Polynomial maps with strongly nilpotent Jacobian matrix and the Jacobian Conjecture

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Abstract

Let \( H : k^n \to k^n \) be a polynomial map. It is shown that the Jacobian matrix \( JH \) is strongly nilpotent (definition 1.1) if and only if \( JH \) is linearly triangularizable if and only if the polynomial map \( F = X + H \) is linearly triangularizable. Furthermore it is shown that for such maps \( F \) is linearizable for almost all \( s \in k \) (except a finite number of roots of unity).

Introduction

In [1] Bass, Connell and Wright and in [7] Yagzhev showed that it suffices to prove the Jacobian Conjecture for polynomial maps \( F : \mathbb{C}^n \to \mathbb{C}^n \) of the form \( F = X + H \), where \( H = (H_1, \ldots, H_n) \) is a cubic homogeneous polynomial map i.e. each \( H_i \) is either zero or homogeneous of degree three. Since \( \det(JF) \in \mathbb{C}^* \) is equivalent to \( JH \) is nilpotent (cf [1, Lemma 4.1]) it follows that the Jacobian Conjecture is equivalent to: if \( F = X + H \) with \( JH \) nilpotent, then \( F \) is invertible. Hence it is clear that understanding nilpotent Jacobian matrices is crucial for the study of the Jacobian Conjecture.

In [6], in an attempt to understand quadratic homogeneous polynomial maps, Meisters and Olech introduced the strongly nilpotent Jacobian matrices: a Jacobian matrix \( JH \) is strongly nilpotent if \( JH(x_1) \ldots JH(x_n) = 0 \) for all vectors \( x_1, \ldots, x_n \in \mathbb{C}^n \). They showed in [6] that for quadratic homogeneous polynomial maps \( JH \) is strongly nilpotent if and only if \( JH \) is nilpotent, if \( n \leq 4 \). However for \( n \geq 5 \) there are counterexamples (cf [4] and [6]).

On the other hand the obvious question: is the Jacobian Conjecture true for arbitrary polynomial maps \( F = X + H \) with \( JH \) strongly nilpotent, remained open.

In this paper we give an affirmative answer to this question. In fact we obtain a much stronger result; in theorem 1.6 we show that the Jacobian matrix \( JH \) is strongly nilpotent if and only if \( JH \) is linearly triangularizable if and only if the
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polynomial map $F = X + H$ is linearly triangularizable. Furthermore we show that
for such maps $F$ the map $sF$ is linearizable for almost all $s \in \mathbb{C}$ (except a finite
number of roots of unity). So for such $F$ the linearization conjecture of Meisters is
true (it turned out to be false in general as was shown in [3]).

1. Definitions and formulation of the first main result

Throughout this paper $k$ denotes an arbitrary field and $k[X] := k[X_1, \ldots, X_n]$
denotes the polynomial ring in $n$ variables over $k$. Let $H = (H_1, \ldots, H_n) : k^n \to k^n$
be a polynomial map i.e. $H_i \in k[X]$ for all $i$. By $JH$ or $JH(X)$ we denote its
Jacobian matrix. So $JH(X) \in M_n(k[X])$.

Now let $Y(1) = (Y(1)_1, \ldots, Y(1)_n), \ldots, Y(n) = (Y(n)_1, \ldots, Y(n)_n)$ be $n$ sets of $n$
new variables. So for each $i$ $JH(Y(i))$ belongs to the ring of $n \times n$ matrices with entries
in the $n^2$ variable polynomial ring $k[Y(i)_{ij}; 1 \leq i, j \leq n]$.

**Definition 1.1.** The Jacobian matrix $JH$ is called strongly nilpotent if and only
if the matrix $JH(Y(1)) \ldots JH(Y(n))$ is the zero matrix.

**Example 1.2.** If $JH$ is upper triangular with zeros on the main diagonal, then
one readily verifies that $JH$ is strongly nilpotent. In fact the main result of this
paper (theorem 1.6 below) asserts that a matrix $JH$ is strongly nilpotent if and
only if it is uppertriangular with zeros on the main diagonal after a suitable linear
change of coordinates!

**Remark 1.3.** One easily verifies that if $k$ is an infinite field, then definition 1.1
is equivalent to $JH(x_1) \ldots JH(x_n) = 0$ for all $x_1, \ldots, x_n \in k^n$. So for $k = \mathbb{R}$ and
$H$ homogeneous of degree two we obtain the strong nilpotence property introduced
by Meisters and Olech in [6]. See also [4].

To formulate the first main result of this paper we need one more definition.

**Definition 1.4.** i) Let $F = X + H$ be a polynomial map. We say that $F$ is in
(upper) triangular form if $H_i \in k[X_{i+1}, \ldots, X_n]$ for all $1 \leq i \leq n - 1$ and
$H_n \in k$.

ii) We say that $F$ is linearly triangularizable if there exists $T \in GL_n(k)$ such that
$T^{-1}FT$ is in upper triangular form.

One easily verifies the following lemma:

**Lemma 1.5.** Let $F = X + H$ be a polynomial map. Then $F$ is in upper triangular
form if and only if $JH$ is upper triangular with zeros on the main diagonal.

Now we are ready to formulate the first main result of this paper:
Theorem 1.6. Let \( H = (H_1, \ldots, H_n) : k^n \to k^n \) be a polynomial map. Then there is equivalence between

i) \( JH \) is strongly nilpotent.
ii) There exists \( T \in GL_n(k) \) such that \( J(T^{-1}HT) \) is upper triangular with zeros on the main diagonal.
iii) \( F := X + H \) is linearly triangularizable.

From this theorem it immediately follows that:

Corollary 1.7. If \( F = X + H \) with \( JH \) strongly nilpotent, then \( F \) is invertible.

2. The proof of theorem 1.6

The proof of theorem 1.6 is based on the following two results.

Lemma 2.1. Let \( JH = \sum_{|\alpha| \leq d} A_\alpha X^\alpha \), where \( d = \max_i (\deg(H_i)) - 1 \) and \( A_\alpha \in M_n(k) \) for all \( \alpha \). Then \( JH \) is strongly nilpotent if and only if \( A_{\alpha(1)} \cdots A_{\alpha(n)} = 0 \), for all multi-indices \( \alpha(i) \) with \( |\alpha(i)| \leq d \).

Proof. By definition 1.1 we obtain

\[
\left( \sum_{|\alpha(1)| \leq d} A_{\alpha(1)} Y_{(1)}^{\alpha(1)} \right) \cdots \left( \sum_{|\alpha(n)| \leq d} A_{\alpha(n)} Y_{(n)}^{\alpha(n)} \right) = 0.
\]

The result then follows by looking at the coefficients of \( Y_{(1)}^{\alpha(1)} \cdots Y_{(n)}^{\alpha(n)} \). \( \square \)

Proposition 2.2. Let \( V \) be a finite dimensional \( k \)-vector space and \( \ell_1, \ldots, \ell_p \) \( k \)-linear maps from \( V \) to \( V \). Let \( r \in \mathbb{N}, \ r \geq 1 \). If \( \ell_{i_1} \circ \cdots \circ \ell_{i_r} = 0 \) for each \( r \)-tuple \( i_1, \ldots, i_r \) with \( 1 \leq i_1, \ldots, i_r \leq p \), then there exists a basis \( (v) \) of \( V \) such that \( \text{Mat}(\ell_i, (v)) = D_i \) where \( D_i \) is an upper triangular matrix with zeros on the main diagonal.

Proof. Let \( d := \dim(V) \). We use induction on \( d \). First let \( d = 1 \). Then the hypothesis implies that \( \ell_i^r = 0 \) for each \( i \). So \( \ell_i = 0 \) for each \( i \) and we are done. So let \( d > 1 \) and assume that the assertion is proved for all \( d - 1 \)-dimensional vector spaces. Now we (also) use induction on \( r \). If \( r = 1 \) then each \( \ell_i = 0 \). So let \( r \geq 2 \). Then for each \( (r - 1) \)-tuple \( \ell_{i_2} \cdots \ell_{i_r} \) with \( 1 \leq i_2, \ldots, i_r \leq p \) we have

\[
\ell_1 \ell_{i_2} \cdots \ell_{i_r} = 0, \ldots, \ell_p \ell_{i_2} \cdots \ell_{i_r} = 0.
\]

(2.1)

If \( \ell_{i_2} \cdots \ell_{i_r} = 0 \) for each such \( (r - 1) \)-tuple we are done by the induction hypothesis on \( r \). So we may assume that for some \( (r - 1) \)-tuple \( i_2, \ldots, i_r \), the map \( \ell_{i_2} \cdots \ell_{i_r} \neq 0 \). So there exists \( v \neq 0, v \in V \) with \( v_1 := \ell_{i_2} \cdots \ell_{i_r} v \neq 0 \). From (2.1) we deduce that \( \ell_i v_1 = 0 \) for all \( i \). Then consider \( V := V/kv_1 \). Since \( l_i v_1 = 0 \) for all \( i \) we get
induced $k$-linear maps $\bar{\ell}_i : \bar{V} \to \bar{V}$. Since $\dim(\bar{V}) = d - 1$ the induction hypothesis implies that there exist $v_2, \ldots, v_r$ in $V$ such that $(\bar{v}_2, \ldots, \bar{v}_r)$ is a $k$-basis of $\bar{V}$ and $\text{Mat}(\bar{\ell}_i, (\bar{v}_2, \ldots, \bar{v}_r))$ is on upper triangular form. Then $(v) = (v_1, v_2, \ldots, v_r)$ is as desired. \hfill \Box

**Corollary 2.3.** Let $A_1, \ldots, A_p \in M_n(k)$. Let $r \in \mathbb{N}$, $r \geq 1$. If $A_{i_1} \cdots A_{i_r} = 0$ for each $r$-tuple $A_{i_1}, \ldots, A_{i_r}$ with $1 \leq i_1, \ldots, i_r \leq p$, then there exists $T \in GL_n(k)$ such that $T^{-1}A_iT = D_i$, where each $D_i$ is an upper triangular matrix with zeros on the main diagonal.

Now we are able to present the proof of theorem 1.6.

**Proof.** $ii) \to iii)$ follows from lemma 1.5. So let’s prove $iii) \to i)$. If $F = X + H$ is linearly triangularizable, then by lemma 1.5 $J(T^{-1}HT)$ is an upper triangular matrix with zeros on the main diagonal. So as remarked in example 1.2 this implies that $J(T^{-1}HT)$ is strongly nilpotent. Finally observe that $J(T^{-1}HT) = T^{-1}JH(TX)T$. So the strong nilpotency of $J(T^{-1}HT)$ implies that $JH(TY_{(1)}) \cdots JH(TY_{(n)}) = 0$, which implies in turn that $JH$ is strongly nilpotent.

Finally we prove $i) \to ii)$. So let $JH$ be strongly nilpotent. Now if we write $JH = \sum_{|\alpha| \leq d} A_\alpha X^\alpha$, then by lemma 2.1 $A_{\alpha(i_1)} \cdots A_{\alpha(n)} = 0$ for all $n$-tuples with $|\alpha(i)| \leq d$. So by corollary 2.3 there exists $T \in GL_n(k)$ such that $T^{-1}A_\alpha T = D_\alpha$ for all $\alpha$ with $|\alpha| \leq d$, where $D_\alpha$ is an upper triangular matrix with zeros on the main diagonal. Consequently so is $T^{-1}JH(X)T = \sum T^{-1}A_\alpha TX^\alpha$ and hence so is $J(T^{-1}HT) = T^{-1}JH(TX)T$, which is obtained by replacing $X$ by $TX$ in $T^{-1}JH(X)T$. \hfill \Box

3. Strongly nilpotent Jacobian matrices and Meisters linearization conjecture

In [2] Deng, Meisters and Zampieri studied dilations of polynomial maps with $\det(JF) \in C^*$. They were able to prove that for large enough $s \in \mathbb{C}$ the map $sF$ is locally linearizable to $sJF(0)X$ by means of an analytic map $\varphi_s$, the so-called Schröder map, which inverse is an entire function and satisfies some nice properties.

Their original aim was to show that $\varphi_s$ is entire analytic, which would imply that $sF$ and hence $F$ is injective, which in turn would imply the Jacobian Conjecture. Although they were not able to prove the ‘entireness’ of $\varphi_s$, calculations of many examples of polynomial maps of the form $X + H$ with $H$ cubic homogeneous showed that in all these cases the Schröder map was even much better than expected, namely it was a polynomial automorphism! (cf [5]) This lead Meisters to the following conjecture:

**Conjecture 3.1.** (Linearization Conjecture, Meisters [5])

Let $F = X + H$ be a cubic homogeneous polynomial map with $JH$ nilpotent. Then
for almost all \( s \in \mathbb{C} \) (except a finite number of roots of unity) there exists a polynomial automorphism \( \varphi_s \) such that \( \varphi_s^{-1}sF \varphi = sX \).

Recently in [3] it was shown by the first author that the conjecture is false if \( n \geq 5 \) and true if \( n \leq 4 \).

In this section we show that Meisters linearization conjecture is true for all \( n \geq 1 \) if we replace ‘\( JH \) is nilpotent’ by ‘\( JH \) is strongly nilpotent’. In fact we even don’t need the assumption that this \( H \) is cubic homogeneous. More precisely we have:

**Theorem 3.2.** Let \( k \) be a field, \( k(s) \) the field of rational functions in one variable and \( F : k^n \to k^n \) a polynomial map of the form \( F = X + H \) with \( F(0) = 0 \) and \( JH \) strongly nilpotent. Then there exists an over \( k \) linearly triangularizable polynomial automorphism \( \varphi_s \in \text{Aut}_{k(s)}(k(s)[X]) \) such that

\[
\varphi_s^{-1}sF \varphi_s = sJF(0)X.
\]

Furthermore, the zeros of the denominators of the coefficients of the \( X \)-monomials appearing in \( \varphi_s \) are roots of unity.

Before we can prove this result we need one definition and some lemmas.

**Definition 3.3.** We say that \( X_{i_1}^{i_1} \ldots X_{i_n}^{i_n} > X_{i_1}'^{i_1} \ldots X_{i_n}'^{i_n} \) if and only if \( \sum_{j=1}^{n} i_j > \sum_{j=1}^{n} i_j' \) or if \( \sum_{j=1}^{n} i_j = \sum_{j=1}^{n} i_j' \) and there exists some \( l \in \{1, 2, \ldots, n\} \) such that \( i_j = i_j' \) for all \( j < l \) and \( i_l > i_l' \).

Furthermore we say that the rank of the monomial \( M := X_{i_1}^{i_1} \ldots X_{i_n}^{i_n} \) is the index of this monomial in the ascending ordered list of all monomials \( M' \) in \( X_1, \ldots, X_n \) with \( \deg(M') \leq \deg(M) \) (total degree).

**Example 3.4.** The rank of \( X_1X_2X_3 \) is 15, since the ascending ordered list of all monomials in \( X_1, X_2 \) and \( X_3 \) of total degree at most three is:

\[
X_3, X_2, X_1, \\
X_2^2, X_2X_3, X_2^2, X_1X_3, X_1X_2, X_1^2, \\
X_3^2, X_2X_3^2, X_2^2X_3, X_2^3, X_1X_2X_3, X_1X_2^2, X_1^2X_3, X_1^2X_2, X_1^3.
\]

**Lemma 3.5.** For each \( 2 \leq j \leq n - 1 \) let \( \ell_j(X_{j+1}, \ldots, X_n) \) be a linear form in \( X_{j+1}, \ldots, X_n \) and let \( \mu \in k \). Then the leading monomial with respect to the order of definition 3.3 in the expansion of

\[
\mu \prod_{j=2}^{n}(sX_j + s\ell_j(X_{j+1}, \ldots, X_n))^{i_j}
\]

is

\[
\mu s^{i_2+\ldots+i_n}X_2^{i_2} \ldots X_n^{i_n}.
\]
Proof. It is obvious that the monomial $\mu s^{i_2 + \ldots + i_n} X_2^{i_2} \ldots X_n^{i_n}$ appears in the expansion of (3.1). Now we have to show that this is really the leading monomial. Note that all monomials in the expansion have the same (total) degree: $i_2 + \ldots + i_n$. For each $j = 2, \ldots, n$ we get a contribution of $(sX_j + s\ell_j(X_{j+1}, \ldots, X_n))^i_j$ that is of the form
\[
\sum_{k=0}^{i_j} \binom{i_j}{k} X_j^k(\ell_j(X_{j+1}, \ldots, X_n))^{i_j-k}
\]
and since $\ell_j$ is a linear term that does not contain $X_j$ it is obvious that we get the highest order monomial if we take $k = i_j$. So if we start with $j = 2$, we see that the highest $X_2$ power is $i_2$. And if we apply this result to $j = 3$ we see that the leading power product must begin with $X_2^{i_2} X_3^{i_3}$. If we do this for all $j$ we see that it is obvious that the leading monomial is $\mu s^{i_2 + \ldots + i_n} X_2^{i_2} \ldots X_n^{i_n}$.

Lemma 3.6. Let $F$ be a polynomial map of the form:
\[
F = \begin{pmatrix}
X_1 + a(X_2, \ldots, X_n) + \ell_1(X_2, \ldots, X_n) \\
X_2 + \ell_2(X_3, \ldots, X_n) \\
\vdots \\
X_{n-1} + \ell_{n-1}(X_n) \\
X_n
\end{pmatrix}
\]
where $a(X_2, \ldots, X_n)$ is a polynomial with leading monomial (with respect to the order of definition 3.3) $\lambda X_2^{i_2} \ldots X_n^{i_n}$ and $i_2 + \ldots + i_n \geq 2$. Furthermore $\ell_i(X_{i+1}, \ldots, X_n)$ are some linear forms. Then there exists a polynomial map $\varphi$ on triangular form such that
\[
\varphi^{-1} sF \varphi = s \begin{pmatrix}
X_1 + \hat{a}(X_2, \ldots, X_n) + \ell_1(X_2, \ldots, X_n) \\
X_2 + \ell_2(X_3, \ldots, X_n) \\
\vdots \\
X_{n-1} + \ell_{n-1}(X_n) \\
X_n
\end{pmatrix}
\]
where the leading monomial of $\hat{a}(X_2, \ldots, X_n)$, say $\hat{\lambda} X_2^{j_2} \ldots X_n^{j_n}$, is of strict lower order than the leading monomial of $a(X_2, \ldots, X_n)$, i.e.:
\[
X_2^{j_2} \ldots X_n^{j_n} < X_2^{i_2} \ldots X_n^{i_n}.
\]

Proof. Let
\[
\varphi = \begin{pmatrix}
X_1 + \mu X_2^{i_2} \ldots X_n^{i_n} \\
X_2 \\
\vdots \\
X_n
\end{pmatrix}
\]
for some $\mu \in k$. It is obvious that $\varphi$ is on triangular form. Proving that the equation
\[(3.2)\) is valid is equivalent with showing that
\[
sF\varphi = \varphi(s)
\begin{pmatrix}
X_1 + \hat{a}(X_2, \ldots, X_n) + \ell_1(X_2, \ldots, X_n) \\
X_2 + \ell_2(X_3, \ldots, X_n) \\
\vdots \\
X_{n-1} + \ell_{n-1}(X_n) \\
X_n
\end{pmatrix}
\]
\[(3.3)\]
is valid. We do this by looking at the \(n\) components. For \(i \geq 2\) it is easy to see that the \(i\)-th component of the left-hand side of \((3.3)\) equals that of the right-hand side of \((3.3)\). Hence our only concern is the first component. Put \(\hat{a}(X_2, \ldots, X_n) := a(X_2, \ldots, X_n) - \lambda X_2^{i_2} \cdots X_n^{i_n}\). On the right-hand side we have:
\[
sF\varphi|_1 = sX_1 + s\mu X_2^{i_2} \cdots X_n^{i_n} + s\lambda X_2^{i_2} \cdots X_n^{i_n} + s\hat{a}(X_2, \ldots, X_n) + s\ell_1(X_2, \ldots, X_n)
\]
and on the right-hand side:
\[
\varphi(s)
\begin{pmatrix}
X_1 + \hat{a}(X_2, \ldots, X_n) + \ell_1(X_2, \ldots, X_n) \\
X_2 + \ell_2(X_3, \ldots, X_n) \\
\vdots \\
X_{n-1} + \ell_{n-1}(X_n) \\
X_n
\end{pmatrix}
|_1
\]
\[(3.5)\]
By subtracting equation \((3.5)\) from equation \((3.4)\) under the assumption that equation \((3.3)\) holds, we get:
\[
s(\mu + \lambda)X_2^{i_2} \cdots X_n^{i_n} + s\hat{a}(X_2, \ldots, X_n) = \mu \prod_{j=2}^{n}(sX_j + s\ell_j(X_{j+1}, \ldots, X_n))^{i_j}
\]
\[(3.6)\]
where \(\hat{a} = \hat{a} - \bar{a}\). Now we have to derive a relation for \(\mu\) to achieve that equation \((3.3)\) indeed holds. We can do this by restricting equation \((3.6)\) to the coefficients of \(X_2^{i_2} \cdots X_n^{i_n}\). With lemma 3.5 we see that the restriction of the right-hand side of \((3.6)\) to \(X_2^{i_2} \cdots X_n^{i_n}\) gives \(\mu s^{i_2+\cdots+i_n}\), so we get:
\[
s\mu + s\lambda = s^{i_2+\cdots+i_n}\mu
\]
and from this equation we can compute \(\mu\):
\[
\mu = \frac{\lambda}{s^{i_2+\cdots+i_n-1} - 1}
\]
Note that we have assumed that \(i_2 + \ldots + i_n \geq 2\) so \(s^{i_2+\cdots+i_n-1} - 1 \neq 0\), hence \(\mu\) is well defined. \(\square\)

Now we are able to give the proof of theorem 3.2.
Proof. By theorem 1.6 we may assume that $F = (F_1, \ldots, F_n)$ is on triangular form. We use induction on $n$. If $n = 1$ $F$ degenerates to the identical map $X_1$ and the theorem follows immediately.

If $n = 2$ we can write $F = \left( X_1 + a(X_2) + \ell_1(X_2) \right)$ where $a = \sum_{i=2}^{m} a_i X_i^i$ and $\ell_1 = a X_2$, the linear part. In particular we have that the leading monomial of $a$ is $a_m X_2^m$. So with lemma 3.6 we know that there exists a map $\varphi_m$ on triangular form such that

$$\varphi_m^{-1} s F \varphi_m = \left( sX_1 + \tilde{a}(X_2) + s\ell_1(X_2) \right) \quad \text{where deg}(\tilde{a}) < m.$$

By applying the same lemma $m$ times (if necessary we can use $\varphi_j$ is the identity) we find a sequence $\varphi_1, \ldots, \varphi_m$ such that

$$\varphi_1^{-1} \ldots \varphi_m^{-1} s F \varphi_m \ldots \varphi_1 = s \left( X_1 + \ell_1(X_2) \right) \quad \text{where deg}(\tilde{a}) < m.$$

So $\varphi_s := \varphi_m \circ \ldots \circ \varphi_1$ is as desired. Now consider $F = (F_1, F_2, \ldots, F_n)$. Put $\tilde{F} := (F_2, \ldots, F_n)$ and $X := (X_2, \ldots, X_n)$. Then by the induction hypothesis we know that there exists an invertible polynomial map $\tilde{\varphi}_s$ such that

$$\tilde{\varphi}_s^{-1} s \tilde{F} \tilde{\varphi}_s = s J X \tilde{F}(0).$$

So with $\chi = (X_1, \tilde{\varphi}_s)$ and with the notation

$$F = (X_1 + a(X_2, \ldots, X_n) + \ell_1(X_2, \ldots, X_n), \tilde{F})$$

we get

$$\chi^{-1} s F \chi = s \left( \begin{array}{c} X_1 + \tilde{a}(X_2, \ldots, X_n) + \ell_1(X_2, \ldots, X_n) \\ X_2 + \ell_2(X_3, \ldots, X_n) \\ \vdots \\ X_{n-1} + \ell_{n-1}(X_n) \\ X_n \end{array} \right)$$

Now we only have to make the first component linear. Let $r$ be the rank of the leading monomial in $\tilde{a}(X_2, \ldots, X_n)$. With lemma 3.6 we know that there exists a $\varphi_r$ such that

$$\varphi_r^{-1} \chi^{-1} s F \chi \varphi_r = s \left( \begin{array}{c} X_1 + \tilde{a}_r(X_2, \ldots, X_n) + \ell_1(X_2, \ldots, X_n) \\ X_2 + \ell_2(X_3, \ldots, X_n) \\ \vdots \\ X_{n-1} + \ell_{n-1}(X_n) \\ X_n \end{array} \right)$$

where the rank of the leading monomial of $\tilde{a}_r(X_2, \ldots, X_n) < r$. So after $r$ applica-
tions of lemma 3.6 we have obtained a sequence \( \varphi_1, \ldots, \varphi_r \) such that

\[
\varphi_1^{-1} \cdots \varphi_r^{-1} s F \chi \varphi_r \cdots \varphi_1 = s \begin{pmatrix}
X_1 + \ell_1(X_2, \ldots, X_n) \\
X_2 + \ell_2(X_3, \ldots, X_n) \\
\vdots \\
X_{n-1} + \ell_{n-1}(X_n) \\
X_n
\end{pmatrix}
\]

which proves the theorem. \( \square \)

References


