DEPARTMENT OF MATHEMATICS

UNIVERSITY OF NIJMEGEN The Netherlands

CENTRAL LIMIT THEOREM FOR THE EDWARDS MODEL

R. van der Hofstad, F. den Hollander and W. König

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIJMEGEN
Toernooiveld
6525 ED Nijmegen
The Netherlands

CENTRAL LIMIT THEOREM FOR

THE EDWARDS MODEL

R. van der Hofstad

Mathematical Institute
University of Utrecht
P.O. Box 80.010
3508 TA Utrecht
The Netherlands

F. den Hollander, W. König[1]

Mathematical Institute
University of Nijmegen
Toernooiveld 1
6525 ED Nijmegen
The Netherlands

Abstract

The Edwards model in one dimension is a transformed path measure for standard Brownian motion discouraging self-intersections. We prove a central limit theorem for the endpoint of the path, extending a law of large numbers proved by Westwater (1984). The scaled variance is characterized in terms of the largest eigenvalue of a one-parameter family of differential operators, introduced and analyzed in van der Hofstad and den Hollander (1994). Interestingly, the scaled variance turns out to be independent of the strength of self-repellence and to be strictly smaller than one.

---

0   Introduction and main result

0.1   The Edwards model

Let $(B_t)_{t \geq 0}$ be standard one-dimensional Brownian motion starting at 0. Let P denote its distribution on path space and E the corresponding expectation. The *Edwards model* is a transformed path measure discouraging self-intersections, defined by the intuitive formula

$$\frac{dP_T^\beta}{dP} = \frac{1}{Z_T^\beta} \exp\left[-\beta \int_0^T ds \int_0^T dt \; \delta(B_s - B_t)\right] \qquad (T > 0). \qquad (0.1)$$

Here δ denotes Dirac's function, $\beta \in (0, \infty)$ is the *strength of self-repellence* and Z_T^β is the normalizing constant.

A rigorous definition of P_T^β is given in terms of Brownian local times as follows. It is well known (see Revuz and Yor (1991), Sect. VI.1) that there exists a jointly continuous version of the Brownian local time process $(L(t,x))_{t \geq 0, x \in \mathbb{R}}$ satisfying the occupation times formula

$$\int_0^t f(B_s)\, ds = \int_\mathbb{R} L(t,x) f(x)\, dx \quad P\text{-a.s.} \qquad (f : \mathbb{R} \to [0, \infty) \text{ Borel}, t \geq 0). \tag{0.2}$$

Think of $L(t,x)$ as the amount of time the Brownian motion spends in x until time t. The Edwards measure in (0.1) may now be defined by

$$\frac{dP_T^\beta}{dP} = \frac{1}{Z_T^\beta} \exp\left[-\beta \int_\mathbb{R} L(T,x)^2\, dx\right], \tag{0.3}$$

where $Z_T^\beta = E(\exp[-\beta \int_\mathbb{R} L(T,x)^2\, dx])$ is the normalizing constant. The random variable $\int_\mathbb{R} L(T,x)^2\, dx$ is called the *self-intersection local time*. Think of this as the amount of time the Brownian motion spends in self-intersection points until time T.

The path measure P_T^β is the continuous analogue of the self-repellent random walk (called the *Domb-Joyce model*), which is a transformed measure for the discrete simple random walk. The latter is used to study the long-time behavior of random polymer chains. The effect of the self-repellence is of particular interest. This effect is known to spread out the path on a linear scale (i.e., B_T is of order T under the law P_T^β as $T \to \infty$). It is the aim of this paper to study the fluctuations of B_T around the linear asymptotics. Our main result appears in Theorem 2 below.

0.2   Theorems

The starting point of our paper is the following law of large numbers:

Theorem 1 *(Westwater (1984)) For every $\beta \in (0, \infty)$ there exists $\theta^*(\beta) \in (0, \infty)$ such that*

$$\lim_{T \to \infty} P_T^\beta\left(\left|\frac{B_T}{T} - \theta^*(\beta)\right| \leq \epsilon \,\Big|\, B_T > 0\right) = 1 \text{ for every } \epsilon > 0. \tag{0.4}$$

2

Theorem 1 says that the self-repellence causes the path to have a ballistic behavior no matter how weak the interaction. Westwater proves this result by applying the Ray-Knight representation for the Brownian local times and using large deviation arguments.

The speed $\theta^*(\beta)$ was characterized by Westwater in terms of the smallest eigenvalue of a certain differential operator. In the present paper, however, we prefer to work with a different operator, introduced and analyzed in van der Hofstad and den Hollander (1995). For $a \in \mathbb{R}$ define $\mathcal{K}^a : L^2(\mathbb{R}_0^+) \cap C^2(\mathbb{R}_0^+) \to C(\mathbb{R}_0^+)$ by

$$\left(\mathcal{K}^a x\right)(u) = 2ux''(u) + 2x'(u) + (au - u^2)x(u) \tag{0.5}$$

for $u \in \mathbb{R}_0^+ = [0, \infty)$. The operator $\mathcal{K}^a$ will play a key role in the present paper.[3] It is symmetric and has a largest eigenvalue $\rho(a)$ with multiplicity 1. The map $a \mapsto \rho(a)$ is real-analytic, strictly convex and strictly increasing, with $\rho(0) < 0$, $\lim_{a \to -\infty} \rho(a) = -\infty$ and $\lim_{a \to \infty} \rho(a) = \infty$.

Define a^*, b^*, $c^* \in (0, \infty)$ by

$$\rho(a^*) = 0, \qquad b^* = \frac{1}{\rho'(a^*)}, \qquad c^{*2} = \frac{\rho''(a^*)}{\rho'(a^*)^3}. \tag{0.6}$$

Our main result is the following central limit theorem:

Theorem 2 *For every $\beta \in (0, \infty)$ there exists $\sigma^*(\beta) \in (0, \infty)$ such that*

$$\lim_{T \to \infty} P_T^\beta \left(\frac{B_T - \theta^*(\beta)T}{\sigma^*(\beta)\sqrt{T}} \leq C \,\Big|\, B_T > 0 \right) = \mathcal{N}((-\infty, C]) \text{ for all } C \in \mathbb{R}, \tag{0.7}$$

where $\mathcal{N}$ denotes the normal distribution with mean 0 and variance 1. The scaled mean and variance are given by

$$\theta^*(\beta) = b^* \beta^{\frac{1}{3}}, \qquad \sigma^*(\beta) = c^*. \tag{0.8}$$

Theorem 2 says that the fluctuations around the asymptotic mean have the classical order $\sqrt{T}$, are symmetric, and even do not depend on the interaction strength.

The numerical values of the constants in (0.6) are

$$a^* = 2.189 \pm 0.001, \qquad b^* = 1.11 \pm 0.01, \qquad c^* = 0.7 \pm 0.1. \tag{0.9}$$

The values for a^* and b^* were obtained in van der Hofstad and den Hollander (1995) by estimating $\rho(a)$ for a range of a-values. This can be done very accurately via a discretization procedure. The same data produce the value for c^*. Note that $c^* < 1$. Apparently, as the path is pushed out to infinity its fluctuations are squeezed compared to those of the free motion with $\theta^*(0) = 0$, $\sigma^*(0) = 1$.

---

[2]By symmetry, (0.4) says that the distribution of B_T/T under P_T^β converges weakly to $\frac{1}{2}(\delta_{\theta^*(\beta)} + \delta_{-\theta^*(\beta)})$ as $T \to \infty$, where δ_θ denotes the Dirac point measure at $\theta \in \mathbb{R}$.

[3]The operator $\mathcal{K}^a$ is a scaled version of the operator $\mathcal{L}^a$ originally analyzed in van der Hofstad and den Hollander (1995), namely $(\mathcal{K}^a x)(u) = (\mathcal{L}^a \bar{x})(u/2)$ where $\bar{x}(u) = x(2u)$.

0.3 Scaling in β

It is noteworthy that the scaled mean depends on β in such a simple manner and that the scaled variance does not depend on β at all. These facts are direct consequences of the Brownian scaling property. Namely, we shall deduce from (0.7) that for every $\beta \in (0, \infty)$

$$\theta^*(\beta) = \theta^*(1)\beta^{\frac{1}{3}}, \qquad \sigma^*(\beta) = \sigma^*(1). \tag{0.10}$$

Indeed, note that for $a, T > 0$

$$\left(B_T, (L(T,x))_{x \in \mathbb{R}}\right) \stackrel{\mathcal{D}}{=} \left(a^{-\frac{1}{2}}B_{aT}, (a^{-\frac{1}{2}}L(aT, a^{\frac{1}{2}}x))_{x \in \mathbb{R}}\right) \tag{0.11}$$

where $\stackrel{\mathcal{D}}{=}$ means equality in distribution (see Revuz and Yor (1991), Ch. VI, Ex. (2.11), 1°)). Apply this to $a = \beta^{\frac{2}{3}}$ to obtain, via (0.3), that

$$P_T^{\beta}(B_T)^{-1} = P_{\beta^{\frac{2}{3}}T}^1\left(\beta^{-\frac{1}{3}}B_{\beta^{\frac{2}{3}}T}\right)^{-1}, \tag{0.12}$$

where we write $\mu(X)^{-1}$ for the distribution of a random variable X under a measure μ. In particular, we have for all $C \in \mathbb{R}$

$$P_T^{\beta}\left(\frac{B_T - \theta^*(1)\beta^{\frac{1}{3}}T}{\sigma^*(1)\sqrt{T}} \le C \,\Big|\, B_T > 0\right) \cdot$$

$$= P_{\beta^{\frac{2}{3}}T}^1\left(\frac{B_{\beta^{\frac{2}{3}}T} - \theta^*(1)\beta^{\frac{2}{3}}T}{\sigma^*(1)\sqrt{\beta^{\frac{2}{3}}T}} \le C \,\Big|\, B_{\beta^{\frac{2}{3}}T} > 0\right). \tag{0.13}$$

The r.h.s. tends to $\mathcal{N}((-\infty, C])$ as $T \to \infty$ (in (0.7) pick $\beta = 1$ and replace T by $\beta^{\frac{2}{3}}T$). Since the pair $(\theta^*(\beta), \sigma^*(\beta))$ is uniquely determined by (0.7), we arrive at (0.10).

0.4 Outline of the proof

Theorem 2 is the continuous analogue of the central limit theorem for the Domb-Joyce model proved by König (1994). We shall be able to use the skeleton of that paper, but the Brownian context will require new ideas and methods. The remaining sections are devoted to the proof of Theorem 2. We give a short outline.

In Section 1, we use the well-known Ray-Knight theorems for the local times of Brownian motion to express the l.h.s. of (0.7) in terms of zero- and two-dimensional squared Bessel processes. Roughly speaking, these processes describe the self-intersections in the areas $(-\infty, 0] \cup [B_T, \infty)$ resp. $[0, B_T]$.

In Section 2, with the help of some analytical properties of the operator $\mathcal{K}^a$ proved in van der Hofstad and den Hollander (1995), we introduce a Girsanov transformation of the two-dimensional squared Bessel process in terms of which the Gaussian behavior becomes transparent.

4

In Section 3, we prove a central limit theorem for an additive functional of the transformed process by applying standard techniques. This will show that the asymptotic normality is determined by those parts of the Brownian path that fall in the area $[0, B_T]$.

Section 4 finishes the proof of Theorem 2 by showing that the influence of the self-intersections in $(-\infty, 0] \cup [B_T, \infty)$ is bounded as $T \to \infty$ and is therefore cancelled by the normalization in the definition of the transformed path measure in (0.3). The methods applied in this section are similar to the ones used in Section 2, but now for the zero-dimensional squared Bessel process.

1 Brownian local times

Since the dependence on β has already been isolated (see (0.13)), we may and shall restrict to the case $\beta = 1$.

Throughout the sequel we shall frequently refer to Revuz and Yor (1991) and to Karatzas and Shreve (1991). We shall therefore adapt the abbreviations RY resp. KS for these references.

The remainder of this paper is devoted to the proof of the following key proposition:

Proposition 1 *There exists an $S \in (0, \infty)$ such that for all $C \in \bar{\bar{\mathbb{R}}}$*

$$\lim_{T \to \infty} e^{a^* T} E\left(e^{-\int_{\mathbf{R}} L(T,x)^2 \, dx} 1_{0 < B_T \le b^* T + C\sqrt{T}}\right) = S \mathcal{N}_{c^{*2}}((-\infty, C]), \tag{1.1}$$

where a^*, b^* and c^* are defined in (0.6), and $\mathcal{N}_{\sigma^2}$ denotes the normal distribution with mean 0 and variance σ^2.

Theorem 2 follows from Proposition 1, since it implies that the distribution of $(B_T - b^* T)/\sqrt{T}$ converges to $\mathcal{N}_{c^{*2}}$ (divide the l.h.s. of (1.1) by the same expression with $C = \infty$ and recall (0.3)).

Subsections 1.1 and 1.2 contain preparatory material. Subsection 1.3 contains the key representation in terms of squared Bessel processes on which the proof of Proposition 1 will be based.

1.1 Ray-Knight theorems

This subsection contains a description of the *time-changed* local time process in terms of squared Bessel processes. The material being fairly standard, our main purpose is to introduce appropriate notations and to prepare for Lemma 1 in Subsection 1.2 and Lemma 2 in Subsection 1.3.

For $u \in \mathbb{R}$ and $h \ge 0$, let τ_h^u denote the time change associated with $L(t, u)$, i.e.,

$$\tau_h^u = \inf\{\, t > 0 : L(t, u) > h \,\}. \tag{1.2}$$

Obviously, the map $h \mapsto \tau_h^u$ is right-continuous and increasing, and therefore makes at most countably many jumps for each $u \in \mathbb{R}$. Moreover, $P(L(\tau_h^u, u) = h$ for all $u \ge$

$0) = 1$ (see RY, Ch. VI). The following lemma contains the well-known Ray-Knight theorems. It identifies the distribution of the local times at the random time τ_h^u, as a process in the spatial variable running forwards resp. backwards from u. We write $C_c^2(\mathbb{R}^+)$ to denote the set of twice continuously differentiable functions on $\mathbb{R}^+ = (0, \infty)$ with compact support.

RK Theorems *Fix $u, h \geq 0$. The random processes $(L(\tau_h^u, u + v))_{v \geq 0}$ and $(L(\tau_h^u, u - v))_{v \geq 0}$ are independent Markov processes, both starting at h.*

(i) *$(L(\tau_h^u, u + v))_{v \geq 0}$ is a zero-dimensional squared Bessel process ($BESQ^0$) with generator*

$$(G^* f)(v) = 2v f''(v) \qquad (f \in C_c^2(\mathbb{R}^+)). \tag{1.3}$$

(ii) *$(L(\tau_h^u, u - v))_{v \in [0, u]}$ is the restriction to the interval $[0, u]$ of a two-dimensional squared Bessel process ($BESQ^2$) with generator*

$$(G f)(v) = 2v f''(v) + 2f'(v) \qquad (f \in C_c^2(\mathbb{R}^+)). \tag{1.4}$$

(iii) *$(L(\tau_h^u, -v))_{v \geq 0}$ has the same transition probabilities as the process in (i).*

Proof. See RY, Sects. XI.1-2 and KS, Sects. 6.3-4. □

1.2 The distribution of $(L(T, x))_{x \in \mathbb{R}}$

The RK theorems give us a nice description of the local time process at certain stopping times. In order to apply them to (0.3), we need to go back to the fixed time T. This causes some complications (e.g. we must handle the global restriction $\int_{\mathbb{R}} L(T, x) \, dx = T$), but these may be overcome by an appropriate conditioning.

This subsection contains a formal description of the joint distribution of the three random processes

$$(L(T, B_T + x))_{x \geq 0}, \quad (L(T, B_T - x))_{x \in [0, B_T]}, \quad (L(T, -x))_{x \geq 0}, \tag{1.5}$$

in terms of the squared Bessel processes. The main intuitive idea is that (recall (1.2))

$$\{ \tau_h^u = T \} = \{ B_T = u, L(T, B_T) = h \} \quad P\text{-a.s.} \tag{1.6}$$

This has two consequences:

(i) Conditioned on $\{ B_T = u, L(T, B_T) = h \}$, the three processes in (1.5) are the squared Bessel processes from the RK theorems conditioned on having total integral equal to T.

(ii) The joint distribution of $(B_T, L(T, B_T))$ can be expressed in terms of τ_h^u and therefore also in terms of the squared Bessel processes.

6

We shall make this precise in Lemma 1 below.

In order to formulate the details, we must first introduce some notation. For the remainder of this paper, let

$$(X_v)_{v \geq 0} = \text{BESQ}^2, \qquad (X_v^\star)_{v \geq 0} = \text{BESQ}^0. \qquad (1.7)$$

Note that $(X_v)_{v \geq 0}$ is recurrent and has 0 as an entrance boundary, while $(X_v^\star)_{v \geq 0}$ is transient and has 0 as an absorbing boundary (see RY, Sect. XI.1). Denote by $\mathbb{P}_h$ and $\mathbb{P}_h^\star$ the distributions of the respective processes conditioned on starting at $h \geq 0$. Denote the corresponding expectations by $\mathbb{E}_h$ resp. $\mathbb{E}_h^\star$. Furthermore, define the following additive functional and its time change:

$$
\begin{aligned}
A(u) &= \int_0^u X_v \, dv \qquad (u \geq 0), \\
A^{-1}(t) &= \inf\{ u > 0 : A(u) > t \} \qquad (t \geq 0).
\end{aligned}
\qquad (1.8)
$$

Note that under $\mathbb{P}_h$ both $u \mapsto A(u)$ and $t \mapsto A^{-1}(t)$ are continuous and strictly increasing towards infinity. So A and A^{-1} are in fact inverse functions of each other. Define Lebesgue densities φ_h and $\psi_{h_1,t}$ by

$$
\begin{aligned}
\varphi_h(t) \, dt &= \mathbb{P}_h^\star \Big(\int_0^\infty X_v^\star \, dv \in dt \Big), \\
\psi_{h_1,t}(u, h_2) \, du \, dh_2 &= \mathbb{P}_{h_1}(A^{-1}(t) \in du, X_{A^{-1}(t)} \in dh_2)
\end{aligned}
\qquad (1.9)
$$

for a.e. $h, t, h_1, u, h_2 \geq 0$. Put the quantities defined in (1.8) and (1.9) equal to zero if any of the variables is negative. Now the joint distribution of the three processes in (1.5) can be described as follows.

Lemma 1 *Fix $T > 0$. For all nonnegative Borel functions Φ_1, Φ_2 and Φ_3 on $C(\mathbb{R}_0^+)$ and for any interval $I \subset [0, \infty)$,*

$$
\begin{aligned}
&E\Big(\Phi_1\big((L(T, B_T + x))_{x \geq 0}\big) \Phi_2\big((L(T, -x))_{x \geq 0}\big) \Phi_3\big((L(T, B_T - x))_{x \in [0, B_T]}\big) \, 1_{B_T \in I} \Big) \\
&= \int_I du \int_{[0,\infty)^4} dt_1 \, dh_1 \, dh_2 \, dt_2 \; \varphi_{h_1}(t_1) \psi_{h_1, T - t_1 - t_2}(u, h_2) \varphi_{h_2}(t_2) \\
&\quad \times \prod_{i=1}^2 \mathbb{E}_{h_i}^\star \Big(\Phi_i\big((X_v^\star)_{v \geq 0}\big) \Big| \int_0^\infty X_v^\star \, dv = t_i \Big) \\
&\quad \times \mathbb{E}_{h_1}\Big(\Phi_3\big((X_v)_{v \in [0,u]}\big) \Big| A^{-1}(T - t_1 - t_2) = u, X_u = h_2 \Big).
\end{aligned}
\qquad (1.10)
$$

Proof. Essentially, Lemma 1 is a formal rewrite using (1.8), (1.9) and the RK-theorems, which say that under $\mathbb{P}_h$ resp. $\mathbb{P}_h^\star$

$$
\begin{aligned}
(X_v)_{v \in [0,u]} &\overset{\mathcal{D}}{=} (L(\tau_h^u, u - v))_{v \in [0,u]} \\
(X_v^\star)_{v \geq 0} &\overset{\mathcal{D}}{=} (L(\tau_h^u, u + v))_{v \geq 0}.
\end{aligned}
\qquad (1.11)
$$

However, the details are far from trivial.

We proceed in four steps, the first of which makes (1.6) precise and is the most technical.

STEP 1 $P(\tau_h^u \in dT)\,du\,dh = P(B_T \in du, L(T, B_T) \in dh)\,dT$ *for a.e.* $u, h, T \geq 0$.

Proof. It follows from (0.2) that P-a.s.

$$|L(s, u) - L(t, u)| \leq |s - t| \quad \text{for all } s, t, u \geq 0. \tag{1.12}$$

This implies that for all $\varepsilon > 0$ and $u, h, T \geq 0$

$$
\begin{aligned}
\frac{1}{\varepsilon}\Big[P(\tau_h^u < T + \varepsilon) - P(\tau_h^u < T)\Big] &= \frac{1}{\varepsilon}P\Big(L(T, u) \leq h < L(T + \varepsilon, u)\Big) \\
&\leq \frac{1}{\varepsilon}P\Big(L(T, u) \in (h - \varepsilon, h]\Big).
\end{aligned}
\tag{1.13}
$$

The r.h.s. is locally bounded in h as $\varepsilon \downarrow 0$. A similar assertion holds for $\varepsilon \uparrow 0$. We may therefore interchange $\int dh$ and $\frac{\partial}{\partial T}$ in the first equality in the next display.

Fix $c_2 > 0$. Compute

$$
\begin{aligned}
\int_0^{c_2} dh\,\frac{\partial}{\partial T}P(\tau_h^u < T) &= \frac{\partial}{\partial T}\int_0^{c_2} dh\,P(\tau_h^u < T) = \frac{\partial}{\partial T}E\Big(\int_0^{c_2} dh\,1_{h < L(T, u)}\Big) \\
&= \frac{\partial}{\partial T}E\Big(L(T, u)1_{L(T, u) \leq c_2} + c_2 1_{L(T, u) > c_2}\Big) \\
&= \Big[\frac{\partial}{\partial t}E\Big(L(t, u)1_{L(T, u) \leq c_2}\Big)\Big]_{t=T} + \Big[\frac{\partial}{\partial t}E\Big([c_2 - L(T, u)]1_{L(t, u) > c_2}\Big)\Big]_{t=T}.
\end{aligned}
\tag{1.14}
$$

The second term in the r.h.s. of (1.14) is zero. Indeed, the right-derivative vanishes because by (1.12)

$$
\begin{aligned}
0 &\leq \limsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}E\Big([c_2 - L(T, u)]1_{L(T+\varepsilon, u) > c_2 \geq L(T, u)}\Big) \\
&\leq \limsup_{\varepsilon \downarrow 0} P\Big(L(T + \varepsilon, u) > c_2 \geq L(T, u)\Big) = 0,
\end{aligned}
\tag{1.15}
$$

and similarly for the left-derivative. Thus only the first term in the r.h.s. of (1.14) survives.

Next, fix $c_1 > 0$. Integrate (1.14) over $u \in [0, c_1]$ and use (1.12) to interchange $\int_0^{c_1} du$ and $\frac{\partial}{\partial t}$. This leads to

$$
\int_0^{c_1} du \int_0^{c_2} dh\,\frac{\partial}{\partial T}P(\tau_h^u < T) = \Big[\frac{\partial}{\partial t}E\Big(\int_0^{c_1} du\,L(t, u)1_{L(T, u) \in [0, c_2]}\Big)\Big]_{t=T}.
\tag{1.16}
$$

8

Now, if we check that the occupation times formula (0.2) may also be applied for the *random* function $f(x) = 1_{x \in [0, c_1]} 1_{L(T, x) \in [0, c_2]}$, then we get

$$E\Big(\int_0^{c_1} du \, L(t, u) 1_{L(T, u) \in [0, c_2]} \Big) = E\Big(\int_0^t ds \, 1_{B_s \in [0, c_1]} 1_{L(T, B_s) \in [0, c_2]} \Big).$$

$$(1.17)$$

Substituting (1.17) into (1.16) we find

$$
\begin{aligned}
\int_0^{c_1} du \int_0^{c_2} dh \, \frac{\partial}{\partial T} P(\tau_h^u < T) \\
= \Big[\frac{\partial}{\partial t} \int_0^t ds \, P\big(B_s \in [0, c_1], L(T, B_s) \in [0, c_2] \big) \Big]_{t=T} \\
= P\big(B_T \in [0, c_1], L(T, B_T) \in [0, c_2] \big),
\end{aligned}
$$

$$(1.18)$$

which proves the claim.

It is not hard to justify (1.17) by appealing to the following property (see KS, Ch. 3, Eq. (6.22)): For every $\gamma \in \big(0, \frac{1}{2} \big)$, P-a.s.

$$u \mapsto L(T, u) \text{ is uniformly Hölder continuos of order } \gamma \text{ on compacts.}$$

$$(1.19)$$

Indeed, one first replaces $\int_0^{c_1} du$ by $\sum_{i=1}^n \int_{c_{i-1,n}}^{c_{i,n}} du$ and $1_{L(T,u) \in [0,c_2]}$ by $1_{L(T,c_{i,n}) \in [0,c_2]}$, with $c_{i,n} = \frac{i}{n} c_1$ for $n \in \mathbb{N}$ and $i = 1, \ldots, n$. The error made in doing so vanishes as $n \to \infty$ by (1.12) and (1.19). For each i and n one can apply (0.2) for $f(x) = f_{i,n}(x) = 1_{x \in [c_{i-1,n}, c_{i,n}]} 1_{L(T,c_{i,n}) \in [0,c_2]}$. Letting $n \to \infty$ and using (1.19) once more, we arrive at (1.17). The details are left to the reader. $\square$

Next, abbreviate for $u, h \geq 0$,

$$\mathcal{Z}_h^u = \Big(\tau_h^u, \int_0^\infty L(\tau_h^u, u+v) \, dv, L(\tau_h^u, 0), \int_0^\infty L(\tau_h^u, -v) \, dv \Big).$$

$$(1.20)$$

Then the distribution of $\mathcal{Z}_h^u$ is identified as:

STEP 2 *For every* $u, h \geq 0$ *and a.e.* T, t_1, h_2, t_2

$$P(\mathcal{Z}_h^u \in d(T, t_1, h_2, t_2)) = \varphi_h(t_1) \psi_{h, T-t_1-t_2}(u, h_2) \varphi_{h_2}(t_2) \, dT \, dt_1 \, dh_2 \, dt_2.$$

$$(1.21)$$

Proof. According to the RK theorems, $(L(\tau_h^u, -x))_{x \geq 0}$ is $BESQ^0$ starting at $L(\tau_h^u, 0)$. Moreover, $L(\tau_h^u, 0)$ itself has distribution $\mathbb{P}_h(X_u)^{-1}$. Furthermore, from (0.2) we have

$$\tau_h^u = \int_0^\infty L(\tau_h^u, u+v) \, dv + \int_0^u L(\tau_h^u, u-v) \, dv + \int_0^\infty L(\tau_h^u, -v) \, dv.$$

$$(1.22)$$

9

Combining these statements with the RK theorems and (1.11), we obtain

$$P(\mathcal{Z}_h^u \in d(T, t_1, h_2, t_2)) = \mathbb{P}_h^\star\Big(\int_0^\infty X_v^\star\, dv \in dt_1 \Big)\, \mathbb{P}_{h_2}^\star\Big(\int_0^\infty X_v^\star\, dv \in dt_2 \Big)$$

$$\times\, \mathbb{P}_h\Big(\int_0^u X_v\, dv \in d(T - t_1 - t_2), X_u \in dh_2 \Big). \qquad (1.23)$$

But the r.h.s. of (1.2) equals the r.h.s. of (1.21) because of (1.9) and the identity $\{A(u) < T - t_1 - t_2\} = \{A^{-1}(T - t_1 - t_2) > u\}$ implied by (1.8). $\qquad\square$

STEP 3 $P(\tau_{L(T,B_T)}^{B_T} = T) = 1.$

Proof. Simply note that $\tau_{L(T,B_T)}^{B_T} - T$ is distributed as the time change τ_0^0 for the process $(B_{T+t} - B_T)_{t \geq 0}$ (recall (1.2)). But $P(\tau_0^0 = 0) = 1$ (see RY, Remark 1°) following Prop. VI.2.5). $\qquad\square$

STEP 4 *Proof of Lemma 1.*

Proof. First condition and integrate the l.h.s. of (1.10) w.r.t. the distribution of $(B_T, L(T, B_T))$, which is identified in Step 1. According to Step 3, we may then replace T by $\tau_{h_1}^u$ on $\{B_T = u, L(T, B_T) = h_1\}$. Next, condition and integrate w.r.t. the conditional distribution of $\mathcal{Z}_{h_1}^u$ given $\{\tau_{h_1}^u = T\}$. Then the l.h.s. of (1.10) becomes

$$\int_I du \int_0^\infty dh_1\, \frac{P(\tau_{h_1}^u \in dT)}{dT} \int_{[0,\infty)^3} \frac{P(\mathcal{Z}_{h_1}^u \in d(T, t_1, h_2, t_2))}{P(\tau_{h_1}^u \in dT)}$$

$$E\Big(\Phi_1\big((L(\tau_{h_1}^u, u + x))_{x \geq 0} \big) \Phi_2\big((L(\tau_{h_1}^u, -x))_{x \geq 0} \big)$$

$$\times\, \Phi_3\big((L(\tau_{h_1}^u, u - x))_{x \in [0,u]} \big) \,\Big|\, \mathcal{Z}_{h_1}^u = (T, t_1, h_2, t_2) \Big). \qquad (1.24)$$

Now use Step 2, apply the description of the local time processes provided by the RK theorems in combination with (1.11) and (1.20), and again use the elementary relation between A and A^{-1} stated at the end of the proof of Step 2. Then we obtain that (1.24) is equal to the r.h.s. of (1.10). $\qquad\square$

$\qquad\square$

1.3 Application to the Edwards model

We are now ready to formulate the key representation of the expectation appearing in the l.h.s. of (1.1). This representation will be the starting point for the proof of Proposition 1 in Sections 2-4. Abbreviate

$$C_T = b^\star T + C\sqrt{T}. \qquad (1.25)$$

Lemma 2 *For all $T > 0$*

$$E\left(e^{-\int_{\mathbf{R}} L(T,x)^2\, dx} 1_{0 < B_T \le C_T}\right)$$

$$= \int_0^{C_T} du \int_{[0,\infty)^4} dh_1\, dh_2\, dt_1\, dt_2$$

$$\prod_{i=1}^{2} \mathbb{E}_{h_i}^{\star}\left(e^{-\int_0^{\infty} X_v^{\star 2}\, dv}\,\bigg|\, \int_0^{\infty} X_v^{\star}\, dv = t_i\right)\varphi_{h_i}(t_i)$$

$$\times\, \mathbb{E}_{h_1}\left(e^{-\int_0^u X_v^2\, dv}\,\bigg|\, A^{-1}(T - t_1 - t_2) = u,\, X_u = h_2\right)\psi_{h_1, T - t_1 - t_2}(u, h_2).$$
$$(1.26)$$

Proof. This follows from Lemma 1. □

Thus, we have expressed the expectation in the l.h.s. of (1.1) in terms of integrals over BESQ^0 and BESQ^2 and their additive functionals. Henceforth we can forget about the underlying Brownian motion and focus on these processes using their generators given in (1.3) and (1.4).

The importance of Lemma 2 is the decomposition into a *product* of three expectations. The main reason to introduce the densities φ_h and $\psi_{h_1,t}$ is the fact that the last factor in (1.26) depends on t_1 and t_2. This dependence will vanish in the limit $T \to \infty$, as we shall see in the sequel. After that the densities φ_h and $\psi_{h_1,t}$ can again be absorbed into the expectations (recall (1.9)). Thus, we shall need little about these densities other than their existence.

2 A transformed Markov process

All we have done so far is to rewrite the key object of Proposition 1 in terms of expectations involving squared Bessel processes. We are now ready for our main attack.

In Subsection 2.1 we use Girsanov's formula to transform BESQ^2 into a new Markov process. The purpose of this transformation is to absorb the exponential factor appearing under the expectation in the last line of (1.26) into the transition probabilities of the new process. In Subsection 2.2 we list some properties of the transformed process, and these are used in Subsection 2.3 to formulate two key propositions on which the proof of Proposition 1 is based. In Subsection 2.4 we complete the proof of Proposition 1 subject to these new propositions.

2.1 Construction of the transformed process

Fix $a \in \mathbb{R}$ (later we shall pick $a = a^{\star}$). Recall from Subsection 0.2 that $\rho(a) \in \mathbb{R}$ is the largest eigenvalue of the operator $\mathcal{K}^a$ defined in (0.5). We denote the corresponding strictly positive and L^2-normalized eigenvector by x_a. From van der Hofstad and den Hollander (1995) we know that $x_a : \mathbb{R}_0^+ \to \mathbb{R}^+$ is real-analytic with

11

$\lim_{u \to \infty} u^{-\frac{3}{2}} \log x_a(u) < 0$, and that $a \mapsto x_a \in L^2(\mathbb{R}_0^+)$ is real-analytic. Define

$$F_a(u) = u^2 - au + \rho(a) \qquad (u \in \mathbb{R}_0^+). \tag{2.1}$$

The following lemma defines the Girsanov transformation of $BESQ^2$ that we shall need later:

Lemma 3 *For $t, h_1, h_2 \geq 0$ let $P_t(h_1, dh_2)$ denote the transition probability function of $BESQ^2$. Then*

$$\widehat{P}_t^a(h_1, dh_2) = \frac{x_a(h_2)}{x_a(h_1)} \, \mathbb{E}_{h_1} \left(e^{-\int_0^t F_a(X_v)\, dv} \,\Big|\, X_t = h_2 \right) P_t(h_1, dh_2) \tag{2.2}$$

defines the transition probability function of a diffusion $(X_v)_{v \geq 0}$ on $\mathbb{R}_0^+$.

Proof. Recall the definition of the generator G of $BESQ^2$ given in (1.4). According to RY, Sect. VIII.3, if $f \in C^2(\mathbb{R}_0^+)$ satisfies the equation

$$G(f) + \frac{1}{2} G(f^2) - f\, G(f) = F_a, \tag{2.3}$$

then

$$(D_t^{f,a})_{t \geq 0} = \left(e^{f(X_T) - f(X_0) - \int_0^t F_a(X_s)\, ds} \right)_{t \geq 0} \tag{2.4}$$

is a local martingale under $\mathbb{P}_h$ for any $h \geq 0$. Substitute $f = \log x$ in the l.h.s. of (2.3). Then an elementary calculation yields that for all $u \geq 0$

$$\begin{aligned}
\left(G(f) + \frac{1}{2} G(f^2) - f\, G(f) \right)(u) &= 2u f''(u) + 2f'(u) + 2u f'(u)^2 \\
&= \frac{2u x''(u) + 2x'(u)}{x(u)}.
\end{aligned} \tag{2.5}$$

We now easily derive from the eigenvalue relation $\mathcal{K}_a x_a = \rho(a) x_a$ (recall (0.5)) that (2.3) is satisfied for $f = f_a = \log x_a$. Hence, $(D_t^{f_a,a})_{t \geq 0}$ is a local martingale under $\mathbb{P}_h$. Since F_a is bounded from below and x_a is bounded from above, each $D_t^{f_a,a}$ is bounded $\mathbb{P}_h$-a.s. Hence $(D_t^{f_a,a})_{t \geq 0}$ is a martingale under $\mathbb{P}_h$. The lemma now follows from RY, Prop. VIII.3.1. $\qquad \square$

We shall denote the distribution of the transformed process, conditioned on starting at $h \geq 0$, by $\widehat{\mathbb{P}}_h^a$ and the corresponding expectation by $\widehat{\mathbb{E}}_h^a$. Note that we have

$$\widehat{\mathbb{E}}_h^a(g(X_t)) = \mathbb{E}_h(D_t^{f_a,a} g(X_t)) \qquad (t \geq 0, g : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \text{ measurable}). \tag{2.6}$$

12

2.2 Properties of the transformed process

We are going to list some properties of the process constructed in the preceding sub-section and of some characteristic quantities related to it.

1. The process introduced in Lemma 3 is a Feller process. According to RY, Prop. VIII.3.4, its generator is given by (recall $f_a = \log x_a$)

$$
\begin{aligned}
(\widehat{G}^a f)(u) &= (Gf)(u) + \big(G(f_a f) - f_a G(f) - f G(f_a)\big)(u) \\
&= (Gf)(u) + 4u f_a'(u) f'(u) \\
&= 2u f''(u) + 2f'(u)\Big(1 + 2u\, \frac{x_a'(u)}{x_a(u)}\Big) \qquad (f \in C_c^2(\mathbb{R}^+)). \qquad (2.7)
\end{aligned}
$$

2. According to KS, Ch. 5, Eq. (5.42), the scale function is given (up to an affine transformation) by

$$
s_a(u) = \int_c^u \frac{dv}{v\, x_a^2(v)} \qquad (c > 0 \text{ arbitrary}). \qquad (2.8)
$$

Since x_a does not vanish at zero and has a subexponential tail at infinity (see the remarks at the beginning of Subsection 2.1), the scale function satisfies

$$
\lim_{u \downarrow 0} s_a(u) = -\infty \qquad \text{and} \qquad \lim_{u \to \infty} s_a(u) = \infty. \qquad (2.9)
$$

3. The probability measure on $\mathbb{R}_0^+$ given by

$$
\mu_a(du) = x_a(u)^2\, du \qquad (2.10)
$$

is the normalized speed measure for the process (see KS, Ch. 5, Eq. (5.51)). Since it has finite mass, and because (2.9) holds, the process converges weakly towards μ_a from any starting point $h \geq 0$ (see KS, Ch. 5, Ex. 5.40), i.e.,

$$
\lim_{t \to \infty} \widehat{\mathbb{E}}_h^a(f(X_t)) = \int_0^\infty f(u)\, \mu_a(du) \text{ for all bounded } f \in C(\mathbb{R}^+). \qquad (2.11)
$$

Using this convergence and the Feller property, one derives in a standard way that μ_a is the invariant distribution for the process. We write

$$
\widehat{\mathbb{P}}^a = \int_0^\infty \widehat{\mathbb{P}}_h^a\, \mu_a(dh) \qquad (2.12)
$$

to denote the distribution of the process starting in the invariant distribution and write $\widehat{\mathbb{E}}^a$ for the corresponding expectation.

2.3 Key steps in the proof of Proposition 1

Using the representation in Lemma 2, we shall rewrite the l.h.s. of (1.1) in terms of the transformed process introduced in Lemma 3. This will be the final reformulation in terms of which the proof of Proposition 1 will be finished in Subsection 2.4.

For $h, t \geq 0$, introduce the abbreviation (recall (1.9))

$$w(h,t) = \mathbb{E}_h^\star \left(e^{-\int_0^\infty F_{a^\star}(X_v^\star)\, dv} \,\Big|\, \int_0^\infty X_v^\star\, dv = t \right) \varphi_h(t). \tag{2.13}$$

Remember that $\widehat{\mathbb{E}}^a$ denotes the distribution of the transformed process under the invariant distribution μ_a given by (2.10).

Lemma 4 *For every $T > 0$*

$$e^{a^\star T} E\left(e^{-\int_{\mathbb{R}} L(T,x)^2\, dx} 1_{0 < B_T \leq C_T} \right)$$

$$= \int_0^\infty dt_1 \int_0^\infty dt_2\, \widehat{\mathbb{E}}^{a^\star} \left(\frac{w(X_0, t_1)}{x_{a^\star}(X_0)} 1_{A^{-1}(T-t_1-t_2) \leq C_T} \frac{w(X_{A^{-1}(T-t_1-t_2)}, t_2)}{x_{a^\star}(X_{A^{-1}(T-t_1-t_2)})} \right). \tag{2.14}$$

Proof. First, from (1.8), (2.1) and from $\rho(a^\star) = 0$ it follows that on $\{ A^{-1}(t) = u \}$

$$a^\star t - \int_0^u X_v^2\, dv = -\int_0^u F_{a^\star}(X_v)\, dv \qquad (t, u \geq 0). \tag{2.15}$$

By an absolute continuous transformation from $\mathbb{P}_h$ to $\widehat{\mathbb{P}}_h^{a^\star}$, we therefore obtain via (2.2) the identity (recall (1.9))

$$e^{a^\star t} \mathbb{E}_{h_1} \left(e^{-\int_0^u X_v^2\, dv} \,\Big|\, A^{-1}(t) = u, X_u = h_2 \right) \psi_{h_1, t}(u, h_2)\, du\, dh_2$$

$$= \widehat{\mathbb{P}}_{h_1}^{a^\star} \left(A^{-1}(t) \in du, X_u \in dh_2 \right) \frac{x_{a^\star}(h_1)}{x_{a^\star}(h_2)} \tag{2.16}$$

for a.e. $u, h_1, h_2, t \geq 0$. Similarly to (2.15), we have on $\{ \int_0^\infty X_v^\star\, dv = t \}$

$$a^\star t - \int_0^\infty (X_v^\star)^2\, dv = -\int_0^\infty F_{a^\star}(X_v^\star)\, dv \qquad (t \geq 0) \tag{2.17}$$

and hence

$$e^{a^\star t_i} \mathbb{E}_{h_i}^\star \left(e^{-\int_0^\infty (X_v^\star)^2\, dv} \,\Big|\, \int_0^\infty X_v^\star\, dv = t_i \right) \varphi_{h_i}(t_i) = w(h_i, t_i) \qquad (i = 1, 2). \tag{2.18}$$

Next, note that the l.h.s. of (2.14) is equal to the l.h.s. of (1.26) times the factor $e^{a^\star T}$. We shall divide this factor into three parts, according to the identity $T = t_1 + (T - t_1 - t_2) + t_2$, and assign them to each of the three expectations in the r.h.s. of (1.26). Substitute (2.16) with $t = T - t_1 - t_2$ and (2.18) into (1.26). Then we obtain that the l.h.s. of (2.14) is equal to

$$\int_{[0,\infty)^4} dh_1\, dh_2\, dt_1\, dt_2\, w(h_1, t_1) w(h_2, t_2) \frac{x_{a^\star}(h_1)}{x_{a^\star}(h_2)}$$

$$\times \widehat{\mathbb{P}}_{h_1}^{a^\star} \left(A^{-1}(T - t_1 - t_2) \leq C_T, X_{A^{-1}(T-t_1-t_2)} \in dh_2 \right). \tag{2.19}$$

Now formally carry out the integration over h_1, h_2, recalling (2.10) and (2.12), to arrive at the r.h.s. of (2.14). $\square$

Roughly speaking, the function w in the r.h.s. of (2.14) describes the contribution to the random variable $\exp[-\int_{\mathbb{R}} L(T,x)^2\,dx]$ coming from the boundary pieces (i.e., the parts of the path in $(-\infty,0] \cup [B_T,\infty)$), while A^{-1} gives the size of the area over which the middle piece (i.e., the parts of the path in $[0, B_T]$) spreads out.

The proof of Proposition 1 now basically requires the following three ingredients:

(1) A CLT for $(A^{-1}(t))_{t\geq 0}$ under $\widehat{\mathbb{P}}^{pa^*}$.

(2) Some integrability properties of w.

(3) An extension of the weak convergence stated in (2.11).

The precise statements we actually need are the following. Let $\langle \cdot, \cdot \rangle_{L^2}$ denote the standard inner product on $L^2(\mathbb{R}_0^+)$.

Proposition 2 *For any $a \in \mathbb{R}$, $R \geq 0$, $C \in \mathbb{R}$ and any continuous function $z \in L^2(\mathbb{R}_0^+)$*

$$\lim_{T\to\infty} \widehat{\mathbb{E}}^a\Big(\frac{z}{x_a}(X_0)\,1_{A^{-1}(T-R)\leq C_{a,T}}\,\frac{z}{x_a}(X_{A^{-1}(T-R)})\Big)$$
$$= \langle z, x_a\rangle_{L^2}^2\,\mathcal{N}_{\sigma^2(a)}((-\infty, C]), \tag{2.20}$$

where

$$\sigma^2(a) = \frac{\rho''(a)}{\rho'(a)^3} \qquad \text{and} \qquad C_{a,T} := \frac{T}{\rho'(a)} + C\sqrt{T}. \tag{2.21}$$

(Note that $\sigma^2(a^*) = c^{*2}$ defined in (0.6) and that $C_{a^*,T} = C_T$ defined in (1.25).)

Proposition 3 *The function $y : \mathbb{R}_0^+ \to \mathbb{R}$ given by*

$$y(h) = \int_0^\infty w(h,t)\,dt = \mathbb{E}_h^*\Big(e^{-\int_0^\infty F_{a^*}(X_v^*)\,dv}\Big) \tag{2.22}$$

is continuous and square-integrable w.r.t. Lebesgue measure.

In Sections 3 and 4 we prove Propositions 2 resp. 3.

2.4 Conclusion of the proof of Proposition 1

In this subsection we finish the proof of Proposition 1 subject to Propositions 2 and 3. We shall show that (1.1) follows from (2.14) and these propositions, with S identified as

$$S = \langle y, x_{a^*}\rangle_{L^2}^2. \tag{2.23}$$

All that we need is to use a cutting argument. Note that the function

$$y_R(h) = \int_R^\infty w(h,t)\,dt = \mathbb{E}_h^*\Big(e^{-\int_0^\infty F_{a^*}(X_v^*)\,dv}1_{\int_0^\infty X_v^*\,dv\geq R}\Big) \qquad (h\geq 0) \tag{2.24}$$

15

converges pointwise to zero as $R \to \infty$. Moreover, it has a square-integrable majorant by Proposition 3. So, with the help of Fubini's theorem, the Cauchy-Schwarz inequality and the stationarity of $(X_v)_{v \geq 0}$ under $\widehat{\mathbb{P}}^{a^*}$ (recall (2.12)), we see that

$$
\begin{aligned}
\sup_{T>0} \int_0^\infty dt_1 \int_R^\infty dt_2 \, \widehat{\mathbb{E}}^{a^*} & \Big(\frac{w(X_0, t_1)}{x_{a^*}(X_0)} \frac{w(X_T, t_2)}{x_{a^*}(X_T)} \Big) \\
&= \sup_{T>0} \widehat{\mathbb{E}}^{a^*} \Big(\frac{y}{x_{a^*}}(X_0) \frac{y_R}{x_{a^*}}(X_T) \Big) \\
&\leq \sup_{T>0} \sqrt{\widehat{\mathbb{E}}^{a^*} \Big(\frac{y}{x_{a^*}}(X_0)^2 \Big)} \sqrt{\widehat{\mathbb{E}}^{a^*} \Big(\frac{y_R}{x_{a^*}}(X_T)^2 \Big)} \\
&= \| y \|_{L^2} \, \| y_R \|_{L^2}.
\end{aligned}
$$

$$(2.25)$$

The r.h.s. tends to zero as $R \to \infty$ by the bounded convergence theorem. A similar calculation can be made for $\int_R^\infty dt_1 \int_0^\infty dt_2$. The proof of Proposition 1 is now easily completed by using Lemma 4 and noting that

$$
1_{A^{-1}(T) \leq C_T} \leq 1_{A^{-1}(T-t_1-t_2) \leq C_T} \leq 1_{A^{-1}(T-2R) \leq C_T} \qquad (t_1, t_2 \in [0, R]).
$$

$$(2.26)$$

Indeed, apply Proposition 2 with $a = a^*$ and with $z = y$ defined in Proposition 3. The details are left to the reader.

3 CLT for the middle piece

This section contains the proof of Proposition 2. First we extend the weak convergence stated in (2.11).

STEP 1 *For any $a, a_T \in \mathbb{R}$ with $\lim_{T \to \infty} a_T = a$ and any continuous function $z \in L^2(\mathbb{R}_0^+)$*

$$
\lim_{T \to \infty} \widehat{\mathbb{E}}^{a_T} \Big(\frac{z}{x_{a_T}}(X_0) \frac{z}{x_{a_T}}(X_T) \Big) = \langle z, x_a \rangle_{L^2}^2.
$$

$$(3.1)$$

Proof. Since $a \mapsto \langle z, x_a \rangle_{L^2}$ is continuous (see the remarks at the beginning of Subsection 2.1), one proves with the help of (2.10–2.12) that for every $R > 0$

$$
\begin{aligned}
\lim_{T \to \infty} \widehat{\mathbb{E}}^{a_T} & \Big(\frac{z}{x_{a_T}}(X_0) \Big[\frac{z}{x_{a_T}} \wedge R \Big](X_T) \Big) \\
&= \lim_{T \to \infty} \int_0^\infty dh \, z(h) \, x_{a_T}(h) \, \widehat{\mathbb{E}}_h^{a_T} \Big(\Big[\frac{z}{x_{a_T}} \wedge R \Big](X_T) \Big) \\
&= \langle z, x_a \rangle_{L^2} \int_0^\infty \Big[\frac{z}{x_a} \wedge R \Big](u) \, \mu_a(du).
\end{aligned}
$$

$$(3.2)$$

The integral tends to $\langle z, x_a \rangle_{L^2}$ as $R \to \infty$. On the other hand, in the same manner as in (2.25) we obtain that

$$\lim_{R \to \infty} \sup_{T > 0} \widehat{\mathbb{E}}^{a_T} \Big(\frac{z}{x_{a_T}}(X_0) \Big(\frac{z}{x_{a_T}} - \Big[\frac{z}{x_{a_T}} \wedge R \Big] \Big)(X_T) \Big) = 0. \tag{3.3}$$

$\square$

Next, introduce the abbreviation (recall (1.8))

$$\bar{A}(T) = \frac{1}{\sqrt{T}} \Big(A(T) - \rho'(a)T \Big) = \frac{1}{\sqrt{T}} \int_0^T \Big(X_v - \rho'(a) \Big) dv \qquad (T > 0). \tag{3.4}$$

In the following step we use the special shape of the diffusion defined in Lemma 3 to establish the convergence of the moment generating function of $\bar{A}(T)$ under $\widehat{\mathbb{P}}^a$.

STEP 2 *For any* $a \in \mathbb{R}$, $\lambda \in \mathbb{R}$ *and any continuous function* $z \in L^2(\mathbb{R}_0^+)$

$$\lim_{T \to \infty} \widehat{\mathbb{E}}^a \Big(\frac{z}{x_a}(X_0) \, e^{\lambda \bar{A}(T)} \, \frac{z}{x_a}(X_T) \Big) = \langle z, x_a \rangle_{L^2}^2 \, e^{-\frac{\lambda^2}{2} \rho''(a)}. \tag{3.5}$$

Proof. Fix $\lambda \in \mathbb{R}$. Let $a_T = a + \lambda/\sqrt{T}$. Then from (2.1) and (3.4) we have

$$\lambda \bar{A}(T) - \int_0^T F_a(X_v) \, dv + \int_0^T F_{a_T}(X_v) \, dv = T(\rho(a_T) - \rho(a)) - \lambda \sqrt{T} \rho'(a). \tag{3.6}$$

According to Taylor's theorem, this is equal to $\frac{1}{2} \lambda^2 \rho''(\xi_T)$ for some $\xi_T \in [a, a_T] \cup [a_T, a]$. Therefore, by a change from $\widehat{\mathbb{E}}^a$ to $\widehat{\mathbb{E}}^{a_T}$ one calculates (See Subsections 2.1 and 2.2) that the expectation in the l.h.s. of (3.5) is equal to

$$\int_0^\infty dh \, z(h) \mathbb{E}_h \Big(e^{-\int_0^T F_{a_T}(X_v) \, dv} z(X_T) \Big) e^{T(\rho(a_T) - \rho(a)) - \lambda \sqrt{T} \rho'(a)}$$

$$= e^{\frac{\lambda^2}{2} \rho''(\xi_T)} \widehat{\mathbb{E}}^{a_T} \Big(\frac{z}{x_{a_T}}(X_0) \, \frac{z}{x_{a_T}}(X_T) \Big). \tag{3.7}$$

Because a_T tends to a, this tends to the r.h.s. of (3.5), by the continuity of ρ'' and by Step 1. $\square$

STEP 3 *Proof of Proposition 2*

Proof. Fix $T > 0$ and a continuous $z \in L^2(\mathbb{R}_0^+)$. Define a (non-Markovian) path measure $\widehat{\mathbb{P}}^{a,z,T}$ by

$$\frac{d\widehat{\mathbb{P}}^{a,z,T}}{d\widehat{\mathbb{P}}^a} = \frac{\frac{z}{x_a}(X_0) \frac{z}{x_a}(X_T)}{\widehat{\mathbb{E}}^a \Big(\frac{z}{x_a}(X_0) \frac{z}{x_a}(X_T) \Big)}. \tag{3.8}$$

Then Steps 1 and 2 show that the moment generating function of $\bar{A}(T)$ under $\widehat{\mathbb{P}}^{a,z,T}$ tends pointwise to the one of the normal distribution with mean 0 and variance $\rho''(a)$. Hence,

$$\lim_{T\to\infty} \widehat{\mathbb{P}}^{a,z,T}(\bar{A}(T) \geq -c) = \mathcal{N}_{\rho''(a)}([-c,\infty)) \text{ for all } c \in \mathbb{R}. \tag{3.9}$$

Next, fix $R \geq 0$ and use (3.4) and (2.21) to see that

$$\{A^{-1}(T-R) \leq C_{a,T}\} = \left\{ \bar{A}(C_{a,T}) \geq -\frac{R}{C_{a,T}} - C\rho'(a)\sqrt{\frac{T}{C_{a,T}}} \right\}. \tag{3.10}$$

Since $\lim_{T\to\infty} C_{a,T} = \infty$ and $\lim_{T\to\infty} T/C_{a,T} = \rho'(a)$, it follows from (3.9) and (3.10) that

$$\lim_{T\to\infty} \widehat{\mathbb{P}}^{a,z,C_{a,T}}(A^{-1}(T-R) \leq C_{a,T}) = \mathcal{N}_{\rho''(a)}([-C\rho'(a)^{\frac{3}{2}},\infty)) \text{ for all } C \in \mathbb{R}. \tag{3.11}$$

This is equal to $\mathcal{N}_{\sigma^2(a)}((-\infty, C])$ (see (2.21)). Now recall the definition of $\widehat{\mathbb{P}}^{a,z,C_{a,T}}$ in (3.8) and apply Step 1 to the denominator in (3.8) to finish the proof of Proposition 2. $\qquad\square$

4 Integrability for the boundary pieces

In Subsection 4.1 we transform BESQ^0 into a new Markov process and list some of its properties. This part is analogous to Subsections 2.1 and 2.2. Using the transformed process, we give the proof of Proposition 3 in Subsection 4.2.

4.1 The transformed process

Fix $a \in \mathbb{R}$. The following lemma introduces a function z_a that will play a role analogous to that of x_a in Section 2.

Lemma 5 *Let $a < 2^{\frac{1}{3}}(-u_1)$ with $u_1 = -2,3381\ldots$ the largest zero of the Airy function. Then there exists a real-analytic and eventually strictly decreasing function $z_a : \mathbb{R}_0^+ \to \mathbb{R}^+$ satisfying $z_a(0) = 1$ and*

$$2z_a''(u) + (a - u) z_a(u) = 0 \qquad (u \geq 0). \tag{4.1}$$

Moreover,

$$\lim_{u\to\infty} u^{-\frac{3}{2}} \log z_a(u) < 0. \tag{4.2}$$

Proof. Let Ai $: \mathbb{R} \to \mathbb{R}$ be the Airy function, i.e., the unique solution (modulo a constant multiple) of the Airy equation

$$x''(u) - ux(u) = 0 \qquad (u \in \mathbb{R}) \tag{4.3}$$

that is bounded on $\mathbb{R}_0^+$. Then

$$z_a(u) = \text{Ai}\left(2^{-\frac{1}{3}}(u-a)\right) \qquad (u \geq 0) \tag{4.4}$$

defines an analytic solution of (4.1). From Abramowitz and Stegun (1970), Table 10.13 (see also p. 450), we know that the largest zero u_1 of Ai is approximately equal to $-2,3381\ldots$. Hence we may assume that $\text{Ai}(u) > 0$ for $u > u_1$. From this and the bound on a it follows that $z_a(u) > 0$ for $u \geq 0$. Therefore, we may also normalize z_a such that $z_a(0) = 1$.

The asymptotics in (4.2) follows from Abramowitz and Stegun (1970), 10.4.59. The fact that $\text{Ai}'(u) < 0$ for large u follows from Table 10.13, so z_a is eventually strictly decreasing. $\qquad\square$

With the help of the function z_a we can now, for sufficiently small a, introduce a transformed process. Let $(P_t^\star)_{t \geq 0}$ denote the transition probability function of BESQ^0 and define

$$F_a^\star(u) = u^2 - au \qquad (u \in \mathbb{R}_0^+). \tag{4.5}$$

Lemma 6 Let $a < 2^{\frac{1}{3}}(-u_1)$. For $t, h_1, h_2 \geq 0$, define

$$\widehat{P}_t^{\star,a}(h_1, dh_2) = \frac{z_a(h_2)}{z_a(h_1)} \mathbb{E}_{h_1}^\star\left(e^{-\int_0^t F_a^\star(X_v^\star)\,dv} \mid X_t^\star = h_2\right) P_t^\star(h_1, dh_2). \tag{4.6}$$

Then $(\widehat{P}_t^{\star,a})_{t \geq 0}$ is the transition probability function of a diffusion $(X_v^\star)_{v \geq 0}$ on $\mathbb{R}_0^+$.

Proof. This is similar to the proof of Lemma 3. An elementary calculation shows that for every strictly positive $z \in C^2(\mathbb{R}_0^+)$ and $f = \log z$

$$\left(G^\star(f) + \frac{1}{2}G^\star(f^2) - f\,G^\star(f)\right)(u) = 2uf''(u) + 2uf'(u)^2 = \frac{2uz''(u)}{z(u)}. \tag{4.7}$$

where $G^\star$ is the generator of BESQ^0 given in (1.3). As is now easily seen from (4.1), the function $f_a^\star = \log z_a$ satisfies the relation

$$G^\star(f_a^\star) + \frac{1}{2}G^\star((f_a^\star)^2) - f_a^\star\,G^\star(f_a^\star) = F_a^\star. \tag{4.8}$$

This implies that

$$(D_t^{\star,a})_{t \geq 0} = \left(\frac{z_a(X_t^\star)}{z_a(X_0^\star)} e^{-\int_0^t F_a^\star(X_v^\star)\,dv}\right)_{t \geq 0} \tag{4.9}$$

is a local martingale under $\mathbb{P}_h^\star$ for every $h \geq 0$. Since z_a is bounded from above and $F_a^\star$ is bounded from below, each $D_t^{\star,a}$ is bounded $\mathbb{P}_h^\star$-a.s. Hence $(D_t^{\star,a})_{t \geq 0}$ is a martingale under $\mathbb{P}_h^\star$. The assertion of the lemma now follows from RY, Prop. VIII.3.1. $\qquad\square$

19

Similarly as in Subsection 2.2, one calculates that the process introduced on Lemma 6 has generator

$$(\widehat{G}^{\star,a} f)(u) = 2uf''(u) + 4uf'(u)\frac{z_a'(u)}{z_a(u)} \qquad (f \in C_c^2(\mathbb{R}^+)) \tag{4.10}$$

and scale function

$$s^{\star,a}(u) = \int_c^u \frac{dv}{z_a^2(v)} \qquad (c \in (0,\infty) \text{ arbitrary}). \tag{4.11}$$

Since $z_a(0) > 0$ and because of (4.2), we have

$$\lim_{u\downarrow 0} s^{\star,a}(u) = s^{\star,a}(0+) > -\infty, \qquad \lim_{u\to\infty} s^{\star,a}(u) = \infty. \tag{4.12}$$

Thus, the transformed process converges almost surely towards its absorbing boundary 0 (see KS, Prop. 5.22).

We denote by $\widehat{\mathbb{P}}_h^{\star,a}$ the distribution of the transformed Markov process, starting at $h \geq 0$. The corresponding expectation is denoted by $\widehat{\mathbb{E}}_h^{\star,a}$.

The following lemma shows that Lemma 6 can be used for $a = a^*$.

Lemma 7 $a^* \leq \frac{3}{2}\pi^{\frac{1}{3}} < 2^{\frac{1}{3}}(-u_1)$.

Proof. The first inequality is proved via the variational representation

$$a^* = \inf_{x \in L^2(\mathbb{R}_0^+)\cap C^2(\mathbb{R}_0^+): x\neq 0} \frac{\int_0^\infty [u^2 x^2(u) + 2ux'(u)^2]\,du}{\int_0^\infty ux^2(u)\,du}. \tag{4.13}$$

This representation stems from the relation (see van der Hofstad and den Hollander (1995))

$$0 = \rho(a^*) = \max_{x \in L^2(\mathbb{R}_0^+)\cap C^2(\mathbb{R}_0^+): \|x\|_{L^2}=1} \langle x, \mathcal{K}^{a^*} x\rangle_{L^2}, \tag{4.14}$$

in which, by (0.5),

$$\langle x, \mathcal{K}^{a^*} x\rangle_{L^2} = \int_0^\infty \left[(a^*u - u^2)x(u)^2 - 2ux'(u)^2\right]du. \tag{4.15}$$

In (4.13), we choose the test function

$$x(u) = \exp\left(-u^2\frac{\pi^{\frac{1}{3}}}{8}\right). \tag{4.16}$$

Elementary computations give that $\int_0^\infty ux^2(u)\,du = 2\pi^{-\frac{1}{3}}$ and $\int_0^\infty u^2x^2(u)\,du = 2$ and $\int_0^\infty ux'(u)^2\,du = \frac{1}{2}$. Substituting this into (4.13), we obtain the bound $a^* \leq \frac{3}{2}\pi^{\frac{1}{3}} = 2.1968\dots$. $\qquad\square$

4.2 Proof of Proposition 3

Since, by Lemmas 5 and 7, $z_{a^\star}$ is analytic and has a subexponentially small tail at infinity, Proposition 3 directly follows from the following identification.

Lemma 8 $y \equiv z_{a^\star}$.

Proof. Fix $h \geq 0$. Recall that $(D_t^{\star,a})_{t \geq 0}$ defined in (4.9) is a martingale under $\mathbb{P}_h^\star$ for $a < 2^{\frac{1}{3}}(-u_1)$ and that this is true, in particular, for $a = a^\star$ by Lemma 7. Since, $\mathbb{P}_h^\star$-a.s., $(X_t^\star)_{t \geq 0}$ tends to the absorbing boundary 0, we see from (4.9) that

$$\lim_{t \to \infty} D_t^{\star,a} = \frac{1}{z_{a^\star}(h)} e^{-\int_0^\infty F_{a^\star}^\star(X_v^\star)\, dv} \tag{4.17}$$

(recall that $z_{a^\star}(0) = 1$). If this convergence were also true in $L^1(\mathbb{P}_h^\star)$, then, by taking expectations in (4.17) and noting that $F_{a^\star} \equiv F_{a^\star}^\star$ (see (2.1) and (4.5)), we would obtain

$$y(h) = \lim_{t \to \infty} \mathbb{E}_h^\star \left(e^{-\int_0^t F_{a^\star}(X_v^\star)\, dv} \right) = z_{a^\star}(h) \lim_{t \to \infty} \mathbb{E}_h^\star(D_t^{\star,a^\star}) = z_{a^\star}(h), \tag{4.18}$$

which proves the claim.

In order to show the $L^1(\mathbb{P}_h^\star)$-convergence in (4.17), it is, according to the martingale convergence theorem, enough to show the $L^p(\mathbb{P}_h^\star)$-boundedness of $(D_t^{\star,a^\star})_{t \geq 0}$ for some $p > 1$. Pick $p > 1$ sufficiently small such that $a = a^\star p < 2^{\frac{1}{3}}(-u_1)$. Then, obviously, $pF_{a^\star}^\star(u) \geq F_a^\star(u)$ for all $u \geq 0$. Hence, by (4.5),

$$\left(D_t^{\star,a^\star} \right)^p \leq \left(\frac{\|z_{a^\star}\|_\infty}{z_{a^\star}(h)} \right)^p \exp\left(-\int_0^t F_a^\star(X_v^\star)\, dv \right) \qquad \mathbb{P}_h^\star\text{-a.s.} \tag{4.19}$$

It is therefore sufficient to show that

$$\lim_{t \to \infty} \mathbb{E}_h^\star \left(e^{-\int_0^t F_a^\star(X_v^\star)\, dv} \right) = z_a(h). \tag{4.20}$$

In order to show (4.20), fix $t, h \geq 0$ and use (4.6) to see that

$$\mathbb{E}_h^\star \left(e^{-\int_0^t F_a^\star(X_v^\star)\, dv} \right) = \widehat{\mathbb{E}}_h^{\star,a} \left(\frac{z_a(h)}{z_a(X_t^\star)} \right) = z_a(h) \int_0^\infty \widehat{\mathbb{P}}_h^{\star,a} \left(\frac{1}{z_a(X_t^\star)} > u \right) du. \tag{4.21}$$

Since, $\widehat{\mathbb{P}}_h^{\star,a}$-a.s., $\lim_{t \to \infty} z_a(X_t^\star) = 1$, the integrand converges to $1_{[0,1)}(u)$ as $t \to \infty$. So in order to prove (4.20) all we need is to derive a bound for the integrand that is uniform in t and integrable in u.

Recall from Lemma 5 that $z_a|_{(u_0,\infty)}$ is strictly decreasing for some $u_0 > 0$ (depending on a). Denote its inverse function by $z_a^{-1}: (0, z_a(u_0)) \to (u_0, \infty)$. Pick u so large that $h < z_a^{-1}(\frac{1}{u})$. Then

$$\sup_{t > 0} \widehat{\mathbb{P}}_h^{\star,a} \left(\frac{1}{z_a(X_t^\star)} > u \right) \leq \widehat{\mathbb{P}}_h^{\star,a} \left(\sup_{t > 0} X_t^\star \geq z_a^{-1}\left(\frac{1}{u}\right) \right)$$

$$= \frac{s^{\star,a}(h) - s^{\star,a}(0+)}{s^{\star,a}\left(z_a^{-1}\left(\frac{1}{u}\right)\right) - s^{\star,a}(0+)}, \tag{4.22}$$

where the last equality uses the defining property of the scale function (see KS, p. 346). In order to show that the latter expression is integrable in u, it is sufficient to show that

$$\liminf_{u \to \infty} s^{\star,a}\left(z_a^{-1}\left(\frac{1}{u}\right)\right) u^{\varepsilon-2} > 0 \qquad \text{for any } \varepsilon > 0. \tag{4.23}$$

From (4.11) we see that

$$s^{\star,a}\left(z_a^{-1}\left(\frac{1}{u}\right)\right) = \int_{z_a(c)}^{\frac{1}{u}} \frac{dv}{v^2\, g_a(v)} \tag{4.24}$$

where $g_a(v) = z_a'(z_a^{-1}(v))$ for small $v > 0$. Use the differential equation in (4.1) to calculate

$$g_a'(v)\, g_a(v) = \frac{v}{2}\left(z_a^{-1}(v) - a\right). \tag{4.25}$$

Deduce from (4.2) that

$$\limsup_{v \downarrow 0} z_a^{-1}(v) v^{2\varepsilon} < \infty \qquad \text{for any } \varepsilon > 0. \tag{4.26}$$

Integrate (4.25) and use (4.26) to obtain

$$\limsup_{v \downarrow 0} g_a^2(v) v^{2\varepsilon-2} < \infty. \tag{4.27}$$

Substitute this into (4.24) to obtain (4.23). $\qquad\qquad\square$

References

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (9th edition), Dover Publications, Inc., New York 1970.

[2] R. van der Hofstad and F. den Hollander, Scaling for a random polymer, Commun. Math. Phys. 169, 397–440 (1995).

[3] I. Karatzas and S.E. Shreve, *Brownian Motion and Stochastic Calculus* (2nd edition), Springer, New York 1991.

[4] W. König, A central limit theorem for a one-dimensional polymer measure. Preprint 1994.

[5] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer, Berlin 1991.

[6] J. Westwater, On Edwards model for polymer chains, in: *Trends and Developments in the Eighties* (S. Albeverio and P. Blanchard, eds.), Bielefeld Encounters in Math. Phys. 4/5, World Scientific, Singapore 1984.