows that for the rest of the paper we can (and will) restrict ourselves to the case of W-invariant wave functions.

Theorem 1.3. *Suppose the coupling parameter $k \in K$ is repulsive, i.e. $k_\alpha \geq 0 \quad \forall \alpha \in R$. For $f \in C_c^\infty(V_{\mathrm{reg}})^W$ we have the inversion formula*

$$f(\xi) = \int_{\lambda \in iV} \left\{ \int_{\eta \in V} f(\eta)\phi(-\lambda, k; \eta) d\mu_E(\eta) \right\} \phi(\lambda, k; \xi) d\mu_P(\lambda) \tag{1.10}$$

with μ_E the Euclidean measure on V, and the Plancherel measure μ_P on iV given by

$$d\mu_P(\lambda) = \frac{(2\pi)^{-n} d\mu_E(\mathrm{Im}(\lambda))}{\tilde{c}(\lambda, k)\tilde{c}(-\lambda, k)} \tag{1.11}$$

The proof of this theorem is sketched in Section 3. We use a contour shift argument due to Van den Ban and Schlichtkrull [2], which is an adaptation of the Helgason-Gangolli-Rozenberg argument in the proof of the Plancherel theorem for a Riemannian symmetric space G/K [15], [11].

We now drop the condition that k is repulsive, and fix $k \in K$ arbitrary. The contour shift forces one to take certain residues into account in this situation. In order to explain the outcome we need some more notations.

For $L \subset V$ an affine subspace we put $R_L = \{\alpha \in R \mid (L, \alpha) = \text{constant}\}$. If $V_L = \mathrm{span}(R_L)$ then it is clear that $R_L = R \cap V_L$ is a parabolic root subsystem of R.

Definition 1.4. *An affine subspace $L \subset V$ is defined to be residual (or more precisely (V, R, k)-residual) by induction on the codimension of L. The space V itself is by definition a residual subspace. The affine subspace $L \subset V$ with positive codimension is called residual if there is a residual subspace $M \subset V$ with $M \supset L$ and $\dim(M) = \dim(L) + 1$ such that*

$$\#\{\alpha \in R_L \backslash R_M \mid (L, \alpha) = k_\alpha\} \geq \#\{\alpha \in R_L \backslash R_M \mid (L, \alpha) = 0\} + 1 \tag{1.12}$$

A residual point is also called a distinguished (or more precisely (V, R, k) distinguished) point.

We have used the terminology residual because these are the subspaces where residues (caused by the poles in the Plancherel measure μ_P given in (1.11)) can be picked up when we shift the contour. The word distinguished is used in accordance with the classification of nilpotent orbits in the semisimple Lie algebras as exposed in Carter's book [6, Ch 5]. Since $w(R_L) = R_{wL} \quad \forall w \in W$ it is clear that the notion of residual subspace is W-invariant. For each affine subspace $L \subset V$ it is clear that $\mathrm{codim}(L) \geq \mathrm{rank}(R_L)$. However by induction on $\mathrm{codim}(L)$ it is easy to see that $\mathrm{codim}(L) = \mathrm{rank}(R_L)$ for $L \subset V$ a residual subspace. If $L \subset V$ is an affine subspace with $\mathrm{codim}(L) = \mathrm{rank}(R_L)$ then $L = c_L + V^L$ with c_L the center of L determined by $\{c_L\} = L \cap V_L$ and V^L the orthogonal complement of V_L in V.

It is easy to see from the above definition that an affine subspace $L \subset V$ is (V, R, k)-residual if and only if $\mathrm{codim}(L) = \mathrm{rank}(R_L)$ and $c_L \in V_L$ is a (V_L, R_L, k_L)-distinguished point. Here $k_L = (k_\alpha)_{\alpha \in R_L}$ is the restriction of the coupling parameter k to R_L. The complete determination of the residual subspaces therefore boils down by induction on $\mathrm{rank}(R)$ to the determination of the distinguished points. In Section 4 we will carry out the classification of distinguished points for each of the irreducible root systems case by case. For R an integral root system and $k_\alpha = k_\beta \quad \forall \alpha, \beta$ this classification is equivalent to the classification of nilpotent orbits in semisimple Lie algebras by their weighted Dynkin diagram. For R of type ADE we recover the tables in [6]. For R of type BFI(even) with 2 coupling parameters and for R of type HI(odd) with 1 coupling parameter these results seem to be new.

There is a twofold reason for actually doing this classification. On the one hand the sum $\sum_L$ in formula (1.14) below becomes more explicit for a given R. On the other hand we are able to prove several properties of residual subspaces-easily stated in general root system terminology and crucially needed in the proof of the result below-only by verification using the classification. Although the concept of residual subspace is simple minded enough it seems that some understanding is lacking.

THEOREM 1.5. *Suppose the coupling parameter k is attractive, i.e. $k_\alpha < 0 \quad \forall \alpha \in R$. For each residual subspace $L \subset V$ the residue formula*

$$\nu_L = (-2\pi i)^{\mathrm{codim}(L)} \mathrm{res}_L(\mu_P) \tag{1.13}$$

defines a nonnegative analytic measure on $c_L + iV^L$, and for $f \in C_c^\infty(V_{\mathrm{reg}})^W$ we have:

$$f(\xi) = \sum_L \int_{c_L + iV^L} \left\{ \int_{\eta \in V} f(\eta) \phi(-\lambda, k; \eta) d\mu_E(\eta) \right\} \phi(\lambda, k; \xi) d\nu_L(\lambda) \tag{1.14}$$

with $\sum_L$ denoting the sum over all the residual subspaces.

The meaning of the residue formula (1.13) will be explained in Section 3, where the theorem is also proved. It follows that the Plancherel measure $\nu_P = \sum_L \nu_L$ is a W-invariant measure on V_c with support contained in $\cup_L \{c_L + iV^L\}$. However the support of ν_P can be strictly smaller. Because the measure ν_L is analytic with respect to the Euclidean measure on $c_L + iV^L$ we have either $\nu_L = 0$ or $\mathrm{supp}(\nu_L) = c_L + iV^L$.

DEFINITION 1.6. *Let $L \subset V$ be a residual subspace. The real affine subspace $c_L + iV^L$ of V_c is called spherical tempered (or more precisely (V, R, k)-spherical tempered) if $\mathrm{supp}(\nu_L) = c_L + iV^L$. If in addition $L = \{c_L\}$ has dimension 0 then c_L is called a spherical cuspidal (or more precisely (V, R, k)-spherical cuspidal) point.*

Being a spherical tempered subspace is clearly W-invariant. Similarly as with the notion of residual subspace we have that $c_L + iV^L$ is a (V, R, k)-spherical tempered subspace if and only if $c_L \in V_L$ is a (V_L, R_L, k_L)-spherical cuspidal point. Therefore the determination of the spherical tempered spectrum reduces by induction on the rank of R to the determination of the spherical cuspidal points. In Section 3 we will show that $\lambda \in V$ is a spherical cuspidal point if and only if $\phi(\lambda, k; \cdot) \in L^2(V, \mu_E)$.

THEOREM 1.7. *If R is an integral root system and $k_\alpha = k_\beta < 0 \quad \forall \alpha, \beta \in R$ then for each residual subspace $L \subset V$ the subspace $c_L + iV^L$ is spherical tempered.*

This theorem follows from the work of Kazhdan and Lusztig on the geometric classification of the irreducible representations of affine Hecke algebras [18]. For $\lambda \in V_{\text{reg}}$ a distinguished point there is an easy criterion for λ to be spherical cuspidal. However for singular λ the actual residue computation can be very cumbersome. For all irreducible root systems with the exception of B_n and H_4 we have been able to give the classification of the spherical cuspidal points. For type B_n we can only handle the case of regular and subregular points and for type H_4 we left the singular distinguished points aside. All these results are given in Section 4. As a consequence of the tables it follows that Theorem 1.7 need no longer be true for R of type H or of type BFI(even) with two possibly distinct negative coupling parameters.

Finally let us return to the case of the symmetric group acting on $\mathbf{R}^n$ by permutations of the coordinates. In this case with an attractive coupling parameter $k < 0$ the $\sum_L$ in the inversion formula (1.14) reduces to a sum over the partitions of n. Each partition $n = n_1 + \cdots + n_r$ gives a separate r-dimensional contribution to the spectrum. The interpretation is that each group of n_j particles is internally bounded and only its center of mass has unbounded motion. This outcome was already obtained by Yang as a result of his computation of the scattering matrix [36]. A mathematically more rigorous derivation of this result was given by Oxford in his thesis [30]. From the point of view of our paper the root system of type A_{n-1} is particularly simple because singular distinguished points are absent. Of the other irreducible root systems only the dihedral type I_2(odd) and the icosahedral type H_3 have the same simplifying feature.

2. Graded Hecke algebras

We keep the notation of the introduction. For $f \in C^\infty(V)$ a smooth function on V define $I(\alpha)f \in C^\infty(V)$ for $\alpha \in R$ by the formula

$$I(\alpha)f(\xi) = \int_0^{(\xi,\alpha^\vee)} f(\xi - t\alpha)dt \quad (\xi \in V) \tag{2.1}$$

Let W act on $C^\infty(V)$ as usual: $wf(\xi) = f(w^{-1}\xi)$. Let $\alpha_1, \dots, \alpha_n$ be the set of simple roots in R_+, and $r_1, \dots, r_n$ the corresponding set of simple reflections. Define operators $Q(r_j, k)$ on $C^\infty(V)$ by $Q(r_j, k) = r_j + k_j I(\alpha_j)$ with $k_j = k_{\alpha_j}$. An easy computation shows that $Q(r_j, k)^2 = 1$.

Theorem 2.1. *If $m_{i,j}$ denotes the order of the element $r_i r_j \in W$ then*

$$Q(r_i, k)Q(r_j, k) \cdots = Q(r_j, k)Q(r_i, k) \dots \quad (i \neq j) \tag{2.2}$$

with $m_{i,j}$ factors on both sides.

In the case of the symmetric group this result goes back to Yang [35] and the general case is due to Gutkin [12]. An immediate consequence of the presentation of W as a Coxeter group on the generators $r_1, \dots, r_n$ (see for example [4] or [16] for the necessary background on reflection groups) is that for $w \in W$ with $w = r_{i_1} \dots r_{i_p}$ a reduced expression, the operator

$$Q(w, k) = Q(r_{i_1}, k) \dots Q(r_{i_p}, k) \tag{2.3}$$

on $C^\infty(V)$ is well defined independently of the choice of the reduced expression. The map $w \to Q(w, k)$ defines a representation of W on $C^\infty(V)$. It is easily verified that

$$Q(r_i, k)\partial(\xi) - \partial(r_i(\xi))Q(r_i, k) = k_i(\xi, \alpha_i^\vee) \tag{2.4}$$

for $r_i \in W$ a simple reflection and $\xi \in V$.

Definition 2.2. *The graded Hecke algebra $\mathcal{H}(R_+, k)$ is the $\mathbf{C}$-vectorspace $S(V_c) \otimes \mathbf{C}[W]$ equiped with the unique associative algebra structure such that $S(V_c) \otimes 1 \simeq S(V_c)$ and $1 \otimes \mathbf{C}[W] \simeq \mathbf{C}[W]$ have their usual algebra structure and*

$$r_i \cdot \xi - r_i(\xi) \cdot r_i = k_i(\xi, \alpha_i^\vee) \tag{2.5}$$

for $r_i \in W$ a simple reflection and $\xi \in V$.

This algebra structure was introduced independently by Drinfeld as the degenerate Hecke algebra [9], by Kostant and Kumar as the nil Hecke ring [20] and by Lusztig as the graded Hecke algebra [22]. In this paper we use the latter terminology. Observe that our notation differs slightly from the one in [29]: positive and negative roots have been interchanged, and we use roots instead of coroots.

Corollary 2.3. The map $w \to Q(w,k)$, $\xi \to \partial(\xi)$ defines a representation of the graded Hecke algebra $\mathcal{H}(R_+,k)$ on $C^\infty(V)$.

To each $f \in C^\infty(V)$ we associate a continuous function $f_+ \in C(V)$ by means of the formula

$$f_+(w^{-1}\xi) = Q(w,k)f(\xi) \tag{2.6}$$

for $w \in W$ and ξ in the closure of V_+. It is easy to see that f_+ is smooth on V_{reg} and satifies the boundary conditions (1.5) and (1.6) along the mirors $\cup V_\alpha$. Moreover $f \to f_+$ is an injective linear map. Define an inner product $(\cdot,\cdot)_k$ on $C^\infty(V)$ depending on k by

$$(f,g)_k = (f_+,g_+) = \sum_w \int_{V_+} Q(w,k)f(\xi)\overline{Q(w,k)g(\xi)}d\mu_E(\xi). \tag{2.7}$$

Here $(\cdot,\cdot)$ denotes the ordinary inner product for functions on V. This turns $\{f \in C^\infty(V) \mid (f,f)_k < \infty\}$ into a pre Hilbert space. Consider the $*$-structure on $\mathcal{H}(R_+,k)$ defined by $w^* = w^{-1}$ for $w \in W$ and $\xi^* = -w_0 \cdot w_0(\xi) \cdot w_0$ for $\xi \in V$ and extended to all of $\mathcal{H}(R_+,k)$ as an anti-linear anti-involution. Here $w_0 \in W$ is the longest element.

Theorem 2.4. *The representation of $\mathcal{H}(R_+,k)$ on the space $\mathcal{C}(V,k) = \{f \in C^\infty(V) \mid (\partial(p)f, \partial(p)f)_k < \infty \quad \forall p \in P(V)\}$ is (pre)unitary.*

Proof. As a consequence of the relations for the graded Hecke algebra (cf. [29], Prop. 1.1) we have

$$Q(w) \cdot \partial(\xi) \cdot Q(w^{-1}) = \partial(w\xi) - \sum_{\alpha>0, w^{-1}\alpha<0} k_\alpha(w\xi, \alpha^\vee)Q(r_\alpha)$$

and

$$Q(ww_0) \cdot \partial(w_0\xi) \cdot Q(w_0w^{-1}) = \partial(w\xi) - \sum_{\alpha>0, w^{-1}\alpha>0} k_\alpha(w\xi, \alpha^\vee)Q(r_\alpha)$$

Hence for $\xi, \eta \in V$ and $f, g \in C^\infty(V)$ we get

$$\begin{aligned}
&\sum_w \Big\{ Q(w)\partial(\xi)f(\eta)\overline{Q(w)g(\eta)} + Q(w)f(\eta)\overline{Q(ww_0)\partial(w_0\xi)Q(w_0)g(\eta)} \Big\} \\
&= \sum_w \Big\{ Q(w)\partial(\xi)Q(w^{-1})Q(w)f(\eta)\overline{Q(w)g(\eta)} \\
&\qquad\qquad + Q(w)f(\eta)Q(ww_0)\partial(w_0\xi)Q(w_0w^{-1})\overline{Q(w)g(\eta)} \Big\}
\end{aligned}$$

$$= \sum_w \Big\{\partial(w\xi)(Q(w)f(\eta))\overline{Q(w)g(\eta)} + Q(w)f(\eta)\partial(w\xi)\overline{Q(w)g(\eta)}\Big\}$$
$$- \sum_w \sum_{\alpha>0, w^{-1}\alpha<0} k_\alpha(w\xi,\alpha^\vee)Q(r_\alpha w)f(\eta)\overline{Q(w)g(\eta)}$$
$$- \sum_w \sum_{\alpha>0, w^{-1}\alpha>0} k_\alpha(w\xi,\alpha^\vee)Q(w)f(\eta)\overline{Q(r_\alpha w)g(\eta)}$$
$$= \sum_w \partial(w\xi)\Big(Q(w)f(\eta)\overline{Q(w)g(\eta)}\Big)$$

using the substitution $w \to r_\alpha w$ in the second term to obtain the cancellation. Hence if $\xi \in V$ and $f, g \in \mathcal{C}(V,k)$ we get (writing $h_w(\eta) = Q(w)f(\eta)\overline{Q(w)g(\eta)}$):

$$\begin{aligned}
&(\partial(\xi)f,g)_k + (f, Q(w_0)\partial(w_0\xi)Q(w_0)g)_k \\
&= \sum_w \int_{V_+} \partial(w\xi)h_w(\eta)d\mu_E(\eta) \\
&= \sum_w \int_{\partial(V_+)} h_w(\eta)(w\xi,\nu)d\sigma_E(\eta)
\end{aligned}$$

by Stokes theorem. Here ν is an outer normal and σ_E the Euclidean volume element for the boundary ∂V_+. In turn this can be rewritten as

$$\begin{aligned}
&\sum_w \sum_{i=1}^n \int_{\overline{V_+}\cap V_{\alpha_i}} h_w(\eta)(w\xi, \frac{\alpha_i}{|\alpha_i|})d\sigma_i(\eta) \\
&= \sum_{i=1}^n \int_{\overline{V_+}\cap V_{\alpha_i}} \Big\{ \sum_{w^{-1}\alpha_i>0} h_w(\eta)(w\xi,\frac{\alpha_i}{|\alpha_i|}) + \sum_{w^{-1}\alpha_i<0} h_w(\eta)(w\xi,\frac{\alpha_i}{|\alpha_i|})\Big\} d\sigma_i(\eta)
\end{aligned}$$

and the two terms cancel using the sustitution $w \to r_i w$ in the second term (taking into account that $Q(r_i)h = h$ on V_{α_i} for $h \in C^\infty(V)$). $\square$

The center of the graded Hecke algebra $\mathcal{H}(R_+,k)$ is equal to $S(V_c)^W$. Therefore the space $E(\lambda) = \{\phi \in C^\infty(V) \mid \partial(p)\phi = p(\lambda)\phi \quad \forall p \in P(V)^W\}$ carries a natural representation of $\mathcal{H}(R_+,k)$, which is called the eigenspace representation of $\mathcal{H}(R_+,k)$ with spectral parameter (or central character) $\lambda \in V_c$. Note that $E(\lambda) = \{\sum_\mu p_\mu \mathrm{e}^\mu \mid p_\mu \text{ is a } W_\mu - \text{harmonic polynomial} \quad \forall \mu \in W\lambda\}$ has dimension $|W|$, and as a $\mathbf{C}[W]$-module (by restriction of the module $E(\lambda)$ to the subalgebra $\mathbf{C}[W]$ of $\mathcal{H}(R_+,k)$) it is equivalent to the regular representation of W. Indeed, this is obvious when $k = 0$ and λ is regular and the representation theory of the finite group W only admits trivial deformations.

For $\lambda \in V_c$ regular one finds the expression

$$\phi(\lambda,k;\cdot) = |W|^{-1}\sum_w Q(w,k)(\mathrm{e}^\lambda) = |W|^{-1}\sum_w \tilde{c}(w\lambda,k)\mathrm{e}^{w\lambda} \tag{2.8}$$

Indeed, it is easy to check by induction on $l(w)$ that

$$Q(w,k)(\mathrm{e}^{\lambda}) = \left\{ \prod_{\alpha>0, w^{-1}\alpha<0} \frac{(w\lambda,\alpha)+k_{\alpha}}{(w\lambda,\alpha)} \right\} \mathrm{e}^{w\lambda}$$

modulo terms $\mathrm{e}^{v\lambda}$ with $v \in W$ and $v < w$ in the Bruhat ordering. Hence the coefficient of $\mathrm{e}^{w_0\lambda}$ in (2.8) is correct, and (2.8) follows by W-invariance in the spectral parameter. Note that the function (2.8) is the unique spherical vector in $E(\lambda)$ normalized to be 1 at the origin. The usual argument shows that the $\mathcal{H}(R_+,k)$-module $U(\lambda,k)$ generated by the spherical vector (2.8) is the unique submodule of $E(\lambda)$. In particular, $U(\lambda,k)$ is irreducible. It will be shown in Section 3 (Corollary 3.8) that the spherical vector $\phi(\lambda,k;\cdot)$ is in $L^2(V,\mu_E)$ if and only if λ is a spherical cuspidal point. Theorem 1.7 therefore states that if R is integral and the root labels are equal and negative then all distinguished points give rise to a spherical cuspidal module $U(\lambda,k)$ for the graded Hecke algebra. As was mentioned before, this is not true in general. One might conjecture that it is still true in general that distinguished points correspond to the existence of cuspidal subquotients of $E(\lambda)$ which are no longer necessarily spherical. Indeed, when λ is regular it is not hard to show this using Rodier's theorem [33].

The content of Theorem 1.2 from the introduction is clear now. The above also justifies the statement made right after this theorem about the reduction of the case of general wave functions to the case of W-invariant ones. Indeed the additional knowledge required is the $\mathbf{C}[W]$-type decomposition of the irreducible modules $U(\lambda,k)$.

3. The contour shift

Let V be a real Euclidean space of dimension n and V_c its complexification. Let $\mathcal{H}$ be a finite affine hyperplane arrangement in V. For each $H \in \mathcal{H}$ choose $(\alpha_H, k_H) \in V \times \mathbf{R}$ such that $H = \{\xi \in V \mid (\xi,\alpha_H) = k_H\}$. Let $\mathcal{L}$ denote the lattice of intersections of elements from $\mathcal{H}$, ordered by inclusion (and containing V itself). For $L \in \mathcal{L}$ the center c_L is defined as the unique point of L with minimal distance to $O = c_V$. Write $\mathcal{C} = \{c_L \mid L \in \mathcal{L}\}$, and let V^L be the linear subspace of V such that $L = c_L + V^L$.

Let ω be a rational n-form on V_c with poles in $\cup H_c$ only. Fix an orientation on V (with an induced orientation on $\gamma + iV \quad \forall \gamma \in V \setminus \cup H$), and consider the linear functional

$$X_{V,\gamma} : PW(V_c) \to \mathbf{C}, \quad X_{V,\gamma}(F) = \int_{\gamma+iV} F\omega \tag{3.1}$$

on the space $PW(V_c)$ of Paley-Wiener functions on V_c (which are rapidly decreasing in the imaginary direction and of exponential type in the real direction).

LEMMA 3.1. *There exists a unique collection of tempered distributions X_c $(c \in \mathcal{C})$ on iV such that*

(1) $\mathrm{supp}(X_c) \subset \cup iV^L$ *(union over $L \in \mathcal{L}$ with $c_L = c$),*
(2) *X_c has finite order,*
(3) $X_{V,\gamma}(F) = \sum_{c\in\mathcal{C}} X_c\Big(F(c+\cdot)\Big) \quad \forall F \in PW(V_c)$.

Proof. The existence follows by induction on $n = \dim(V)$. If $n = 0$ there is nothing to prove. Suppose the lemma holds for $\dim(V) = n-1$. Choose a path in V from γ to the origin which intersects each $H \in \mathcal{H}$ transversally in at most one point γ_H. We may assume that $\gamma_H \notin H' \quad \forall H' \in \mathcal{H},\ H' \neq H$ if $\gamma_H \neq O$. When we pass a hyperplane H at γ_H we apply Cauchy's theorem to obtain an extra contribution of the form (with $d+1$ the pole order of ω along H):

$$\sum_{j=0}^{d} X^j_{H,\gamma_H}(\partial(\alpha_H)^j F|_{H_c})$$

with

$$X^j_{H,\gamma_H}(G) = \int_{\gamma_H + iV^H} G\omega_j$$

for some rational $(n-1)$-form ω_j on H_c which is regular outside $\cup_{H'\neq H}(H' \cap H)_c$. The induction hypothesis takes care of these contributions. Finally when we approach O along the path we have to take a boundary value of a meromorphic function with moderate growth.

We now prove the uniqueness. Suppose we are given a collection of tempered ditributions Y_c $(c \in \mathcal{C})$ on iV such that

(1) $\mathrm{supp}(Y_c) \subset \cup iV^L$ (union over $L \in \mathcal{L}$ with $c_L = c$),
(2) Y_c has finite order,
(3) $\sum_{c\in\mathcal{C}} Y_c\Big(F(c+\cdot)\Big) = 0 \quad \forall F \in PW(V_c)$.

We show that $Y_c = 0$ for $c \in \mathcal{C}$ by induction on $|c|$. Assume $c \in \mathcal{C}$ and $Y_{c'} = 0 \quad \forall c' \in \mathcal{C}$ with $|c'| < |c|$. For each $L \in \mathcal{L}$ with $c_L \neq c$ and $|c_L| \geq |c|$ we can choose $(\beta_L, l_L) \in V \times \mathbf{R}$ such that $(L, \beta_L) = l_L$ and $(c, \beta_L) \neq l_L$. Hence the polynomial $p(\cdot) = \prod((\cdot, \beta_L) - l_L)$ with the product taken over all such L satisfies $p(c + i\lambda) \neq 0 \quad \forall \lambda \in V$ and $p(L_c) = 0$ for all $L \in \mathcal{L}$ with $c_L \neq c$ and $|c_L| \geq |c|$. Hence if $N \in \mathbf{N}$ is large enough we get $\forall F \in PW(V_c)$:

$$0 = \sum_{c'\in\mathcal{C}} Y_{c'}\Big(p^N F(c' + \cdot)\Big) = Y_c\Big(p^N F(c + \cdot)\Big)$$

which in turn implies $Y_c = 0$. $\square$

Remark 3.2. We call X_c $(c \in \mathcal{C})$ the local contribution at c for the contour shift of the integral (3.1). If $U \in V$ is a ball containing $\mathcal{C}$ and γ then it is clear that the above lemma also holds for functions F of the form $F = rG$ with $G \in PW(V_c)$ and r rational and regular inside the tube $U + iV$. This can be used to calculate the local contribution X_c at c as follows. Let U be a small ball with center c such that $H \cap U = \emptyset$ for $H \in \mathcal{H}$ with $c \notin H$. Let γ' and O' be the images of γ and O under a central contraction with center c, such that $\gamma',\ O' \in U$. When we take paths from γ to γ' and from O to O' and carry out the contour shift as in the above lemma we will get no contributions to X_c. Indeed, by choosing appropriate paths we only pass hyperplanes $H \in \mathcal{H}$ with $c \notin H$. It follows that we can calculate X_c by applying Lemma 3.1 to

$$\int_{\gamma'+iV} F'\omega'$$

with respect to the new origin O'. Here $\omega = r\omega'$ with r regular inside $U + iV$ and containing all poles of ω outside $U + iV$, and $F' = rF$. The conclusion is that in order to calculate the local contribution X_c it suffices to consider the associated central arrangement $\{H \in \mathcal{H} \mid c \in H\}$ only.

Lemma 3.3. *Let $\mathcal{H} = \{H\}$ be a finite hyperplane arrangement in V, $\mathcal{L} = \{L\}$ its intersection lattice, and $\mathcal{C} = \{c_L \mid L \in \mathcal{L}\}$ the centers as before. Assume that for each $L \in \mathcal{L}$ one has $c_L \in H$ for some $H \in \mathcal{H}$ if and only if $L \subset H$ (in particular $O = c_V$ lies outside $\cup H$). If $\mathcal{H}' = \{H \in \mathcal{H} \mid H \text{ separates } \gamma \text{ and } O\}$ and $\mathcal{H}'' = \mathcal{H} \backslash \mathcal{H}'$, then for $c \in \mathcal{C}$ we have $X_c = 0$ unless $c \in \sum_{H \in \mathcal{H}'} \mathbf{R}_+ c_H + \sum_{H \in \mathcal{H}''} \mathbf{R}_- c_H$.*

Proof. By the previous remark it suffices to consider the case that $\mathcal{H}$ is a central arrangement with center c. Moreover we can also assume that $\cap H = \{c\}$, and that ω has the form

$$\omega = \frac{d\lambda}{\prod_H ((\lambda, \alpha_H) - k_H)^{d_H}} \tag{3.2}$$

for certain integers $d_H \geq 1$. In fact we can assume that $d_H = 1 \quad \forall H$, and $\cup H$ is a divisor with normal crossings. Indeed, the differential form

$$\omega_\epsilon = \frac{d\lambda}{\prod_H \prod_{j=1}^{d_H} ((\lambda, \alpha_H) - k_H - j\epsilon_H)}$$

with $\epsilon = (\epsilon_H) \in \mathbf{R}^{\mathcal{H}}$ a perturbation parameter satisfies

$$\lim_{\epsilon \to 0} \int_{\gamma+iV} F\omega_\epsilon = \int_{\gamma+iV} F\omega$$

for all $F \in PW(V_c)$. For ϵ generic this reduces (again using Remark 3.2) to the case that $\cup H$ is a divisor with normal crossings and ω a form with simple poles along $\mathcal{H}$.

Let $\mathcal{D} = \{D\}$ be the hyperplane arrangement centered at c dual to $\mathcal{H}$: $D \in \mathcal{D} \Leftrightarrow c \in D$ and $D \perp L$ for some $L \in \mathcal{L}$ with $\dim(L) = 1$. Again $\cup D$ is a divisor with normal crossings. Both $V \backslash \cup H$ and $V \backslash \cup D$ consist of 2^n connected components (called hyperoctants), which are open convex simplicial cones. These two sets of hyperoctants are in natural duality. Clearly the outcome of X_c as far as γ is concerned depends only on the hyperoctant C_1 of $V \backslash \cup H$ containing γ (Cauchy). On the other hand if the origin moves in the hyperoctant C_2 of $V \backslash \cup D$ containing O then the points c_L move on $\cup L$ without confluence. This implies that as far as O is concerned, X_c depends only on the hyperoctant C_2 (Cauchy). Also observe that it follows from our assumptions that O actually lies in the complement of $\cup D$.

We claim that the local contribution $X_c = 0$ unless C_1 and C_2 are antidual hyperoctants: $c+\lambda \in C_1$ for some $\lambda \in V \Leftrightarrow (\lambda, \mu) < 0 \quad \forall \mu \in V$ with $c+\mu \in C_2$. Indeed if C_1 and C_2 are not antidual then there exists $L \in \mathcal{L}$ with $\dim(L) = 1$ and $c_L \in \overline{C_1} \backslash \{c\}$. Let $D \in \mathcal{D}$ with $D \perp L$ and D' the hyperplane in V through c_L parallel to D. Following the path $[\gamma, c_L] \cup [c_L, O]$ the computation is reduced to one in the hyperplane D'. The only residues possibly picked up under the contour shift are those whose centers lie in D'. Hence $X_c = 0$. $\square$

Remark 3.4. Remark In the notation of the proof of the lemma suppose that $(\gamma, \alpha_H) < k_H \quad \forall H \in \mathcal{H}$ and that $\cup H$ is a divisor with normal crossings such that $\cap H = \{c\}$. Number the elements of $\mathcal{H}$ and assume the basis $\{\alpha_H \mid H \in \mathcal{H}\}$ is positively oriented with respect to the fixed orientation on V. Taking for $d\lambda$ the positively oriented Euclidean n-form $\Big(\det(\alpha_H, \alpha_{H'})\Big)^{-1/2} \wedge_H d\alpha_H$ the outcome of the local contribution X_c in the case where C_1 and C_2 are antidual hyperoctants is given by (with ω given by (3.2) and $d_H = 1 \quad \forall H$):

$$X_c\Big(F(c+\cdot)\Big) = (-2\pi i)^n \Big(\det(\alpha_H, \alpha_{H'})\Big)^{-1/2} F(c) \tag{3.3}$$

$\forall F \in PW(V_c)$. For example for $n = 1$ we have indeed

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{F(z)dz}{\alpha z - k} = (-2\pi i)\mathrm{res}_c\left(\frac{F(z)}{\alpha z - k}\right) + \int_{-i\infty}^{+i\infty} \frac{F(z)dz}{\alpha z - k}$$

if $\alpha > 0$ and $\gamma < c = k/\alpha < 0$.

Now let us consider the Fourier-Yang transform

$$\mathcal{F}(k)f(\lambda) = \int_{\eta \in V} f(\eta)\phi(-\lambda, k, ; \eta)d\mu_E(\eta) \tag{3.4}$$

for $f \in C_c^\infty(V_{\text{reg}})^W$, and the candidate inversion operator

$$\mathcal{J}(k)F(\xi) = (2\pi)^{-n} \int_{\lambda \in \gamma + iV} F(\lambda) e^{(\lambda,\xi)} \frac{d\mu_E(\text{Im}\lambda)}{\tilde{c}(-\lambda, k)} \tag{3.5}$$

for $F \in PW(V_c)^W$. Here $\xi \in \overline{V_+}$ and $\gamma \in V_-$ far away from walls, and $\mathcal{J}(k)F$ is extended to all of V as a W-invariant function. For $f \in C_c^\infty(V_{\text{reg}})^W$ it is clear from the Euclidean Paley-Wiener theorem that $\mathcal{F}(k)f \in PW(V_c)$. Moreover if $\mathcal{K}(k)$ denotes the composition $\mathcal{J}(k) \circ \mathcal{F}(k)$ then $\mathcal{K}(k)f$ is smooth on $\overline{V_+}$. As in Helgason's proof of the Paley-Wiener theorem for Riemannian symmetric spaces [15], sending γ off to infinity shows that the support of $\mathcal{K}(k)f$ has to be contained in the convex hull of the support of f. Suppose now that we are in the attractive case $k_\alpha > 0 \quad \forall \alpha \in R$. In this situation we are also allowed to simply shift γ towards the origin without picking up residues. It is easy to see that we may now rewrite (3.5) as follows:

$$\mathcal{J}(k)F(\xi) = \int_{\lambda \in iV} F(\lambda)\phi(\lambda, k; \xi) d\mu_P(\lambda) \tag{3.6}$$

by the W-invariace of F and μ_P. From (3.6) we easily derive the formula

$$(\mathcal{K}(k)f, g) = \int_{\lambda \in iV} \mathcal{F}(k)f(\lambda)\overline{\mathcal{F}(k)g(\lambda)} d\mu_P(\lambda) \tag{3.7}$$

for $f, g \in C_c^\infty(V_{\text{reg}})^W$, which shows that $\mathcal{K}(k)$ is a (formally) symmetric operator. Together with the above mentioned Paley-Wiener theorem this shows that in the repulsive case $\mathcal{K}(k)$ is a support preserving operator. By Peetre's theorem [31] we now know that $\mathcal{K}(k)$ is a differential operator on V_{reg}. It is clear that $\mathcal{K}(k)$ commutes with all W-invariant differential operators on V, and therefore $\mathcal{K}(k)$ is itself a constant coefficient differential operator. Finally a scaling argument shows that $\mathcal{K}(k) = \text{Id}$. This proves Theorem 1.3. For more details on this argument of Van den Ban and Schlichtkrull see [2], [14], [13] and [29]. Let us now return to the general, not necessarily attractive case. Clearly the formulas (3.6) and (3.7) are no longer valid now because we have to take into account the residues that one picks up when moving the contour of integration. However the inversion formula still holds:

Proposition 3.5. *$\mathcal{K}(k) = \text{Id} \quad \forall k \in K$.*

Proof. It is easy to see that $\mathcal{J}(k)F$ is holomorphic in k ($\forall F \in PW(V_c)^W$ fixed) and that $\mathcal{F}(k)f$ is a polynomial in k ($\forall f \in C_c^\infty(V_{\text{reg}})^W$ fixed). Hence the general result follows from the attractive case. $\square$

In the remainder of this section we shall derive the formulas that replace (3.6) and (3.7) when we are dealing with the purely attractive case $k_\alpha < 0 \quad \forall \alpha \in R$. Hence from now on in this section we shall assume we are in the purely attractive case. We are going to study the linear functionals X_c and Y_c on $PW(V_c)$ defined by ($\gamma \in V_{\mathrm{reg}}$):

$$X_{V,\gamma}(F) = \int_{\lambda\in\gamma+iV} F(\lambda)\frac{d\mu_E(\mathrm{Im}\lambda)}{\tilde{c}(-\lambda,k)} \tag{3.8}$$

(cf. (3.5)) and

$$Y_{V,\gamma}(F) = \int_{\lambda\in\gamma+iV} F(\lambda)\frac{d\mu_E(\mathrm{Im}\lambda)}{\tilde{c}(\lambda,k)\tilde{c}(-\lambda,k)} \tag{3.9}$$

Let $H_\alpha = \{\lambda \in V \mid (\lambda,\alpha) = k_\alpha\}$ for $\alpha \in R$, and put $\mathcal{H} = \{H_\alpha \mid \alpha \in R\}$. Clearly $\mathcal{H} = \mathcal{H}_+ \cup \mathcal{H}_-$ with $\mathcal{H}_+ = \{H_\alpha \mid \alpha \in R_+\}$ and $\mathcal{H}_- = \{H_\alpha \mid \alpha \in R_-\}$. Write $\mathcal{L}$, $\mathcal{L}_+$ and $\mathcal{C}$, $\mathcal{C}_+$ for the intersection lattices and their centers of $\mathcal{H}$ and $\mathcal{H}_+$ respectively. Clearly $\mathcal{H}$, $\mathcal{L}$, and $\mathcal{C}$ are W-invariant, and $\mathcal{C} \cap \overline{V_-} = \mathcal{C}_+ \cap \overline{V_-}$ (indeed, $H_\alpha \cap \overline{V_-} = \emptyset$ for $\alpha \in R_-$ since $k_\alpha < 0$). For $c \in \mathcal{C}$ let X_c and Y_c denote as before the local contributions of (3.8) and (3.9) at c (with the convention $X_c = 0$ for $c \in \mathcal{C}\backslash\mathcal{C}_+$). For $c \in V$ let W_c denote the stabilizer subgroup of c in W, and let A_c denote the following operator on meromorphic functions:

$$A_cF(\lambda) = |W_c|^{-1}\sum_{w\in W_c}\tilde{c}(w\lambda,k)F(w\lambda) \tag{3.10}$$

Notice that if F is holomorphic on a small tubular neighbourhood $U + iV$ of $c + iV$ then A_cF also extends holomorphically on this tubular neighbourhood $U + iV$.

Proposition 3.6. *For $c \in \mathcal{C} \cap \overline{V_-}$ and $w \in W$ we have*

$$X_{wc} = Y_c \circ w^{-1} \circ A_{wc} \tag{3.11}$$

Proof. Clearly both sides of (3.11) depend only on the left coset of w modulo W_c, and therefore we can assume w to be a minimal length representative in this coset. The segment $[\gamma, w\gamma]$ only intersects those $H_\alpha \in \mathcal{H}_+$ for which $w^{-1}\alpha \in R_-$. For these α we get $(wc,\alpha) = (c, w^{-1}\alpha) \geq 0$ since $c \in \overline{V_-}$, and so $wc \notin H_\alpha$ since $k_\alpha < 0$. Hence the local contributions of $X_{V,\gamma}$ and $X_{V,w\gamma}$ at wc are the same. On the other hand the local contribution of $Y_{V,w\gamma}$ at wc is equal to $Y_c \circ w^{-1}$ with Y_c the local contribution of $Y_{V,\gamma}$ at c. Therefore it suffices to show that

$$X_{V,w\gamma'} = Y_{V,w\gamma'} \circ A_{wc}$$

if γ' is a point of the form $\gamma' = \epsilon\gamma + (1-\epsilon)c$ with ϵ very small (cf. Remark 3.2). Now if $F \in PW(V_c)$ then we have:

$$X_{V,w\gamma'} = \int_{w\gamma'+iV} F(\lambda)\frac{d\mu_E(\mathrm{Im}\lambda)}{\tilde{c}(-\lambda,k)}$$

$$\begin{aligned}
&= |W_c|^{-1} \int_{\cup_{v\in W_{wc}}(vw\gamma'+iV)} F(\lambda)\frac{d\mu_E(\mathrm{Im}\lambda)}{\tilde{c}(-\lambda,k)} \\
&= |W_c|^{-1} \int_{\cup_{v\in W_{wc}}(vw\gamma'+iV)} \tilde{c}(\lambda,k)F(\lambda)\frac{d\mu_E(\mathrm{Im}\lambda)}{\tilde{c}(\lambda,k)\tilde{c}(-\lambda,k)} \\
&= \int_{w\gamma'+iV} A_{wc}F(\lambda)\frac{d\mu_E(\mathrm{Im}\lambda)}{\tilde{c}(\lambda,k)\tilde{c}(-\lambda,k)} \\
&= Y_{V,w\gamma'}(A_{wc}F)
\end{aligned}$$

Here we have used that all points $vw\gamma'$ lie in the same connected component of $V\backslash \cup H_\alpha$ (union over $\alpha \in R_+$ for which $c \in H_\alpha$), and that $A_{wc}(F)$ is holomorphic near $wc + iV$. This completes the proof of the proposition. □

Corollary 3.7. For $c \in \mathcal{C} \cap \overline{V_-}$ write ${}_-V^c = \sum_{\alpha:(\lambda,\alpha)=k_\alpha} \mathbf{R}_-\alpha$. Observe that ${}_-V^c \subset \overline{{}_-V}$ if $\overline{{}_-V}$ denotes the closure of the antidual ${}_-V = \sum_{\alpha>0} \mathbf{R}_-\alpha$ of the positive chamber V_+. Let $c \in \mathcal{C} \cap \overline{V_-}$ and $w \in W$ with $wc \notin {}_-V^c$. If $\lambda \in c + \mathrm{supp}(Y_c)$ then $A_{wc}F(w\lambda) = 0 \quad \forall F \in PW(V_c)$.

Proof. Suppose $A_{wc}F(w\lambda) \neq 0$ for some $F \in PW(V_c)$. Then the W_c-invariant distribution $A_{wc}F(w(c+\cdot))Y_c(\cdot)$ does not vanish identically on iV, and therefore $Y_c(A_{wc}F(w(c+\cdot))G(w(c+\cdot))) \neq 0$ for some $G \in PW(V_c)^{W_{wc}}$. However, if $wc \notin {}_-V^c$ then

$$\begin{aligned}
Y_c(A_{wc}F(w(c+\cdot))G(w(c+\cdot))) &= Y_c(w^{-1}(A_{wc}(FG))(wc+\cdot)) \\
&= X_{wc}(FG(wc+\cdot)) = 0
\end{aligned}$$

by (3.11), and Lemma 3.3. It should be remarked here that we have not checked the validity of the technical assumption on the hyperplane arrangement that is necessary in order to apply Lemma 3.3. This verification is not straightforward and depends on our classification of distinguished points. This point will be addressed in Remark 3.14. □

Corollary 3.8. Write the wave function $\phi(\lambda,k;\xi)$ for $\xi \in \overline{V_+}$ as

$$\phi(\lambda,k;\xi) = \sum_{\mu\in W\lambda} a(\mu,k;\xi)\mathrm{e}^{(\mu,\xi)} \tag{3.12}$$

with $a(\lambda,k;\xi) \in PV$ a W_μ-harmonic polynomial given by

$$a(\mu,k;\xi) = |W|^{-1} \lim_{\epsilon\to 0} \sum_{w\in W_\mu} \tilde{c}(\mu+w\epsilon,k)\mathrm{e}^{(w\epsilon,\xi)} \tag{3.13}$$

If $\lambda \in c + \mathrm{supp}(Y_c)$ for $c \in \mathcal{C} \cap \overline{V_-}$ then $a(\mu,k;\cdot) = 0$ for all $\mu \in W\lambda$ and $\mathrm{Re}(\mu) \notin {}_-V^c$ (In particular, $\phi(\lambda,k;\xi)$ has at most moderate growth in ξ in this situation. If $\lambda = c$, a distinguished point for which $Y_c \neq 0$, then $\phi(\lambda,k;\xi)$ even has exponential decay).

Proof. Let $\lambda \in c + \operatorname{supp}(Y_c)$ for $c \in \mathcal{C} \cap \overline{V_-}$ and $w \in W$ with $wc \notin {}_-V^c$. Choose $F \in PW(V_c)^{W_{wc}}$ with $F(w\lambda) \neq 0$. By the previous corollary we get for all $\xi \in V$:

$$0 = A_{wc}(F(\cdot)\mathrm{e}^{(\cdot,\xi)})(w\lambda) = F(w\lambda)\sum(a(\mu,k;\xi)\mathrm{e}^{(\mu,\xi)})$$

with the sum over all $\mu \in W\lambda$ with $\mathrm{Re}(\mu) = wc$. Hence $a(\mu,k;\cdot) = 0$ for all such μ. $\square$

At this moment we only know that Y_c is a distribution with support contained in $\cup iV^L$ (union over $L \in \mathcal{L}$ with $c_L = c$). The following two results play a crucial role to arrive at the conclusion that Y_c is in fact a nonnegative measure. Recall the concepts of residual subspace and distinguished points in V as given in Definition 1.4.

THEOREM 3.9. *If $M \subset V$ is a residual subspace then*

$$\#\{\alpha \in R_L \backslash R_M \mid (L,\alpha) = k_\alpha\} \leq \#\{\alpha \in R_L \backslash R_M \mid (L,\alpha) = 0\} + 1 \tag{3.14}$$

for each affine subspace $L \subset M$ with $\dim(L) = \dim(M) - 1$.

THEOREM 3.10. *For $L \subset V$ a residual subspace we have $-c_L \in W(R_L)c_L$.*

Apparently if $M \subset V$ is residual subspace and $L \subset M$ is an affine subspace of codimension one then L then L is residual if and only if

$$\#\{\alpha \in R_L \backslash R_M \mid (L,\alpha) = k_\alpha\} = \#\{\alpha \in R_L \backslash R_M \mid (L,\alpha) = 0\} + 1 \tag{3.15}$$

By induction on $\operatorname{codim}(L)$ it follows that

$$\#\{\alpha \in R_L \mid (L,\alpha) = k_\alpha\} = \#\{\alpha \in R_L \mid (L,\alpha) = 0\} + \operatorname{codim}(L) \tag{3.16}$$

for each residual subspace $L \subset V$, and in particular for $L = \{c\}$ a distinguished point we find

$$\#\{\alpha \in R \mid (c,\alpha) = k_\alpha\} = \#\{\alpha \in R \mid (c,\alpha) = 0\} + n \tag{3.17}$$

Remark 3.11. It is quite likely that for all points $c \in V$ we have

$$\#\{\alpha \in R \mid (c,\alpha) = k_\alpha\} \leq \#\{\alpha \in R \mid (c,\alpha) = 0\} + n \tag{3.18}$$

with equality if and only if c is a distinguished point. For R an integral root system and $k_\alpha = k_\beta \quad \forall \alpha, \beta \in R$ this can be derived from Richardson's dense orbit theorem [6, Ch 5]. In turn this would imply that for each subspace $L \subset V$ we have

$$\#\{\alpha \in R_L \mid (L,\alpha) = k_\alpha\} \leq \#\{\alpha \in R_L \mid (L,\alpha) = 0\} + \operatorname{codim}(L) \tag{3.19}$$

with equality if and only if L is a residual subspace.

Remark 3.12. It is also quite likely that the map $L \to c_L$ is a bijection between residual subspaces and their centers. Once again, for R integral and $k_\alpha = k_\beta \quad \forall \alpha, \beta \in R$ this is known to be true.

In the next section we shall carry out the classification of the finite set of distinguished points for each of the irreducible root systems case by case, and thereby obtain a proof of the above theorems by inspection. In principle it should be possible to also check the questions posed in the two above remarks by a case by case analysis. However the amount of work becomes still more elaborate, and since the results of Theorem 3.9 and Theorem 3.10 are sufficient for our purposes we have left these questions aside.

THEOREM 3.13. *For $c \in \mathcal{C} \cap \overline{V_-}$ the local contribution Y_c of (3.9) at c can be written as*

$$Y_c = \sum_{L \in \mathcal{L}, c_L = c} Y_L \tag{3.20}$$

with Y_L an analytic measure on iV^L, and $Y_L = 0$ unless L is a residual subspace. If Y_{R_L, c_L} denotes the local contribution at the R_L-distinguished point $c = c_L \in V_L$ of the lower rank integral $Y_{R_L, V_L, \gamma}$, and $Y_{R_L, c_L}(\{0\})$ denotes its total mass, then

(3.21)

$$Y_L(F) = Y_{R_L, c_L}(\{0\}) \int_{\lambda \in V^L} F(i\lambda) \prod_{\alpha \in R_+ \setminus R_L} \frac{(c_L, \alpha)^2 + (\lambda, \alpha)^2}{((c_L, \alpha) - k_\alpha)^2 + (\lambda, \alpha)^2} d\mu_E(\lambda)$$

for all test functions F on iV (here μ_E denotes the Lebesgue measure on V^L).

Proof. It is clear from the proof of Lemma 3.1 and by Theorem 3.9 that the only $L \in \mathcal{L}$ for which nonzero residues are picked up are the residual subspaces. Now let L be a residual subspace with $c_L \in \overline{V_-}$ (and let R_L, $V = V_L \oplus V^L$, $L = c_L + V^L$, be as before). For $\lambda \in V^L$ we have

$$\begin{aligned}
&\prod_{\alpha \in R \setminus R_L} \frac{(c_L + i\lambda, \alpha)}{(c_L + i\lambda, \alpha) + k_\alpha} \\
&= \prod_{\alpha \in R \setminus R_L} \frac{(c_L + i\lambda, \alpha)}{(c_L + i\lambda, \alpha) - k_\alpha} \\
&= \prod_{\alpha \in R_+ \setminus R_L} \frac{((c_L, \alpha) + i(\lambda, \alpha))((c_L, -\alpha) + i(\lambda, -\alpha))}{((c_L, \alpha) - k_\alpha + i(\lambda, \alpha))((c_L, -\alpha) - k_\alpha + i(\lambda, -\alpha))} \\
&= \prod_{\alpha \in R_+ \setminus R_L} \frac{((c_L, \alpha) + i(\lambda, \alpha))((c_L, w_L\alpha) - i(\lambda, w_L\alpha))}{((c_L, \alpha) - k_\alpha + i(\lambda, \alpha))((c_L, w_L\alpha) - k_\alpha - i(\lambda, w_L\alpha))}
\end{aligned}$$

$$= \prod_{\alpha\in R_+\backslash R_L} \frac{((c_L,\alpha)+i(\lambda,\alpha))((c_L,\alpha)-i(\lambda,\alpha))}{((c_L,\alpha)-k_\alpha+i(\lambda,\alpha))((c_L,\alpha)-k_\alpha-i(\lambda,\alpha))}$$

$$= \prod_{\alpha\in R_+\backslash R_L} \frac{(c_L,\alpha)^2+(\lambda,\alpha)^2}{((c_L,\alpha)-k_\alpha)^2+(\lambda,\alpha)^2} \geq 0 \tag{3.22}$$

since the longest element w_L of $W(R_L)$ satisfies $w_Lc_L = -c_L$ (by Theorem 3.10), $w_L\lambda = \lambda$ and $w_L(R_+\backslash R_L) = R_+\backslash R_L$. We claim that the expression (3.22) is smooth for $\lambda \in V^L$. If $R^z_+ = \{\alpha \in R_+\backslash R_L \mid (c_L,\alpha) = 0\}$ and $R^p_+ = \{\beta \in R_+\backslash R_L \mid (c_L,\beta) = k_\beta\}$ we have to show that the function

$$\prod_{\alpha\in R^z_+}(\lambda,\alpha)^2 \prod_{\beta\in R^p_+}(\lambda,\beta)^{-2}$$

is smooth for $\lambda \in V^L$. The only way this can happen is when the denominator of this rational function divides the numerator. Writing $V^L_\alpha = \{\lambda \in V^L \mid (\lambda,\alpha) = 0\}$ for $\alpha \in R\backslash R_L$ we have $V^L_\alpha = V^L_\beta \Leftrightarrow \beta \in (R \cap (\mathbf{R}\alpha + V_L))\backslash R_L$. Hence the parabolic subsystem $S = (R \cap (\mathbf{R}\beta + V_L))$ of R (containing R_L as a corank one subsystem) for $\beta \in R^p_+$ is the relevant root system to consider for the above question of divisibility. Replacing R by S we can assume that $\dim(V^L) = 1$, and the divisibility holds if and only if $\#(R^z_+) \geq \#(R^p_+)$. By Theorem 3.9 we have

$$\#\{\beta \in R\backslash R_L \mid (c_L,\beta) = k_\beta\} \leq \#\{\beta \in R\backslash R_L \mid (c_L,\beta) = 0\} + 1$$

and since $-w_L$ fixes c_L and interchanges $R_+\backslash R_L$ and $R_-\backslash R_L$ we find $2\#(R^p_+) \leq 2\#(R^z_+) + 1 \Leftrightarrow \#(R^z_+) \geq \#(R^p_+)$. Hence (3.22) is smooth indeed for $\lambda \in V^L$. When we actually carry out the contour shift in (3.9) by moving γ through the hyperplanes $H \in \mathcal{H}$ with $L \in H$ it suffices by the above to only consider the local contribution Y_{R_L,c_L} of the lower rank integral $Y_{R_L,V_L,\gamma}$ at the R_L-distinguished point $c = c_L \in V_L$. If this is a measure with support at the origin of V_L then clearly Y_L is given by (3.21). In the remaining case of a distinguished point the inequality (3.17) ensures that the local contribution is indeed a measure with support in the origin (cf. Algorithm 3.15), and this finishes the proof of this theorem. □

Remark 3.14. If $L \subsetneq M$ are both residual subspaces then $|c_L| > |c_M|$ (in particular $c_L \neq c_M$). This is clear from the fact that (3.22) is smooth for $\lambda \in V^L$. This justifies the use of Lemma 3.3 in the proof of Corollary 3.7.

Algorithm 3.15. Assume $c \in \overline{V_-}$ is a distinguished point. If $R^z = \{\alpha \in R \mid (c,\alpha) = 0\}$ and $R^p = \{\beta \in R \mid (c,\beta) = k_\beta\}$ then $\#R^p = \#R^z + n$ with $n = \dim(V)$. The local contribution Y_c of (3.9) at c can now be computed by induction on $\#R^z$. The case $\#R^z = 0$ yields a residue computation for the normal crossings situation as dicussed in Remark 3.4. If $\#R^z \geq 1$ then take $\alpha \in R^z$ and write $\alpha = \sum c_j \beta_j$ with $c_j \in \mathbf{R}$ and $\{\beta_1, \ldots, \beta_n\} \subset R^p$ a basis of V. Substitution in the integrand yields a sum of at most n similar local contribution computations but with $\#R^z$ diminished by one. Iterating this procedure we can therefore compute the local contribution Y_c as a sum over at most $n^{\#R^z}$ normal crossings situations. In principle this algorithm for computing Y_c is simple , but in practice it can be very cumbersome (if $\#R^z$ is large). For example if R is of type E_8 there exists a c with $\#R^z = 32$.

Example 3.16. Let $c \in V_-$ be a regular distinguished point, and put $B = \{\beta \in R_+ \mid (c, \beta) = k_\beta\} = \{\beta_1, \ldots, \beta_n\}$. If we write

$$c = l_1 \beta_1 + \cdots + l_n \beta_n \tag{3.23}$$

with $l_1, \ldots, l_n \in \mathbf{R}$ then $Y_c = 0$ unless $l_1, \ldots, l_n < 0$. In the latter case we find using Remark 3.4 that $\forall F \in PW(V_c)$:

$$Y_c(F(c + \cdot)) = \frac{(-2\pi)^n F(c) \prod_{\alpha > 0} (c, \alpha)}{(\det(\beta_i, \beta_j))^{1/2} \tilde{c}(c, k) \prod_{\beta \in R_+ \backslash B} ((c, \beta) - k_\beta)} \tag{3.24}$$

Notice that $d\mu_E(\mathrm{Im}\lambda)$ is the measure associated to the n-form $(-i)^n d\lambda$.

Definition 3.17. *For $L \subset V$ a residual subspace let ν_L be the unique measure on V_c with support inside $c_L + iV^L$ and also formally denoted by*

$$\nu_L = (-2\pi i)^{\mathrm{codim}(L)} \mathrm{res}_L(\mu_P) \tag{3.25}$$

characterized by $\int F d\nu_L = (2\pi)^{-n} Y_L(F(c_L + \cdot)) \quad \forall F \in PW(V_c)$ if $c_L \in \overline{V_-}$ and by the requirement that $\nu_P = \sum_L \nu_L$ is a W-invariant measure.

The next theorem will give a proof of formula (1.14) when combined with Proposition 3.5.

Theorem 3.18. *For $F \in PW(V_c)^W$ the inversion operator (3.5) can be written in the symmetric form*

$$\mathcal{J}(k)F(\xi) = \sum_L \int_{c_L + iV^L} F(\lambda) \phi(\lambda, k; \xi) d\nu_L(\lambda) \tag{3.26}$$

Proof. Indeed, for $F \in PW(V_c)^W$ and $\xi \in \overline{V_+}$ we get

$$\begin{aligned} \mathcal{J}(k)F(\xi) &= (2\pi)^{-n} \sum_{c \in \mathcal{C}_+} X_c\Big(F(c + \cdot) \mathrm{e}^{(c + \cdot, \xi)}\Big) \\ &= (2\pi)^{-n} \sum_{c \in \mathcal{C} \cap \overline{V_-}} Y_c\Big(\sum_{w \in W/W_c} A_{wc}\big(F(w(c + \cdot)) \mathrm{e}^{(w(c + \cdot), \xi)}\big)\Big) \end{aligned}$$

$$
\begin{aligned}
&= (2\pi)^{-n} \sum_{c \in \mathcal{C} \cap \overline{V_-}} Y_c\Big(F(c+\cdot)|W_c|^{-1} \sum_{w \in W} \tilde{c}(w(c+\cdot),k)\mathrm{e}^{(w(c+\cdot),\xi)}\Big) \\
&= (2\pi)^{-n} \sum_{c \in \mathcal{C} \cap \overline{V_-}} \frac{|W|}{|W_c|} Y_c\Big(F(c+\cdot)\phi(c+\cdot,k;\xi)\Big) \\
&= (2\pi)^{-n} \sum_{c \in \mathcal{C} \cap \overline{V_-}} \Big\{ \sum_{L, c_L = c} \frac{|W|}{|W_c|} Y_L\Big(F(c+\cdot)\phi(c+\cdot,k;\xi)\Big)\Big\} \\
&= \sum_L \int_{c_L + iV^L} F(\lambda)\phi(\lambda,k;\xi)d\nu_L(\lambda)
\end{aligned}
$$

which proves the theorem. □

Corollary 3.19. For $f, g \in C_c^\infty(V_{\mathrm{reg}})^W$ we get

$$
\int_V f(\xi)\overline{g(\xi)}d\mu_E(\xi) = \sum_L \int_{c_L+iV^L} \mathcal{F}(k)f(\lambda)\overline{\mathcal{F}(k)g(\lambda)}d\nu_L(\lambda). \tag{3.27}
$$

Proof. Theorem 3.10 implies that $\overline{\phi(\lambda,k;\xi)} = \phi(-\lambda,k;\xi)$ for $\lambda \in c_L + iV^L$. Now use Proposition 3.5 in order to write

$$
\begin{aligned}
\int_V f(\xi)\overline{g(\xi)}d\mu_E(\xi) &= \int_V \big(\mathcal{K}(k)f(\xi)\big)\overline{g(\xi)}d\mu_E(\xi) \\
&= \int_V \Big(\mathcal{J}(k)\big(\mathcal{F}(k)f(\xi)\big)\Big)\overline{g(\xi)}d\mu_E(\xi)
\end{aligned}
$$

Now use the previous theorem and change the order of integration (which is allowed as one easily checks). □

In order to complete the proof of Theorem 1.5 it remains to be shown that the measures ν_L are nonnegative. This will also allow us to interpret Corollary 3.19 as a Plancherel formula. From the positivity of (3.22) it follows that it is sufficient to show that $\nu_c \geq 0$ for c a distinguished point.

THEOREM 3.20. *If c is a distinguished point and $\nu_c \neq 0$ then $\phi(c,k;\cdot) \in L^2(V,\mu_E)$ and*

$$
\sum_{d \in Wc} \nu_d(\{d\}) = (\phi(c,k;\cdot),\phi(c,k;\cdot))^{-1} \tag{3.27}
$$

Proof. By induction on the rank of R together with the positivity of (3.22) we may assume that $\nu_L \geq 0$ for all L a residual subspace with $\dim(L) \geq 1$. Let $c_1, \ldots, c_N$ be the set of distinguished points in $\overline{V_-}$ with $\nu_{c_i} \neq 0$, and put $\phi_i = \phi(c_i,k;\cdot)$ for $i = 1, \ldots, N$. By Corollary 3.8 we know that ϕ_i has exponential decay, and in particular lies in $L^2(V,\mu_E)$. Put

$$
C_{c,0}^\infty = \{f \in C_c^\infty(V_{\mathrm{reg}})^W \mid (f,\phi_i) = 0 \quad \forall i = 1,\ldots,N\}
$$

Now it follows from (3.27) that if $\{f_n\}$ is a L^2-converging sequence in $C^\infty_{c,0}$ then the sequence $\{\mathcal{F}(k)f_n|_{c_L+iV^L}\}$ converges in $L^2(c_L+iV^L,\nu_L)$ if L is a residual subspace of positive dimension for which $\nu_L>0$. And of course we have that $\mathcal{F}(k)f_n(c_i)=0 \quad \forall i$ by the very definition of $C^\infty_{c,0}$. We can choose $\phi^i\in C^\infty_c(V_{\mathrm{reg}})^W$ such that $(\phi_i,\phi^j)=\delta_{i,j}$. Indeed, choose $\tilde{\phi}^i\in C^\infty_c(V_{\mathrm{reg}})^W$ such that $(\phi_i,\tilde{\phi}^j)$ is a nonsingular matrix, which is possible bychoosing $\tilde{\phi}^i$ close to ϕ_i in $L^2(V,\mu_E)$. Now take the basis dual to the linear functionals $(\cdot,\phi_j)$ in the space $\oplus_i\mathbf{C}\tilde{\phi}^i\cong\mathbf{C}^N$.

Choose a sequence $\{f_{i,n}\}\subset C^\infty_c(V_{\mathrm{reg}})^W$ such that $\phi f_{i,n}\to\phi\phi_i$ in $L^2(V,\mu_E)$ for each function ϕ which has moderate growth (we can do this because ϕ_i has exponential decay). Then $\mathcal{F}(k)f_{i,n}(\lambda)\to 0$ for each $\lambda\in c_L+iV^L$ if L is a residual subspace of positive dimension for which $\nu_L>0$. We claim that in fact $\mathcal{F}(k)f_{i,n}|_{c_L+iV^L}\to 0$ in $L^2(c_L+iV^L,\nu_L)$ for such L.

To see this consider the sequence $\tilde{f}_{i,n}{=}f_{i,n}{-}\sum_j(f_{i,n},\phi_j)\phi^j{\in}C^\infty_{c,0}$ converging to $\phi_i{-}(\phi_i,\phi_i)\phi^i$ in $L^2(V,\mu_E)$. Hence the sequence $\{\mathcal{F}(k)\tilde{f}_{i,n}|_{c_L+iV^L}\}$ converges in $L^2(c_L+iV^L,\nu_L)$. Therefore the original sequence $\{\mathcal{F}(k)f_{i,n}|_{c_L+iV^L}\}$ has to converge in $L^2(c_L+iV^L,\nu_L)$ as well.

On the one hand $(f_{i,n},f_{i,n})\to(\phi_i,\phi_i)$, and on the other hand $(f_{i,n},f_{i,n})\to |W/W_{c_i}|\nu_{c_i}(\{c_i\})(\phi_i,\phi_i)^2$. This proves the theorem. □

Remark 3.21. It follows that the Fourier-Yang transform extends to a unitary injection of Hilbert spaces

$$L^2(V,\mu_E)^W \xrightarrow{\mathcal{F}(k)} L^2(V_c,\nu_P)^W := \Big(\bigoplus_L L^2(c_L+iV^L,\nu_L)\Big)^W \tag{3.28}$$

with the direct sum taken over those residual subspaces L for which $\nu_L>0$ as a measure on c_L+iV^L. It is quite likely that (3.29) is in fact a unitary isomorphism of Hilbert spaces.

Example 3.22. Define the vector $\rho(k)\in V$ by

$$2\rho(k)=\sum_{\alpha>0}k_\alpha\alpha=l_1(k)\alpha_1+\cdots+l_n(k)\alpha_n \tag{3.29}$$

with $\{\alpha_i\}=B$ a basis of simple roots and $l_i(k)\in\mathbf{R}_-$. Now it is easy to see that $\rho(k)$ is a distinguished point, and

$$\phi(\rho(k),k,\xi)=\mathrm{e}^{(\rho(k),\xi)} \quad \forall\xi\in\overline{V_+} \tag{3.30}$$

This wave function is square integrable as it should be since $\nu_{\rho(k)}>0$ by direct computation. The L^2-norm of this function can be computed in two different ways now. The first way is a direct evaluation using the formula $\int_0^\infty \mathrm{e}^{lx}dx = -l^{-1}$ if $l<0$. The second way is by doing the residue computation at $\rho(k)$ as in (3.24) and using (3.28). Comparison of the two answers yields a nontrivial identity. In case R is a normalized root system and $k_\alpha = k_\beta \quad \forall \alpha, \beta \in R$ one finds:

$$\det(\alpha_i, \alpha_j) l_1 \dots l_n = |W| \prod_{\alpha \in R_+ \setminus B} \frac{\mathrm{ht}(\alpha)}{\mathrm{ht}(\alpha) - 1} \tag{3.31}$$

with $2\rho = \sum_{\alpha>0} \alpha = \sum_i l_i \alpha_i$, and $\mathrm{ht}(\alpha = \sum_i x_i \alpha_i) = \sum_i x_i$. For R integral this identity is an exercise in [4, Ch VI, Sec. 4, Ex. 6] with the invitation to the reader to do the excercise case by case!

Remark 3.23. For R of type BFI(even) we have two independent coupling parameters, one for each orbit of roots. We hope that the method of this section can be suitably adapted so as to also cover the case with one positive and one negative coupling parameter.

4. Distinguished points and spherical cuspidal points

In this section we will classify the distinguished points for each of the individual irreducible root systems case by case. The method uses induction on the rank of R, and therefore the collection of residual lines is assumed to be known. Now for each point L on a given residual line M we just verify that (with $R_L = R$):

$$\#\{\alpha \in R_L \setminus R_M \mid (L, \alpha) = k_\alpha\} \leq \#\{\alpha \in R_L \setminus R_M \mid (L, \alpha) = 0\} + 1$$

and the points $L \in M$ for which equality holds are by definition the distinguished points. This is how Theorem 3.9 is proved, and in the end Theorem 3.10 is easily checked by going through the list of distinguished points.

Proposition 4.1. *Let $V = \mathbf{R}^n$ with standard basis $e_1, \dots, e_n$. Let $R = R(A_{n-1}) = \{\alpha \in \mathbf{Z}^n \mid (\alpha, \alpha) = 2,\ (\alpha, \sum e_i) = 0\} = \{e_i - e_j \mid i \neq j\}$ and $W = W(A_{n-1}) = S_n$. For $k \in K$, $k \neq 0$ there are no distinguished points and up to the action of S_n there is just a single residual line*

$$L = \{x = (nk + t, (n-1)k + t, \dots, k + t) \mid t \in \mathbf{R}\}. \tag{4.1}$$

Proof. The first statement is clear since the rank of R is $n - 1$. By induction on n it follows that the residual planes are conjugated by S_n to planes of the form

$$M = \{x = (pk + t, (p-1)k + t, \dots, k + t, qk + s, \dots, k + s) \mid s, t \in \mathbf{R}\}$$

with $p, q \geq 1$ and $p + q = n$. Observe that R_M has type $A_{p-1} + A_{q-1}$ and $R \setminus R_M = \{\pm(e_i - e_j) \mid 1 \leq i < p, p + 1 \leq j \leq n\}$. The lines L in M we have to analyze are those for which $ik + t - jk - s = k \Leftrightarrow s = (i - j - 1)k + t$ for some $i = 1, \dots, p$ and $j = 1, \dots, q$. Assume that exactly r coordinates of the first p and the last q coordinates coincide for some $r \geq 0$. Then we find that $\#\{\alpha \in R_L \backslash R_M \mid (L, \alpha) = k_\alpha\} = r + 1$ (if $r < p$, $r < q$), r (if $r = p$, $r < q$ or $r < p$, $r = q$), $r - 1$ (if $r = p = q$), and $\#\{\alpha \in R_L \backslash R_M \mid (L, \alpha) = 0\} = 2r$. Clearly $r + 1 \leq 2r + 1$ with equality if and only if $r = 0$. Hence the only distinguished line we find up to the action of S_n is (4.1) $\square$

Definition 4.2. *Let $V = \mathbf{R}^n$ with standard basis $e_1, \dots, e_n$. Let $R = R(B_n) = R(D_n) \cup \{\pm e_1, \dots, \pm e_n\} = \{\alpha \in \mathbf{Z}^n \mid (\alpha, \alpha) = 1$ or $2\}$ and $W = W(B_n) = C_2^n \rtimes S_n$ the hyperoctahedral group. The coupling parameter $(k, k') \in K$ with $k = k_{e_i \pm e_j}$ $(i \neq j)$ and $k' = k_{e_i}$ is called generic if*

$$kk' \prod_{j=1}^{2(n-1)} (jk + 2k')(jk - 2k') \neq 0 \tag{4.2}$$

Proposition 4.3. *For generic coupling parameters the distinguished point of type B_n are conjugated under the action of W to the points*

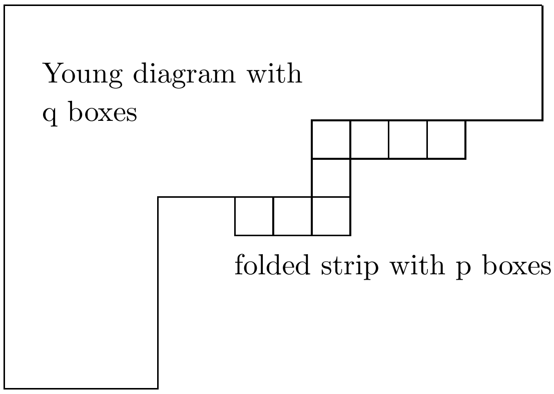
$$c(\lambda, k, k') \in \mathbf{R}^n, \quad c(\lambda, k, k')_x = c(x)k + k' \tag{4.3}$$

where λ ranges over the set of partitions of weight n and $x = (i, j) \in \lambda$ ranges over the set of boxes of λ. If $\lambda = (\lambda_1, \dots, \lambda_r)$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 1$ is a partition of length $l(\lambda) = r$ and weight $|\lambda| = \sum \lambda_i = n$ then we identify λ with its Young diagram (with λ_1 boxes in the first row, λ_2 boxes in the second row, etc.) For $x = (i, j) \in \lambda \Leftrightarrow 1 \leq j \leq \lambda_i$ the number $c(x) := j - i$ is the content of the box x. For example if $\lambda = (5, 4, 4, 1)$ then $c(\lambda, k, k') = (4k+k', 3k+k', 2k+k', 2k+k', k+k', k+k', k+k', k', k', k', -k+k', -k+k', -2k+k', -3k+k') \in \mathbf{R}^{14}$.

0	1	2	3	4
-1	0	1	2	
-2	-1	0	1	
-3				

Proof. By induction on the rank we have to consider the situation of a parabolic subsystem of type $A_{p-1} + B_q$ with $p + q = n$.

We have to consider a diagram as indicated below, composed of a Young diagram with q boxes and a folded strip of p boxes. Let m_i be the multiplicity of the content i in the boxes of this new diagram.



Now with $c(k,k') \in \mathbf{R}^n$ as before we have

$$\#\{\alpha \in R \mid (c(k,k'),\alpha) = k_\alpha\} = m_0 + \sum_i m_i m_{i+1}$$

and

$$\#\{\alpha \in R \mid (c(k,k'),\alpha) = 0\} = \sum_i m_i(m_i - 1).$$

Therefore we have to verify that

$$m_0 + \sum_i m_i m_{i+1} \le n + \sum_i m_i(m_i-1) = \sum_i m_i^2$$

with equality if and only if the new diagram is a Young diagram (i.e. $m_{i+1} = m_i$ or $m_i - 1$ if $i \ge 0$, and $m_{i-1} = m_i$ or $m_i - 1$ if $i \le 0$). This will be an immediate consequence of the following lemma. □

LEMMA 4.4. *Let $m_i \in \mathbf{N}$ for $i \in \mathbf{Z}$ with $m_i = 0$ for $|i|$ large. Then we have*

$$\max(m_i) + \sum_i m_i m_{i+1} \le \sum_i m_i^2$$

with equality if and only if (say $m_0 = \max(m_i)$ by shifting the index set) $m_{i+1} = m_i$ or $m_i - 1$ if $i \ge 0$, and $m_{i-1} = m_i$ or $m_i - 1$ if $i \le 0$.

Proof. Since $2\sum_i m_i^2 - 2\sum_i m_i m_{i+1} = \sum_i (m_i - m_{i+1})^2$ the statement follows from

$$a^2 + b^2 + c^2 + \dots \ge a + b + c + \dots$$

if $a, b, c, \dots$ are integers, with equality if and only if $a, b, c, \dots \in \{0, 1\}$. □

PROPOSITION 4.5. *If $k' = (q + \frac{1}{2})k$, $k \neq 0$ for some $q = 0, 1, \dots, p$ and $m = (m_{p+\frac{1}{2}}, \dots, m_{\frac{1}{2}}) \in \mathbf{N}^{p+1}$ with $|m| = \sum_i m_i = n$ then the point*

$$c(m,k,k' = (q + \frac{1}{2})k)$$

$$= ((\underbrace{(p+\frac{1}{2})k,\ldots,(p+\frac{1}{2})k}_{m_{p+\frac{1}{2}}\text{ times}},(p-\frac{1}{2})k,\ldots,\frac{3}{2}k,\underbrace{\frac{1}{2}k,\ldots,\frac{1}{2}k}_{m_{\frac{1}{2}}\text{ times}}) \in \mathbf{R}^n \tag{4.4}$$

is distinguished if and only if $m_{i+1} = m_i$ *or* $m_i - 1$ *for* $i \geq q + \frac{1}{2}$ *(with the convention that* $m_{p+\frac{1}{2}} = 0$ *and* $m_i = 0$ *for* $i > p + \frac{1}{2}$*) and* $m_{i-1} = m_i$ *or* $m_i - 1$ *for* $i = \frac{3}{2}, \ldots, q + \frac{1}{2}$*. All distinguished points for these coupling parameters are obtained in this way up to the action of* W*.*

PROPOSITION 4.6. *If* $k'=0,\ k\neq 0$ *(*R *of type* D_n*) and* $m = (m_p, \ldots, m_0) \in \mathbf{N}^{p+1}$ *with* $|m| = n$ *then the point*

$$c(m,k,0) = (\underbrace{pk,\ldots,pk}_{m_p\text{ times}},(p-1)k,\ldots,k,\underbrace{0,\ldots,0}_{m_0\text{ times}}) \in \mathbf{R}^n \tag{4.5}$$

is distinguished if and only if and only if $m_p = 1$ *and* $m_{i+1} = m_i$ *or* $m_i - 1$ *for* $i \geq 1$ *and* $m_0 = [\frac{1}{2}(m_1 + 1)]$*. All distinguished points for these coupling parameters are obtained in this way up to action of* W*.*

PROPOSITION 4.7. *If* $k' = qk,\ k \neq 0$ *for some* $q = 1, \ldots, p$ *and* $m = (m_p, \ldots, m_0) \in \mathbf{N}^{p+1}$ *with* $|m| = n$ *then the point (4.5) is distinguished if and only if* $m_p = 1$ *and* $m_{i+1} = m_i$ *or* $m_i - 1$ *for* $i \geq q$ *and* $m_{i-1} = m_i$ *or* $m_i - 1$ *for* $i = 2, \ldots, q$ *and* $m_0 = [\frac{1}{2}m_1]$*. All distinguished points for these coupling parameters are obtained in this way up to action of* W*.*

The proof of these propositions is similar to the proof of Proposition 4.3, and therefore will be skipped. The case $k' = \frac{1}{2}k$ corresponds to the split C_n-case, and $k' = k$ corresponds to the split B_n-case. For these two cases the outcome can be compared with the results of [1] or [6, p. 174-175]. For type E_n the list of distinguished points can be derived directly from the tables in [6, p. 176-177]. For $k \neq 0$ there are 3, 6 and 11 distinguished points for n =6, 7 and 8 respectively (modulo the action of W).

DEFINITION 4.8. *For* R *of type* F_4 *let* $k = k_\alpha$ *for* α *long and* $k' = k_\alpha$ *for* α *short. The coupling parameter* (k, k') *is called generic if*

$$\begin{aligned} &kk'(3k \pm k')(2k \pm k')(3k \pm 2k')(k \pm k')(5k \pm 6k')(3k \pm 4k')\cdot \\ &\cdot (2k \pm 3k')(3k \pm 5k')(k \pm 2k')(k \pm 3k')(k \pm 4k')(k \pm 6k') \neq 0 \end{aligned} \tag{4.6}$$

PROPOSITION 4.9. *For generic* (k, k') *of type* F_4 *there are 8 distinguished points as given in Table 4.10 (with* $\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \alpha_3 = e_3,\ \alpha_4 = \frac{1}{2}(-e_1 - e_2 - e_3 + e_4)$ *the simple roots and* $\omega_1 = e_1 + e_4,\ \omega_2 = e_1 + e_2 + 2e_4,\ \omega_3 = e_1 + e_2 + e_3 + 3e_4,\ \omega_4 = 2e_4$ *the dual basis of fundamental coweights). For nongeneric* (k, k') *there are no other distinguished points than those obtained as limit of a generic distinguished point.*

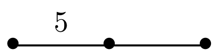
The proof is by direct (though rather lengthy) computation, and will be skipped (since it does not seem to be very instructive).

Table 4.10. The distinguished points for type F_4.

No	$c(k,k')$	$c(k,k')$ distinguished iff
1.	$k\omega_1 + k\omega_2 + k'\omega_3 + k'\omega_4$	$(2k+3k')(3k+4k')$ $(3k+5k')(5k+6k') \neq 0$
2.	$k\omega_1 + k\omega_2 + (-k+k')\omega_3 + k'\omega_4$	$(k \pm 6k')k' \neq 0$
3.	$k\omega_1 + k\omega_2 + (-k+k')\omega_3 + k\omega_4$	$(3k+2k')(k+3k')$ $(2k+3k')(3k+4k') \neq 0$
4.	$k\omega_1 + k\omega_2 + (-2k+k')\omega_3 + k'\omega_4$	$(2k-3k')(3k-4k')$ $(3k-5k')(5k-6k') \neq 0$
5.	$k\omega_1 + k\omega_2 + (-2k+k')\omega_3 + 2k\omega_4$	$(3k \pm 2k')(k \pm 3k') \neq 0$
6.	$k\omega_1 + k\omega_2 + (-2k+k')\omega_3 + k\omega_4$	$(3k-2k')(k-3k')$ $(2k-3k')(3k-4k') \neq 0$
7.	$k\omega_1 + k\omega_2 + (-2k+k')\omega_3 + (3k-k')\omega_4$	$k(3k \pm k') \neq 0$
8.	$k\omega_2 + (-k+k')\omega_4$	$kk' \neq 0$

Remark 4.11. For type F_4 the map $(k,k') \to (2k',k)$ is a natural involution of the situation corresponding to the interchange of long and short roots. For R of type D_4 we have two distinguished points $(3k,2k,k,0)$ and $(2k,k,k,0)$ for $k \neq 0$. They can be viewed as the specialization $k'=0$ of No 1 and No 3 respectively.

PROPOSITION 4.12. *For $k \neq 0$ and R of type H_3 there are 4 distinguished points, which are all regular. For $k \neq 0$ and R of type H_4 there are 17 distinguished points, 12 of which are regular. The results are listed in Tables 4.13 and 4.14. Here the numbering of the basis $\omega_1,\omega_2,\omega_3(,\omega_4)$ dual to the basis $\alpha_1,\alpha_2,\alpha_3(,\alpha_4)$ of simple roots is according to the nodes from left to right in the Coxeter diagrams*



and

respectively, and $\tau = \frac{1}{2}(1+\sqrt{5})$.

Table 4.13. Distinguished points for type H_3.

No	point $c(k)$
1.	$k\omega_1 + k\omega_2 + k\omega_3$
2.	$(1+\tau)^{-1}(k\omega_1 + k\omega_2 + k\tau\omega_3)$
3.	$(1+\tau)^{-1}(k\omega_1 + k\omega_2 + k(1+\tau)\omega_3)$
4.	$(2+3\tau)^{-1}(k(1+\tau)\omega_1 + k\tau\omega_2 + k\omega_3)$

Table 4.14. Distinguished points for type H_4.

No	point $c(k)$
1.	$k\omega_1 + k\omega_2 + k\omega_3 + k\omega_4$
2.	$(1+\tau)^{-1}(k\omega_1 + k\omega_2 + k\tau\omega_3 + k\omega_4)$
3.	$(1+\tau)^{-1}(k\omega_1 + k\omega_2 + k\tau\omega_3 + k(1+\tau)\omega_4)$
4.	$(1+\tau)^{-1}(k\omega_1 + k\omega_2 + k(1+\tau)\omega_3 + k(1+\tau)\omega_4)$
5.	$(2+3\tau)^{-1}(k(1+\tau)\omega_1 + k\tau\omega_2 + k\omega_3 + k(1+2\tau)\omega_4)$
6.	$(2+3\tau)^{-1}(k(1+\tau)\omega_1 + k\tau\omega_2 + k\omega_3 + k(1+3\tau)\omega_4)$
7.	$(2+3\tau)^{-1}(k(1+\tau)\omega_1 + k\tau\omega_2 + k\omega_3 + k(2+3\tau)\omega_4)$
8.	$(3+5\tau)^{-1}(k(1+2\tau)\omega_1 + k\tau\omega_2 + k\tau\omega_3 + k\tau\omega_4)$
9.	$(2+4\tau)^{-1}(k\omega_1 + k\tau\omega_2 + k\tau\omega_3 + k\omega_4)$
10.	$(2+3\tau)^{-1}(k\omega_1 + k\omega_2 + k\tau\omega_3 + k\omega_4)$
11.	$(3+5\tau)^{-1}(k\tau\omega_1 + k\tau\omega_2 + k\omega_3 + k\tau\omega_4)$
12.	$(5+8\tau)^{-1}(k\omega_1 + k(1+2\tau)\omega_2 + k\omega_3 + k\tau\omega_4)$
13.	$(1+2\tau)^{-1}(k\omega_2 + k\tau\omega_3 + k\tau\omega_4)$
14.	$(2+3\tau)^{-1}(k\tau\omega_2 + k\tau\omega_3 + k\omega_4)$
15.	$(1+\tau)^{-1}(k\omega_1 + k\omega_2 + k(1+\tau)\omega_4)$
16.	$(1+2\tau)^{-1}(k\omega_2 + k\tau\omega_3)$
17.	$(1+\tau)^{-1}k\omega_2$

PROPOSITION 4.15. *Let R be the normalized dihedral root system of type $I_2(m)$ with simple roots α_1, α_2. For $j = 1, 2, \ldots, [\frac{m}{2}]$ let $\beta_1, \beta_2 \in R_+$ be defined by*

$$\sin\frac{\pi}{m}\beta_1 = \sin\frac{\pi j}{m}\alpha_1 + \sin\frac{\pi(j-1)}{m}\alpha_2$$
$$\sin\frac{\pi}{m}\beta_2 = \sin\frac{\pi(j-1)}{m}\alpha_1 + \sin\frac{\pi j}{m}\alpha_2$$

with dual basis β_1^*, β_2^* of the form

$$2\sin^2\frac{\pi(2j-1)}{m}\beta_1^* = \beta_1 + \cos\frac{\pi(2j-1)}{m}\beta_2$$
$$2\sin^2\frac{\pi(2j-1)}{m}\beta_2^* = \cos\frac{\pi(2j-1)}{m}\beta_1 + \beta_2$$

For $k_1 = k_{\beta_1},\ k_2 = k_{\beta_2}$ *with* $(k_1 + k_2 \cos\frac{\pi(2j-1)}{m})(k_1 \cos\frac{\pi(2j-1)}{m} + k_2) \neq 0$ *the point*

$$c(k_1, k_2) = k_1\beta_1^* + k_2\beta_2^* \tag{4.7}$$

is distinguished, and all distinguished points are conjugated under W *to these.*

Proof. This is straightforward. □

As mentioned before, with the complete enumeration of the distinguished points for each of the irreducible root systems at hand the proofs of Theorem 3.9 and Theorem 3.10 can be carried out by inspection. We now discuss which of these distinguished points are spherical cuspidal, i.e. correspond to a square integrable wave function. *For the rest of this section we will assume that* $k_\alpha < 0 \quad \forall\alpha \in R$.

If $c \in V$ is a regular distinguished point the criterium for c to be spherical cuspidal is easy, and was described in Example 3.16. However for singular distinguished points it can be very difficult in our approach to actually check whether the residue vanishes or not.

PROPOSITION 4.16. *Let* λ *be the partition* $\lambda = (i+1, 1^j)$ *with* $i+j = n-1$ *and* $i \geq 0, j \geq 0, n \geq 2$. *The distinguished point* $c(\lambda, k, k')$ *given by (4.3) is spherical cuspidal if and only if in case* $j = 0$ *(i.e.* $c(\lambda, k, k') = \rho(k, k')$*)*

$$k' < \min(-\frac{1}{2}(n-1)k, -(n-1)k), \tag{4.8}$$

and in case $j \geq 1$

$$\frac{1}{2}(j+1)k < k' < \frac{1}{2}(j-i)k. \tag{4.9}$$

Let μ *be the partition* $\mu = (i+1, 2, 1^{j-1})$ *with* $i + j = n-2$ *and* $i \geq 1,\ j \geq 1,\ n \geq 4$. *The distinguished point* $c(\lambda, k, k')$ *given by (4.3) is spherical cuspidal if and only if*

$$\frac{1}{2}jk < k' < \min(\frac{1}{2}(j-i)k, 0). \tag{4.10}$$

Proof. For the partition λ this is clear from Example 3.16. For the partition μ just use Algorithm 3.15. Details are left to the reader. □

PROPOSITION 4.17. *Let* R *be of type* F_4. *For which* (k, k') *the previously found distinguished points are spherical cuspidal is given in the next table. Note that for a given No 1 up to 8 the point* $c(k, k')$ *is spherical cuspidal for* (k, k') *in a nonempty open convex cone.*

Proof. Again we skip the proof which is quite long but altogether straightforward. □

Table 4.18. The spherical cuspidal points for type F_4. Each regular point (so all cases except No 8) $c(k,k')$ is displayed by its coordinates with respect to the set of roots $\{\beta_1,\beta_2,\beta_3,\beta_4\}$ defined by $\{\beta_1,\beta_2,\beta_3,\beta_4\}=\{\beta\in R \mid (c(k,k'),\beta)=k_\beta \quad \forall k,k'\}$

No	$c(k,k')$	spherical cuspidal iff
1.	$((5k+6k'),3(3k+4k'),6(2k+3k'),2(3k+5k'))$	$5k+6k'<0$, $3k+5k'<0$.
2.	$((k+6k'),(k-6k'),18k',10k')$	$k-6k'<0$, $k'<0$.
3.	$((3k+4k'),(3k+2k'),2(k+3k'),2(2k+3k'))$	$3k+2k'<0$, $k+3k'<0$.
4.	$(3(3k-4k'),(5k-6k'),6(-2k+3k'),2(-3k+5k'))$	$3k-4k'<0$, $-2k+3k'<0$.
5.	$((3k-2k'),(3k+2k'),2(-k+3k'),2(k+3k'))$	$3k-2k'<0$, $-k+3k'<0$.
6.	$((3k-2k'),(3k-4k'),2(-2k+3k'),2(-k+3k'))$	$3k-4k'<0$, $-2k+3k'<0$.
7.	$(9k,5k,2(-3k+k'),2(3k+k'))$	$k<0$, $-3k+k'<0$.
8.		$k<0, k'<0$.

Proposition 4.19. *For R of type H_3 and $k<0$ the 3 points 1,3 and 4 of Table 4.13 are spherical cuspidal, and 2 is not spherical cuspidal. Let R be of type H_4 and $k<0$. The following are the regular spherical cuspidal points: 1, 2, 4, 5, 7, 8, 9, 12. At present we have not checked the singular ones* (*the points 13 to 17*) *for spherical cuspidality.*

Proposition 4.20. *Let R be of type $I_2(m)$. The point* (*4.7*) *is spherical cuspidal if and only if*

$$k_1+k_2\cos\frac{\pi(2j-1)}{m}<0,\quad k_1\cos\frac{\pi(2j-1)}{m}+k_2<0 \tag{4.11}$$

In particular this is the case if $k_1=k_2<0$ (*eg. if m is odd*).

Proof. This is easy using the formulas in Proposition 4.15. □

The simplest criterion for spherical cuspidality is Theorem 1.7. How this follows from the work of Kazhdan and Lusztig will be indicated in the next section.

5. Perspectives

Consider the following tabeau for hypergeometry associated with a root system R.

$$\begin{tikzcd}
 & \boxed{\text{1. The q-hypergeometric functions for } R} \arrow[dl, "t=q^k,q\to 1"'] \arrow[dr, "q=0,t=q^{-1}"] & \\
\boxed{\text{2. Ordinary hypergeometric functions for } R} \arrow[d] & & \boxed{\text{3. Elementary spherical functions for the affine Hecke algebra}} \arrow[d] \\
\boxed{\text{4. Bessel functions for } R} & & \boxed{\text{5. Elementary wave functions for Yang's system}}
\end{tikzcd}$$

Boxes 1,2,3 make sense for R an integral root system, and boxes 4,5 make sense for R arbitrary (but finite). The nonreduced root system BC_n admits some additional flexibility, and a few extra boxes can be added [19], [34]. In the first box we have the theory of Macdonald's orthogonal q-polynomials for root systems [24]. From the work of Cherednik the pivotal role of the affine Hecke algebra as an indispensable tool has now become clear [7], [8], [26]. In the second box we have the theory of hypergeometric functions for root systems as developed by the authors (see [14] for a survey, and [29] for some recent results), and which contains the theory of spherical functions on a real semisimple Lie group. In the third box we have the theory of spherical functions for the regular representation of the affine Hecke algebra, containing (for q a prime power) the theory of spherical functions on a semisimple group of p-adic type [25],[27]. The fourth box deals with a local version of the second box near the identity element, and contains the theory of spherical functions for Cartan motion groups [10], [17], [28]. Finally in the fifth box we have the theory dealt with in this paper. Just as box 4 is the infinitesimal version of box 2 one should think of box 5 as the infinitesimal version of box 3. The affine Hecke algebra plays a role in box 1 and box 3, and this role is taken over by the graded Hecke algebra in box 2 and box 5. Each of the boxes has its own $\tilde{c}$-function and one can speculate about the applicability of the method ofthis paper in a larger context.

In box 3 there are no problems whatsoever, and the whole theory can be applied without serious changes. Let F be a nonarchimedean local field and let $\mathcal{O}$ denote the ring of integers of F. The cardinality of the residue field is denoted by q. Let G be a semisimple algebraic group defined over F, which is assumed to be of adjoint type. Let $G(F)$ denote the group of F rational points of G, which we assume to be split (for sake of simplicity). We choose an Iwahori subgroup $\mathcal{I} \subset G(\mathcal{O})$ and normalize the Haar measure on $G(F)$ so that $\mathrm{Vol}(\mathcal{I}) = 1$. Denote by ${}^dG$ the Langlands dual group, and let T be a maximal torus of ${}^dG$. Let $R \subset \mathrm{Lie}(T)^*$ denote the set of roots of ${}^dG$ with respect to T. The character lattice of T is the weight lattice P of R, and if $\lambda \in P$ we denote the corresponding character by e^λ. The theory of elementary $G(\mathcal{O})$-spherical functions on $G(F)$ leads to an explicit Plancherel formula with completely continuous spectrum which was studied in [25]. The Plancherel measure μ has support on the compact form T_c of T, and if we normalize the spherical functions so that their value at the identity equals 1 then this measure is given explicitly by:

$$d\mu(t) = |W|^{-1}q^{-N}\frac{\prod_{\alpha\in R}(\mathrm{e}^\alpha(t)-1)}{\prod_{\alpha\in R}(q^{-1}\mathrm{e}^\alpha(t)-1)}dt \tag{5.1}$$

where dt is the normalized Haar measure on T_c, and N is the cardinality of R_+. We are to use the explicit formula of Macdonald as a starting point, analogous to Theorem 1.3. Replace q by its reciprocal q^{-1}. If we apply the contour shift argument as explained in this paper we encounter (among other tempered families) spherical cuspidal representations of the specialization of the affine Hecke algebra at q^{-1} at points of T where a point residue is picked up. Via the involution i of the affine Hecke algebra defined by sending $q \to q^{-1}$ and $T_i \to -q^{-1}T_i$ these correspond to certain cuspidal representations of the specialization of the affine Hecke algebra at q, and all these modules share in common the property that they contain the sign representation of the Hecke algebra of the finite Weyl group W. From (5.1) it is clear that the eligible residual points s of T have to satisfy:

$$\#\{\alpha \in R \mid \mathrm{e}^\alpha(s) = 1\} + \dim(T) = \#\{\alpha\in R \mid \mathrm{e}^\alpha(s) = q^{-1}\}. \tag{5.2}$$

But these points s are in one to one correspondence with the distinguished unipotent orbits of those semisimple subgroups H of ${}^dG$ which are the centralizer of a semisimple element of ${}^dG$. From the geometric classification of the irreducible modules of the affine Hecke algebra by Kazhdan and Lusztig [18] it is known that these are precisely the central characters for which there exist cuspidal modules. Moreover, it is known that to each of those points there belongs exactly one cuspidal module that contains the sign representation of the Hecke algebra of W. In the classification of [18] these are denoted by $\mathcal{M}^{\mathcal{I}}_{s,1}$, and the corresponding cuspidal representations $\mathcal{M}_{s,1}$ of $G(F)$ are called the generic Iwahori spherical cuspidal representations. When s is a real point of type (5.2), then clearly $\log s$ is a $(\mathrm{Lie}(T_v), R, k)$-distinguished point if we set the root labels k_α all equal to $-\log q$. Here $T = T_v T_c$ is the polar decomposition of the complex torus T, and $\mathrm{Lie}(T_v)$ is considered as euclidean space with respect to some W-invariant inner product (for example the Killing form). Hence there exists a spherical cuspidal representation of the graded Hecke algebra for this infinitesimal central character and value of k, namely the module of the graded Hecke algebra corresponding to $(\mathcal{M}_{s,1}^{\mathcal{I}})^i$ (here $(\mathcal{M}_{s,1}^{\mathcal{I}})^i$ denotes the module of the specialization of the affine Hecke algebra at q^{-1} obtained from the module $\mathcal{M}_{s,1}^{\mathcal{I}}$ using the involution i defined above). This proves Theorem 1.7.

But there are also important applications in the context of this box 3 itself, all based on the analogue of Theorem 3.20. The analogue of Example 3.22 will give the explicit formula of Bott and Macdonald for the Poincaré series of affine Weyl groups [3], [23]. In general, this Theorem 3.20 provides us with a method to compute the formal degree of the generic cuspidal representations, up to an absolute constant. We use a formula of Li's [21] saying essentially that there exists a matrix coeficient of $\mathcal{M}_{s,1}$ which is obtained from the K-spherical function at s by replacing q by q^{-1}. As was explained in Reeder [32] we need to calculate the reciprocal of the square norm of this matrix coefficient in order to obtain the formal degree, and this we do by appealing to the analogue of Theorem 3.20. The resulting formula explains why the formal degree has such a nice factorization in the examples that were calculated by Reeder [32]. We shall give the precise statement in the following theorem:

THEOREM 5.1. *There exists an absolute constant $c \neq 0$ such that the formal degree of $M_{s,1}$ is given by:*

$$\deg(M_{s,1}) = cq^N \frac{\prod'_{\alpha\in R}(\mathrm{e}^\alpha(s)-1)}{\prod'_{\alpha\in R}(q\mathrm{e}^\alpha(s)-1)}$$

where $\prod'$ is the product over all nonzero factors, and N is the number of positive roots.

It is quite likely that the methods of this paper can also be transfered to box 2. However there are some technical difficulties to overcome now, due to the fact that the special functions are more complicated. Once these difficulties are resolved the theory will yield a proof of the main result of [5] along the same lines as the proof of the formula of Bott and Macdonald mentioned above (which in [5] was used as just one of the ingredients of the proof). More importantly, the theory will yield the L^2-norm computations of other highly transcendental functions for which the method used in [5] fails.

Finally one may even hope that the methods of this paper apply to the first box, but at the moment this is merely speculation.

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