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On Quasi-Stable Sets

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Abstract: In this paper it is shown that for a bimatrix game each quasi-stable set is finite.

1 Introduction

Over the last decades several attempts have been made to find refinements of the Nash equilibrium concept in order to eliminate equilibria with undesirable properties. In this field of game theory Kohlberg and Mertens (1986) have started a new approach by considering set-valued solution concepts and by formulating a list of properties that a proper solution concept should satisfy. Furthermore these authors introduced several stability concepts for closed sets of equilibria. They call a set stable if all slightly perturbed games have an equilibrium close to this set. By specifying the kind of perturbation and the meaning of the word ‘slightly’, they consider successively hyperstable, fully stable and stable sets. We call the last mentioned sets KM-stable to distinguish them from other kinds of stable sets. None of these concepts however satisfies all of the requirements formulated by them. Some years later first Mertens (1989, 1991) and then Hillas (1990) proposed new kinds of stability satisfying indeed all the Kohlberg-Mertens properties. In his paper, Hillas introduced amongst others quasi-stable sets and showed that such sets satisfy most of the important requirements. Unfortunately these sets need not be invariant or connected. In this paper it is shown that, for bimatrix games, these quasi-stable sets are in fact finite. As such, this result can be seen as an extension of the finiteness of the minimal KM-stable sets of bimatrix games as was proved by Jansen, Jurg and Borm (1994). Wilson (1992) used this result to develop an algorithm to determine so-called simple stable sets. It is possible to construct a three-person game with a quasi-stable set consisting of a line segment. Hence our result is not correct for games with more than two players.

In Section 2 we define quasi-stable sets as minimal Q-sets and characterize for later purposes Q-sets in terms of sequences. In Section 3 equivalence relations on
the strategy set of each player are introduced. These relations turn out to define
a finite partition of the strategy spaces. If we select from a given Q-set one point of
each (nonempty) intersection of the Q-set with a product of equivalence classes,
we obtain, as we prove, another finite Q-set. Therefore minimal Q-sets are finite
and the purpose of this paper is achieved.

Notation: For a \( k \in \mathbb{N} := \{1, 2, \ldots\} \), \( \mathbb{R}^k \) is the vector space of \( k \)-tuples of real
numbers and \( \Delta_k := \{ p \in \mathbb{R}^k | p \geq 0, \sum_{i=1}^k p_i = 1 \} \). For a vector \( p \in \Delta_k, C(p) := \{ i | p_i > 0 \} \)
is the carrier of \( p \). The unit vectors in \( \mathbb{R}^k \) are denoted by \( e_1, e_2, \ldots, e_k \). For \( x, y \in \mathbb{R}^k \),
\( \langle x, y \rangle := \sum_{i=1}^k x_i y_i \) and \( \| x \|_\infty := \max_{i=1,2,\ldots,k} | x_i | \). If \( S \subseteq \{1, 2, \ldots, k\} \), \( e_S \in \mathbb{R}^k \) is the
vector with \( (e_S)_i := \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases} \)

2 Preliminaries

Given an ordered pair of two \( m \times n \)-matrices \( A \) and \( B \), the \( m \times n \)-bimatrix game
\( (A, B) \) is the two-person game \( \langle \Delta_m, \Delta_n, K, L \rangle \) in strategic form with strategy spaces
\( \Delta_m \) and \( \Delta_n \) for player 1 and player 2, respectively and payoff functions
\( K: \Delta_m \times \Delta_n \to \mathbb{R} \) and \( L: \Delta_m \times \Delta_n \to \mathbb{R} \), where \( K(p, q) := p A q \) and \( L(p, q) := p B q \) for
all \( (p, q) \in \Delta_m \times \Delta_n \). A pair of strategies \( (\bar{p}, \bar{q}) \in \Delta_m \times \Delta_n \) is an equilibrium of the
bimatrix game \( (A, B) \) if \( \bar{p} A \bar{q} \geq p A \bar{q} \) and \( \bar{p} B \bar{q} \geq p B \bar{q} \), for all \( (p, q) \in \Delta_m \times \Delta_n \).

For an \( m \times n \)-bimatrix game we introduce the sets \( M := \{1, 2, \ldots, m\} \) and
\( N := \{1, 2, \ldots, n\} \). The collection of all non-empty proper subsets of \( M \) (\( N \)) is
denoted by \( 2^M \) (\( 2^N \)). For mappings \( \delta: 2^M \to (0,1) \) and \( \varepsilon: 2^N \to (0,1) \) the \((\delta, \varepsilon)\)-
restricted game \( (A, B, \delta, \varepsilon) \) corresponding to \( (A, B) \) is defined as the game of which
the strategy spaces are restricted to the polytopes

\[
\Delta_m(\delta) := \{ p \in \Delta_m | \langle e_S, p \rangle \geq \delta(S) \text{ for all } S \in 2^M \}
\]

and

\[
\Delta_n(\varepsilon) := \{ q \in \Delta_n | \langle e_T, q \rangle \geq \varepsilon(T) \text{ for all } T \in 2^N \},
\]

respectively. Note that the set \( E(A, B, \delta, \varepsilon) \) of equilibria of this restricted game is
non-empty if \( \| \delta \| := \max \{ \delta(S) | S \in 2^M \} \) and \( \| \varepsilon \| := \max \{ \varepsilon(T) | T \in 2^N \} \) are small
enough.

These restricted games play an important role in Hillas' definition of quasi-
stable sets.

Definition 1: A closed set \( C \) of equilibria of \( (A, B) \) is called a Q-set if for any open
set \( V \) containing \( C \) there is a number \( \eta > 0 \) such that each \((\delta, \varepsilon)\)-restricted game
with \( \| \delta \| < \eta \) and \( \| \varepsilon \| < \eta \) has an equilibrium in \( V \). A Q-set that does not properly
contain another Q-set is called a quasi-stable set.
On Quasi-Stable Sets

In the lemma $Q$-sets are characterized in terms of sequences. The proof is easy and left to the reader.

**Lemma 1:** Let $(A, B)$ be an $m \times n$-bimatrix game. A closed set $C$ of equilibria of $(A, B)$ is a $Q$-set if and only if $\lim \sup_{k} E(A, B, \delta_{k}, e_{k}) \cap C \neq \emptyset$ for every sequence $(\delta_{k}, e_{k})_{k \in \mathbb{N}}$ converging to zero.

3 Two Equivalence Relations

In this section we introduce two equivalence relations on the strategy space of each player, best reply equivalence and direction equivalence. We prove that there are finitely many equivalence classes under direction equivalence and that direction equivalence implies best reply equivalence. Hence, there are finitely many equivalence classes under best reply equivalence. This is the result we need in the next section. First we need some technical definitions.

For an extreme point $q$ of the restricted strategy space $\Delta_{n}(e)$ we define

$$
\mathcal{S}_{q}(q) := \{ T \in 2^{\mathbb{N}} | \langle e_{T}, q \rangle = u(T) \}.
$$

Furthermore, we associate with such an extreme point $q$ the polyhedral cone $K_{q}(q)$ consisting of those points $x$ in $\mathbb{R}^{n}$ for which

$$
\langle e_{T}, x \rangle \geq 0 \quad \text{for all } T \in \mathcal{S}_{q}(q)
$$

$$
\langle -e_{N}, x \rangle \geq 0
$$

and

$$
\langle e_{N}, x \rangle \geq 0.
$$

Note that every cone $K_{q}(q)$ has finitely many extreme directions and that only finitely many cones of type $K_{q}(q)$ exist. This implies that the set $\mathcal{R}_{2}$ of all extreme directions of length one (w.r.t. $\| \cdot \|_{\infty}$) over all possible cones $K_{q}(q)$ is finite. The set $\mathcal{R}_{1}$, defined in a similar way, is also finite. Furthermore, $x \in K_{q}(q)$ if and only if $q + \alpha x \in \Delta_{n}(e)$ for some small positive number $\alpha$.

Note that $\mathcal{R}_{1}$ and $\mathcal{R}_{2}$ are just the sets of (normalized) directions of the edges of all possible restricted strategy spaces. So obviously all vectors of the form $e_{i} - e_{j}$ for some $i$ and $j$ with $i \neq j$ are extreme directions. In order to show that also extreme directions of a different form are possible, we consider for a $2 \times 5$-bimatrix game the mapping $e: 2^{\mathbb{N}} \rightarrow (0, 1)$ with for $T \in 2^{\mathbb{N}}$

$$
60 e(T) := \begin{cases} 
2 & \text{if } |T| = 1 \\
6 & \text{if } |T| = 2 \\
10 & \text{if } |T| = 4 \\
1 & \text{otherwise}.
\end{cases}
$$

The strategies $q := \frac{1}{30}(25, 1, 1, 1, 2)$ and $q' := \frac{1}{30}(23, 2, 2, 2, 1)$ are neighboring
extreme points of \( \Delta_n(\varepsilon) \) as the subsets \( \{2, 5\}, \{3, 5\} \) and \( \{4, 5\} \) are in \( \mathcal{T}_n(q) \cap \mathcal{T}_n(q') \). By normalizing the vector \( q - q' \) we find \( (1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) \) as an extreme direction.

**Definition 2:** For an \( m \times n \)-bimatrix game \((A, B)\) two strategies \( p \) and \( \tilde{p} \) of player 1 are direction-equivalent, denoted as \( p \sim_{\text{dir}} \tilde{p} \), if \( C(p) = C(\tilde{p}) \) and for all \( r \in \mathcal{R}_2 \)

\[
pBr > 0 \text{ if and only if } \tilde{p}Br > 0.
\]

A similar equivalence relation can be defined on \( A_n \).

Now since both \( \mathcal{R}_i \)'s are finite sets, there are only finitely many equivalence classes corresponding with the relation \( \sim_{\text{dir}} \). Next we come to a second pair of equivalence relations that play a central role in the rest of this paper.

**Definition 3:** For an \( m \times n \)-bimatrix game \((A, B)\) two strategies \( p \) and \( \tilde{p} \) of player 1 are best-reply-equivalent, denoted as \( p \sim_{\text{BR}} \tilde{p} \), if \( C(p) = C(\tilde{p}) \) and \( p \) and \( \tilde{p} \) have the same set of best replies in every restricted strategy space \( \Delta_n(\varepsilon) \). A similar equivalence relation can be defined for the strategies of player 2.

In order to show that the number of equivalence classes corresponding with the relation \( \sim_{\text{BR}} \) is also finite, we need the following

**Lemma 2:** Direction-equivalent strategies are best-reply-equivalent.

**Proof:** Assume that \( p \sim_{\text{dir}} \tilde{p} \) and that \( q \) is a best reply in \( \Delta_n(\varepsilon) \) to \( p \). We only show that \( q \) is also a best reply to \( \tilde{p} \). Because the set of best replies in \( \Delta_n(\varepsilon) \) to \( p \) is exactly the set of points in \( \Delta_n(\varepsilon) \) where the linear function \( q \rightarrow pBq \) attains its maximum, we may assume without loss of generality that \( q \) is an extreme point of \( \Delta_n(\varepsilon) \).

Let \( q' \) be any point of \( \Delta_n(\varepsilon) \). Then \( q' - q \) is a nonnegative linear combination of extreme directions \( r \) of the cone \( K_\varepsilon(q) \). For each of these directions \( r \) we have \( pBr \leq 0 \) and therefore, \( \tilde{p}Br \leq 0 \) by direction equivalence of \( p \) and \( \tilde{p} \). Consequently, \( \tilde{p}B(q' - q) \leq 0 \) and \( q \) is an at least as good response as any point \( q' \in \Delta_n(\varepsilon) \).

**Corollary 1:** There are finitely many equivalence classes under best response equivalence.

The equivalence classes in \( \Delta_m \) corresponding to \( \sim_{\text{BR}} \) are denoted by \( \mathcal{V}_1, \ldots, \mathcal{V}_K \) and those in \( \Delta_n \) by \( \mathcal{W}_1, \ldots, \mathcal{W}_L \).

### 4 On the Equivalence Classes

The purpose of this section is to prove

**Theorem 1:** Let \((A, B)\) be an \( m \times n \)-matrix game and let \((p, q)\) be an element of \( \limsup_k E(A, B, \delta_k, \varepsilon_k) \) for some sequence \( \{(\delta_k, \varepsilon_k)\}_{k \in \mathbb{N}} \) converging to zero. If
(p, q) \in \mathcal{V}^* \times \mathcal{W}^* for some equivalence classes \mathcal{V}^* and \mathcal{W}^* corresponding to \sim_{BR}, then

\mathcal{V}^* \times \mathcal{W}^* \subset \limsup_{k} E(A, B, \delta_k, \varepsilon_k).

If the conditions mentioned in the theorem are satisfied, we may suppose, without loss of generality, that there exists a sequence \{ (p^k, q^k) \}_{k \in \mathbb{N}} converging to \( p, q \) such that, for every \( k \in \mathbb{N} \), \( (p^k, q^k) \) is an equilibrium of the game \( (A, B, \delta^k, \varepsilon^k) \). Since there are only finitely many equivalence classes corresponding with \sim_{BR}, this sequence can be taken in such a way that all points \( p^k \) are contained in some equivalence class, say \( \mathcal{V} \), corresponding with \sim_{BR}, and all \( q^k \) are contained in some equivalence class, say \( \mathcal{W} \), corresponding with \sim_{BR}.

In order to prove that an arbitrarily chosen point \((\bar{p}, \bar{q}) \in \mathcal{V}^* \times \mathcal{W}^* \) is an element of \( \limsup_k E(A, B, \delta_k, \varepsilon_k) \), we introduce, for each \( k \in \mathbb{N} \), the strategies

\[ \bar{p}^k := p^k + (\bar{p} - p) \quad \text{and} \quad \bar{q}^k := q^k + (\bar{q} - q). \]

Obviously, \( (\bar{p}^k, \bar{q}^k) \to (\bar{p}, \bar{q}) \) as \( k \to \infty \). Furthermore, one can show that \( (\bar{p}^k, \bar{q}^k) \in F_k(\delta_k) \times G_k(\varepsilon_k) \) for large \( k \). So the proof of Theorem 1 is complete if we can show that \( (\bar{p}^k, \bar{q}^k) \in E(A, B, \delta^k, \varepsilon^k) \) for large \( k \). To that purpose we consider, for each \( k \in \mathbb{N} \), the minimal face \( F_k \) of \( A, B, \delta_k, \varepsilon_k \) containing \( p^k \) and the minimal face \( G_k \) of \( A, B, \delta_k, \varepsilon_k \) containing \( q^k \).

In the following two lemmas two properties are described for the strategies introduced before.

**Lemma 3:** \( \bar{p}^k \in F_k \) and \( \bar{q}^k \in G_k \) for large \( k \).

**Proof:** In order to prove that \( \bar{q}^k \) is an element of \( G_k \) for large \( k \), we will show that, for large \( k \), \( \mathcal{F} \), \( q^k = \mathcal{F} \), \( \mathcal{F} \). Now suppose \( e_T, q^k = e_T(\bar{q}) \). If \( T \cap C(q) \neq \emptyset \), then \( e_T, q^k > e_T(\bar{q}) \) for large \( k \). So we may conclude that \( T \cap C(q) = \emptyset \). In that case \( e_T, q^k = e_T, q^k + \bar{q} - q = e_T, e_T = e_k(T) \). The first inclusion can be proved in a similar way. \( \square \)

**Lemma 4:** \( \bar{p}^k \in \mathcal{V}^* \) and \( \bar{q}^k \in \mathcal{W}^* \) for large \( k \).

**Proof:** We only show that, for large \( k \), \( \bar{p}^k \in \mathcal{V}^* \), i.e. \( \bar{p}^k \sim_{BR} p^k \). First note that \( C(\bar{p}^k) = C(p^k) \) for large \( k \), because both strategies are completely mixed.

Let \( e:2^\mathbb{N} \to (0, 1) \) be such that \( \Delta(\varepsilon) \neq \emptyset \) and let \( q \) be a best reply in \( \Delta(\varepsilon) \neq \emptyset \) to \( p^k \). The proof is complete if we can show that \( q \) also is a best reply to \( \bar{p}^k \) for large \( k \).

In order to show that for large \( k \)

\[ \bar{p}^kB(q - q') \geq 0 \quad \text{for all} \quad q' \in \Delta(\varepsilon), \]

we consider two cases.
(1) Suppose that \( p^k B(q - q') > 0 \) for some \( q' \in \Lambda_n(e) \).

Since \( p^k \in \mathcal{V} \) for all \( k \), \( q \) is a best reply to \( p^k \) for all \( k \). This implies that \( q \) is a best reply to \( p \). Hence \( q \) is also a best reply to \( \tilde{p} \), because \( p \sim_{BR} \tilde{p} \). So \( \tilde{p} B(q - q') \geq 0 \).

If \( \tilde{p} B(q - q') > 0 \), then \( \tilde{p}^k B(q - q') > 0 \) for large \( k \), because \( \tilde{p}^k \to \tilde{p} \) as \( k \to \infty \).

If \( \tilde{p} B(q - q') = 0 \), then \( q' \) is also a best reply to \( \tilde{p} \). In that case \( q' \) is also a best reply to \( p \) and \( p B(q - q') = 0 \). This implies that

\[
\tilde{p}^k B(q - q') = p^k B(q - q') + \tilde{p} B(q - q') - p B(q - q') = p^k B(q - q') > 0.
\]

(2) Suppose that \( p^k B(q - q') = 0 \) for some \( q' \in \Lambda_n(e) \).

As in part (1) one can show that also in this case \( \tilde{p} B(q - q') = p B(q - q') = 0 \). Hence, \( \tilde{p}^k B(q - q') = p^k B(q - q') = 0 \). □

With the help of the Lemmas 3 and 4, we get

\[
(\tilde{p}^k, \tilde{q}^k) \in (F_k \cap \mathcal{V}) \times (G_k \cap \mathcal{W}) \text{ for large } k.
\]

Hence, \( (\tilde{p}^k, \tilde{q}^k) \in E(A, B, \delta_k, e_k) \) for large \( k \) as is implied by

**Lemma 5:** For all \( k \in \mathbb{N} \), \( (F_k \cap \mathcal{V}) \times (G_k \cap \mathcal{W}) \) is a subset of \( E(A, B, \delta_k, e_k) \).

**Proof:** Let \( k \in \mathbb{N} \). Take \( (p', q') \in (F_k \cap \mathcal{V}) \times (G_k \cap \mathcal{W}) \). Because both \( p^k \) and \( p' \) are elements of \( \mathcal{V} \), \( p^k \sim_{BR} p' \). Since \( (p^k, q^k) \in E(A, B, \delta_k, e_k) \), \( q^k \) is a best reply in \( \Lambda_n(e_k) \) to \( p^k \). So by definition, \( q^k \) is also a best reply in \( \Lambda_n(e_k) \) to \( p' \). Then however all elements of \( G_k \)– the minimal face containing \( q^k \) – are best replies in \( \Lambda_n(e_k) \) to \( p' \). In particular \( q' \) is a best reply to \( p' \). Similarly, \( p' \) is a best reply to \( q' \), so \( (p', q') \in E(A, B, \delta_k, e_k) \). □

5 The Finiteness of Quasi-Stable Sets

In this section we show that for a bimatrix game each \( Q \)-set contains a finite \( Q \)-set. In particular this implies that quasi-stable sets are finite.

Let \( C \) be a \( Q \)-set for a bimatrix game \( (A, B) \). If for some equivalence classes \( \mathcal{V} \) and \( \mathcal{W} \)

\[
C \cap (\mathcal{V} \times \mathcal{W}) \neq \emptyset
\]

we select one point in this intersection. The set of equilibria selected in this way is denoted to \( C^* \).

Because there are only finitely many equivalence classes, it’s obvious that \( C^* \) is a finite set. Since \( C \) is a \( Q \)-set, we know that \( C \cap \lim \sup_{k} E(A, B, \delta_k, e_k) \neq \emptyset \) for a given sequence \( \{(\delta_k, e_k)\}_{k \in \mathbb{N}} \) converging to zero. Take a point \((p', q')\) in this
intersection. If $\mathcal{V}$ and $\mathcal{W}$ are the equivalence classes containing $p'$ and $q'$, respectively, then it's obvious that also $C \cap (\mathcal{V} \times \mathcal{W})$ is non-empty, since $(p', q')$ is an element of this intersection. Hence, by construction, $C^*$ contains an element, say $(p^*, q^*)$, of $\mathcal{V} \times \mathcal{W}$. By Theorem 1, $(p^*, q^*)$ is also an element of $\limsup_{k \in \mathbb{N}} E(A, B, \delta_k, e_k)$. Consequently, $C^* \subset \limsup_{k \in \mathbb{N}} E(A, B, \delta_k, e_k)$ is non-empty. Since the sequence $\{(\delta_k, e_k)\}_{k \in \mathbb{N}}$ was chosen arbitrarily, by Lemma 1, $C^*$ is a $Q$-set. Hence we have proved

**Theorem 2:** Every $Q$-set contains a finite $Q$-set.

In particular Theorem 2 implies that a quasi-stable set contains a finite $Q$-set. So

**Corollary 2:** For a bimatrix game every quasi-stable set is finite.

Furthermore Theorem 2 implies that the set of all equilibria of a bimatrix game contains a finite $Q$-set. This provides a new proof of

**Corollary 3:** Every bimatrix game possesses a quasi-stable set.

Finally, we give an example of a game with a unique quasi-stable set.

**Example:** For the $2 \times 3$-bimatrix game

$$(A, B) = \begin{bmatrix} (0, 3) & (1, 0) & (1, 2) \\ (1, 0) & (0, 3) & (1, 2) \end{bmatrix}$$

the set of equilibria equals $\{p|\frac{1}{3} \leq p_1 \leq \frac{2}{3}\} \times \{e_3\}$, while

$$\{(\frac{1}{3}, \frac{2}{3}), (\frac{1}{2}, \frac{1}{2}), (\frac{2}{3}, \frac{1}{3})\} \times \{e_3\}$$

is the only quasi-stable set for this game.

**References**


Received September 1993

Revised version July 1994