D_n(A) for a Class of Polynomial Automorphisms and Stably Tameness

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In this paper we introduce a set, denoted by D_n(A), for every commutative ring A and every positive integer n. It is shown that the elements of this set can be used to give an explicit description of the class H_n(A) introduced in van den Essen and Hubbers [J. Algebra 187 (1997), 214–226]. We deduce that each polynomial map of the form F = X + H with H ∈ H_n(A) can be written as a finite product of automorphisms of the form exp(D), where each D is a locally nilpotent derivation satisfying D^i(X) = 0 for all i. Furthermore we deduce that all such F's are stably tame.

1. NOTATION, DEFINITIONS, AND AN EXPLICIT DESCRIPTION OF THE CLASS H_n(A)

1.1. Notation

Throughout this paper A denotes an arbitrary commutative ring and A[X] := A[X_1, ..., X_n] denotes the polynomial ring in n variables over A. Furthermore if G = (G_1, ..., G_n) ∈ A[X]^n and S = (S_i(X)) ∈ M_p,q(A[X]) then S(G) or S|_G denotes the p × q matrix (S_i(G_1, ..., G_n)). In particular if F ∈ A[X]^n (M_n(A[X])) then the composition of the polynomial maps F and G, denoted F ∘ G, is equal to F(G).

Matrix multiplication will be denoted by the symbol '⋆'. So if S, T ∈ M_n(A[X]) then the matrix product of S and T is denoted by S ⊙ T. By X we denote the column vector (X_1, ..., X_n). In the sequel we also need another multiplication in M_n(A[X]), which we denote by △. This multipli-
cation is defined as follows:
\[ S \triangle T := S(T * X) * T \]
for all \( S, T \in M_n(A[X]) \).

One easily verifies that this multiplication is associative, so it makes sense to write
\[ S_1 \triangle S_2 \triangle \cdots \triangle S_n \]
for each \( n \)-tuple \( S_1, \ldots, S_n \) in \( M_n(A[X]) \). Sometimes we need to extend a vector of length \( 1 \leq p \leq n - 1 \) or a \( p \times p \) matrix to, respectively, a vector of length \( n \), or an \( n \times n \) matrix. This is done as follows: let \( 1 \leq p \leq n - 1 \), \( c \in A[X]^p \), and \( T \in M_p(A[X]) \). Then \( \vec{c}^n \) denotes the vector
\[ \vec{c}^n = \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in A[X]^n, \]
obtained by extending \( c \) by \( n - p \) zeros, and \( \vec{T}^n \) denotes the matrix
\[ \vec{T}^n = \begin{pmatrix} T & 0 \\ 0 & I_{n-p} \end{pmatrix} \in M_n(A[X]), \]
obtained by extending \( T \) with the \((n-p)\times(n-p)\) identity matrix. To simplify the notation we drop the superscript \( n \) and write \( \vec{c}^n \) and \( \vec{T}^n \), even sometimes when it is clear from the context that we mean \( \vec{c}^{n-1} \), respectively, \( \vec{T}^{n-1} \) instead of \( \vec{c}^n \), respectively, \( \vec{T}^n \).

Finally the adjoint of a matrix \( T \) is denoted by \( \text{Adj}(T) \) and if \( a_1, \ldots, a_p \) are elements of a (nonnecessary commutative) ring then \( \prod_{i=1}^{p} a_i \) denotes the element \( a_1 \cdots a_p \).

1.2. \( D_n(A) \) and the class \( H_n(A) \)

In [6] we introduced a new class of polynomial maps, denoted by \( H_n(A) \), and showed that for each \( H \in H_n(A) \) the jacobian matrix \( JH \) is nilpotent and that the polynomial map \( F = X + H \) is invertible over \( A \) with \( \det(JF) = 1 \).

Let us recall the definition of \( H_n(A) \).

**Definition 1.1.** First if \( n = 1 \) we define \( H_1(A) = A \). If \( n \geq 2 \) we define \( H_n(A) \) inductively as follows: Let \( H \in A[X]^n \). Then \( H \in H_n(A) \) if and only if there exist \( T \in M_n(A) \), \( c \in A^n \), and \( H_* \in H_{n-1}(A[X]) \) such that
\[ H = \text{Adj}(T) * \begin{pmatrix} H_* \\ 0 \end{pmatrix}_{T * X} + c. \quad (1) \]
The main aim of this section is to give an explicit description of the elements of $H_n(A)$. Therefore we introduce some useful objects.

**Definition 1.2.** Let $n \geq 2$. Then $D_n(A)$ is the set of $(2n-1)$-tuples $(T, c) := (T_2, \ldots, T_n, c_1, \ldots, c_n)$, where $T_i \in M_i(A)$, $T_i \in M_i(A[X_{i+1}, \ldots, X_n])$ for all $2 \leq i \leq n-1$, $c_i \in A^a$ and $c_i \in M_i(A[X_{i+1}, \ldots, X_n])$ for all $1 \leq i \leq n-1$.

If $n \geq 3$ we get a natural map $\pi: D_n(A) \to D_{n-1}(A[X_n])$ defined by

$$\pi((T_2, \ldots, T_n, c_1, \ldots, c_n)) = (T_2, \ldots, T_{n-1}, c_1, \ldots, c_{n-1}).$$

Instead of $\pi((T, c))$ we often write $(T', c')$.

**Definition 1.3.** Let $n \geq 2$ and $0 \leq p \leq n - 2$. Then

$$E_{n,p}: D_n(A) \to A[X]^n$$

is given by

1. $E_{n,0}((T, c)) := \text{Adj}(T_n) * c_{n-1, T_n} X$ for all $(T, c) \in D_n(A)$.
2. If $n \geq 3$ and $1 \leq p \leq n - 2$, then inductively (with respect to $n$)

$$E_{n,p}((T, c)) := \text{Adj}(T_n) * \left( \begin{array}{c} E_{n-1,p-1}(T', c') \\ 0 \end{array} \right)_{|T_n \times X},$$

Instead of $E_{n,p}((T, c))$ we simply write $E_{n,p}(T, c)$.

Now we are able to give the main result of this section.

**Proposition 1.4.** Let $n \geq 2$ and $H \in A[X]^n$. Then $H \in H_n(A)$ if and only if there exists $(T, c) \in D_n(A)$ such that

$$H = \sum_{p=0}^{n-2} E_{n,p}(T, c) + c_n.$$

**Proof.** The proof is by induction on $n$. The case $n = 2$ is obvious, so let $n \geq 3$. Then

$$H = \text{Adj}(T_n) \left( \begin{array}{c} H_* \\ 0 \end{array} \right)_{|T_n \times X} + c_n,$$

where $T_n \in M_n(A)$, $c_n \in A^n$, and $H_* \in H_{n-1}(A[X_n])$. So by the induction hypothesis we have

$$H_* = \sum_{p=0}^{n-3} E_{n-1,p}(T^*, c^*) + c_{n-1}.$$
for some \((T^*,c^*) \in D_{n-1}(A[X])\). Put \((T,c) := (T^*,T_n,c^*,c_n)\) and observe that \((T,c) \in D_n(A)\) and \((T',c') = (T^*,c^*)\). So

\[
H = \sum_{p=0}^{n-3} \text{Adj}(T_n) * E_{n-1,p}(T',c')_{|T_n} * X + \text{Adj}(T_n) * \left( \begin{array}{c} c_n^* \\ 0 \end{array} \right)_{|T_n} * X + c_n
\]

\[
= \sum_{p=1}^{n-2} E_{n,p}(T,c) + E_{n,0}(T,c) + c_n
\]

\[
= \sum_{p=0}^{n-2} E_{n,p}(T,c) + c_n.
\]

**Proposition 1.5.** Let \(n \geq 2\), \(0 \leq p \leq n - 2\), and \((T,c) \in D_n(A)\). Then

\[
E_{n,p}(T,c) = \text{Adj}(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n) * \tilde{c}_{n-p-1}(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n) * X.
\]

**Proof.** The proof is by induction on \(p\). The case \(p = 0\) is obvious. So let \(p \geq 1\). Then

\[
E_{n,p}(T,c) = \text{Adj}(T_n) * \left( \begin{array}{c} E_{n-1,p-1}(T',c') \\ 0 \end{array} \right)_{|T_n} * X
\]

\[
= \text{Adj}(T_n) * \left[ \text{Adj}(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1})_{|T_n} * X
\]

\[
* \left( \tilde{c}_{n-p-1}(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1})_{|T_n} * X \right)_{|T_n} * X
\]

(by the induction hypothesis)

\[
= \text{Adj}(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1})_{|T_n} * T_n
\]

\[
* \tilde{c}_{n-p-1}(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1})_{|T_n} * T_n * X
\]

\[
= \text{Adj}(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n) * \tilde{c}_{n-p-1}(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n) * X.
\]

**Example 1.6.** Consider the polynomial map \(F := X + H : \mathbb{C}^4 \to \mathbb{C}^4\), where \(H\) equals

\[
\begin{pmatrix}
-X_4X_2^2 - e_4X_2^2X_4 - g_4X_2X_3X_4 - k_4X_3^2 - m_4X_4 - m_4X_3^2 \\
-X_4X_2^2 - e_4X_2^2X_4 + g_4X_2X_3X_4 - k_4X_3^2 + m_4X_2X_3 + g_4X_2X_3 + 1X_4^3 \\
0 \\
\end{pmatrix}
\]
and \( e_3, k_3, e_4, g_4, k_4, m_4 \in \mathbb{C} \) and \( g_4 \neq 0 \). This \( F \) is invertible. In fact if we take \( P = P^{-1} = (X_4, X_3, X_2, X_1) \), we have that \( PFP \) is one of the eight representatives of the cubic homogeneous maps in dimension 4 as given by Hubbers [7] and also published in [4, Theorem 2.10].

Now consider the following element \((T, c)\) of \( D_4(\mathbb{C}) \), where

\[
T = \begin{pmatrix}
1 & 0 \\
g_4^2 X_3 & g_4 X_4 + m_4 X_3
\end{pmatrix}, \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

and

\[
c = \begin{pmatrix}
-\frac{1}{g_4} X_2, -X_3^2 (e_4 X_4 + k_4 X_3) \\
X_3^2 X_4^2 - e_4 X_4 X_3^2 - k_3 X_3^2 \\
-\frac{1}{3} X_4^3
\end{pmatrix}
\]

Our claim is that

\[
H = \sum_{p=0}^{2} E_{4,p}(T, c) + c_4.
\]

To prove this we will compute \( E_{4,0}, E_{4,1}, \) and \( E_{4,2} \) by the method of Proposition 1.5. Note that \( c_4 = 0 \). Since \( \tilde{T}_4 = \tilde{T}_3 = I_4 \), \( E_{4,0} \) and \( E_{4,1} \) are easy:

\[
E_{4,0} = \text{Adj}(T_4) * \tilde{c}_{3|T_4}*X = \tilde{c}_3 = \begin{pmatrix}
0 \\
0 \\
-\frac{1}{3} X_4^3
\end{pmatrix},
\]

\[
E_{4,1} = \text{Adj}(\tilde{T}_3 \triangle T_4) * \tilde{c}_{2|\tilde{T}_3 \triangle T_4}*X = \tilde{c}_2 = \begin{pmatrix}
-\frac{1}{3} X_4^3 (e_4 X_4 + k_4 X_3) \\
X_3^2 X_4^2 - e_4 X_4 X_3^2 - k_3 X_3^2 \\
0
\end{pmatrix}.
\]
Before we compute $E_{4,2}$ we present the following identities:

$$\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4 = \tilde{T}_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
g_2^2 X_3 & g_4 X_4 + m_4 X_3 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix};$$

$$\text{Adj}(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4)$$

$$= \begin{pmatrix}
g_4 X_4 + m_4 X_3 & 0 & 0 & 0 \\
-g_2^2 X_3 & 1 & 0 & 0 \\
0 & 0 & g_4 X_4 + m_4 X_3 & 0 \\
0 & 0 & 0 & g_4 X_4 + m_4 X_3
\end{pmatrix};$$

$$(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4) \ast X = \begin{pmatrix}
X_1 \\
g_2^2 X_1 X_3 + g_4 X_2 X_4 + m_4 X_2 X_3 \\
X_3 \\
X_4
\end{pmatrix};$$

$$\tilde{c}_{1(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4)} \ast X = \begin{pmatrix}
-X_1 X_3 - \frac{1}{g_4} X_2 X_3 - \frac{m_4}{g_4} X_2 X_3 \\
0 \\
0 \\
0
\end{pmatrix};$$

and finally

$$E_{4,2} = \text{Adj}(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4) \ast \tilde{c}_{1(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4)} \ast X$$

$$= \begin{pmatrix}
-\left(X_4 + \frac{m_4}{g_4} X_3\right) \left(g_4 X_1 X_3 + X_2 X_4 + \frac{m_4}{g_4} X_2 X_3\right) \\
X_3 \left(g_2^2 X_1 X_3 + g_4 X_2 X_4 + m_4 X_2 X_3\right) \\
0 \\
0
\end{pmatrix}. $$

It is easy to verify that $H = E_{4,0} + E_{4,1} + E_{4,2} + c_4$, which was our claim.

2. NICE DERIVATIONS

Let $B := A[x_1, \ldots, x_n]$ be a finitely generated $A$-algebra and $D$ a subset of $\text{Der}_A(B)$. By $B^D$ we denote the set of all $b \in B$ such that $d(b) = 0$ for all $d \in D$. 
**Definition 2.1.** Let \( D \subset \text{Der}_A(B) \) be a finite subset and \( \tau \in \text{Der}_A(B) \).

1. We say that \( \tau \) is derived from \( D \) in at most one step if \( \tau \) is of the form \( \tau = \sum_{d \in D} b_d d \), where \( b_d \in B^0 \) for all \( d \in D \).

2. Let \( m \geq 2 \). We say that \( \tau \) is derived from \( D \) in at most \( m \) steps if there exists a sequence of finite subsets \( D = D_0, D_1, D_2, \ldots, D_m \) of \( \text{Der}_A(B) \) such that \( \tau \in D_m \) and all elements of \( D_i \) are derived from \( D_{i-1} \) in at most one step, for all \( 1 \leq i \leq m \). If furthermore the elements of \( D \) satisfy \( d_1 d_2(x_i) = 0 \) for all \( d_1, d_2 \in D \) and all \( i \), then \( \tau \) is called nice of order \( \leq m \), with respect to \( x_1, \ldots, x_n \) and \( D \).

**Proposition 2.2.** The notation is as in Definition 2.1. If \( d_1 d_2(x_i) = 0 \) for all \( d_1, d_2 \in D \) and all \( i \), then \( d_1 d_2(x_i) = 0 \) for all \( d_1, d_2 \in D_m \) and all \( i \). In particular \( d(x_i) = 0 \) for every nice derivation.

**Proof.** We use induction on \( m \). The case \( m = 0 \) is obvious since \( D_0 = D \). Now let \( m \geq 1 \). Then \( d_1 = \sum_{d \in D_{m-1}} b_d d \), \( d_2 = \sum_{d' \in D_{m-1}} b_{d'} d' \) with \( b_d, b_{d'} \in B^{D_{m-1}} \). Then

\[
\sum_{d, d'} b_d b_{d'} d d'(x_i) = \sum_{d, d'} b_d b_{d'} d d'(x_i) + \sum_{d, d'} b_d b_{d'} d d'(x_i) - \sum_{d, d'} b_d b_{d'} d d'(x_i).
\]

Now observe that \( d(b_{d'}) = 0 \) since \( b_{d'} \in B^{D_{m-1}} \) and \( d \in D_{m-1} \). Finally the induction hypothesis gives \( d d'(x_i) = 0 \) for all \( d, d' \in D_{m-1} \) and all \( i \), so \( 2 \) implies \( d_1 d_2(x_i) = 0 \). \( \blacksquare \)

We demonstrate these aspects by the so-called Winkelmann derivation. See [11].

**Example 2.3.** Let \( \tau = (1 + X_4 X_2 - X_3 X_3) \partial_{X_1} + X_2 \partial_{X_2} + X_4 \partial_{X_4} \) a derivation on \( B := A[X_1, X_2, X_3, X_4, X_5] \). Let \( D = \{ \partial_{X_1}, \partial_{X_2}, \partial_{X_3} \} \). Then \( \tau \) is nice of order 2 with respect to \( X_1, X_2, X_3, X_4, X_5 \) and \( D \). To show that this is true, we present a sequence of finite subsets of \( \text{Der}_A(B) \),

\[
D = D_0, D_1, D_2.
\]

Take \( D_1 := \{ \partial_{X_1}, X_5 \partial_{X_2} + X_4 \partial_{X_3} \} \) and \( D_2 := \{ \tau \} \). Note that in Definition
2.1 it is not demanded that the set $D_i$ of this sequence is a subset of $D_{i+1}$. The only demand is that each $D_i$ is a finite subset of $\text{Der}_A(B)$. Since $X_4, X_5 \in B^D$ it follows immediately that $\partial_{X_1}$ and $X_2 \partial_{X_2} + X_4 \partial_{X_1}$ are derived from $D$ in one step. And $1 + X_4 X_2 - X_2 X_5 \in B^D$, it follows that $\tau$ is derived from $D_1$ in one step. Obviously we have $d_1 d_2 (X_i) = 0$ for all $d_1, d_2 \in D$ and hence with Proposition 2.2 also $\tau^2(X_i) = 0$.

3. DERIVATIONS ASSOCIATED WITH POLYNOMIAL MAPS

The main aim of this section is to show that for each $0 \leq p \leq n - 2$ the polynomial map $X + E_{n,p}(T, c)$ (where $(T, c) \in D_n(A)$) is of the form $\exp(d)$, for some nice $A$-derivation $d$ of $A[X]$. Observe that $d$ is locally nilpotent if $d$ is nice with respect to $X_1, \ldots, X_n$, since $d^2(X_i) = 0$ for all $i$, by Proposition 2.2.

In order to prove this result (see Theorem 3.3), we need to generalise some of the notions of Sect. 1 to arbitrary finitely generated $A$-algebras. So let $B := A[x_1, \ldots, x_n]$ be a finitely generated $A$-algebra, and let $\varphi: A[X_1, \ldots, X_n] \rightarrow B$ be the $A$-ring homomorphism defined by $\varphi(X_i) = x_i$ for all $i$. For each $p, q \geq 1$ consider the natural extension $\varphi: M_{p,q}(A[X_1, \ldots, X_n]) \rightarrow M_{p,q}(B)$.

Then for each $(T, c) \in D_n(A)$ we define

$$E_{n,p}(T, c)(x) := \varphi(E_{n,p}(T, c)) \in B^n.$$

Now let $(\vartheta_1, \ldots, \vartheta_n)$ be an $n$-tuple of $A$-derivations of $B$. With each vector $b = (b_1, \ldots, b_n)^t \in B^n$ we associate the following $A$-derivation of $B$:

$$D(b; \vartheta_1, \ldots, \vartheta_n) := b_1 \vartheta_1 + \cdots + b_n \vartheta_n$$

To formulate the next lemma we need some more notation: Let $(T, c) \in D_n(A)$. Put

$$x_i := T_n \ast (x_1, \ldots, x_n)^t,$$

$$(\vartheta_1, \ldots, \vartheta_n) := (A \ast (T_n))^t \ast (\vartheta_1, \ldots, \vartheta_n)^t,$$

$$x'^n := (x_1, \ldots, x_{n-1}),$$

$$(T'^n, c'^n) := (T'(x_n = x'_n), c'(x_n = x'_n)) \in D_{n-1}(A[x'_n]).$$
Lemma 3.1. Let \( n \geq 3 \) and \( 1 \leq p \leq n - 2 \). Then

\[
D(E_{n,p}(T, c)(x); \partial_1, \ldots, \partial_n) = D(E_{n-1,p-1}(T^n, c^n)(x^n); \partial'_1, \ldots, \partial'_{n-1}).
\]

Proof.

\[
D(E_{n,p}(T, c)(x); \partial_1, \ldots, \partial_n)
= (E_{n,p}(T, c)(x))^t \left( \begin{array}{c} \partial_1 \\ \vdots \\ \partial_n \end{array} \right)
= \left( (E_{n-1,p-1}(T', c')(x'))^t 0 \right)^t \ast (\text{Adj}(T_n))^t \ast \left( \begin{array}{c} \partial_2 \\ \vdots \\ \partial_n \end{array} \right)
= \left( (E_{n-1,p-1}(T^n, c^n)(x^n))^t 0 \right)^t \ast \left( \begin{array}{c} \partial'_1 \\ \vdots \\ \partial'_{n-1} \end{array} \right)
= D(E_{n-1,p-1}(T^n, c^n)(x^n); \partial'_1, \ldots, \partial'_{n-1}).
\]

Lemma 3.2. The notation is as above. Let \( a \in A \) and let \( \partial_1, \ldots, \partial_n \) be \( A \)-derivations of \( B \) such that \( \partial_i(x_j) = a \delta_{ij} \) for all \( i, j \). Then

\[
\partial'_i(x'_j) = a \det(T_n) \delta_{ij}
\]

for all \( i, j \).

Proof. Denote the \( i \)th column of \( \text{Adj}(T_n) \) by \( (t^u_{1i}, \ldots, t^u_{ni})^t \) and the \( j \)th row of \( T_n \) by \( (t_{1j}, \ldots, t_{nj}) \). Then

\[
\partial'_i(x'_j) = \left( \sum_{s=1}^n t^u_{is} \partial_s \right) \left( \sum_{s=1}^n t_{js} x_s \right)
= \sum_{s=1}^n a t^u_{is} t_{js}
= a (T_n \ast \text{Adj}(T_n))_{ji}
= a \det(T_n) \delta_{ij}.
\]
Now we are able to prove:

**Theorem 3.3.** Let \( \partial_1, \ldots, \partial_n \) be \( A \)-derivations on \( A[x_1, \ldots, x_n] \) such that there exists an element \( a \in A \) such that \( \partial_i(x_j) = a \delta_{ij} \) for all \( i, j \). Let \( (T, c) \in D_0(A) \). Then the \( A \)-derivation \( d := D(E_{n, p}(T, c)(x); \partial_1, \ldots, \partial_n) \) is nice with respect to \( x_1, \ldots, x_n \) and \( D_0 := \{ \partial_1, \ldots, \partial_n \} \), for all \( n \geq 2 \) and all \( 0 \leq p \leq n - 2 \).

**Proof.**

1. The hypotheses on the \( \partial_i \) imply that \( dd'(x_i) = 0 \) for all \( d, d' \in D_0 \) and all \( i \).

2. First we consider the case \( p = 0 \). Then

\[
E_{n, 0}(T, c) = \text{Ad}(T_n) * \tilde{c}_{n-1} | T_n * x.
\]

So

\[
d = (\tilde{c}_{n-1} | T_n * x)^t * (\text{Ad}(T_n))^t \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix}.
\]

Write \( \tilde{c}_{n-1} = (y_1(x_n), \ldots, y_{n-1}(x_n), 0) \). Then the definition of \( x_n' \) and the \( \partial_j' \) imply that

\[
d = (y_1(x_n'), \ldots, y_{n-1}(x_n'), 0)^t * (\partial_1', \ldots, \partial_n')^t = \sum_{i=1}^{n-1} y_i(x_n') \partial_i'.
\]

Put \( D_1 := \{ \partial_1', \ldots, \partial_{n-1}' \} \) and observe that \( D_1 \subset \text{Der}_A(B) \) and that each element of \( D_1 \) is derived from \( D_0 \) in at most one step. Finally since \( \partial_i'(x_n') = 0 \) for all \( 1 \leq i \leq n - 1 \) (by Lemma 3.2) we get that \( y_i(x_n') \in B^{D_1} \) for all \( 1 \leq i \leq n - 1 \). So (3) implies that \( d \) is derived from \( D_1 \) in at most one step. Consequently \( d \) is derived from \( D_0 \) in at most two steps. So \( d \) is nice with respect to \( x_1, \ldots, x_n \) and \( D_0 \) by case 1.

3. Now we prove the theorem by induction on \( n \). If \( n = 2 \), then \( p = 0 \) and we are in case 2. So let \( n \geq 3 \). By case 2 we may assume that \( p \geq 1 \). Then by Lemma 3.1 we have

\[
d = D(E_{n-1, p-1}(T', c')(x'); \partial_1', \ldots, \partial_{n-1}')
\]

with \( (T', c') \in D_{n-1}(A[x_n']) \). By Lemma 3.2 we can apply the induction hypothesis to the ring \( A[x_n'] \) and the \((n-1)-\)tuple of \( A[x_n']\)-derivations \( \partial_1', \ldots, \partial_{n-1}' \) on the \( A[x_n'] \)-algebra \( B' := A[x_n'] \langle x_1', \ldots, x_{n-1}' \rangle \). So the \( A[x_n']\)-derivation \( d \) on \( B' \) is nice with respect to \( D_0 := \{ \partial_1', \ldots, \partial_{n-1}' \} \) and \( x_1', \ldots, x_{n-1}' \). So there exists a sequence

\[
D_0, D_1', \ldots, D_m'
\]
of finite subsets of $\text{Der}_{d_i}(B')$ such that $d \in D_i'$ and $D_i'$ is derived from $D_{i-1}'$ in at most one step for all $1 \leq i \leq m$. Now observe that $D'_0 \subseteq \text{Der}_d(B)$ and that $B' \subseteq B$ since by definition obviously $x'_i \in B$ for all $i$. Consequently if $d'$ is an $A[x'_i]$-derivation of $B'$ derived from $D'_0$ in at most one step, then $d' \in \text{Der}_d(B)$. Hence $D'_1 \subseteq \text{Der}_d(B)$. Arguing in a similar way we conclude by induction on $i$ that $D_i' \subseteq \text{Der}_d(B)$ for all $0 \leq i \leq m$. Since as remarked in case 2 above, all elements of $D_0'$ (as $D_1$ in case 2) are derived from $D_0$ in at most one step we deduce that $d$ is derived from $D_0$ in at most $m + 1$ steps. Just define $D_i := D_{i-1}'$ for all $1 \leq i \leq m + 1$. Hence $d$ is nice with respect to $x_1, \ldots, x_n$ and $D_0$ by 1.

**Corollary 3.4.** Let $(T, c) \in D_n(A)$ and $0 \leq p \leq n - 2$. Put

$$D := D\left(E_{n,p}(T, c); \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}\right).$$

Then $D$ is nice with respect to $X_1, \ldots, X_n$ and $\left(\frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}\right)$. Furthermore we have $\exp(D) = X + E_{n,p}(T, c)$ and the inverse map is given by $\exp(-D) = X - E_{n,p}(T, c)$.

**Proof.** The first part is an immediate consequence of Theorem 3.3. Furthermore $D^2(X) = 0$ by Proposition 2.2. So $\exp(D)(X) = X + E_{n,p}(T, c)$ and the inverse map is given by $\exp(-D)(X) = X - E_{n,p}(T, c)$.

**4. The Main Theorem**

In this section we show that for every $H \in H_n(A)$ the polynomial map $F = X + H$ is a product of $n$ polynomial automorphisms of the form $\exp(D)$, where each $D$ is a nice derivation on $A[X]$. More precisely

**Theorem 4.1.** Let $F = X + H$, where $H = \sum_{p=0}^{n-2} E_{n,p}(T, c) + c_n$, for some $(T, c) \in D_n(A)$. Then

$$F = \exp\left(D\left(c_n; \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}\right)\right)$$

$$\times \prod_{p=0}^{n-2} \exp\left(D\left(E_{n,p}(T, c); \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}\right)\right).$$
Proof. Observe that
\[
\exp \left( -D \left( \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n} \right) \right) \circ F = \sum_{p=0}^{n-2} E_{n, p}(T, c).
\]
So the case \( n = 2 \) follows from Corollary 3.4. Hence we may assume that \( n \geq 3 \). Now Theorem 4.1 follows directly from Proposition 4.2 below and Corollary 3.4.

**Proposition 4.2.** Let \( n \geq 3, 0 \leq p \leq n - 3 \), and \((T, c) \in D_n(A)\). Then
\[
\exp\left( -D(E_{n, p}(T, c)) \right) \circ \left( X + \sum_{q=p}^{n-2} E_{n, q}(T, c) \right) = X + \sum_{q=p+1}^{n-2} E_{n, q}(T, c).
\]

*Proof.* Put \( G := \exp( -D(E_{n, p}(T, c)) ) \). So \( G = X - E_{n, p}(T, c) \) (by Corollary 3.4). Hence if we put
\[
U := T_{n-p} \triangle \cdots \triangle T_{n-1} \triangle T_n
\]
then by Proposition 1.4 we get
\[
G = X - \text{Ad}(U) * \tilde{c}_{n-p-1}(U * X).
\]
So if we put
\[
f := X + \sum_{q=p}^{n-2} E_{n, q}(T, c)
\]
then
\[
G \circ f = f - \text{Ad}(U(f)) * \tilde{c}_{n-p-1}(U(f) * f).
\]
Since \( U(f) = f \) (by Corollary 4.4 below, with \( j = 0 \)) we get
\[
G \circ f = f - \text{Ad}(U) * \tilde{c}_{n-p-1}(U * f).
\]
Now observe that each component of \( \tilde{c}_{n-p-1} \) belongs to \( A[X_{n-p}, \ldots, X_n] \) and that for each \( i \geq n - p \) \((U * f)_i = (U * X)_i \) (by Lemma 4.3 below). So \( \tilde{c}_{n-p-1}(U * f) = \tilde{c}_{n-p-1}(U * X) \) and hence
\[
G \circ f = f - \text{Ad}(U) * \tilde{c}_{n-p-1}(U * X)
\]
\[
= f - E_{n, p}(T, c)
\]
(by Proposition 1.4) \( \blacksquare \)
Lemma 4.3. Let \( n \geq 3, 0 \leq p \leq n - 2, 0 \leq j \leq p, \) and \((T, c) \in \mathbb{D}_a(A) \).
Put \( f := X + \sum_{q=p}^{n-2} E_{n, q}(T, c) \). Then

\[
\left[ \left( \tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n \right) * f \right]_i = \left[ \left( \tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n \right) * X \right]_i
\]

for all \( i \geq n - p + j \).

Proof. Put \( U := \tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n \). It suffices to show that for each \( q \geq p \)

\[
\left[ U * E_{n, q}(T, c) \right]_i = 0
\]

for all \( i \geq n - p + j \). So let \( q \geq p \). Then \( q \geq p - j \).

1. We first treat the case that \( q = p - j \). Then \( j = 0 \) and \( q = p \). Consequently \( U = \tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n \), \( E_{n, q}(T, c) = E_{n, p}(T, c) \), and hence by Proposition 1.4

\[
U * E_{n, q}(T, c) = U * \text{Adj}(U) * \tilde{c}_{n-p-1 | U * X} = \text{det}(U) * \tilde{c}_{n-p-1 | U * X}.
\]

Since the last \( p + 1 \) coordinates of \( \tilde{c}_{n-p-1} \) are zero, we obtain that

\[
\left[ U * E_{n, q}(T, c) \right]_i = 0
\]

for all \( i \geq n - p \), which proves the case that \( q = p - j \).

2. Now assume that \( q \geq p - j + 1 \). So \( n - q \leq n - p + j - 1 \). Put \( V := \tilde{T}_{n-q} \triangle \cdots \triangle \tilde{T}_{n-p+j-1} \). Then by Proposition 1.4 we can write

\[
E_{n, q}(T, c) = \text{Adj}(V * U) * \tilde{c}_{n-q-1 | (V * U) * X} = \text{Adj}(V * X * U) * \tilde{c}_{n-q-1 | (V * U) * X} = \text{Adj}(U) * \text{Adj}(V * X) * \tilde{c}_{n-q-1 | (V * U) * X}.
\]

Consequently

\[
U * E_{n, q}(T, c) = \text{det}(U) * \text{Adj}(V * X) * \tilde{c}_{n-q-1 | (V * U) * X}.
\]

Note that \( V \), and hence \( V * X \), is of the form \( \tilde{B} \) for some \( B \in M_{n-p+j-1}(A[X]) \). Furthermore \( \langle \tilde{c}_{n-q-1} \rangle_i = 0 \) if \( i \geq n - q \), which implies that \( \langle \tilde{c}_{n-q-1 | (V * U) * X} \rangle_i = 0 \) if \( i \geq n - p + j \) (since \( n - p + j > n - q \)).

Now the desired result (4) follows from (5). 

Corollary 4.4. The notation is as in Lemma 4.3. Then

$$
\left( \tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n \right)(f) = \tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n.
$$

Proof. The proof is by induction on $N := p - j$. If $N = 0$ the result is obvious. So let $N \geq 1$. Then

$$
\left( \tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n \right)(f)
$$

$$
= \tilde{T}_{n-p+j}(\tilde{T}_{n-p+j+1} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n)(f) \ast \left( \tilde{T}_{n-p+j+1} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n \right)(f)
$$

$$
= \tilde{T}_{n-p+j}(\tilde{T}_{n-p+j+1} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n)(f) \ast \left( \tilde{T}_{n-p+j+1} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n \right)(f)
$$

by the induction hypothesis. Finally observe that the matrix elements of $\tilde{T}_{n-p+j}$ depend only on $X_{n-p+j}, \ldots, X_n$. The result follows immediately from Lemma 4.3 (with $j + 1$ instead of $j$). 

5. STABLY TAMENESS

With Theorem 4.1 we are now able to prove the stably tame generators conjecture for all maps in our class $\mathcal{H}_n(A)$, and, we will also show that this result is “sharp”: We give an example of an element of our class which is not tame, so in general we cannot get a better result than this stable tameness.

First let us recall the conjecture (it has already been mentioned in [1–4] and [8]):

Conjecture 5.1. For every invertible polynomial map $F : k^n \to k^n$ over a field $k$ there exist $t_1, \ldots, t_m$ such that

$$
\exp(aD), t) = \rho^{-1} \exp(-tD) \rho \exp(tD).
$$
COROLLARY 5.4. Let $D$ and $a$ be as in Proposition 5.3. If $D$ is conjugate by a tame automorphism to a triangular derivation, then $(\exp(aD), t)$ is tame.

LEMMA 5.5. Let $\tau$ be a nice derivation of order $m$ with respect to $X_1, \ldots, X_n$ and $D := (\partial/\partial X_1, \ldots, \partial/\partial X_n)$ on $A[X]$. Then $\exp(a\tau)$ is stably tame for all $a \in \ker(\tau)$.

Proof. We use induction on $m$. Consider the case that $m = 1$. Then $\tau = \sum_{d \in D} b_d d$ with $b_d \in A[X]^D = \cap_{d \in D} \ker(d) = A$. Hence $\tau(X_i) \in A$ and clearly $\tau$ is on triangular form. So now we can apply Corollary 5.4 and find that $\exp(a\tau)$ is stably tame.

Now consider the case $m > 1$. We may assume that for all nice derivations $\sigma \in \text{Der}_A(\mathcal{A}[X])$ of order $m - 1$ with respect to $D$ and $X_1, \ldots, X_n$ and for any commutative ring $A$ we have that $\exp(a\sigma)$ is stably tame for all $a \in \ker(\sigma)$. Let $\tau$ be nice of order $m$. Define $\rho$ and extend $\tau$ to $A[X][t]$ as in Proposition 5.3 (in fact we extend all derivations of $D_i$ to $A[X][t]$ in this way). Now from

$$\exp(a\tau), t = \rho^{-1} \exp(-t\tau) \rho \exp(t\tau)$$

it follows that it suffices to see that $\exp(t\tau)$ is stably tame. Now we see that $t\tau = \sum_{d \in D_{m-1}} b_d d$ with $b_d \in A[X][t]^{D_{m-1}}$. But from this it follows that

$$\exp(t\tau) = \exp\left(\sum_{d \in D_{m-1}} b_d d\right) = \prod_{d \in D_{m-1}} \exp(b_d d).$$

This last equation follows from Proposition 1.5. Obviously it suffices to prove that each $\exp(b_d d)$ is stably tame to conclude that $\exp(t\tau)$ is stably tame. But $d$ is a nice derivation of order $m - 1$, $b_d \in \ker(d)$, and hence we can apply the induction hypothesis to the ring $A[t]$ and find that $\exp(t\tau)$ is stably tame and hence $\exp(a\tau)$ is stably tame.

Proof of Theorem 5.2. Now if we look at Theorem 4.1 we see that each $F = X + H$ with $H \in H_n(A)$ can be written as the product of a finite number of $\exp(a_i D_i)s$, where each $D_i$ is a nice derivation with respect to $X_1, \ldots, X_n$ and $(\partial/\partial X_1, \ldots, \partial/\partial X_n)$ and $a_i \in \ker(D_i)$. Applying Lemma 5.5 $n$ times gives us the desired result: $F$ is stably tame.

Remark 5.6. Note that we do not give an indication of the value of $m$ in Conjecture 5.1. As can be seen from the proof above, this $m$ can be very high. At the highest level we have $n \exp(a_i D_i)$s, but each of these factors can give rise to a great number of extra variables, depending on the “order of niceness” of each $D_i$.
To conclude this paper we show that in general the automorphisms $F = X + H$ with $H \in H_2(A)$ need not be tame. Actually, this idea was already presented by Nagata [9].

Example 5.7. Let $A$ be a domain, but not a principle ideal domain. Let $a, b \in A$ such that $Aa + Ab$ is not a principal ideal. Let $f(T) \in A[T]$ with $\deg(f) \geq 2$ and let $F = X + H$ with

$$H = \begin{pmatrix}
bf(aX_1 + bX_2) \\
-af(aX_1 + bX_2)
\end{pmatrix}$$

Since $H \in H_2(A)$ $F$ is an automorphism of $A[X_1, X_2]$. However, it is shown in [9] that $F$ is not tame.

References

2. L. M. Drużkowski, The Jacobian conjecture, Preprint 492, Institute of Mathematics, Polish Academy of Sciences, IMPAN, Sniadekich 8, P.O. Box 137, 00-950 Warsaw, Poland, 1991.