PDF hosted at the Radboud Repository of the Radboud University Nijmegen

The following full text is a publisher's version.

For additional information about this publication click this link.
http://hdl.handle.net/2066/28453

Please be advised that this information was generated on 2020-01-27 and may be subject to change.
In this paper we introduce a set, denoted by $D_n(A)$, for every commutative ring $A$ and every positive integer $n$. It is shown that the elements of this set can be used to give an explicit description of the class $H_n(A)$ introduced in van den Essen and Hubbers [J. Algebra 187 (1997), 214–226]. We deduce that each polynomial map of the form $F = X + H$ with $H \in H_n(A)$ can be written as a finite product of automorphisms of the form $\exp(D)$, where each $D$ is a locally nilpotent derivation satisfying $D^2(X) = 0$ for all $i$. Furthermore we deduce that all such $F$’s are stably tame.

1. Notation, Definitions, and An Explicit Description of the Class $H_n(A)$

1.1. Notation

Throughout this paper $A$ denotes an arbitrary commutative ring and $A[X] := A[X_1, \ldots, X_n]$ denotes the polynomial ring in $n$ variables over $A$. Furthermore if $G = (G_1, \ldots, G_n) \in A[X]^n$ and $S = (S_i(X)) \in M_{p,q}(A[X])$ then $S(G)$ or $S^G$ denotes the $p \times q$ matrix $(S_i(G_1, \ldots, G_n))_{i,j}$. In particular if $F \in A[X]^n$ (i.e., $M_{n,1}(A[X])$) then the composition of the polynomial maps $F$ and $G$, denoted $F \circ G$, is equal to $F(G)$.

Matrix multiplication will be denoted by the symbol ‘$\cdot$’. So if $S, T \in M_{n,1}(A[X])$ then the matrix product of $S$ and $T$ is denoted by $S \cdot T$. By $X$ we denote the column vector $(X_1, \ldots, X_n)^\top$. In the sequel we also need another multiplication in $M_{n,1}(A[X])$, which we denote by $\triangle$. This multipli-
cation is defined as follows:

\[ S \triangle T := S(T \ast X) \ast T \]

for all \( S, T \in M_n(A[X]) \).

One easily verifies that this multiplication is associative, so it makes sense to write

\[ S_1 \triangle S_2 \triangle \cdots \triangle S_n \]

for each \( n \)-tuple \( S_1, \ldots, S_n \) in \( M_n(A[X]) \). Sometimes we need to extend a vector of length \( 1 \leq p \leq n - 1 \) or a \( p \times p \) matrix to, respectively, a vector of length \( n \), or an \( n \times n \) matrix. This is done as follows: let \( 1 \leq p \leq n - 1 \), \( c \in A[X]^p \), and \( T \in M_p(A[X]) \). Then \( \tilde{c}^n \) denotes the vector

\[ \tilde{c}^n = \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in A[X]^n, \]

obtained by extending \( c \) by \( n - p \) zeros, and \( \tilde{T}^n \) denotes the matrix

\[ \tilde{T}^n = \begin{pmatrix} T & 0 \\ 0 & I_{n-p} \end{pmatrix} \in M_n(A[X]), \]

obtained by extending \( T \) with the \( (n-p) \times (n-p) \) identity matrix. To simplify the notation we drop the superscript \( n \) and write \( \tilde{c} \) and \( \tilde{T} \), even sometimes when it is clear from the context that we mean \( \tilde{c}^{n-1} \), respectively, \( \tilde{T}^{n-1} \) instead of \( \tilde{c}^n \), respectively, \( \tilde{T}^n \).

Finally the adjoint of a matrix \( T \) is denoted by \( \text{Adj}(T) \) and if \( a_1, \ldots, a_p \) are elements of a (nonnecessary commutative) ring then \( \prod_{i=1}^p a_i \) denotes the element \( a_1 \cdots a_p \).

1.2. \( D_n(A) \) and the class \( H_n(A) \)

In [6] we introduced a new class of polynomial maps, denoted by \( H_n(A) \), and showed that for each \( H \in H_n(A) \) the Jacobian matrix \( JH \) is nilpotent and that the polynomial map \( F = X + H \) is invertible over \( A \) with \( \det(JF) = 1 \).

Let us recall the definition of \( H_n(A) \).

**Definition 1.1.** First if \( n = 1 \) we define \( H_1(A) = A \). If \( n \geq 2 \) we define \( H_n(A) \) inductively as follows: Let \( H \in A[X]^n \). Then \( H \in H_n(A) \) if and only if there exist \( T \in M_n(A), c \in A^n \), and \( H_\ast \in H_{n-1}(A[X]) \) such that

\[ H = \text{Adj}(T) \ast \begin{pmatrix} H_\ast \\ 0 \end{pmatrix}_{(T \ast X)} + c. \]
The main aim of this section is to give an explicit description of the elements of $H_n(A)$. Therefore we introduce some useful objects.

**Definition 1.2.** Let $n \geq 2$. Then $D_n(A)$ is the set of $(2n-1)$-tuples

$$(T, c) := (T_2, \ldots, T_n, c_1, \ldots, c_n),$$

where $T_n \in M_n(A)$, $T_i \in M_i(A[X_{i+1}, \ldots, X_n])$ for all $2 \leq i \leq n-1$, $c_n \in A^n$ (i.e., $M_n(A)$) and $c_i \in M_{i, i}(A[X_{i+1}, \ldots, X_n])$ for all $1 \leq i \leq n-1$.

If $n \geq 3$ we get a natural map $\pi : D_n(A) \to D_{n-1}(A[X_n])$ defined by

$$\pi((T_2, \ldots, T_n, c_1, \ldots, c_n)) = (T_2, \ldots, T_{n-1}, c_1, \ldots, c_{n-1}).$$

Instead of $\pi((T, c))$ we often write $(T', c')$.

**Definition 1.3.** Let $n \geq 2$ and $0 \leq p \leq n - 2$. Then

$$E_{n, p} : D_n(A) \to A[X]^n$$

is given by

1. $E_{n, 0}(T, c) := \text{Adj}(T_n)^*_{\eta} c_{n-1}/T_n$ for all $(T, c) \in D_n(A)$.
2. If $n \geq 3$ and $1 \leq p \leq n - 2$, then inductively (with respect to $n$)

$$E_{n, p}(T, c) := \text{Adj}(T_n)^* \left( E_{n-1, p-1}(T', c') \right)_{|T_n \times X},$$

Instead of $E_{n, p}(T, c)$ we simply write $E_{n, p}(T, c)$.

Now we are able to give the main result of this section.

**Proposition 1.4.** Let $n \geq 2$ and $H \in A[X]^n$. Then $H \in H_n(A)$ if and only if there exists $(T, c) \in D_n(A)$ such that

$$H = \sum_{p=0}^{n-2} E_{n, p}(T, c) + c_n.$$

**Proof.** The proof is by induction on $n$. The case $n = 2$ is obvious, so let $n \geq 3$. Then

$$H = \text{Adj}(T_n)^* \left( H^* \right)_{|T_n \times X} + c_n,$$

where $T_n \in M_n(A)$, $c_n \in A^n$, and $H^* \in H_{n-1}(A[X_n])$. So by the induction hypothesis we have

$$H^* = \sum_{p=0}^{n-3} E_{n-1, p}(T^*, c^*) + c_{n-1}.$$
for some \((T^*, c^*) \in D_{n-\{A[X^n]\}}\). Put \((T, c) := (T^*, T_n, c^*, c_n)\) and observe that \((T, c) \in D_\alpha(A)\) and \((T', c') = (T^*, c^*)\). So

\[
H = \sum_{p=0}^{n-3} \text{Adj}(T_n) \ast E_{n-1,p}(T', c')_{|T_n \ast X} + \text{Adj}(T_n) \ast \left( \begin{array}{cc}
          c_n^* \\
          0
        \end{array} \right)_{|T_n \ast X} + c_n
\]

\[
= \sum_{p=1}^{n-2} E_{n,p}(T, c) + E_{n,0}(T, c) + c_n
\]

\[
= \sum_{p=0}^{n-2} E_{n,p}(T, c) + c_n. \quad \Box
\]

**Proposition 1.5.** Let \(n \geq 2, 0 \leq p \leq n - 2\), and \((T, c) \in D_\alpha(A)\). Then

\[E_{n,p}(T, c) = \text{Adj}\left( \tilde{T}_{n-p} \triangledown \cdots \triangledown \tilde{T}_{n-1} \triangledown T_n \right) \ast \tilde{c}_{n-p-1}(\tilde{T}_{n-p} \triangledown \cdots \triangledown \tilde{T}_{n-1} \triangledown T_n)_{|T_n \ast X}.\]

**Proof.** The proof is by induction on \(p\). The case \(p = 0\) is obvious. So let \(p \geq 1\). Then

\[E_{n,p}(T, c) = \text{Adj}(T_n) \ast \left( \begin{array}{cc}
          E_{n-1,p-1}(T', c') \\
          0
        \end{array} \right)_{|T_n \ast X}
\]

\[
= \text{Adj}(T_n) \ast \left[ \text{Adj}(\tilde{T}_{n-p} \triangledown \cdots \triangledown \tilde{T}_{n-1})_{|T_n \ast X}

\ast \left( \tilde{c}_{n-p-1}(\tilde{T}_{n-p} \triangledown \cdots \triangledown \tilde{T}_{n-1})_{|T_n \ast X} \right) \ast T_n_{|T_n \ast X} \right]
\]

(by the induction hypothesis)

\[
= \text{Adj}(\tilde{T}_{n-p} \triangledown \cdots \triangledown \tilde{T}_{n-1})_{|T_n \ast X} \ast \tilde{c}_{n-p-1}(\tilde{T}_{n-p} \triangledown \cdots \triangledown \tilde{T}_{n-1})_{|T_n \ast X} \ast T_n_{|T_n \ast X}
\]

\[\Box\]

**Example 1.6.** Consider the polynomial map \(F := X + H: \mathbb{C}^4 \rightarrow \mathbb{C}^4\), where \(H\) equals

\[
\left|
\begin{array}{c}
X_2X_1^2 - e_4X_3^2X_4 - \frac{m_2}{g_4}X_2X_3X_4 - g_6X_2X_3X_4 - k_4X_3 - \frac{m_4}{g_4}X_2X_3^2 - m_4X_3X_4^2
\end{array}
\right|
\]

\[
\left|
\begin{array}{c}
-X_4X_1^2 - e_5X_3^2X_4 + g_6X_2X_3X_4 - k_3X_4 + m_4X_2X_4^2 + g_4^2X_4
\end{array}
\right|
\]

\[
- \frac{1}{3}X_3^3
\]

\[
0
\]
and $e_3, k_3, e_4, g_4, k_4, m_4 \in \mathbb{C}$ and $g_4 \neq 0$. This $F$ is invertible. In fact if we take $P = P^{-1} = (X_4, X_3, X_2, X_1)$, we have that $PFP$ is one of the eight representatives of the cubic homogeneous maps in dimension 4 as given by Hubbers [7] and also published in [4, Theorem 2.10].

Now consider the following element $(T, c)$ of $D_4(\mathbb{C})$, where

$$T = \left( \begin{array}{ccc}
1 & 0 & 0 \\
g_4^2 X_3 & g_4 X_4 + m_4 X_3 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right), \quad \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{array} \right)$$

and

$$c = \left( \begin{array}{ccc}
-1 & -X_3^2 (e_4 X_4 + k_4 X_3) \\
g_4^2 X_2 & -X_3 X_4^2 - e_4 X_4 X_3^2 - k_3 X_3^3 \\
0 & 0 & -1 \\
0 & 0 & 0
\end{array} \right), \quad \left( \begin{array}{ccc}
0 & 0 \\
0 & 0 \\
0 & 0
\end{array} \right).$$

Our claim is that

$$H = \sum_{p=0}^2 E_{4,p}(T, c) + c_4.$$

To prove this we will compute $E_{4,0}$, $E_{4,1}$, and $E_{4,2}$ by the method of Proposition 1.5. Note that $c_4 = 0$. Since $T_4 = T_3 = I_4$, $E_{4,0}$ and $E_{4,1}$ are easy:

$$E_{4,0} = \text{Adj}(T_4) \ast \tilde{c}_{3 | T_4} \ast X = \tilde{c}_3 = \left( \begin{array}{c}
0 \\
0 \\
-1 \\
3 X_4^2
\end{array} \right).$$

$$E_{4,1} = \text{Adj}(T_3 \triangle T_4) \ast \tilde{c}_{2 | (T_3 \triangle T_4)} \ast X = \tilde{c}_2 = \left( \begin{array}{c}
-X_3^2 (e_4 X_4 + k_4 X_3) \\
-X_3 X_4^2 - e_4 X_4 X_3^2 - k_3 X_3^3 \\
0 \\
0
\end{array} \right).$$
Before we compute $E_{4,2}$ we present the following identities:

\[ \tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4 = \tilde{T}_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
g_2^2X_3 & g_4X_4 + m_4X_3 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \]

\[ \text{Adj}(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4) \]

\[ = \begin{pmatrix}
g_4X_4 + m_4X_3 & 0 & 0 & 0 \\
-g_2^2X_3 & 1 & 0 & 0 \\
0 & 0 & g_4X_4 + m_4X_3 & 0 \\
0 & 0 & 0 & g_4X_4 + m_4X_3
\end{pmatrix}; \]

\[ (\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4)^* X = \begin{pmatrix}
X_1 \\
g_2^2X_1X_3 + g_4X_2X_4 + m_4X_2X_3 \\
X_3 \\
X_4
\end{pmatrix}; \]

\[ \tilde{c}_{1(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4)^*} X = \begin{pmatrix}
-X_1X_3 - \frac{1}{g_4}X_2X_4 - \frac{m_4}{g_4}X_2X_3 \\
0 \\
0 \\
0
\end{pmatrix}; \]

and finally

\[ E_{4,2} = \text{Adj}(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4)^* \tilde{c}_{1(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4)^*} X \]

\[ = \begin{pmatrix}
-X_4 + \frac{m_4}{g_4}X_3 \left( g_4X_1X_3 + X_2X_4 + \frac{m_4}{g_4}X_2X_3 \right) \\
X_3 \left( g_2^2X_1X_3 + g_4X_2X_4 + m_4X_2X_3 \right) \\
0 \\
0
\end{pmatrix}. \]

It is easy to verify that $H = E_{4,0} + E_{4,1} + E_{4,2} + c_{4}$, which was our claim.

2. NICE DERIVATIONS

Let $B := A[x_1, \ldots, x_n]$ be a finitely generated $A$-algebra and $D$ a subset of $\text{Der}_A(B)$. By $B^D$ we denote the set of all $b \in B$ such that $d(b) = 0$ for all $d \in D$. 
**Definition 2.1.** Let $D \subseteq \text{Der}_A(B)$ be a finite subset and $\tau \in \text{Der}_A(B)$.

1. We say that $\tau$ is derived from $D$ in at most one step if $\tau$ is of the form $\tau = \sum_{d \in D} b_d d$, where $b_d \in B^D$ for all $d \in D$.

2. Let $m \geq 2$. We say that $\tau$ is derived from $D$ in at most $m$ steps if there exists a sequence of finite subsets $D = D_0, D_1, D_2, \ldots, D_m$ of $\text{Der}_A(B)$ such that $\tau \in D_m$ and all elements of $D_i$ are derived from $D_{i-1}$ in at most one step, for all $1 \leq i \leq m$. If furthermore the elements of $D$ satisfy $d_1 d_2(x_i) = 0$ for all $d_1, d_2 \in D$ and all $i$, then $\tau$ is called nice of order $\leq m$, with respect to $x_1, \ldots, x_n$ and $D$.

**Proposition 2.2.** The notation is as in Definition 2.1. If $d_1 d_2(x_i) = 0$ for all $d_1, d_2 \in D$ and all $i$, then $d_1 d_2(x_i) = 0$ for all $d_1, d_2 \in D_m$ and all $i$. In particular $d^2(x_i) = 0$ for every nice derivation.

**Proof.** We use induction on $m$. The case $m = 0$ is obvious since $D_0 = D$. Now let $m \geq 1$. Then $d_1 = \sum_{d \in D_{m-1}} b_d d$, $d_2 = \sum_{d' \in D_{m-1}} b'_d d'$ with $b'_d, b_d \in B^{D_{m-1}}$. Then

$$d_1 d_2(x_i) = \sum_{d, d'} bd(b'_d) d(x_i) + \sum_{d, d'} b_d b'_d, dd'(x_i).$$

(2)

Now observe that $d(b'_d) = 0$ since $b'_d \in B^{D_{m-1}}$ and $d \in D_{m-1}$. Finally the induction hypothesis gives $dd'(x_i) = 0$ for all $d, d' \in D_{m-1}$ and all $i$, so (2) implies $d_1 d_2(x_i) = 0$.

We demonstrate these aspects by the so-called Winkelmann derivation. See [11].

**Example 2.3.** Let $\tau = (1 + X_4 X_2 - X_5 X_3) \partial_{X_4} + X_5 \partial_{X_4} + X_4 \partial_{X_5}$, a derivation on $B := A[X_1, X_2, X_3, X_4, X_5]$. Let $D = \{ \partial_{X_1}, \partial_{X_2}, \partial_{X_3}, \partial_{X_4}, \partial_{X_5} \}$. Then $\tau$ is nice of order 2 with respect to $X_1, X_2, X_3, X_4, X_5$, and $D$. To show that this is true, we present a sequence of finite subsets of $\text{Der}_A(B)$,

$$D = D_0, D_1, D_2.$$

Take $D_1 := \{ \partial_{X_1}, X_5 \partial_{X_2} + X_4 \partial_{X_3} \}$ and $D_2 := \{ \tau \}$. Note that in Definition
2.1 it is not demanded that the set \( D_i \) of this sequence is a subset of \( D_{i+1} \). The only demand is that each \( D_i \) is a finite subset of \( \text{Der}_A(B) \). Since \( X_4, X_5 \in B^D \) it follows immediately that \( \partial X_4 \) and \( X_2 \partial X_2 + X_4 \partial X_1 \) are derived from \( D \) in one step. And from \( 1 + X_4 X_2 - X_2 X_3 \in B^D \) it follows that \( \tau \) is derived from \( D_1 \) in one step. Obviously we have \( d_1 d_2 (X_i) = 0 \) for all \( d_1, d_2 \in D \) and hence with Proposition 2.2 also \( \tau^2(X_i) = 0 \).

3. DERIVATIONS ASSOCIATED WITH POLYNOMIAL MAPS

The main aim of this section is to show that for each \( 0 \leq p \leq n - 2 \) the polynomial map \( X + E_{n,p}(T, c) \) (where \( (T, c) \in D_n(A) \)) is of the form \( \exp(d) \), for some nice \( A \)-derivation \( d \) of \( A[X] \). Observe that \( d \) is locally nilpotent if \( d \) is nice with respect to \( X_1, \ldots, X_n \), since \( d^2(X_i) = 0 \) for all \( i \), by Proposition 2.2.

In order to prove this result (see Theorem 3.3), we need to generalise some of the notions of Sect. 1 to arbitrary finitely generated \( A \)-algebras. So let \( B := A[x_1, \ldots, x_n] \) be a finitely generated \( A \)-algebra, and let \( \varphi: A[X_1, \ldots, X_n] \to B \) be the \( A \)-ring homomorphism defined by \( \varphi(X_i) = x_i \) for all \( i \). For each \( p, q \geq 1 \) consider the natural extension

\[
\varphi: M_{p,q}(A[X_1, \ldots, X_n]) \to M_{p,q}(B).
\]

Then for each \( (T, c) \in D_n(A) \) we define

\[
E_{n,p}(T, c)(x) := \varphi(E_{n,p}(T, c)) \in B^n.
\]

Now let \( (\partial_1, \ldots, \partial_n) \) be an \( n \)-tuple of \( A \)-derivations of \( B \). With each vector \( b = (b_1, \ldots, b_n)^T \in B^n \) we associate the following \( A \)-derivation of \( B \):

\[
D(b; \partial_1, \ldots, \partial_n) := b_1 \partial_1 + \cdots + b_n \partial_n = b^T \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix}.
\]

To formulate the next lemma we need some more notation: Let \( (T, c) \in D_n(A) \). Put

\[
(x_1', \ldots, x_n') := T_n \ast (x_1, \ldots, x_n),
\]

\[
(\partial_1', \ldots, \partial_n') := \text{adj}(T_n)^\dagger \ast (\partial_1, \ldots, \partial_n),
\]

\[
x'' := (x_1', \ldots, x'_{n-1}),
\]

\[
(T'', c'') := (T'(X_n = x_n'), c'(X_n = x_n')) \in D_{n-1}(A[x_n']).
\]
Lemma 3.1. Let \( n \geq 3 \) and \( 1 \leq p \leq n - 2 \). Then

\[
D\left( E_{n,p}(T, c)(x); \partial_1, \ldots, \partial_n \right) = D\left( E_{n-1,p-1}(T''', c''')(x'''); \partial'_1, \ldots, \partial'_{n-1} \right).
\]

Proof.

\[
\begin{align*}
D\left( E_{n,p}(T, c)(x); \partial_1, \ldots, \partial_n \right) & = \left( E_{n,p}(T, c)(x) \right)^{\prime} \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \\
& = \left( \left( E_{n-1,p-1}(T', c')|_{T_{n-1} \times x} \right)^{\prime} 0 \right) \ast \left( \text{Adj}(T_n) \right)^{\prime} \begin{pmatrix} \partial_2 \\ \vdots \\ \partial_n \end{pmatrix} \\
& = \left( \left( E_{n-1,p-1}(T'', c'')(x'') \right)^{\prime} 0 \right) \begin{pmatrix} \partial'_1 \\ \vdots \\ \partial'_{n-1} \end{pmatrix} \\
& = D\left( E_{n-1,p-1}(T'', c'')(x''); \partial'_1, \ldots, \partial'_{n-1} \right).
\end{align*}
\]

Lemma 3.2. The notation is as above. Let \( a \in A \) and let \( \partial_1, \ldots, \partial_n \) be \( A \)-derivations of \( B \) such that \( \partial_i(x_j) = a \delta_{ij} \) for all \( i, j \). Then

\[
\partial'_i(x'_j) = a \det(T_n) \delta_{ij}
\]

for all \( i, j \).

Proof. Denote the \( i \)-th column of \( \text{Adj}(T_n) \) by \((t_{1i}, \ldots, t_{ni})^{\prime}\) and the \( j \)-th row of \( T_n \) by \((t_{1j}, \ldots, t_{nj})\). Then

\[
\partial'_i(x'_j) = \sum_{s=1}^{n} t_{si}^{u} \delta_{s} \left( \sum_{s=1}^{n} t_{js} x_{s} \right) \\
= \sum_{s=1}^{n} at_{si} t_{js} \\
= a(T_n \ast \text{Adj}(T_n))_{ji} \\
= a \det(T_n) \delta_{ij}.
\]
Now we are able to prove:

**Theorem 3.3.** Let \( \partial_1, \ldots, \partial_n \) be \( A \)-derivations on \( A[x_1, \ldots, x_n] \) such that there exists an element \( a \in A \) such that \( \partial_i(x) = a \delta_j \) for all \( i, j \). Let \( (T, c) \in \mathbb{D}_n(A) \). Then the \( A \)-derivation \( d := D(E_n, p)(T, c)(x); \partial_1, \ldots, \partial_n \) is nice with respect to \( x_1, \ldots, x_n \) and \( D_0 := \{ \partial_1, \ldots, \partial_n \} \), for all \( n \geq 2 \) and all \( 0 \leq p \leq n - 2 \).

**Proof.**
1. The hypotheses on the \( \partial_i \) imply that \( dd'(x_i) = 0 \) for all \( d, d' \in D_0 \) and all \( i \).

2. First we consider the case \( p = 0 \). Then
\[
E_n, 0(T, c) = A d(T_n) \circ c_{n-1_{T_n, x}}.
\]
So
\[
d = (c_{n-1_{T_n, x}}) \circ (A d(T_n))^{t} = \begin{pmatrix}
\partial_1 \\
\vdots \\
\partial_n
\end{pmatrix}.
\]

Write \( c_{n-1} = (\gamma(X_n), \ldots, \gamma_{n-1}(X_n), 0) \). Then the definition of \( x_n \) and the \( \partial_i \) imply that
\[
d = (\gamma(x_n'), \ldots, \gamma_{n-1}(x_n'), 0) \circ (\partial_1, \ldots, \partial_n)^t = \sum_{i=1}^{n-1} \gamma_i(x_n') \partial_i'.
\]

Put \( D_1 := \{ \partial_1', \ldots, \partial_{n-1}' \} \) and observe that \( D_1 \subset \text{Der}_n(B) \) and that each element of \( D_1 \) is derived from \( D_0 \) in at most one step. Finally since \( \partial_i'(x_n') = 0 \) for all \( 1 \leq i \leq n - 1 \) (by Lemma 3.2) we get that \( \gamma(x_n') \in B^D_1 \) for all \( 1 \leq i \leq n - 1 \). So (3) implies that \( d \) is derived from \( D_1 \) in at most one step. Consequently \( d \) is derived from \( D_0 \) in at most two steps. So \( d \) is nice with respect to \( x_1, \ldots, x_n \) and \( D_0 \) by case 1.

3. Now we prove the theorem by induction on \( n \). If \( n = 2 \), then \( p = 0 \) and we are in case 2. So let \( n \geq 3 \). By case 2 we may assume that \( p \geq 1 \). Then by Lemma 3.1 we have
\[
d = D(E_{n-1}, p-1(T^{''}, c^{''})(x^{''}); \partial_1, \ldots, \partial_{n-1})
\]
with \( (T'', c'') \in \mathbb{D}_{n-1}(A[x'_n]) \). By Lemma 3.2 we can apply the induction hypothesis to the ring \( A[x'_n] \) and the \( (n - 1) \)-tuple of \( A[x'_n] \)-derivations \( \partial_1, \ldots, \partial_{n-1} \) on the \( A[x'_n] \)-algebra \( B' := A[x'_n][x'_1, \ldots, x'_{n-1}] \). So the \( A[x'_n] \)-derivation \( d \) on \( B' \) is nice with respect to \( D'_0 := \{ \partial_1', \ldots, \partial_{n-1}' \} \) and \( x'_1, \ldots, x'_{n-1} \). So there exists a sequence
\[
D'_0, D'_1, \ldots, D'_m
\]
of finite subsets of $\text{Der}_{A_{x_i}(B')} \{B'\}$ such that $d \in D'_i$ and $D'_i$ is derived from $D'_i$ in at most one step for all $1 \leq i \leq m$. Now observe that $D'_0 \subset \text{Der}_A(B)$ and that $B' \subset B$ since by definition obviously $x'_i \in B$ for all $i$. Consequently if $d'$ is an $A[x'_i]$-derivation of $B'$ derived from $D'_0$ in at most one step, then $d' \in \text{Der}_A(B)$. Hence $D'_i \subset \text{Der}_A(B)$. Arguing in a similar way we conclude by induction on $i$ that $D'_i \subset \text{Der}_A(B)$ for all $0 \leq i \leq m$. Since as remarked in case 2 above, all elements of $D'_0$ (= $D_1$ in case 2) are derived from $D_0$ in at most one step we deduce that $d$ is derived from $D_0$ in at most $m + 1$ steps. Just define $D_i := D'_{i-1}$ for all $1 \leq i \leq m + 1$. Hence $d$ is nice with respect to $x_1, \ldots, x_n$ and $D_0$ by 1. 

**Corollary 3.4.** Let $(T, c) \in D_n(A)$ and $0 \leq p \leq n - 2$. Put

$$D := D \left( E_{n, p}(T, c); \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n} \right).$$

Then $D$ is nice with respect to $X_1, \ldots, X_n$ and $(\partial/\partial X_1, \ldots, \partial/\partial X_n)$. Furthermore we have $\exp(D) = X + E_{n, p}(T, c)$ and the inverse map is given by $\exp(-D) = X - E_{n, p}(T, c)$.

**Proof.** The first part is an immediate consequence of Theorem 3.3. Furthermore $D^2(X) = 0$ by Proposition 2.2. So $\exp(D)(X) = X + E_{n, p}(T, c)$ and the inverse map is given by $\exp(-D)(X) = X - E_{n, p}(T, c)$.

**4. THE MAIN THEOREM**

In this section we show that for every $H \in H_n(A)$ the polynomial map $F = X + H$ is a product of $n$ polynomial automorphisms of the form $\exp(D)$, where each $D$ is a nice derivation on $A[X]$. More precisely

**Theorem 4.1.** Let $F = X + H$, where $H = \sum_{p = 0}^{n-2} E_{n, p}(T, c) + c_n$, for some $(T, c) \in D_n(A)$. Then

$$F = \exp \left( D \left( c_n; \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n} \right) \right) \times \prod_{p = 0}^{n-2} \exp \left( D \left( E_{n, p}(T, c); \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n} \right) \right).$$
Proof. Observe that
\[
\exp\left(-D\left(c_n; \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}\right)\right) \circ F = \sum_{p=0}^{n-2} E_{n,p}(T, c).
\]
So the case \(n = 2\) follows from Corollary 3.4. Hence we may assume that \(n \geq 3\). Now Theorem 4.1 follows directly from Proposition 4.2 below and Corollary 3.4.

**Proposition 4.2.** Let \(n \geq 3\), \(0 \leq p \leq n - 3\), and \((T, c) \in D_n(A)\). Then
\[
\exp(-D(E_{n,p}(T, c))) \circ \left(X + \sum_{q=p}^{n-2} E_{n,q}(T, c)\right) = X + \sum_{q=p+1}^{n-2} E_{n,q}(T, c).
\]

**Proof.** Put \(G := \exp(-D(E_{n,p}(T, c)))\). So \(G = X - E_{n,p}(T, c)\) (by Corollary 3.4). Hence if we put
\[
U := \tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n
\]
then by Proposition 1.4 we get
\[
G = X - \text{Adj}(U) \ast \tilde{c}_{n-p-1|U \ast X}.
\]
So if we put
\[
f := X + \sum_{q=p}^{n-2} E_{n,q}(T, c)
\]
then
\[
G \circ f = f - \text{Adj}(U(f)) \ast \tilde{c}_{n-p-1|U(f) \ast f}.
\]
Since \(U(f) = f\) (by Corollary 4.4 below, with \(j = 0\)) we get
\[
G \circ f = f - \text{Adj}(U) \ast \tilde{c}_{n-p-1|U \ast f}.
\]
Now observe that each component of \(\tilde{c}_{n-p-1} \ast f\) belongs to \(A[X_{n-p}, \ldots, X_n]\) and that for each \(i \geq n - p\) \((U \ast f)_i = (U \ast X)_i\) (by Lemma 4.3 below). So \(\tilde{c}_{n-p-1|U \ast X} = \tilde{c}_{n-p-1|U \ast X}\) and hence
\[
G \circ f = f - \text{Adj}(U) \ast \tilde{c}_{n-p-1|U \ast X}
\]
\[
= f - E_{n,p}(T, c)
\]
(by Proposition 1.4) \(\blacksquare\).
**Lemma 4.3.** Let $n \geq 3$, $0 \leq p \leq n - 2$, $0 \leq j \leq p$, and $(T, c) \in \mathcal{D}_s(A)$. Put $f := X + \sum_{q=p}^{n-2} E_{n,q}(T, c)$. Then

\[
\left[\left(\tilde{T}_{n+p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n\right) * f\right]_i = \left[\left(\tilde{T}_{n+p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n\right) * X\right]_i
\]

for all $i \geq n - p + j$.

**Proof.** Put $U := \tilde{T}_{n+p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n$. It suffices to show that for each $q \geq p$

\[
[U * E_{n,q}(T, c)]_i = 0
\]

for all $i \geq n - p + j$. So let $q \geq p$. Then $q \geq p + j$.

1. We first treat the case that $q = p - j$. Then $j = 0$ and $q = p$. Consequently $U = \tilde{T}_{n+p} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n$, $E_{n,q}(T, c) = E_{n,p}(T, c)$, and hence by Proposition 1.4

\[
U * E_{n,q}(T, c) = U * A \text{adj}(U) * c_{n-p-1(U*X)}
\]

\[
= \det(U) * c_{n-p-1(U*X)}.
\]

Since the last $p + 1$ coordinates of $c_{n-p-1}$ are zero, we obtain that

\[
[U * E_{n,q}(T, c)]_i = 0
\]

for all $i \geq n - p$, which proves the case that $q = p - j$.

2. Now assume that $q \geq p - j + 1$. So $n - q \leq n - p + j - 1$. Put $V := \tilde{T}_{n-q} \triangle \cdots \triangle \tilde{T}_{n-p+j-1}$. Then by Proposition 1.4 we can write

\[
E_{n,q}(T, c) = A \text{adj}(V \triangle U) * c_{n-q-1(V \triangle U)*X}
\]

\[
= A \text{adj}(V_{U*X} * U) * c_{n-q-1(V \triangle U)*X}
\]

\[
= A \text{adj}(U) * A \text{adj}(V_{U*X}) * c_{n-q-1(V \triangle U)*X}.
\]

Consequently

\[
U * E_{n,q}(T, c) = \det(U) * A \text{adj}(V_{U*X}) * c_{n-q-1(V \triangle U)*X}. \tag{5}
\]

Note that $V$, and hence $V_{U*X}$, is of the form $\tilde{B}$ for some $B \in M_{n-p+j-1}(A[X])$. Furthermore $(c_{n-q-1})_i = 0$ if $i \geq n - q$, which implies that $(c_{n-q-1(V \triangle U)*X})_i = 0$ if $i \geq n - p + j$ (since $n - p + j > n - q$). Now the desired result (4) follows from (5). \qed
**Corollary 4.4.** The notation is as in Lemma 4.3. Then

\[
\left(\tilde{T}_{n-p+j} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n\right)(f) = \tilde{T}_{n-p+j} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n.
\]

*Proof.* The proof is by induction on \(N := p - j\). If \(N = 0\) the result is obvious. So let \(N \geq 1\). Then

\[
\left(\tilde{T}_{n-p+j} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n\right)(f) = \tilde{T}_{n-p+j}(\tilde{T}_{n-p+j+1} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n)(f) * (\tilde{T}_{n-p+j+1} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n)(f) = \tilde{T}_{n-p+j}(\tilde{T}_{n-p+j+1} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n)(f)
\]

by the induction hypothesis. Finally observe that the matrix elements of \(\tilde{T}_{n-p+j}\) depend only on \(X, \ldots, X_n\). The result follows immediately from Lemma 4.3 (with \(j + 1\) instead of \(j\)).

---

**5. STABLY TAMENESS**

With Theorem 4.1 we are now able to prove the stably tame generators conjecture for all maps in our class \(H_n(A)\), and, we will also show that this result is “sharp”: We give an example of an element of our class which is not tame, so in general we cannot get a better result than this stable tameness.

First let us recall the conjecture (it has already been mentioned in \([1-4]\) and \([8]\)):

**Conjecture 5.1.** For every invertible polynomial map \(F: k^n \to k^n\) over a field \(k\) there exist \(t_1, \ldots, t_m\) such that

\[
F^{[m]} = (F_t, t_1, \ldots, t_m): k^{n+m} \to k^{n+m}
\]

is tame, i.e., \(F\) is stably tame.

**Theorem 5.2.** Let \(F = X + H\) with \(H \in H_n(A)\). Then \(F\) is stably tame.

To do this we use the following result due to Martha Smith \([10]\):

**Proposition 5.3.** Let \(D\) be a locally nilpotent derivation of \(A[X]\). Let \(a \in \ker(D)\). Extend \(D\) to \(A[X][t]\) by setting \(D(t) = 0\). Note that \(tD\) is locally nilpotent. Define \(\rho \in \text{Aut}_A A[X][t]\) by \(\rho(X_i) = X_i, i = 1, \ldots, n\), and \(\rho(t) = t + a\). Then

\[
(\exp(aD), t) = \rho^{-1} \exp(-tD) \rho \exp(tD).
\]
Corollary 5.4. Let $D$ and $a$ be as in Proposition 5.3 If $D$ is conjugate by a tame automorphism to a triangular derivation, then $(\exp(aD, t))$ is tame.

Lemma 5.5. Let $\tau$ be a nice derivation of order $m$ with respect to $X_1, \ldots, X_n$ and $D := (\partial/\partial X_1, \ldots, \partial/\partial X_n)$ on $A[X]$. Then $\exp(\alpha\tau)$ is stably tame for all $\alpha \in \ker(\tau)$.

Proof. We use induction on $m$. Consider the case that $m = 1$. Then

$$\tau = \sum_{d \in D} b_d d$$

with $b_d \in A[X]^D = \cap_{d \in D} \ker(d) = A$. Hence $\tau(X_i) \in A$ and clearly $\tau$ is on triangular form. So now we can apply Corollary 5.4 and find that $\exp(\alpha\tau)$ is stably tame.

Now consider the case $m > 1$. We may assume that for all nice derivations $\sigma \in \Der(A[X])$ of order $m - 1$ with respect to $D$ and $X_1, \ldots, X_n$ and for any commutative ring $A$ we have that $\exp(\alpha\sigma)$ is stably tame for all $\alpha \in \ker(\sigma)$. Let $\tau$ be nice of order $m$. Define $\rho$ and extend $\tau$ to $\Der(A[X][t])$ as in Proposition 5.3 (in fact we extend all derivations of $D_i$ to $A[X][t]$ in this way). Now from

$$(\exp(\alpha\tau), t) = \rho^{-1} \exp(-t\tau) \rho \exp(t\tau)$$

it follows that it suffices to see that $\exp(t\tau)$ is stably tame. Now we see that

$$t\tau = \sum_{d \in D_{m-1}} tb_d d$$

with $tb_d \in A[X][t]^{D_{m-1}}$. But from this it follows that

$$\exp(t\tau) = \exp\left(\sum_{d \in D_{m-1}} tb_d d\right)$$

$$= \prod_{d \in D_{m-1}} \exp(tb_d d).$$

This last equation follows from Proposition 1.5. Obviously it suffices to prove that each $\exp(tb_d d)$ is stably tame to conclude that $\exp(t\tau)$ is stably tame. But $d$ is a nice derivation of order $m - 1$, $tb_d \in \ker(d)$, and hence we can apply the induction hypothesis to the ring $A[t]$ and find that $\exp(t\tau)$ is stably tame and hence $\exp(\alpha\tau)$ is stably tame.

Proof of Theorem 5.2. Now if we look at Theorem 4.1 we see that each $F = X + H$ with $H \in H_0(A)$ can be written as the product of a finite number of $\exp(a_iD_i)s$, where each $D_i$ is a nice derivation with respect to $X_1, \ldots, X_n$ and $(\partial/\partial X_1, \ldots, \partial/\partial X_n)$ and $a_i \in \ker(D_i)$. Applying Lemma 5.5 $n$ times gives us the desired result: $F$ is stably tame.

Remark 5.6. Note that we do not give an indication of the value of $m$ in Conjecture 5.1. As can be seen from the proof above, this $m$ can be very high. At the highest level we have $n \exp(a_iD_i)s$, but each of these factors can give rise to a great number of extra variables, depending on the “order of niceness” of each $D_i$. 


To conclude this paper we show that in general the automorphisms $F = X + H$ with $H \in H_2(A)$ need not be tame. Actually, this idea was already presented by Nagata [9].

**Example 5.7.** Let $A$ be a domain, but not a principle ideal domain. Let $a, b \in A$ such that $Aa + Ab$ is not a principal ideal. Let $f(T) \in A[T]$ with $\deg(f) \geq 2$ and let $F = X + H$ with

$$H = \begin{pmatrix} bf(aX_1 + bX_2) \\ -af(aX_1 + bX_2) \end{pmatrix}$$

Since $H \in H_2(A)$ $F$ is an automorphism of $A[X_1, X_2]$. However, it is shown in [9] that $F$ is not tame.

**References**

2. L. M. Drużkowski, The Jacobian conjecture, Preprint 492, Institute of Mathematics, Polish Academy of Sciences, IM PAN, Sniadeckich 8, P.O. Box 137, 00-950 Warsaw, Poland, 1991.