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for a Class of Polynomial Automorphisms and Stably Tameness

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In this paper we introduce a set, denoted by $D_n(A)$, for every commutative ring $A$ and every positive integer $n$. It is shown that the elements of this set can be used to give an explicit description of the class $H_n(A)$ introduced in van den Essen and Hubbers [J. Algebra 187 (1997), 214–226]. We deduce that each polynomial map of the form $F = X + H$ with $H \in H_n(A)$ can be written as a finite product of automorphisms of the form $\exp(D)$, where each $D$ is a locally nilpotent derivation satisfying $D^2(X) = 0$ for all $i$. Furthermore we deduce that all such $F$s are stably tame.

1. NOTATION, DEFINITIONS, AND AN EXPLICIT DESCRIPTION OF THE CLASS $H_n(A)$

1.1. Notation

Throughout this paper $A$ denotes an arbitrary commutative ring and $A[X] := A[X_1, \ldots, X_n]$ denotes the polynomial ring in $n$ variables over $A$. Furthermore if $G = (G_1, \ldots, G_n) \in A[X]^n$ and $S = (S_i(X)) \in M_{p,q}(A[X])$ then $S(G)$ or $S^G$ denotes the $p \times q$ matrix $(S_i(G_1, \ldots, G_n))_{i,j}$.

In particular if $F \in A[X]^n$ ($= M_n(A[X])$) then the composition of the polynomial maps $F$ and $G$, denoted $F \circ G$, is equal to $F(G)$.

Matrix multiplication will be denoted by the symbol ‘*’. So if $S, T \in M_n(A[X])$ then the matrix product of $S$ and $T$ is denoted by $S \circ T$. By $X$ we denote the column vector $(X_1, \ldots, X_n)'$. In the sequel we also need another multiplication in $M_n(A[X])$, which we denote by $\triangle$. This multipli-
cution is defined as follows:

\[ S \triangle T := S(T \ast X) \ast T \]

for all \( S, T \in M_n(A[X]) \).

One easily verifies that this multiplication is associative, so it makes sense to write

\[ S_1 \triangle S_2 \triangle \cdots \triangle S_n \]

for each \( n \)-tuple \( S_1, \ldots, S_n \) in \( M_n(A[X]) \). Sometimes we need to extend a vector of length \( 1 \leq p \leq n - 1 \) or a \( p \times p \) matrix to, respectively, a vector of length \( n \), or an \( n \times n \) matrix. This is done as follows: let \( 1 \leq p \leq n - 1 \), \( c \in A[X]^p \), and \( T \in M_p(A[X]) \). Then \( \tilde{c}^n \) denotes the vector

\[
\begin{pmatrix}
   c \\
   0 \\
   \vdots \\
   0
\end{pmatrix} \in A[X]^n,
\]

obtained by extending \( c \) by \( n - p \) zeros, and \( \tilde{T}^n \) denotes the matrix

\[
\begin{pmatrix}
   T \\
   0 \\
   I_{n-p}
\end{pmatrix} \in M_n(A[X]),
\]

obtained by extending \( T \) with the \((n-p) \times (n-p)\) identity matrix. To simplify the notation we drop the superscript \( n \) and write \( \tilde{c} \) and \( \tilde{T} \), even sometimes when it is clear from the context that we mean \( \tilde{c}^{n-1} \), respectively, \( \tilde{T}^{n-1} \) instead of \( \tilde{c}^n \), respectively, \( \tilde{T}^n \).

Finally the adjoint of a matrix \( T \) is denoted by \( \text{Adj}(T) \) and if \( a_1, \ldots, a_p \) are elements of a (nonnecessary commutative) ring then \( \prod_{i=1}^p a_i \) denotes the element \( a_1 \cdots a_p \).

1.2. \( D_n(A) \) and the class \( H_n(A) \)

In [6] we introduced a new class of polynomial maps, denoted by \( H_n(A) \), and showed that for each \( H \in H_n(A) \) the Jacobian matrix \( JH \) is nilpotent and that the polynomial map \( F = X + H \) is invertible over \( A \) with \( \det(JF) = 1 \).

Let us recall the definition of \( H_n(A) \).

**Definition 1.1.** First if \( n = 1 \) we define \( H_1(A) = A \). If \( n \geq 2 \) we define \( H_n(A) \) inductively as follows: Let \( H \in A[X]^n \). Then \( H \in H_n(A) \) if and only if there exist \( T \in M_n(A), c \in A^n \), and \( H_* \in H_{n-1}(A[X]) \) such that

\[ H = \text{Adj}(T) \ast \begin{pmatrix} H_* \\ 0 \end{pmatrix}_{[T \ast X]} + c. \tag{1} \]
The main aim of this section is to give an explicit description of the elements of $H_0(A)$. Therefore we introduce some useful objects.

**Definition 1.2.** Let $n \geq 2$. Then $D_n(A)$ is the set of $(2n - 1)$-tuples

$$(T, c) := (T_2, \ldots, T_n, c_1, \ldots, c_n),$$

where $T_i \in M_i(A)$, $T_i \in M_i(A[X_i+1, \ldots, X_n])$ for all $2 \leq i \leq n - 1$, $c_n \in A^n$ ($= M_{n,1}(A)$) and $c_i \in M_{i,1}(A[X_i+1, \ldots, X_n])$ for all $1 \leq i \leq n - 1$.

If $n \geq 3$ we get a natural map $\pi : D_n(A) \to D_{n-1}(A[X_n])$ defined by

$$\pi((T_2, \ldots, T_n, c_1, \ldots, c_n)) = (T_2, \ldots, T_{n-1}, c_1, \ldots, c_{n-1}).$$

Instead of $\pi((T, c))$ we often write $(T', c')$.

**Definition 1.3.** Let $n \geq 2$ and $0 \leq p \leq n - 2$. Then

$$E_{n,p} : D_n(A) \to A[X]^n$$

is given by

1. $E_{n,0}((T, c)) := A \text{adj}(T_n) \ast c_{n-1}/T_n \ast X$ for all $(T, c) \in D_n(A)$.
2. If $n \geq 3$ and $1 \leq p \leq n - 2$, then inductively (with respect to $n$)

$$E_{n,p}((T, c)) := A \text{adj}(T_n) \ast \left( E_{n-1,p-1}(T', c') \right) \bigg|_{T_n \ast X}$$

Instead of $E_{n,p}((T, c))$ we simply write $E_{n,p}(T, c)$.

Now we are able to give the main result of this section.

**Proposition 1.4.** Let $n \geq 2$ and $H \in A[X]^n$. Then $H \in H_n(A)$ if and only if there exists $(T, c) \in D_n(A)$ such that

$$H = \sum_{p=0}^{n-2} E_{n,p}(T, c) + c_n.$$

**Proof.** The proof is by induction on $n$. The case $n = 2$ is obvious, so let $n \geq 3$. Then

$$H = A \text{adj}(T_n) \ast \left( H_\ast \right) \bigg|_{T_n \ast X} + c_n,$$

where $T_n \in M_n(A)$, $c_n \in A^n$, and $H_\ast \in H_{n-1}(A[X_n])$. So by the induction hypothesis we have

$$H_\ast = \sum_{p=0}^{n-3} E_{n-1,p}(T^*, c^*) + c_{n-1}^n.$$
for some \((T^*, c^*) \in D_{n-2}(A[X_n])\). Put \((T, c) := (T^*, T_n, c^*, c_n)\) and observe that \((T, c) \in D_n(A)\) and \((T', c') = (T^*, c^*)\). So

\[
H = \sum_{p=0}^{n-3} \text{Adj}(T_n) \ast E_{n-1, p}(T', c')|_{T_n \ast X} + \text{Adj}(T_n) \ast \left( \begin{array}{c} c_{n-1}^* \\ 0 \end{array} \right) |_{T_n \ast X} + c_n
\]

\[
= \sum_{p=1}^{n-2} E_{n, p}(T, c) + E_{n, 0}(T, c) + c_n
\]

\[
= \sum_{p=0}^{n-2} E_{n, p}(T, c) + c_n.
\]

**Proposition 1.5.** Let \(n \geq 2, 0 \leq p \leq n - 2\), and \((T, c) \in D_n(A)\). Then

\[
E_{n, p}(T, c) = \text{Adj}\left(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n\right) \ast \tilde{c}_{n-p-1}(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n) \ast X.
\]

**Proof.** The proof is by induction on \(p\). The case \(p = 0\) is obvious. So let \(p \geq 1\). Then

\[
E_{n, p}(T, c) = \text{Adj}(T_n) \ast \left( \begin{array}{c} E_{n-1, p-1}(T', c') \\ 0 \end{array} \right) |_{T_n \ast X}
\]

\[
= \text{Adj}(T_n) \ast \left[ \text{Adj}\left(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1}\right) |_{T_n \ast X} \right. \\
\left. \ast \left( \tilde{c}_{n-p-1}(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1}) \ast X \right) |_{T_n \ast X} \right]
\]

(by the induction hypothesis)

\[
= \text{Adj}\left(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1}\right) |_{T_n \ast X} \ast T_n \\
\ast \tilde{c}_{n-p-1}(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1}) \ast X \\
= \text{Adj}\left(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n\right) \ast \tilde{c}_{n-p-1}(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n) \ast X.
\]

**Example 1.6.** Consider the polynomial map \(F := X + H: \mathbb{C}^4 \to \mathbb{C}^4\), where \(H\) equals

\[
\begin{pmatrix}
-X_2^2 + e_4 X_3 X_4 - \frac{m_2}{g_4} X_2 X_3 X_4 - g_3 X_2 X_3 X_4 - k_4 X_3^3 - \frac{m_3}{g_4} X_2 X_3^2 - m_4 X_2 X_3^2 \\
-X_1^2 - e_3 X_2 X_4 + g_3 X_2 X_3 X_4 - k_2 X_3^3 + m_4 X_2 X_3^2 + g_4 X_3 X_4 \\
-\frac{1}{3} X_4^3 \\
0
\end{pmatrix}
\]
and $e_3, k_3, e_4, g_4, k_4, m_4 \in \mathbb{C}$ and $g_4 \neq 0$. This $F$ is invertible. In fact if we take $P = P^{-1} = (X_3, X_3, X_3, X_3)$, we have that $PFP$ is one of the eight representatives of the cubic homogeneous maps in dimension 4 as given by Hubbers [7] and also published in [4, Theorem 2.10].

Now consider the following element $(T, c)$ of $D_4(\mathbb{C})$, where

$$T = \begin{pmatrix}
1 & 0 & 0 & 0 \\
g_4^2 X_3 & g_4 X_4 + m_4 X_3 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
$$

and

$$c = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
g_4 & -X_3^3 (e_4 X_4 + k_4 X_3) & 0 & 0 \\
-X_3^2 (e_4 X_4 - k_3 X_3) & -X_3^2 (e_4 X_4 + k_4 X_3) & -1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$

Our claim is that

$$H = \sum_{p=0}^{2} E_{4,p}(T, c) + c_4.$$

To prove this we will compute $E_{4,0}$, $E_{4,1}$, and $E_{4,2}$ by the method of Proposition 1.5. Note that $c_4 = 0$. Since $T_4 = T_3 = I_4$, $E_{4,0}$ and $E_{4,1}$ are easy:

$$E_{4,0} = \text{Adj}(T_4) \ast \tilde{c}_3 \ast T_4 \ast \text{x} = \tilde{c}_3 = \begin{pmatrix}
0 \\
0 \\
-1 \\
3 X_4^3 \\
0
\end{pmatrix},$$

$$E_{4,1} = \text{Adj}(T_3 \times T_4) \ast \tilde{c}_2 \ast (T_3 \times T_4) \ast \text{x} = \tilde{c}_2 = \begin{pmatrix}
-X_3^2 (e_4 X_4 + k_4 X_3) \\
-X_3^2 (e_4 X_4 - k_3 X_3) \\
0
\end{pmatrix}.$$
Before we compute $E_{4,2}$ we present the following identities:

$$
\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4 = \tilde{T}_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
g_4^2X_3 & g_4X_4 + m_4X_3 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix};
$$

$\text{Adj}(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4)$

$$
= \begin{pmatrix}
g_4X_4 + m_4X_3 & 0 & 0 & 0 \\
-g_4^2X_3 & 1 & 0 & 0 \\
0 & 0 & g_4X_4 + m_4X_3 & 0 \\
0 & 0 & 0 & g_4X_4 + m_4X_3
\end{pmatrix};
$$

$$(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4)^* X = \begin{pmatrix}
X_3 \\
g_4^2X_1X_3 + g_4X_2X_4 + m_4X_2X_3 \\
X_3 \\
X_4
\end{pmatrix};
$$

$$(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4)^* \bar{c}_{1}(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4)^* X = \begin{pmatrix}
-X_1X_3 - \frac{1}{g_4}X_2X_4 - \frac{m_4}{g_4^2}X_2X_3 \\
0 \\
0 \\
0
\end{pmatrix};
$$

and finally

$$
E_{4,2} = \text{Adj}(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4)^* \bar{c}_{1}(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4)^* X
$$

$$
\begin{pmatrix}
-X_1X_3 + \frac{m_4}{g_4^2}X_3 & \left(g_4X_1X_3 + X_2X_4 + \frac{m_4}{g_4}X_2X_3\right) \\
X_3(g_4^2X_1X_3 + g_4X_2X_4 + m_4X_2X_3) \\
0 \\
0
\end{pmatrix}.
$$

It is easy to verify that $H = E_{4,0} + E_{4,1} + E_{4,2} + c_4$, which was our claim.

2. NICE DERIVATIONS

Let $B := A[x_1, \ldots, x_n]$ be a finitely generated $A$-algebra and $D$ a subset of $\text{Der}_A(B)$. By $B^D$ we denote the set of all $b \in B$ such that $d(b) = 0$ for all $d \in D$. 
**Definition 2.1.** Let $D \subseteq \text{Der}_A(B)$ be a finite subset and $\tau \in \text{Der}_A(B)$.

1. We say that $\tau$ is derived from $D$ in at most one step if $\tau$ is of the form $\tau = \sum d \in D b_d d$, where $b_d \in B^D$ for all $d \in D$.

2. Let $m \geq 2$. We say that $\tau$ is derived from $D$ in at most $m$ steps if there exists a sequence of finite subsets $D = D_0, D_1, D_2, \ldots, D_m$ of $\text{Der}_A(B)$ such that $\tau \in D_m$ and all elements of $D_i$ are derived from $D_{i-1}$ in at most one step, for all $1 \leq i \leq m$. If furthermore the elements of $D$ satisfy $d_1 d_2(x_i) = 0$ for all $d_1, d_2 \in D$ and all $i$, then $\tau$ is called nice of order $\leq m$, with respect to $x_1, \ldots, x_n$ and $D$.

**Proposition 2.2.** The notation is as in Definition 2.1. If $d_1 d_2(x_i) = 0$ for all $d_1, d_2 \in D$ and all $i$, then $d_1 d_2(x_i) = 0$ for all $d_1, d_2 \in D_m$ and all $i$. In particular $d^2(x_i) = 0$ for every nice derivation.

**Proof.** We use induction on $m$. The case $m = 0$ is obvious since $D_0 = D$. Now let $m \geq 1$. Then $d_1 = \sum d \in D_{m-1} b_d d$, $d_2 = \sum d' \in D_{m-1} b'_d d'$ with $b'_d, b_d \in B^{D_{m-1}}$. Then

$$d_1 d_2(x_i) = \sum_{d, d'} b_d (b'_d) d(x_i) + \sum_{d, d'} b_d b'_d d' d(x_i).$$

Now observe that $d(b'_d) = 0$ since $b'_d \in B^{D_{m-1}}$ and $d \in D_{m-1}$. Finally the induction hypothesis gives $dd'(x_i) = 0$ for all $d, d' \in D_{m-1}$ and all $i$, so (2) implies $d_1 d_2(x_i) = 0$.

We demonstrate these aspects by the so-called Winkelmann derivation. See [11].

**Example 2.3.** Let $\tau = (1 + X_4 X_3 - X_5 X_3) \partial_{X_4} + X_4 \partial_{X_3} + X_4 \partial_{X_5}$, a derivation on $B := A[X_1, X_2, X_3, X_4, X_5]$. Let $D = \{ \partial_{X_1}, \partial_{X_2}, \partial_{X_3} \}$. Then $\tau$ is nice of order 2 with respect to $X_1, X_2, X_3, X_4, X_5$, and $D$. To show that this is true, we present a sequence of finite subsets of $\text{Der}_A(B)$,

$$D = D_0, D_1, D_2.$$

Take $D_1 := \{ \partial_{X_1}, X_5 \partial_{X_2} + X_4 \partial_{X_1} \}$ and $D_2 := \{ \tau \}$. Note that in Definition
2.1 it is not demanded that the set $D_i$ of this sequence is a subset of $D_{i+1}$.
The only demand is that each $D_i$ is a finite subset of $\text{Der}_A(B)$. Since
$X_4, X_3 \in B^D$ it follows immediately that $\partial_{X_3}$ and $X_5 \partial_{X_4} + X_4 \partial_{X_3}$ are
derived from $D$ in one step. And from $1 + X_4 X_2 - X_2 X_3 \in B^D$, it follows
that $\tau$ is derived from $D_1$ in one step. Obviously we have $d_1 d_2(X_i) = 0$ for
all $d_1, d_2 \in D$ and hence with Proposition 2.2 also $\tau^2(X_i) = 0$.

3. DERIVATIONS ASSOCIATED WITH POLYNOMIAL MAPS

The main aim of this section is to show that for each $0 \leq p \leq n - 2$ the
polynomial map $X + E_{n,p}(T, c)$ (where $(T, c) \in D_n(A)$) is of the form
$\text{exp}(d)$, for some nice $A$-derivation $d$ of $A[X]$. Observe that $d$ is locally
nilpotent if $d$ is nice with respect to $X_1, \ldots, X_n$ since $d^2(X_i) = 0$ for all $i$,
by Proposition 2.2.

In order to prove this result (see Theorem 3.3), we need to generalise
some of the notions of Sect. 1 to arbitrary finitely generated $A$-algebras.
So let $B := A[X_1, \ldots, X_n]$ be a finitely generated $A$-algebra, and let
$\varphi: A[X_1, \ldots, X_n] \to B$ be the $A$-ring homomorphism defined by $\varphi(X_i) = x_i$
for all $i$. For each $p, q \geq 1$ consider the natural extension
$\varphi: M_{p,q}(A[X_1, \ldots, X_n]) \to M_{p,q}(B)$.

Then for each $(T, c) \in D_n(A)$ we define
$E_{n,p}(T, c)(x) := \varphi(E_{n,p}(T, c)) \in B^n$.

Now let $(\partial_1, \ldots, \partial_n)$ be an $n$-tuple of $A$-derivations of $B$. With each vector
$b = (b_1, \ldots, b_n) \in B^n$ we associate the following $A$-derivation of $B$:
$D(b; \partial_1, \ldots, \partial_n) := b_1 \partial_1 + \cdots + b_n \partial_n$
$= b^t \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix}$.

To formulate the next lemma we need some more notation: Let $(T, c) \in D_n(A)$. Put
$(x_1', \ldots, x_n') := T_n'(x_1, \ldots, x_n)$,
$(\partial_1', \ldots, \partial_n') := (A d)(T_n)' \ast (\partial_1, \ldots, \partial_n)'$,
$x'^n := (x_1', \ldots, x_{n-1}')$,
$(T^n, c'^n) := (T'(X_n = x'_n), c'(X_n = x'_n)) \in D_{n-1}(A[x'_n])$. 
Lemma 3.1. Let \( n \geq 3 \) and \( 1 \leq p \leq n - 2 \). Then
\[
D(E_{n,p}(T,c)(x); \partial_1, \ldots, \partial_n) = D(E_{n-1,p-1}(T''', c''')(x'''); \partial'_1, \ldots, \partial'_{n-1}).
\]

Proof.
\[
D(E_{n,p}(T,c)(x); \partial_1, \ldots, \partial_n)
= (E_{n,p}(T,c)(x))^t \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix}
= \left( (E_{n-1,p-1}(T'', c'')(x'')(T_n)^t \cdot x) \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \right)^t \cdot (\text{Adj}(T_n))^t \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix}
= D(E_{n-1,p-1}(T'', c''')(x'''); \partial'_1, \ldots, \partial'_{n-1}).
\]

Lemma 3.2. The notation is as above. Let \( a \in A \) and let \( \partial_1, \ldots, \partial_n \) be \( A \)-derivations of \( B \) such that \( \partial_i(x_j) = a \delta_{ij} \) for all \( i, j \). Then
\[
\partial'_i(x_j) = a \det(T_n) \delta_{ij}
\]
for all \( i, j \).

Proof. Denote the \( i \)-th column of \( \text{Adj}(T_n) \) by \( (t^*_i, \ldots, t^*_n)^t \) and the \( j \)-th row of \( T_n \) by \( (t_{1j}, \ldots, t_{nj}) \). Then
\[
\partial'_i(x_j) = \left( \sum_{s=1}^n t^*_i x_s \right) \left( \sum_{s=1}^n t_{js} x_s \right)
= \sum_{s=1}^n at^*_i t_{js}
= a(T_n \ast \text{Adj}(T_n))_{ji}
= a \det(T_n) \delta_{ij}.
\]
Now we are able to prove:

**Theorem 3.3.** Let $\partial_1, \ldots, \partial_n$ be $A$-derivations on $A[x_1, \ldots, x_n]$ such that there exists an element $a \in A$ such that $\partial_i(x_i) = a \delta_i$ for all $i, j$. Let $(T, c) \in \mathcal{D}_n(A)$. Then the $A$-derivation $d := D(E_{n, p}(T, c)(x_1, \ldots, x_n))$ is nice with respect to $x_1, \ldots, x_n$ and if $D_0 := \{\partial_1, \ldots, \partial_n\}$, for all $n \geq 2$ and all $0 \leq p \leq n - 2$.

**Proof.** 1. The hypotheses on the $\partial_i$ imply that $dd'(x_i) = 0$ for all $d, d' \in D_0$ and all $i$.

2. First we consider the case $p = 0$. Then

$$E_{n, 0}(T, c) = \text{Ad}(T_n) \ast \tilde{c}_{n-1} \ast x_n.$$  

So

$$d = \left(\tilde{c}_{n-1} \ast x_n\right) \ast \left(\text{Ad}(T_n)\right) \ast \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix}.$$  

Write $\tilde{c}_{n-1} = (\gamma_1(x_n), \ldots, \gamma_{n-1}(x_n), 0)$. Then the definition of $x_n'$ and the $\partial_i'$ imply that

$$d = (\gamma_1(x_n'), \ldots, \gamma_{n-1}(x_n'), 0) \ast (\partial_1', \ldots, \partial_n') = \sum_{i=1}^{n-1} \gamma_i(x_n') \partial_i'.$$

Put $D_1 := \{\partial_1', \ldots, \partial_{n-1}'\}$ and observe that $D_1 \subset \text{Der}_n(B)$ and that each element of $D_1$ is derived from $D_0$ in at most one step. Finally since $\partial_i'(x_n') = 0$ for all $1 \leq i \leq n - 1$ (by Lemma 3.2) we get that $\gamma_i(x_n') \in B^{D_1}$ for all $1 \leq i \leq n - 1$. So (3) implies that $d$ is derived from $D_1$ in at most one step. Consequently $d$ is derived from $D_0$ in at most two steps. So $d$ is nice with respect to $x_1, \ldots, x_n$ and $D_0$ by case 1.

3. Now we prove the theorem by induction on $n$. If $n = 2$, then $p = 0$ and we are in case 2. So let $n \geq 3$. By case 2 we may assume that $p \geq 1$. Then by Lemma 3.1 we have

$$d = D(E_{n-1, p-1}(T'', c'')(x''); \partial_1', \ldots, \partial_{n-1}')$$

with $(T'', c'') \in \mathcal{D}_{n-1}(A[x_n'])$. By Lemma 3.2 we can apply the induction hypothesis to the ring $A[x_n']$ and the $(n - 1)$-tuple of $A[x_n']$-derivations $\partial_1', \ldots, \partial_{n-1}'$ on the $A[x_n']$-algebra $B' := A[x_n'][x_1', \ldots, x_{n-1}']$. So the $A[x_n']$-derivation $d$ on $B'$ is nice with respect to $D_0' := \{\partial_1', \ldots, \partial_{n-1}'\}$ and $x_1', \ldots, x_{n-1}'$. So there exists a sequence

$$D_0', D_1', \ldots, D_m'$$
of finite subsets of $\text{Der}_{\mathcal{A}}(B')$ such that $d \in D^m_0$ and $D'_i$ is derived from $D'_{i-1}$ in at most one step for all $1 \leq i \leq m$. Now observe that $D^m_0 \subset \text{Der}_{\mathcal{A}}(B)$ and that $B' \subset B$ since by definition obviously $x'_i \in B$ for all $i$. Consequently if $d'$ is an $\mathcal{A}[x'_i]$-derivation of $B'$ derived from $D^m_0$ in at most one step, then $d' \in \text{Der}_{\mathcal{A}}(B)$. Hence $D'_i \subset \text{Der}_{\mathcal{A}}(B)$. Arguing in a similar way we conclude by induction on $i$ that $D'_i \subset \text{Der}_{\mathcal{A}}(B)$ for all $0 \leq i \leq m$. Since as remarked in case 2 above, all elements of $D^m_0$ ($= D^m_1$ in case 2) are derived from $D^m_0$ in at most one step we deduce that $d$ is derived from $D^m_0$ in at most $m + 1$ steps. Just define $D_i := D'_i$ for all $1 \leq i \leq m + 1$. Hence $d$ is nice with respect to $x_1, \ldots, x_n$ and $D_0$ by 1.  

**Corollary 3.4.** Let $(T, c) \in D_n(A)$ and $0 \leq p \leq n - 2$. Put

$$D := D \left( E_{n, p}(T, c); \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n} \right).$$

Then $D$ is nice with respect to $X_1, \ldots, X_n$ and $\{ \partial/\partial X_1, \ldots, \partial/\partial X_n \}$. Furthermore we have $\exp(D) = X + E_{n, p}(T, c)$ and the inverse map is given by $\exp(-D) = X - E_{n, p}(T, c)$. 

**Proof.** The first part is an immediate consequence of Theorem 3.3. Furthermore $D^2(X) = 0$ by Proposition 2.2. So $\exp(D)(X) = X + E_{n, p}(T, c)$ and the inverse map is given by $\exp(-D)(X) = X - E_{n, p}(T, c)$.  

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4. THE MAIN THEOREM

In this section we show that for every $H \in H_n(A)$ the polynomial map $F = X + H$ is a product of $n$ polynomial automorphisms of the form $\exp(D)$, where each $D$ is a nice derivation on $\mathcal{A}[X]$. More precisely

**Theorem 4.1.** Let $F = X + H$, where $H = \sum_{p=0}^{n-2} E_{n, p}(T, c) + c_n$, for some $(T, c) \in D_n(A)$. Then

$$F = \exp \left( D \left( c_n; \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n} \right) \right) \times \prod_{p=0}^{n-2} \exp \left( D \left( E_{n, p}(T, c); \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n} \right) \right).$$
Proof. Observe that
\[
\exp \left( -D \left( c_{n}, \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n} \right) \right) \circ F = \sum_{p=0}^{n-2} E_{n, p}(T, c).
\]

So the case \( n = 2 \) follows from Corollary 3.4. Hence we may assume that \( n \geq 3 \). Now Theorem 4.1 follows directly from Proposition 4.2 below and Corollary 3.4.

**Proposition 4.2.** Let \( n \geq 3 \), \( 0 \leq p \leq n - 3 \), and \((T, c) \in D_n(A)\). Then
\[
\exp(-D(E_{n, p}(T, c))) \circ \left( X + \sum_{q=p}^{n-2} E_{n, q}(T, c) \right) = X + \sum_{q=p+1}^{n-2} E_{n, q}(T, c).
\]

*Proof.* Put \( G := \exp(-D(E_{n, p}(T, c))) \). So \( G = X - E_{n, p}(T, c) \) (by Corollary 3.4). Hence if we put
\[
U := \tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n
\]
then by Proposition 1.4 we get
\[
G = X - \text{Ad}(U) \ast \tilde{c}_{n-p-1|U \ast X}.
\]

So if we put
\[
f := X + \sum_{q=p}^{n-2} E_{n, q}(T, c)
\]
then
\[
G \circ f = f - \text{Ad}(U(f)) \ast \tilde{c}_{n-p-1|U(f) \ast f}.
\]

Since \( U(f) = f \) (by Corollary 4.4 below, with \( j = 0 \)) we get
\[
G \circ f = f - \text{Ad}(U) \ast \tilde{c}_{n-p-1|U \ast f}.
\]

Now observe that each component of \( \tilde{c}_{n-p-1|U \ast f} \) belongs to \( A[X_{n-p}, \ldots, X_n] \) and that for each \( i \geq n - p \) \((U \ast f)_i = (U \ast \hat{X})_i \) (by Lemma 4.3 below). So \( \tilde{c}_{n-p-1|U \ast f} = \tilde{c}_{n-p-1|U \ast X} \) and hence
\[
G \circ f = f - \text{Ad}(U) \ast \tilde{c}_{n-p-1|U \ast X}
\]
\[
= f - E_{n, p}(T, c)
\]
(by Proposition 1.4).
**Lemma 4.3.** Let \( n \geq 3, 0 \leq p \leq n - 2, 0 \leq j \leq p \), and \((T, c) \in \mathcal{D}_p(A)\). Put \( f := X + \sum_{q=p}^{n-2} E_{n,q}(T, c) \). Then

\[
\left[ \left( \tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n \right) \ast f \right]_i = \left[ \left( \tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n \right) \ast X \right]_i
\]

for all \( i \geq n - p + j \).

**Proof.** Put \( U := \tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n \). It suffices to show that for each \( q \geq p \)

\[
\left[ U \ast E_{n,q}(T, c) \right]_i = 0 \quad (4)
\]

for all \( i \geq n - p + j \). So let \( q \geq p \). Then \( q \geq p - j \).

1. We first treat the case that \( q = p - j \). Then \( j = 0 \) and \( q = p \). Consequently \( U = \tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n \). Then by Proposition 1.4

\[
U \ast E_{n,q}(T, c) = U \ast \text{adj}(U) \ast \tilde{c}_{n-p-1(U \ast X)} = \det(U) \ast \tilde{c}_{n-p-1(U \ast X)}.
\]

Since the last \( p + 1 \) coordinates of \( \tilde{c}_{n-p-1} \) are zero, we obtain that

\[
\left[ U \ast E_{n,q}(T, c) \right]_i = 0
\]

for all \( i \geq n - p \), which proves the case that \( q = p - j \).

2. Now assume that \( q \geq p - j + 1 \). So \( n - q \leq n - p + j - 1 \). Put \( V := \tilde{T}_{n-q} \triangle \cdots \triangle \tilde{T}_{n-p+j-1} \). Then by Proposition 1.4 we can write

\[
E_{n,q}(T, c) = \text{adj}(V \triangle U) \ast \tilde{c}_{n-q-1(V \triangle U) \ast X} = \text{adj}(V_{U \ast X} \ast U) \ast \tilde{c}_{n-q-1(V \triangle U) \ast X} = \text{adj}(U) \ast \text{adj}(V_{U \ast X} \ast U) \ast \tilde{c}_{n-q-1(V \triangle U) \ast X}.
\]

Consequently

\[
U \ast E_{n,q}(T, c) = \det(U) \ast \text{adj}(V_{U \ast X}) \ast \tilde{c}_{n-q-1(V \triangle U) \ast X}. \quad (5)
\]

Note that \( V \), and hence \( V_{U \ast X} \), is of the form \( \tilde{B} \) for some \( B \in M_{n-p+j-1}(A[X]) \). Furthermore \( \tilde{c}_{n-q-1} \), \( = 0 \) if \( i \geq n - q \), which implies that \( \tilde{c}_{n-q-1(V \triangle U) \ast X} = 0 \) if \( i \geq n - p + j \) (since \( n - p + j > n - q \)).

Now the desired result (4) follows from (5). \( \blacksquare \)
**Corollary 4.4.** The notation is as in Lemma 4.3. Then

\[
\left( \tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n \right)(f) = T_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n.
\]

**Proof.** The proof is by induction on \( N := p - j. \) If \( N = 0 \) the result is obvious. So let \( N \geq 1. \) Then

\[
\left( \tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n \right)(f)
= \tilde{T}_{n-p+j}(\tilde{T}_{n-p+j+1} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n)(f) \circ f \left( \tilde{T}_{n-p+j+1} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n \right)(f)
= \tilde{T}_{n-p+j}(\tilde{T}_{n-p+j+1} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n)(f)
\]

by the induction hypothesis. Finally observe that the matrix elements of \( \tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n \) depend only on \( X_{n-p+j+1}, \ldots, X_n. \) The result follows immediately from Lemma 4.3 (with \( j + 1 \) instead of \( j). \) \[\blacksquare\]

**5. STABLY TAMENESS**

With Theorem 4.1 we are now able to prove the stably tame generators conjecture for all maps in our class \( H_0(A), \) and, we will also show that this result is “sharp”: We give an example of an element of our class which is not tame, so in general we cannot get a better result than this stable tameness.

First let us recall the conjecture (it has already been mentioned in [1–4] and [8]):

**Conjecture 5.1.** For every invertible polynomial map \( F: k^n \rightarrow k^n \) over a field \( k \) there exist \( t_1, \ldots, t_m \) such that

\[ F^{[m]} = (F, t_1, \ldots, t_m): k^{n+m} \rightarrow k^{n+m} \]

is tame, i.e., \( F \) is stably tame.

**Theorem 5.2.** Let \( F = X + H \) with \( H \in H_0(A). \) Then \( F \) is stably tame.

To do this we use the following result due to Martha Smith [10]:

**Proposition 5.3.** Let \( D \) be a locally nilpotent derivation of \( A[X]. \) Let \( a \in \ker(D). \) Extend \( D \) to \( A[X][t] \) by setting \( D(t) = 0. \) Note that \( tD \) is locally nilpotent. Define \( \rho \in \text{Aut}_A A[X][t] \) by \( \rho(X_i) = X_i, \) \( i = 1, \ldots, n, \) and \( \rho(t) = t + a. \) Then

\[
(\exp(aD), t) = \rho^{-1} \exp(-tD) \rho \exp(tD).
\]
**Corollary 5.4.** Let $D$ and $a$ be as in Proposition 5.3. If $D$ is conjugate by a tame automorphism to a triangular derivation, then $(\exp(aD, t))$ is tame.

**Lemma 5.5.** Let $\tau$ be a nice derivation of order $m$ with respect to $X_1, \ldots, X_n$ and $D := (\partial/\partial X_1, \ldots, \partial/\partial X_n)$ on $A[X]$. Then $\exp(\alpha \tau)$ is stably tame for all $\alpha \in \ker(\tau)$.

**Proof.** We use induction on $m$. Consider the case that $m = 1$. Then $\tau = \sum_{d \in D} b_d d$ with $b_d \in A[X]^D = \cap_{d \in D} \ker(d) = A$. Hence $\tau(X_1) \in A$ and clearly $\tau$ is on triangular form. So now we can apply Corollary 5.4 and find that $\exp(\alpha \tau)$ is stably tame.

Now consider the case $m > 1$. We may assume that for all nice derivations $\sigma \in \text{Der}_A(A[X])$ of order $m - 1$ with respect to $D$ and $X_1, \ldots, X_n$ and for any commutative ring $A$ we have that $\exp(\alpha \sigma)$ is stably tame for all $\alpha \in \ker(\sigma)$. Let $\tau$ be nice of order $m$. Define $\rho$ and extend $\tau$ to $A[X[i]]$ as in Proposition 5.3 (in fact we extend all derivations of $D_i$ to $A[X[i]]$ in this way). Now from

$$(\exp(\alpha \tau), t) = \rho^{-1} \exp(-t \tau) \rho \exp(t \tau)$$

it follows that it suffices to see that $\exp(t \tau)$ is stably tame. Now we see that $t \tau = \sum_{d \in D_{m-1}} t b_d d$ with $t b_d \in A[X[i]]^{D_{m-1}}$. But from this it follows that

$$\exp(t \tau) = \exp\left( \sum_{d \in D_{m-1}} t b_d d \right)$$

$$= \prod_{d \in D_{m-1}} \exp(t b_d d).$$

This last equation follows from Proposition 1.5. Obviously it suffices to prove that each $\exp(t b_d d)$ is stably tame to conclude that $\exp(t \tau)$ is stably tame. But $d$ is a nice derivation of order $m - 1$, $t b_d \in \ker(d)$, and hence we can apply the induction hypothesis to the ring $A[i]$ and find that $\exp(t \tau)$ is stably tame and hence $\exp(\alpha \tau)$ is stably tame. 

**Proof of Theorem 5.2.** Now if we look at Theorem 4.1 we see that each $F = X + H$ with $H \in H_n(A)$ can be written as the product of a finite number of $\exp(a D)_i$, where each $D_i$ is a nice derivation with respect to $X_1, \ldots, X_n$ and $(\partial/\partial X_1, \ldots, \partial/\partial X_n)$ and $a_i \in \ker(D_i)$. Applying Lemma 5.5 $n$ times gives us the desired result: $F$ is stably tame.

**Remark 5.6.** Note that we do not give an indication of the value of $m$ in Conjecture 5.1. As can be seen from the proof above, this $m$ can be very high. At the highest level we have $n \exp(a_i D_i)_i$, but each of these factors can give rise to a great number of extra variables, depending on the “order of niceness” of each $D_i$. 

To conclude this paper we show that in general the automorphisms $F = X + H$ with $H \in H_2(A)$ need not be tame. Actually, this idea was already presented by Nagata [9].

**Example 5.7.** Let $A$ be a domain, but not a principle ideal domain. Let $a, b \in A$ such that $aA + Ab$ is not a principal ideal. Let $f(T) \in A[T]$ with $\deg(f) \geq 2$ and let $F = X + H$ with

$$H = \begin{pmatrix} bf(aX_1 + bX_2) \\ -af(aX_1 + bX_2) \end{pmatrix}$$

Since $H \in H_2(A)$ $F$ is an automorphism of $A[X_1, X_2]$. However, it is shown in [9] that $F$ is not tame.

**References**

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