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On Quasi-Stable Sets

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Abstract: In this paper it is shown that for a bimatrix game each quasi-stable set is finite.

1 Introduction

Over the last decades several attempts have been made to find refinements of the Nash equilibrium concept in order to eliminate equilibria with undesirable properties. In this field of game theory Kohlberg and Mertens (1986) have started a new approach by considering set-valued solution concepts and by formulating a list of properties that a proper solution concept should satisfy. Furthermore these authors introduced several stability concepts for closed sets of equilibria. They call a set stable if all slightly perturbed games have an equilibrium close to this set. By specifying the kind of perturbation and the meaning of the word 'slightly', they consider successively hyperstable, fully stable and stable sets. We call the last mentioned sets KM-stable to distinguish them from other kinds of stable sets. None of these concepts however satisfies all of the requirements formulated by them. Some years later first Mertens (1989, 1991) and then Hillas (1990) proposed new kinds of stability satisfying indeed all the Kohlberg-Mertens properties. In his paper, Hillas introduced amongst others *quasi-stable* sets and showed that such sets satisfy most of the important requirements. Unfortunately these sets need not be invariant or connected. In this paper it is shown that, for bimatrix games, these quasi-stable sets are in fact finite. As such, this result can be seen as an extension of the finiteness of the minimal KM-stable sets of bimatrix games as was proved by Jansen, Jurg and Borm (1994). Wilson (1992) used this result to develop an algorithm to determine so-called simple stable sets. It is possible to construct a three-person game with a quasi-stable set consisting of a line segment. Hence our result is not correct for games with more than two players.

In Section 2 we define quasi-stable sets as minimal Q-sets and characterize for later purposes Q-sets in terms of sequences. In Section 3 equivalence relations on

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the strategy set of each player are introduced. These relations turn out to define a finite partition of the strategy spaces. If we select from a given Q-set one point of each (nonempty) intersection of the Q-set with a product of equivalence classes, we obtain, as we prove, another *finite* Q-set. Therefore minimal Q-sets are finite and the purpose of this paper is achieved.

Notation: For a $k \in \mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{R}^k$ is the vector space of k-tuples of real numbers and $\Delta_k := \{p \in \mathbb{R}^k | p \geq 0, \sum_{i=1}^k p_i = 1\}$. For a vector $p \in \Delta_k$, $C(p) := \{i | p_i > 0\}$ is the *carrier* of p. The unit vectors in $\mathbb{R}^k$ are denoted by $e_1, e_2, \ldots, e_k$. For $x, y \in \mathbb{R}^k$, $\langle x, y \rangle := \sum_{i=1}^k x_i y_i$ and $\| x \|_\infty := \max_{i=1,2,\ldots,k} |x_i|$. If $S \subset \{1, 2, \ldots, k\}$, $e_S \in \mathbb{R}^k$ is the vector with $(e_S)_i := \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases}$

2 Preliminaries

Given an ordered pair of two $m \times n$-matrices A and B, the $m \times n$-*bimatrix game* (A, B) is the two-person game $\langle \Delta_m, \Delta_n, K, L \rangle$ in strategic form with strategy spaces Δ_m and Δ_n for player 1 and player 2, respectively and payoff functions $K: \Delta_m \times \Delta_n \to \mathbb{R}$ and $L: \Delta_m \times \Delta_n \to \mathbb{R}$, where $K(p, q) := pAq$ and $L(p, q) := pBq$ for all $(p, q) \in \Delta_m \times \Delta_n$. A pair of strategies $(\bar{p}, \bar{q}) \in \Delta_m \times \Delta_n$ is an *equilibrium* of the bimatrix game (A, B) if $\bar{p}A\bar{q} \geq pA\bar{q}$ and $\bar{p}B\bar{q} \geq \bar{p}Bq$, for all $(p, q) \in \Delta_m \times \Delta_n$.

For an $m \times n$-bimatrix game we introduce the sets $M := \{1, 2, \ldots, m\}$ and $N := \{1, 2, \ldots, n\}$. The collection of all non-empty *proper* subsets of M (N) is denoted by 2^M (2^N). For mappings $\delta: 2^M \to (0, 1)$ and $\varepsilon: 2^N \to (0, 1)$ the (δ, ε)-*restricted game* $(A, B, \delta, \varepsilon)$ corresponding to (A, B) is defined as the game of which the strategy spaces are restricted to the polytopes

$$\Delta_m(\delta) := \{p \in \Delta_m | \langle e_S, p \rangle \geq \delta(S) \text{ for all } S \in 2^M\}$$

and

$$\Delta_n(\varepsilon) := \{q \in \Delta_n | \langle e_T, q \rangle \geq \varepsilon(T) \text{ for all } T \in 2^N\},$$

respectively. Note that the set $E(A, B, \delta, \varepsilon)$ of equilibria of this restricted game is non-empty if $\| \delta \| := \max\{\delta(S) | S \in 2^M\}$ and $\| \varepsilon \| := \max\{\varepsilon(T) | T \in 2^N\}$ are small enough.

These restricted games play an important role in Hillas' definition of quasi-stable sets.

Definition 1: A closed set C of equilibria of (A, B) is called a *Q-set* if for any open set V containing C there is a number $\eta > 0$ such that each (δ, ε)-restricted game with $\| \delta \| < \eta$ and $\| \varepsilon \| < \eta$ has an equilibrium in V. A Q-set that does not properly contain another Q-set is called a *quasi-stable* set.

In the lemma Q-sets are characterized in terms of sequences. The proof is easy and left to the reader.

Lemma 1: Let (A,B) be an $m \times n$-bimatrix game. A closed set C of equilibria of (A,B) is a Q-set if and only if $\limsup_k E(A,B,\delta_k,\varepsilon_k) \cap C \neq \phi$ for every sequence $\{(\delta^k,\varepsilon^k)\}_{k\in\mathbb{N}}$ converging to zero.

3 Two Equivalence Relations

In this section we introduce two equivalence relations on the strategy space of each player, *best reply equivalence* and *direction equivalence*. We prove that there are finitely many equivalence classes under direction equivalence and that direction equivalence implies best reply equivalence. Hence, there are finitely many equivalence classes under best reply equivalence. This is the result we need in the next section. First we need some technical definitions.

For an extreme point q of the restricted strategy space $\Delta_n(\varepsilon)$ we define

$$\mathscr{T}_\varepsilon(q) := \{T \in 2^N \mid \langle e_T, q\rangle = \varepsilon(T)\}.$$

Furthermore, we associate with such an extreme point q the polyhedral cone $K_\varepsilon(q)$ consisting of those points x in R^n for which

$$\langle e_T, x\rangle \geq 0 \quad \text{for all } T \in \mathscr{T}_\varepsilon(q)$$

$$\langle -e_N, x\rangle \geq 0$$

and

$$\langle e_N, x\rangle \geq 0.$$

Note that every cone $K_\varepsilon(q)$ has finitely many extreme directions and that only finitely many cones of type $K_\varepsilon(q)$ exist. This implies that the set $\mathscr{R}_2$ of all extreme directions of length one (w.r.t. $\|\ \|_\infty$) over all possible cones $K_\varepsilon(q)$ is finite. The set $\mathscr{R}_1$, defined in a similar way, is also finite. Furthermore, $x \in K_\varepsilon(q)$ if and only if $q + \lambda x \in \Delta_n(\varepsilon)$ for some small positive number λ.

Note that $\mathscr{R}_1$ and $\mathscr{R}_2$ are just the sets of (normalized) directions of the edges of all possible restricted strategy spaces. So obviously all vectors of the form $e_i - e_j$ for some i and j with $i \neq j$ are extreme directions. In order to show that also extreme directions of a different form are possible, we consider for a 2×5-bimatrix game the mapping $\varepsilon: 2^N \to (0,1)$ with for $T \in 2^N$

$$60\varepsilon(T) := \begin{cases} 2 & |T| = 1 \\ 6 & |T| = 2 \\ 10 & |T| = 4 \\ 1 & \text{otherwise.} \end{cases} \quad \text{if}$$

The strategies $q := \frac{1}{30}(25, 1, 1, 1, 2)$ and $q' := \frac{1}{30}(23, 2, 2, 2, 1)$ are neighboring extreme points of $\Delta_5(\varepsilon)$ as the subsets $\{2, 5\}$, $\{3, 5\}$ and $\{4, 5\}$ are in $\mathcal{T}_\varepsilon(q) \cap \mathcal{T}_\varepsilon(q')$. By normalizing the vector $q - q'$ we find $(1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ as an extreme direction.

Definition 2: For an $m \times n$-bimatrix game (A, B) two strategies p and $\tilde{p}$ of player 1 are *direction-equivalent*, denoted as $p \sim_{dir} \tilde{p}$, if $C(p) = C(\tilde{p})$ and for all $r \in \mathcal{R}_2$

$pBr > 0$ if and only if $\tilde{p}Br > 0$.

A similar equivalence relation can be defined on Δ_n.

Now since both $\mathcal{R}_i$'s are finite sets, there are only finitely many equivalence classes corresponding with the relation $\sim_{dir}$. Next we come to a second pair of equivalence relations that play a central role in the rest of this paper.

Definition 3: For an $m \times n$-bimatrix game (A, B) two strategies p and $\tilde{p}$ of player 1 are *best-reply-equivalent*, denoted as $p \sim_{BR} \tilde{p}$, if $C(p) = C(\tilde{p})$ and p and $\tilde{p}$ have the same set of best replies in every restricted strategy space $\Delta_n(\varepsilon)$. A similar equivalence relation can be defined for the strategies of player 2.

In order to show that the number of equivalence classes corresponding with the relation $\sim_{BR}$ is also finite, we need the following

Lemma 2: Direction-equivalent strategies are best-reply-equivalent.

Proof: Assume that $p \sim_{dir} \tilde{p}$ and that q is a best reply in $\Delta_n(\varepsilon)$ to p. We only show that q is also a best reply to $\tilde{p}$. Because the set of best replies in $\Delta_n(\varepsilon)$ to p is exactly the set of points in $\Delta_n(\varepsilon)$ where the linear function $q \mapsto pBq$ attains its maximum, we may assume without loss of generality that q is an extreme point of $\Delta_n(\varepsilon)$.

Let q' be any point of $\Delta_n(\varepsilon)$. Then $q' - q$ is a nonnegative linear combination of extreme directions r of the cone $K_\varepsilon(q)$. For each of these directions r we have $pBr \leq 0$ and therefore, $\tilde{p}Br \leq 0$ by direction equivalence of p and $\tilde{p}$. Consequently, $\tilde{p}B(q' - q) \leq 0$ and q is an at least as good response as any point $q' \in \Delta_n(\varepsilon)$. □

Corollary 1: There are finitely many equivalence classes under best response equivalence.

The equivalence classes in Δ_m corresponding to $\sim_{BR}$ are denoted by $\mathcal{V}_1, \ldots, \mathcal{V}_K$ and those in Δ_n by $\mathcal{W}_1, \ldots, \mathcal{W}_L$.

4 On the Equivalence Classes

The purpose of this section is to prove

Theorem 1: Let (A, B) be an $m \times n$-matrix game and let (p, q) be an element of $\limsup_k E(A, B, \delta_k, \varepsilon_k)$ for some sequence $\{(\delta_k, \varepsilon_k)\}_{k \in \mathbb{N}}$ converging to zero. If

$(p,q) \in \mathscr{V}^* \times \mathscr{W}^*$ for some equivalence classes $\mathscr{V}^*$ and $\mathscr{W}^*$ corresponding to $\sim_{BR}$, then

$$\mathscr{V}^* \times \mathscr{W}^* \subset \limsup_k E(A, B, \delta_k, \varepsilon_k).$$

If the conditions mentioned in the theorem are satisfied, we may suppose, without loss of generality, that there exists a sequence $\{(p^k, q^k)\}_{k \in \mathbb{N}}$ converging to (p,q) such that, for every $k \in \mathbb{N}$, (p^k, q^k) is an equilibrium of the game $(A, B, \delta^k, \varepsilon^k)$. Since there are only finitely many equivalence classes corresponding with $\sim_{BR}$, this sequence can be taken in such a way that *all* points p^k are contained in some equivalence class, say $\mathscr{V}$, corresponding with $\sim_{BR}$, and *all* q^k are contained in some equivalence class, say $\mathscr{W}$, corresponding with $\sim_{BR}$.

In order to prove that an arbitrarily chosen point $(\tilde{p}, \tilde{q}) \in \mathscr{V}^* \times \mathscr{W}^*$ is an element of $\limsup_k E(A, B, \delta_k, \varepsilon_k)$, we introduce, for each $k \in \mathbb{N}$, the strategies

$$\tilde{p}^k := p^k + (\tilde{p} - p) \quad \text{and} \quad \tilde{q}^k := q^k + (\tilde{q} - q).$$

Obviously, $(\tilde{p}^k, \tilde{q}^k) \to (\tilde{p}, \tilde{q})$ as $k \to \infty$. Furthermore, one can show that $(\tilde{p}^k, \tilde{q}^k) \in \Delta_m(\delta_k) \times \Delta_n(\varepsilon_k)$ for large k. So the proof of Theorem 1 is complete if we can show that $(\tilde{p}^k, \tilde{q}^k) \in E(A, B, \delta^k, \varepsilon^k)$ for large k. To that purpose we consider, for each $k \in \mathbb{N}$, the minimal face F_k of $\Delta_m(\delta_k)$ containing p^k and the minimal face G_k of $\Delta_n(\varepsilon_k)$ containing q^k.

In the following two lemmas two properties are described for the strategies introduced before.

Lemma 3: $\tilde{p}^k \in F_k$ and $\tilde{q}^k \in G_k$ for large k.

Proof: In order to prove that $\tilde{q}^k$ is an element of G_k for large k, we will show that, for large k, $\mathscr{T}_{\varepsilon_k}(q^k) = \mathscr{T}_{\varepsilon_k}(\tilde{q}^k)$. Now suppose $\langle e_T, q^k \rangle = \varepsilon_k(T)$. If $T \cap C(q) \neq \phi$, then $\langle e_T, q^k \rangle > \varepsilon_k(T)$ for large k. So we may conclude that $T \cap C(q) = \phi$. In that case $\langle e_T, \tilde{q}^k \rangle = \langle e_T, q^k + \tilde{q} - q \rangle = \langle e_T, q^k \rangle = \varepsilon_k(T)$. The first inclusion can be proved in a similar way. □

Lemma 4: $\tilde{p}^k \in \mathscr{V}$ and $\tilde{q}^k \in \mathscr{W}$ for large k.

Proof: We only show that, for large k, $\tilde{p}^k \in \mathscr{V}$, i.e. $\tilde{p}^k \sim_{BR} p^k$. First note that $C(\tilde{p}^k) = C(p^k)$ for large k, because both strategies are completely mixed.

Let $\varepsilon : 2^N \to (0,1)$ be such that $\Delta_n(\varepsilon) \neq \phi$ and let q be a best reply in $\Delta_n(\varepsilon) \neq \phi$ to p^k. The proof is complete if we can show that q is also a best reply to $\tilde{p}^k$ for large k.

In order to show that for large k

$$\tilde{p}^k B(q - q') \geq 0 \quad \text{for all } q' \in \Delta_n(\varepsilon),$$

we consider two cases.

(1) Suppose that $p^k B(q-q')>0$ for some $q' \in \Delta_n(\varepsilon)$.

Since $p^k \in \mathscr{V}$ for all k, q is a best reply to p^k for all k. This implies that q is a best reply to p. Hence q is also a best reply to $\tilde{p}$, because $p \sim_{BR} \tilde{p}$. So $\tilde{p}B(q-q') \geq 0$.

If $\tilde{p}B(q-q')>0$, then $\tilde{p}^k B(q-q')>0$ for large k, because $\tilde{p}^k \rightarrow \tilde{p}$ as $k \rightarrow \infty$.

If $\tilde{p}B(q-q')=0$, then q' is also a best reply to $\tilde{p}$. In that case q' is also a best reply to p and $pB(q-q')=0$. This implies that

$$\tilde{p}^k B(q-q') = p^k B(q-q') + \tilde{p}B(q-q') - pB(q-q') = p^k B(q-q') > 0.$$

(2) Suppose that $p^k B(q-q')=0$ for some $q' \in \Delta_n(\varepsilon)$.

As in part (1) one can show that also in this case $\tilde{p}B(q-q') = pB(q-q') = 0$. Hence, $\tilde{p}^k B(q-q') = p^k B(q-q') = 0$. □

With the help of the Lemmas 3 and 4, we get

$$(\tilde{p}^k, \tilde{q}^k) \in (F_k \cap \mathscr{V}) \times (G_k \cap \mathscr{W}) \text{ for large } k.$$

Hence, $(\tilde{p}^k, \tilde{q}^k) \in E(A, B, \delta_k, \varepsilon_k)$ for large k as is implied by

Lemma 5: For all $k \in \mathbb{N}$, $(F_k \cap \mathscr{V}) \times (G_k \cap \mathscr{W})$ is a subset of $E(A, B, \delta_k, \varepsilon_k)$.

Proof: Let $k \in \mathbb{N}$. Take $(p', q') \in (F_k \cap \mathscr{V}) \times (G_k \cap \mathscr{W})$. Because both p^k and p' are elements of $\mathscr{V}$, $p^k \sim_{BR} p'$. Since $(p^k, q^k) \in E(A, B, \delta_k, \varepsilon_k)$, q^k is a best reply in $\Delta_n(\varepsilon_k)$ to p^k. So by definition, q^k is also a best reply in $\Delta_n(\varepsilon_k)$ to p'. Then however all elements of G_k – the minimal face containing q^k – are best replies in $\Delta_n(\varepsilon_k)$ to p'. In particular q' is a best reply to p'. Similarly, p' is a best reply to q', so $(p', q') \in E(A, B, \delta_k, \varepsilon_k)$. □

5 The Finiteness of Quasi-Stable Sets

In this section we show that for a bimatrix game each Q-set contains a finite Q-set. In particular this implies that quasi-stable sets are finite.

Let C be a Q-set for a bimatrix game (A, B). If for some equivalence classes $\mathscr{V}$ and $\mathscr{W}$

$$C \cap (\mathscr{V} \times \mathscr{W}) \neq \phi$$

we select one point in this intersection. The set of equilibria selected in this way is denoted to C^*.

Because there are only finitely many equivalence classes, it's obvious that C^* is a finite set. Since C is a Q-set, we know that $C \cap \limsup_k E(A, B, \delta_k, \varepsilon_k) \neq \phi$ for a given sequence $\{(\delta_k, \varepsilon_k)\}_{k \in \mathbb{N}}$ converging to zero. Take a point (p', q') in this intersection. If $\mathscr{V}$ and $\mathscr{W}$ are the equivalence classes containing p' and q', respectively, then it's obvious that also $C \cap (\mathscr{V} \times \mathscr{W})$ is non-empty, since (p', q') is an element of this intersection. Hence, by construction, C^* contains an element, say (p^*, q^*), of $\mathscr{V} \times \mathscr{W}$. By Theorem 1, (p^*, q^*) is also an element of $\limsup_k E(A, B, \delta_k, \varepsilon_k)$. Consequently, $C^* \cap \limsup_k E(A, B, \delta_k, \varepsilon_k) \neq \phi$. Since the sequence $\{(\delta_k, \varepsilon_k)\}_{k \in \mathbb{N}}$ was chosen arbitrarily, by Lemma 1, C^* is a Q-set. Hence we have proved

Theorem 2: Every Q-set contains a finite Q-set.

In particular Theorem 2 implies that a quasi-stable set contains a finite Q-set. So

Corollary 2: For a bimatrix game every quasi-stable set is finite.

Furthermore Theorem 2 implies that the set of all equilibria of a bimatrix game contains a finite Q-set. this provides a new proof of

Corollary 3: Every bimatrix game possesses a quasi-stable set.

Finally, we give an example of a game with a unique quasi-stable set.

Example: For the 2×3-bimatrix game

$$(A, B) = \begin{bmatrix} (0,3) & (1,0) & (1,2) \\ (1,0) & (0,3) & (1,2) \end{bmatrix}$$

the set of equilibria equals $\{p \mid \frac{1}{3} \leq p_1 \leq \frac{2}{3}\} \times \{e_3\}$, while

$$\{(\tfrac{1}{3}, \tfrac{2}{3}), (\tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{2}{3}, \tfrac{1}{3})\} \times \{e_3\}$$

is the only qausi-stable set for this game.

References

Hillas J (1990) On the definition of the strategic stability of equilibria. Econometrica 58: 1365–1391

Jansen MJM, Jurg AP, Borm PEM (1994) On strictly perfect sets. Games and Economic Behavior 6: 400–415

Kohlberg E, Mertens JF (1986) On strategic stability of equilibria. Econometrica 54: 1003–1037

Mertens JF (1989) Stable equilibria – a reformulation. Part I: Definitions and basic properties. Math Oper Res 14: 575–625

Mertens JF (1991) Stable equilibria – a reformulation. Part II: Discussion of the definition and further results. Math Oper Res 16: 694–753

Wilson R (1992) Computing simply stable equilibria. Econometrica 60: 1039–1070

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