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Characterization of weak convergence for smoothed empirical and quantile processes under φ-mixing

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Abstract

Let $\{X_n, n \geqslant 1\}$ be a sequence of φ-mixing random variables having a smooth common distribution function F. The smoothed empirical distribution function is obtained by integrating a kernel type density estimator. In this paper we provide necessary and sufficient conditions for the central limit theorem to hold for smoothed empirical distribution functions and smoothed sample quantiles. Also, necessary and sufficient conditions are given for weak convergence of the smoothed empirical process and the smoothed uniform quantile process.

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1. Introduction

Let $\{X_n, n \geqslant 1\}$ be a φ-mixing sequence of identically distributed r.v.'s with continuous d.f. F and density f. By definition, $\{X_n, n \geqslant 1\}$ is φ-mixing if for each $k > 0$ and for each $n \geqslant 1$, $A \in \mathscr{M}_0^k$ and $B \in \mathscr{M}_{k+n}^\infty$,

$$|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \leqslant \varphi(n)\mathbb{P}(A),$$

where φ is a nonnegative function of positive integers and $\mathscr{M}_a^b$ denotes the σ-algebra generated by $X_a, X_{a+1}, \ldots, X_b$, for $a \leqslant b$.

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The classical estimator of F is the empirical d.f. defined as

$$F_n(x) = n^{-1}\sum_{i=1}^{n} u(x-X_i),$$

where u is the d.f. of the unit mass at 0. Given the information that F is absolutely continuous, it is natural to consider smooth estimators $\widehat{F}_n$ of F rather than the classical step function F_n in order to obtain a better asymptotic performance. In particular, Falk (1983) showed in the i.i.d. case that under appropriate conditions, $\widehat{F}_n(x)$ has a better asymptotic performance as an estimator of $F(x)$ than $F_n(x)$, in the sense of asymptotic deficiency. The smoothed empirical d.f. is defined by

$$\widehat{F}_n(x) = n^{-1}\sum_{i=1}^{n} K_n(x-X_i), \tag{1.1}$$

where K_n, $n = 1,2,\ldots$, is a sequence of continuous d.f.'s such that $K_n \xrightarrow{\mathrm{w}} u$. A natural way to obtain such a K_n is by integrating a kernel density estimator

$$\hat{f}_n(x) := n^{-1}\sum_{i=1}^{n} a_n^{-1}k((x - X_i)/a_n),$$

such that

$$K_n(x) = \int_{-\infty}^{x} a_n^{-1}k(t/a_n)\,\mathrm{d}t =: \int_{-\infty}^{x} k_n(t)\,\mathrm{d}t,$$

where $k \geqslant 0$, $\int_{-\infty}^{\infty} k(t)\,\mathrm{d}t = 1$ and $\{a_n, n \geqslant 1\}$ is a positive sequence such that $a_n \downarrow 0$ as $n \to \infty$. Furthermore, let $\xi_p = \inf\{x : F(x) \geqslant p\}$ denote the quantile of order p. As an estimator of ξ_p we shall consider the smoothed sample quantile, defined by $\hat{\xi}_{np} = \inf\{x : \widehat{F}_n(x) \geqslant p\}$.

Nadaraya (1964) initiated the study of the asymptotic behavior of $\widehat{F}_n(x)$ and $\hat{\xi}_{np}$, and proved in the i.i.d. case that under certain regularity conditions both $\widehat{F}_n(x)$ and $\hat{\xi}_{np}$ have an asymptotic normal distribution. Related asymptotic results for $\widehat{F}_n(x)$ in the i.i.d. case can be found in Fernholz (1991), Puri and Ralescu (1986) and Lea and Puri (1988) and for $\hat{\xi}_{np}$ in Ralescu and Sun (1993). An extension of the central limit theorem for $\hat{\xi}_{np}$ to m-dependent r.v.'s has been obtained by Sun (1993).

Recently, Yukich (1992) established weak convergence of the smoothed empirical process for i.i.d. r.v.'s under the assumption that F is Lipschitz. More precisely, he proved that the following condition is necessary and sufficient for weak convergence:

$$\forall \varepsilon > 0\colon\ n^{1/2} \sup_{x} \int_{|t|>\varepsilon/\sqrt{n}} |F(x-t) - F(x)|k_n(t)\,\mathrm{d}t \to 0. \tag{1.2}$$

In this paper we will provide necessary and sufficient conditions for the asymptotic normality of smoothed empirical d.f.'s and of smoothed sample quantiles, as well as necessary and sufficient conditions for the functional central limit theorem to hold for the smoothed empirical process and the smoothed uniform quantile process under φ-mixing. Moreover, we will establish a relationship between (1.2) and a_n for kernels satisfying $\int_{\infty}^{\infty} |y|k(y)\,\mathrm{d}y < \infty$, also called first-order kernels. We also show that the order of a_n can be reduced for second-order kernels (which have the property that $\int_{\infty}^{\infty} yk(y)\,\mathrm{d}y = 0$ and $\int_{\infty}^{\infty} y^2k(y)\,\mathrm{d}y < \infty$), in which case condition (1.2) will no longer be necessary and sufficient.

Our main results are stated in the next section; the proofs are given in Section 3.

2. Results

Let $x \in \mathbb{R}$ be fixed and let by definition the set $\{y \in \mathbb{R} : 0 < F(y) < 1\}$ be the support of F. We introduce the following sets of regularity conditions.

(A_x) (i) f is differentiable with bounded derivative f' on the support of F. f' is continuous in a neighborhood of x and $f'(x) \neq 0$.

(ii) $\int_{-\infty}^{\infty} tk(t)\,\mathrm{d}t = 0$ and $\int_{-\infty}^{\infty} t^2k(t)\,\mathrm{d}t < \infty$.

(A) There exists an $x' \in \mathbb{R}$ such that f and k satisfy $\mathrm{A}_{x'}$ (i.e. $\mathrm{A} = \bigcup_{y\in\mathbb{R}} \mathrm{A}_y$).

Let $\|g\| = \sup_t |g(t)|$ be the supremum norm. We now state a central limit theorem for the smoothed empirical process at a fixed point x.

Theorem 2.1. *Let $\{x_n, n \geqslant 1\}$ be a sequence of real numbers such that $x_n \to x$ and suppose $\sum_{n=0}^{\infty} n^2(\varphi(n))^{1/2} < \infty$.*

(a) *Suppose f and k satisfy* (A_x). *Then we have*

$$n^{1/2}\left(\frac{\widehat{F}_n(x_n) - F(x_n)}{\sigma(x)}\right) \xrightarrow{\mathrm{d}} \mathrm{N}(0,1), \tag{2.1}$$

iff $n^{1/4}a_n \to 0$, where

$$\sigma^2(x) = F(x)(1 - F(x)) + 2\sum_{k=1}^{\infty}[\mathbb{E}(u(x - X_1)u(x - X_{k+1})) - F^2(x)].$$

(b) *Suppose $\|f\| < \infty$. Then* (2.1) *holds iff*

$$\forall \varepsilon > 0:\ n^{1/2}\int_{|t|>\varepsilon/\sqrt{n}} (F(x_n - t) - F(x_n))k_n(t)\,\mathrm{d}t \to 0. \tag{2.2}$$

The next theorem gives the analogue of Theorem 2.1 for the standardized smoothed quantile process at a fixed point.

Theorem 2.2. *Let $f(\xi_p) > 0$ and suppose $\sum_{n=0}^{\infty} n^2(\varphi(n))^{1/2} < \infty$.*

(a) *Suppose that f and k satisfy* (A_x) *with $x = \xi_p$. Then we have*

$$n^{1/2}f(\xi_p)\left(\frac{\hat{\xi}_{np} - \xi_p}{\sigma(\xi_p)}\right) \xrightarrow{\mathrm{d}} \mathrm{N}(0,1), \tag{2.3}$$

iff $n^{1/4}a_n \to 0$, *where*

$$\sigma^2(\xi_p) = p(1-p)+2\sum_{k=1}^{\infty}[\mathbb{E}(u(\xi_p-X_1)u(\xi_p-X_{k+1})) - p^2].$$

(b) *Suppose that* $\|f\| < \infty$. *Then* (2.3) *holds iff for all* $y \in \mathbb{R}$, (2.2) *holds with* $x_n = \xi_p + y\sigma(\xi_p)/(n^{1/2}f(\xi_p))$.

In order to formulate the next theorem, we introduce some notation. Let

$$\widehat{U}_n(x) = n^{1/2}(\widehat{F}_n(x) - F(x)), \quad x \in \mathbb{R},$$

denote the smoothed empirical process. The uniform empirical process on $[0,1]$ is defined as

$$\alpha_n(t) = n^{1/2}(\Gamma_n(t)-t), \quad 0 \leqslant t \leqslant 1,$$

where $\Gamma_n(t) = n^{-1}\sum_{i=1}^n \mathbf{1}_{[0,t]}(U_i)$ and $U_i = F(X_i)$ is uniformly distributed on $[0,1]$. Moreover, as a special case of (1.1), let us define the smoothed uniform empirical distribution function as $\widehat{\Gamma}_n(x) = n^{-1}\sum_{i=1}^n K_n(x - U_i)$, $x \in \mathbb{R}$, and let $\widehat{\beta}_n$ denote the smoothed uniform quantile process, that is

$$\widehat{\beta}_n(t) = n^{1/2}(\widehat{\Gamma}_n^{-1}(t)-t), \quad 0 \leqslant t \leqslant 1.$$

Depending on our needs, $D[0,1]$ (respectively $D[-\infty,\infty]$) will denote either the space of all right-continuous functions with left-hand limits or the space of all left-continuous functions with right-hand limits on $[0,1]$ (the extended real line $[-\infty,\infty]$) endowed with the Skorohod metric ρ. The corresponding σ-field of Borel sets will be written as $\mathscr{D}$. Furthermore, let $C[0,1]$ (respectively $C[-\infty,\infty]$) be the space of all continuous functions on $[0,1]$ ($[-\infty,\infty]$) endowed with the uniform metric and let $\mathscr{C}$ be the σ-field of Borel sets. By definition we set $G(-\infty) = 0$ and $G(\infty) = 1$, for any distribution function G. Then clearly $\widehat{U}_n$ takes values in the space $C[-\infty,\infty]$. The symbol $\xrightarrow{\mathscr{L}}$ will be used to denote weak convergence in $D[0,1]$, $D[-\infty,\infty]$, $C[0,1]$ or $C[-\infty,\infty]$.

Theorem 2.3. *Let* $\sum_{n=0}^{\infty} n^2(\varphi(n))^{1/2} < \infty$.

(a) *Suppose that* f *and* k *satisfy* (A). *Then we have*

$$\widehat{U}_n \xrightarrow{\mathscr{L}} U(F) \text{ in } C[-\infty,\infty], \tag{2.4}$$

iff $n^{1/4}a_n \to 0$, *where* U *is the Gaussian random process on* $[0,1]$ *specified by*

$$\mathbb{E}U(t) = 0$$

and

$$\mathbb{E}U(s)U(t) = \mathbb{E}g_s(U_1)g_t(U_1) + \sum_{k=1}^{\infty}\mathbb{E}g_s(U_1)g_t(U_{k+1}) + \sum_{k=1}^{\infty}\mathbb{E}g_s(U_{k+1})g_t(U_1).$$

Here the function g_t *is defined by* $g_t(x) = \mathbf{1}_{[0,t]}(x)-t$.

(b) *Suppose that* $\|f\| < \infty$. *Then* (2.4) *holds iff* (1.2) *holds.*

A natural way to continue would be to prove a result similar to (2.4) for the standardized smoothed quantile process $\widehat{V}_n(t) = n^{1/2} f(F^{-1}(t))(\widehat{F}_n^{-1}(t) - F^{-1}(t))$, $t \in (0,1)$. Some remarks have to be made in this respect. First of all, for proving weak convergence for the classical standardized quantile process $V_n(t) = n^{1/2} f(F^{-1}(t))(F_n^{-1}(t) - F^{-1}(t))$, $t \in (0,1)$, several regularity conditions have to be imposed on the underlying distribution, see e.g. Shorack and Wellner (1986, p. 645). Other complications arise from the fact that, in general, $n^{-1/2} V_n$ does not even satisfy the Glivenko–Cantelli property. In case of the smoothed quantile process one can even expect more difficulties. Finally, $\widehat{F}_n^{-1}$ cannot easily be written as a composition of F^{-1} and the smoothed uniform quantile function. In order to avoid these technical complications we give an analogue of Theorem 2.3 for the uniform case.

Theorem 2.4. *Suppose that* $\sum_{n=0}^{\infty} n^2(\varphi(n))^{1/2} < \infty$. *Then we have*

$$\widehat{\beta}_n \xrightarrow{\mathscr{D}} -U \text{ in } C[0,1], \tag{2.5}$$

iff

$$\forall \varepsilon > 0: \ n^{1/2} \int_{|t| > \varepsilon/\sqrt{n}} |t| k_n(t)\,\mathrm{d}t \to 0. \tag{2.6}$$

Remark. Expressions (1.2) and (2.2) are a combination of conditions on F, k and a_n. Under appropriate conditions on f and k, a closer examination of these expressions gives us precise information about the needed order of a_n. We introduce the following set of regularity conditions:

(B_x) (i) f is continuous in a neighborhood of x, $f(x) > 0$ and f is bounded.
(ii) $\int_{-\infty}^{\infty} |t| k(t)\,\mathrm{d}t < \infty$ but $\int_{-\infty}^{\infty} t k(t)\,\mathrm{d}t \neq 0$.

For first-order kernels, we can establish the following relations between the expressions mentioned above and the sequence of bandwidths $\{a_n, n \geqslant 1\}$:

(a) Suppose $\|f\| < \infty$ and $\int_{-\infty}^{\infty} |t| k(t)\,\mathrm{d}t < \infty$, then

$$(1.2) \Longleftrightarrow n^{1/2} a_n \to 0. \tag{2.7}$$

(b) Let $x \in \mathbb{R}$ be fixed and suppose f and k satisfy (B_x), for some $x \in \mathbb{R}$. Then

$$(2.2) \Longleftrightarrow n^{1/2} a_n \to 0. \tag{2.8}$$

Statements (a) and (b) can easily be proved by means of a Taylor expansion and using Lebesgue's dominated convergence theorem.

For second-order kernels and sufficiently smooth f, the following analogue of (2.8) holds:

(c) Let f and k satisfy (A_x) for some $x \in \mathbb{R}$ and suppose

$$\forall \varepsilon > 0: \; n^{1/2} \int_{|t| > \varepsilon / n^{1/2}} t k_n(t) \, \mathrm{d}t \to 0. \tag{2.9}$$

Then

$$(2.2) \Longleftrightarrow n^{1/4} a_n \to 0. \tag{2.10}$$

Hence, in order to let (2.2) be necessary and sufficient for (2.1) or (2.3), we need the extra condition (2.9). Clearly, (2.9) is satisfied if k is symmetric.

It is also worth noting that no such relation can be established between (1.2) and bandwidths of order $o(n^{-1/4})$ when dealing with second order kernels: it can easily be checked that (2.7) still holds if f and k satisfy conditions (A). Hence, by Theorem 2.3 we conclude that the use of higher-order kernels allows us to reduce the order of the bandwidth a_n to $o(n^{-1/4})$.

3. Proofs

Proof of Thorem 2.1. By integration by parts we have

$$\begin{aligned} n^{1/2}(\widehat{F}_n(x_n) - F(x_n)) &= n^{1/2} \int_{-\infty}^{\infty} (F_n(x_n - t) - F(x_n - t)) \, \mathrm{d}K_n(t) \\ &\quad + n^{1/2} \int_{-\infty}^{\infty} F(x_n - t)(\mathrm{d}K_n(t) - \mathrm{d}u(t)) \\ &:= A_n(x_n) + B_n(x_n). \end{aligned} \tag{3.1}$$

We will first show that $A_n(x_n) \xrightarrow{\mathrm{d}} \mathrm{N}(0, \sigma(x))$ follows from the fact that $a_n \downarrow 0$. Let $h_n, h : D[0,1] \to \mathbb{R}$ be defined as

$$h(y) = \pi_{1/2}(y) = y(\tfrac{1}{2}) \tag{3.2}$$

and

$$h_n(y) = \int_{-\infty}^{\infty} y(\Phi(t)) \, \mathrm{d}K_n(t), \tag{3.3}$$

where Φ is the d.f. of the standard normal distribution. Obviously we have

$$A_n(x_n) = h_n(\widetilde{\alpha}_n), \tag{3.4}$$

where

$$\widetilde{\alpha}_n(t) = \begin{cases} \alpha_n(F(x_n - \Phi^{-1}(t))) & \text{for } t \in (0,1), \\ 0 & \text{for } t \in \{0,1\}. \end{cases} \tag{3.5}$$

Next, let us consider the following subspace of $D[0,1]$:

$$\widetilde{D}[0,1] = \{f \in D[0,1]\colon 0 \leqslant f \leqslant 1 \text{ and } f \text{ is nonincreasing}\}. \tag{3.6}$$

The σ-field of Borel sets of $\widetilde{D}[0,1]$ is denoted by $\widetilde{\mathscr{D}}$, where $\widetilde{\mathscr{D}} = \widetilde{D}[0,1] \cap \mathscr{D}$. Obviously, the functions g_n and g defined by

$$g_n(t) = F(x_n - \Phi^{-1}(t)), \quad t \in [0,1], \tag{3.7}$$

and

$$g(t) = F(x - \Phi^{-1}(t)), \quad t \in [0,1], \tag{3.8}$$

are elements of $\widetilde{D}[0,1]$. Moreover, since

$$\|g_n - g\| \leqslant |x_n - x| \; \|f\|, \tag{3.9}$$

we have

$$g_n \to g \text{ in } \widetilde{D}[0,1]. \tag{3.10}$$

Also, application of Theorem 22.1 in Billingsley (1968) gives us

$$\alpha_n \xrightarrow{\mathscr{L}} U \text{ in } D[0,1],$$

where U is the Gaussian random process as defined in Theorem 3. Consequently we have

$$(\alpha_n, g_n) \xrightarrow{\mathscr{L}} (U, g) \text{ in } D[0,1] \times \widetilde{D}[0,1], \tag{3.11}$$

see e.g. Billingsley (1968, Theorem 4.4). Hence, since $g \in C[0,1]$ and $\mathbb{P}(U \in C[0,1]) = 1$ we can apply the argument on p. 145 in Billingsley (1968) in order to obtain

$$\widetilde{\alpha}_n = \alpha_n(g_n) \xrightarrow{\mathscr{L}} U(g) = \widetilde{U} \text{ in } D[0,1]. \tag{3.12}$$

Here, $\widetilde{U}(t) = U(F(x - \Phi^{-1}(t)))$ for $t \in (0,1)$ and 0 otherwise, provided that the function $\Psi : D[0,1] \times \widetilde{D}[0,1] \to D[0,1]$ defined by

$$\Psi(y, \varphi) = y(\varphi) \tag{3.13}$$

is measurable with respect to $\mathscr{D}$ and $\mathscr{D} \times \widetilde{\mathscr{D}}$. This however follows from Billingsley (1968, p. 232). Thus, since $\mathbb{P}(\widetilde{U} \in C[0,1]) = 1$, it follows from Theorem 5.5 in Billingsley (1968) that

$$h_n(\widetilde{\alpha}_n) \xrightarrow{\mathrm{d}} h(\widetilde{U}), \tag{3.14}$$

once we have shown that $\mathbb{P}(E) = 0$, where

$$E = \{y \in C[0,1] \mid \lim_{n\to\infty} h_n(y_n) \neq h(y) \text{ for some } \{y_n\} \text{ with } y_n \in D[0,1], \text{ and } \lim_{n\to\infty} \rho(y_n, y) = 0\}. \tag{3.15}$$

So, let y be an element of $C[0,1]$ and suppose $\|y\| \leqslant M_0$. Let $\varepsilon > 0$, and choose $M_1 = M_1(\varepsilon) > 0$ such that

$$\int_{-\infty}^{-M_1} k(t)\,\mathrm{d}t \leqslant \frac{\varepsilon}{4M_0} \text{ and } \int_{M_1}^{\infty} k(t)\,\mathrm{d}t \leqslant \frac{\varepsilon}{4M_0}. \tag{3.16}$$

Then we have

$$\begin{aligned}
|h_n(y_n)-h(y)| &= \left|\int_{-\infty}^{\infty} (y_n(\Phi(t)) - y(\tfrac{1}{2}))\,\mathrm{d}K_n(t)\right| \\
&\leqslant \sup_{t\in[0,1]} |y_n(\Phi(t))-y(\Phi(t))| + \int_{-\infty}^{\infty} |y(\Phi(t)) - y(\tfrac{1}{2})|\,\mathrm{d}K_n(t) \\
&\leqslant \sup_{t\in[0,1]} |y_n(t)-y(t)| + \int_{-M_1}^{M_1} |y(\Phi(a_n t)) - y(\tfrac{1}{2})|\,k(t)\,\mathrm{d}t \\
&\quad + \int_{-\infty}^{-M_1} |y(\Phi(a_n t)) - y(\tfrac{1}{2})|\,k(t)\,\mathrm{d}t \\
&\quad + \int_{M_1}^{\infty} |y(\Phi(a_n t)) - y(\tfrac{1}{2})|\,k(t)\,\mathrm{d}t \\
&\leqslant \sup_{t} |y_n(t)-y(t)| + \sup_{|t|\leqslant M_1} \left|y(\Phi(a_n t)) - y\left(\tfrac{1}{2}\right)\right| + \varepsilon.
\end{aligned} \tag{3.17}$$

Since $y \in C[0,1]$, convergence in the Skorohod metric implies uniform convergence and thus

$$\lim_{n\to\infty} |h_n(y_n)-h(y)| \leqslant \varepsilon. \tag{3.18}$$

Since $\varepsilon > 0$ can be chosen as small as desired, we find that $E = \emptyset$, and consequently $\mathbb{P}(E) = 0$.

Using the fact that $\pi_{1/2}(\widetilde{U}) = U(F(x))$ and $U(F(x)) \stackrel{\mathrm{d}}{=} \mathrm{N}(0, \sigma(x))$ we immediately obtain from (3.14)

$$A_n(x_n) \xrightarrow{\mathrm{d}} \mathrm{N}(0, \sigma(x)) \text{ as } n \to \infty. \tag{3.19}$$

Next, we will show that under conditions (A_x), $n^{1/4}a_n \longrightarrow 0$ is necessary and sufficient for $B_n(x_n) \longrightarrow 0$.

By means of a Taylor expansion we find

$$\begin{aligned}
B_n(x_n) &= n^{1/2} \int_{-\infty}^{\infty} (F(x_n - a_n t) - F(x_n))k(t)\,\mathrm{d}t \\
&= n^{1/2} a_n^2 \int_{-\infty}^{\infty} f'(\xi_{t,x_n}) t^2 k(t)\,\mathrm{d}t,
\end{aligned} \tag{3.20}$$

where ξ_{t,x_n} lies between $x_n - a_n t$ and x_n. From this it is immediately clear that

$$B_n(x_n) \longrightarrow 0 \quad \text{iff} \quad n^{1/4}a_n \longrightarrow 0. \tag{3.21}$$

The result (2.1) now follows from (3.1), (3.19) and (3.21).

In order to prove the second part it suffices to show that

$$B_n(x_n) \longrightarrow 0 \quad \text{iff} \quad (2.2) \quad \text{holds.} \tag{3.22}$$

However, this follows immediately from

$$B_n(x_n) = n^{1/2} \int_{|t| \leqslant \varepsilon/\sqrt{n}} (F(x_n - t) - F(x_n))k_n(t)\,\mathrm{d}t + n^{1/2} \int_{|t| > \varepsilon/\sqrt{n}} (F(x_n - t) - F(x_n))k_n(t)\,\mathrm{d}t \tag{3.23}$$

and the fact that

$$n^{1/2} \int_{|t| \leqslant \varepsilon/\sqrt{n}} |F(x_n - t) - F(x_n)|k_n(t)\,\mathrm{d}t \leqslant n^{1/2}\|f\| \int_{|t| \leqslant \varepsilon/\sqrt{n}} |t|k_n(t)\,\mathrm{d}t \leqslant \|f\|\varepsilon. \qquad \square \tag{3.24}$$

Proof of Thorem 2.2. We have

$$\begin{aligned} \mathbb{P}\left(n^{1/2} f(\xi_p)\left(\frac{\hat{\xi}_{np} - \xi_p}{\sigma(\xi_p)}\right) \leqslant y\right) &= \mathbb{P}\left(\hat{\xi}_n(p) \leqslant \frac{y\sigma(\xi_p)}{n^{1/2} f(\xi_p)} + \xi_p\right) \\ &= \mathbb{P}(\widehat{F}_n(c_n) \geqslant p) \\ &= \mathbb{P}\left(n^{1/2}\left(\frac{\widehat{F}(c_n) - F(c_n)}{\sigma(\xi_p)}\right) \geqslant t_n\right) \\ &=: 1 - G_n(t_n), \end{aligned} \tag{3.25}$$

where $c_n = y\sigma(\xi_p)/(n^{1/2} f(\xi_p)) + \xi_p$ and $t_n = \frac{n^{1/2}(p - F(c_n))}{\sigma(\xi_p)}$. Since $c_n \to \xi_p$ we have from Theorem 2.1

$$G_n \xrightarrow{\mathrm{w}} \Phi. \tag{3.26}$$

Finally $t_n \to -y$ yields

$$G_n(t_n) \longrightarrow \Phi(-y), \tag{3.27}$$

by Slutsky's theorem. $\square$

Proof of Thorem 2.3. From Theorem 22.1 in Billingsley (1968) we have

$$\alpha_n(F) \xrightarrow{\mathscr{L}} U(F) \text{ in } D[-\infty, \infty]. \tag{3.28}$$

Let $\delta > 0$. Then there exist $\varepsilon > 0$ and $N = N(\varepsilon, \delta)$ such that for all $n \geqslant N$

$$\mathbb{P}\left(\sup_{|F(x) - F(y)| \leqslant \varepsilon n^{-1/2}} |U_n(x) - U_n(y)| > \delta\right) < \delta, \tag{3.29}$$

where $U_n(x) = n^{1/2}(F_n(x) - F(x))$, $x \in \mathbb{R}$, see Billingsley (1968, p. 198). The proof of the first part can now be carried out in the same way as the proof of Theorem 2.1 and Theorem 4.2 in Yukich (1992).

For the proof of the second part, note that pointwise convergence follows from weak convergence and hence the necessary part follows from Theorem 2.1. For the sufficiency part it suffices to show that $n^{1/4}a_n \to 0$ implies

$$n^{1/2}\|\widehat{F}_n - F_n\| \xrightarrow{\mathrm{p}} 0. \tag{3.30}$$

Therefore we write

$$\begin{aligned} n^{1/2}\|\widehat{F}_n - F_n\| &= n^{1/2}\|\widehat{F}_n - \mathbb{E}\widehat{F}_n - F_n + F\| + n^{1/2}\|\mathbb{E}\widehat{F}_n - F\| \\ &= A_n + B_n. \end{aligned} \tag{3.31}$$

It is easy to prove that

$$A_n \xrightarrow{p} 0, \tag{3.32}$$

see e.g. Yukich (1992, Lemma 4.1). Furthermore, by Taylor's theorem we have

$$\begin{aligned} B_n &= \sup_x n^{1/2}\left|\int (F(x) - F(x-y))k_n(y)\,\mathrm{d}y\right| \\ &= \sup_x n^{1/2}\left|\int \left(yf(x) - \frac{y^2}{2}f'(t_y)\right)k_n(y)\,\mathrm{d}y\right| \\ &\leqslant \sup_x n^{1/2}a_n f(x)\left|\int yk(y)\,\mathrm{d}y\right| + \frac{\|f'\|}{2}n^{1/2}a_n^2\left|\int y^2k(y)\,\mathrm{d}y\right| \\ &= \mathrm{O}(n^{1/2}a_n^2), \end{aligned} \tag{3.33}$$

which proves (3.30). □

Proof of Thorem 2.4. We show that (2.5) is an immediate consequence of Theorem 2.3 and a lemma by Vervaat (1972). For this we use the Skorokhod construction in order to construct $\widehat{\Gamma}_n^*$ and U^* such that

$$\widehat{\Gamma}_n^* \stackrel{\mathrm{d}}{=} \widehat{\Gamma}_n, \quad U \stackrel{\mathrm{d}}{=} U^* \tag{3.34}$$

and

$$\widehat{\alpha}_n^* := n^{1/2}(\widehat{\Gamma}_n^* - .) \xrightarrow{\|\cdot\|} U^* \quad \text{a.s.} \tag{3.35}$$

Next we apply Lemma 1 in Vervaat (1972) in order to obtain

$$\widehat{\beta}_n^* := n^{1/2}(\widehat{\Gamma}_n^{*\,-1} - .) \xrightarrow{\|\cdot\|} -U^* \quad \text{a.s.} \tag{3.36}$$

Moreover, (3.35) and (3.36) are equivalent. From (1.40) it then follows that

$$\widehat{\beta}_n \xrightarrow{\mathscr{L}} -U, \tag{3.37}$$

iff (3.36) holds. □

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