Chaotic Polynomial Automorphisms; counterexamples to several conjectures

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Abstract

We give a polynomial counterexample to a discrete version of the Markus-Yamabe Conjecture and a conjecture of Deng, Meisters and Zampieri, asserting that if $F: \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map with $\det(JF) \in \mathbb{C}^*$, then for all $\lambda \in \mathbb{R}$ large enough $\lambda F$ is global analytic linearizable. These counterexamples hold in any dimension $\geq 4$.

Introduction

In [4] a new approach to the Jacobian Conjecture is introduced. The authors conjecture that if $F: \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map with $F(0) = 0$ and $JF(0) = I$, then for all $\lambda > 1$, $\lambda$ large enough there exists an analytic automorphism $\varphi_\lambda: \mathbb{C}^n \to \mathbb{C}^n$ such that $\varphi_\lambda^{-1} \circ \lambda F \circ \varphi_\lambda = \lambda I$ i.e. $\varphi_\lambda$ conjugates $\lambda F$ to its linear part. We also say that $\lambda F$ is analytic linearizable to its linear part. We call this conjecture the DMZ-conjecture (after Deng, Meisters and Zampieri). Of course this conjecture, if true, would imply the Jacobian Conjecture since it follows readily that $F$ and hence $\lambda F$ is injective. The local existence of $\varphi_\lambda$ is guaranteed by the Poincaré-Siegel theorem (cf. [1, section 25, p. 193]) since if $\lambda > 1$ the eigenvalues of $\lambda I$ are non-resonant. Furthermore $\varphi_\lambda(0) = 0$ and $\varphi_\lambda$ is unique if we assume that $J\varphi_\lambda(0) = I$, which we can do without loss of generality. It was shown in [4] that $\varphi_\lambda^{-1}$ is entire, however the convergence of $\varphi_\lambda$ could only be proved in some neighbourhood of 0. Meisters in [8] restricted the problem to polynomial maps of the form $F = X + H$ with $H$ cubic homogeneous and $\det(JF) = 1$ (or equivalently $JH$ nilpotent) and conjectured that for such maps $\lambda F$ can be conjugated to its linear part $\lambda I$ by means of polynomial automorphisms $\varphi_\lambda$, for almost all $\lambda \in \mathbb{C}$, except a finite number of roots of unity. In [5] the first author gave a
counterexample to this conjecture for any dimension $\geq 4$. On the other hand it was recently shown by Gorni and Zampieri in [7] that this example can be conjugated to its linear part for all $\lambda$ with $|\lambda| \neq 1$ by means of an analytic automorphism $\varphi_{\lambda}$! So the $DMZ$-conjecture remained open.

Another proof of the fact that the counterexample of [5] satisfies the $DMZ$-conjecture was even more recently given by Deng in [3]. In his very elegant and short paper he proves that an analytic map $F : \mathbb{C}^n \to \mathbb{C}^n$ with $F(0) = 0$ can be analytically conjugated to its linear part if and only if $F$ is an analytic automorphism of $\mathbb{C}^n$ and $0$ is a global attractor of $F$ (i.e. for every $x \in \mathbb{C}$ the sequence $x, F(x), F^2(x), \ldots$ tends to $0$). In the same paper he conjectured that if $F = X + H$ with $H$ cubic homogeneous and $JH$ nilpotent then $0$ is a global attractor of $F \circ \lambda$ for all $\lambda$ with $|\lambda| < 1$. (In fact in the argument he gave to motivate this conjecture he does not use that $H$ is of degree $3$.)

A similar kind of question was brought up independently by Cima, Gasull and Mañosa in [2]. They studied the problem that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial map with $F(0) = 0$ and such that the eigenvalues of $JF(x)$ are smaller then $1$ in absolute value for all $x \in \mathbb{R}^n$, then $0$ is a global attractor of $F$. They call it the discrete Markus-Yamabe Question and show that this problem implies the Jacobian Conjecture and that it is true for triangular maps.

In this paper we give a counterexample to the $DMZ$-conjecture of the form $F = X + H$, where $H$ is homogeneous of degree $5$ in any dimension $n \geq 4$. Furthermore we show that if $0 < \lambda < 1$ then $\lambda F$ is a counterexample to the discrete Markus-Yamabe Question.

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1 A counterexample to the discrete Markus-Yamabe Question

Let $n \geq 4$ and consider the polynomial ring $\mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n]$. In $\mathbb{R}[X]$ define the element

$$d(X) := X_3X_1 + X_4X_2$$

**Theorem 1.1** Let $n \geq 4$ and $m \in \mathbb{N}, m \geq 1$. Define the polynomial automorphism

$$F = (X_1 + X_4d(X))^2, X_2 - X_3d(X)^2, X_3 + X_4^m, X_4, \ldots, X_n).$$

Then for each $0 < \lambda < 1 \lambda F$ is a counterexample to the discrete Markus-Yamabe Question. More precisely, if $0 < \lambda < 1$ and $a \in \mathbb{R}$ is such that $a\lambda > 1$ then the first component of $(\lambda F)^k(a, a, \ldots, a)$ tends to infinity if $k$ tends to infinity.

**Definition 1.2** For each $\lambda > 0$ and $a > 0$ we put $(\lambda F)^k(a) := (\lambda F)^k(a, a, \ldots, a)$ and denote the first component of this vector by $f_k(\lambda, a)$. So

$$f_k(\lambda, a) := ((\lambda F)^k(a))_1,$$

for all $k \geq 1$. Furthermore we put

$$d_k(\lambda, a) := d((\lambda F)^k(a)),$$

for all $k \geq 1$.

**Lemma 1.3**

i). $d(\lambda F(X)) = \lambda^2[X_4^{m+1}d(X)^2 + d(X) + X_4^mX_1]$

ii). $d_{k+1}(\lambda, a) \geq \lambda^2(\lambda^k a)^{m+1}(d_k(\lambda, a))^2$, for all $k \geq 1$.

iii). $f_{k+1}(\lambda, a) \geq \lambda^{k+1}a(d_k(\lambda, a))^2$, for all $k \geq 1$.

**Proof.** i) is easy to verify. Consequently, since all monomials in $d(\lambda F(X))$ have positive coefficients, we get

$$d_{k+1}(\lambda, a) = d((\lambda F)(\lambda F)^k(a))$$
$$\geq \lambda^2((\lambda F)^k(a))^{m+1}d((\lambda F)^k(a))^2$$
$$= \lambda^2(\lambda^k a)^{m+1}(d_k(\lambda, a))^2$$
since the fourth component of $(\lambda F)^k(a)$ equals $\lambda^k a$. This proves ii). Finally

$$f_{k+1}(\lambda, a) = (\lambda F)_{11}((\lambda F)^k(a)) \geq \lambda((\lambda F)^k(a))_4 d((\lambda F)^k(a))$$

(assuming that $(\lambda F)_1 = \lambda X_4 d(X)^2 + \lambda X_1$). So $f_{k+1}(\lambda, a) \geq \lambda^{k+1} a(d_k(\lambda, a))^2$, which proves iii).

**Proposition 1.4** We have:

$$f_k(\lambda, a) \geq \lambda^{p_k a^{p_k + (2m+1)(k-1)+4}}$$
$$d_k(\lambda, a) \geq \lambda^{p_k + m(k-1)+1} a^{p_k + (2m+1)(k-1)+m+4}$$

for all $k \geq 1$, where $p_1 = 1$ and $p_{k+1} = 2p_k + (2m + 1)(k-1) + 4$ for all $k \geq 1$.

**Proof.** Use induction on $k$. Details are left to the reader.

**Proof of theorem 1.1.** It follows immediately from the estimation of $f_k(\lambda, a)$ in proposition 1.4 that $\lim_{k \to \infty} f_k(\lambda, a) = \infty$ if $\lambda a > 1$. Furthermore one easily verifies that $\lambda F = \lambda X + H$ with $JH$ nilpotent. So for all $x \in \mathbb{R}^n$ the eigenvalues of $JF(x)$ are equal to $\lambda$.

**Corollary 1.5** Let $m = 5$ and $0 < \lambda < 1$. Put $\tilde{F} := \lambda F\lambda^{-1}$. Then $\tilde{F} = X + H$ with $H$ homogeneous of degree 5 and $JH$ is nilpotent. However 0 is not a global attractor of $\tilde{F} \circ \lambda (= \lambda F)$.

2 A counterexample to the DMZ-conjecture

Let $n \geq 4$ and consider the polynomial ring $\mathbb{C}[X] := \mathbb{C}[X_1, \ldots, X_n]$. In $\mathbb{C}[X]$ define the element $d(X) := X_3 X_1 + X_4 X_2$.

**Theorem 2.1** Let $n \geq 4$ and $m \geq 3$, $m$ odd. Define the polynomial automorphism

$$F = (X_1 + X_4 d(X)^2, X_2 - X_3 d(X)^2, X_3 + X_4^m, X_4, \ldots, X_n).$$

Then $F$ is a counterexample to the DMZ-conjecture. More precisely, for every $\lambda > 0$, $\lambda \neq 1$, $\lambda F$ is not global analytic linearisable to $\lambda X$. 

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The proof of this theorem is based on the following observation which is
due to Bo Deng (cf [3]).

**Lemma 2.2** Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be an analytic map with $F(0) = 0$. Put $A := JF(0)$ and suppose that the eigenvalues of $A$ are smaller than 1 in absolute value. If $F$ is global analytic linearisable to its linear part $A$ then 0 is a global attractor of $F$.

**Proof.** Let $x \in \mathbb{C}^n$ and let $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ be the analytic automorphism of $\mathbb{C}^n$ such that $\varphi^{-1}F\varphi = A$. Then $F = \varphi A \varphi^{-1}$ and hence $F^k(x) = \varphi A^k \varphi^{-1}(x)$, for all $k \geq 1$. By the hypothesis on the eigenvalues of $A$ it follows that $A^k \varphi^{-1}(x) \to 0$ if $k \to \infty$. Consequently $F^k(x) = \varphi(A^k \varphi^{-1}(x)) \to 0$ if $k \to \infty$. □

**Proof of theorem 2.1.** i). From lemma 2.2 and theorem 1.1 it follows that $\lambda F$ is not analytic linearisable if $0 < \lambda < 1$.

ii). Now let $\lambda > 1$. Suppose that $\lambda F$ is analytic linearisable. We derive a contradiction. Then $(\lambda F)^{-1} = F^{-1} \circ \lambda^{-1}$ is also analytic linearisable. Put $\mu := \lambda^{-1}$ and $G := F^{-1}$. So $G \circ \mu$ is analytic linearisable. One easily verifies that

$$G = (X_1 - X_4 \tilde{d}(X)^2, X_2 + (X_3 - X_4^m) \tilde{d}(X)^2, X_3 - X_4^m, X_4, \ldots, X_n) \quad (1)$$

where

$$\tilde{d}(X) := d(X) - X_4^m X_1. \quad (2)$$

Since $0 < \mu < 1$ it follows from lemma 2.2 that 0 is a global attractor of $G \circ \mu$. However we will show below (corollary 2.6) that for every $0 < \mu < 1$ 0 is not a global attractor of $G \circ \mu$. Hence we have derived a contradiction. □

So it remains to show that 0 is not a global attractor of $G \circ \mu$. First we show that 0 is not a global attractor of $\mu G$ if $0 < \mu < 1$. To prove this we need some lemmas. So let $G$ and $\tilde{d}(X)$ be as in (1) resp. (2).

For each $a > 0$ let $a^* := (a, -a, a, -a, a, \ldots, a) \in \mathbb{R}^n$. Then we define for each $a > 0$ and $\mu > 0$:

$$g_k(\mu, a) := ((\mu G)^k(a^*))_1$$

$$\tilde{d}_k(\mu, a) := \tilde{d}((\mu G)^k(a^*))$$

for all $k \geq 1$. 5
Lemma 2.3  
i).  \(d(G(X)) = \tilde{d}(X)\).

ii).  \(\tilde{d}((\mu G)(X)) = \mu^2 \tilde{d}(X) - \mu^{m+1}X_4^mX_1 + \mu^{m+1}X_4^{m+1}\tilde{d}(X)^2\).

iii).  \(\tilde{d}_{k+1}(\mu, a) = (\mu^{k+1}a)^{m+1}(\tilde{d}_k(\mu, a))^2 + \mu^2 \tilde{d}_k(\mu, a) + \mu(\mu^{k+1}a)^m g_k(\mu, a)\) for all \(k \geq 1\).

iv).  \(g_{k+1}(\mu, a) = \mu^{k+1}a(\tilde{d}_k(\mu, a))^2 + \mu g_k(\mu, a)\) for all \(k \geq 1\).

Proof.  The proofs of i) and ii) are straightforward and left to the reader. From ii) we deduce that

\[
\tilde{d}_{k+1}(\mu, a) = \tilde{d}((\mu G)^{k+1}(a^*))
\]
\[
= \tilde{d}((\mu G)((\mu G)^k(a^*)))
\]
\[
= \mu^2 \tilde{d}((\mu G)^k(a^*)) - \mu^{m+1}(((\mu G)^k(a^*))_4)^m((\mu G)^k(a^*))_1
\]
\[
+ \mu^{m+1}(((\mu G)^k(a^*))_4)^{m+1}\tilde{d}((\mu G)^k(a^*))^2
\]

Now observe that \(((\mu G)^k(a^*))_4 = \mu^k(-a)\), hence since \(m\) is odd \(((\mu G)^k(a^*))_4^m = -(\mu^ka)^m\). So we get

\[
\tilde{d}_{k+1}(\mu, a) = \mu^2 \tilde{d}_k(\mu, a) + \mu^{m+1}(\mu^k a)^m g_k(\mu, a) + \mu^{m+1}(\mu^k a)^{m+1}(\tilde{d}_k(\mu, a))^2
\]
\[
= (\mu^{k+1}a)^{m+1}(\tilde{d}_k(\mu, a))^2 + \mu(\mu^{k+1}a)^m g_k(\mu, a) + \mu^2 \tilde{d}_k(\mu, a)
\]

which proves iii). Finally

\[
g_{k+1}(\mu, a) = ((\mu G)^{k+1}(a^*))_1
\]
\[
= (\mu G)_1((\mu G)^k(a^*))
\]
\[
= \mu((\mu G)^k(a^*))_1 - \mu((\mu G)^k(a^*))_4(\tilde{d}((\mu G)^k(a^*))^2
\]
\[
= \mu g_k(\mu, a) - \mu \cdot \mu^k(-a)(\tilde{d}_k(\mu, a))^2
\]
\[
= \mu g_k(\mu, a) + \mu^{k+1}a(\tilde{d}_k(\mu, a))^2
\]

which proves iv).  

Corollary 2.4  
i).  \(\tilde{d}_{k+1}(\mu, a) \geq (\mu^{k+1}a)^{m+1}(\tilde{d}_k(\mu, a))^2\) for all \(k \geq 1\).

ii).  \(g_{k+1}(\mu, a) \geq \mu^{k+1}a(\tilde{d}_k(\mu, a))^2\) for all \(k \geq 1\).

Proof.  By induction on \(k\) one readily verifies that for all \(k \geq 1\) both \(\tilde{d}_k(\mu, a)\) and \(g_k(\mu, a)\) are polynomials in \(\mu\) and \(a\) with coefficients in \(\mathbb{N}\). Then the result follows from lemma 2.3 iii) an iv).  

\[\square\]
Proposition 2.5 We have:
\[
g_k(\mu, a) \geq \mu^{q_k(m+1)+k} a^{q_k+2k(m+1)+1} \\
d_k(\mu, a) \geq \mu^{q_k+k(m+1)} a^{q_k+2k+1}(m+1)
\]
for all \( k \geq 1 \), where \( q_1 = 0 \) and \( q_{k+1} = 2q_k + 2k \) for all \( k \geq 1 \).

Proof. Use induction on \( k \). \( \square \)

Corollary 2.6 If \( \mu a > 1 \) and \( a > 1 \) then \( \lim_{k \to \infty} ((G \circ \mu)^k(G(a^*))_1 = \infty \). So 0 is not a global attractor of \( G \circ \mu \).

Proof. Observe that \( (G\mu)^k(G(a^*)) = \mu^{-1}(\mu G)^{k+1}(a^*) \). Then apply proposition 2.5. \( \square \)

References


