Minimal bi-invariant Hilbert subspaces of distributions on nilpotent Lie groups

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Introduction and notation

Let $G$ be a unimodular Lie group. Spaces of test functions and distributions on $G$ (in the sense of Schwartz [13]) will be denoted by $D(G)$ and $D'(G)$ respectively. Between the spaces $D'(G)$ and $D(G)$ an antiduality $\langle \cdot, \cdot \rangle$ is set up by

$$\langle T, \varphi \rangle = T(\tilde{\varphi}), \quad T \in D'(G), \varphi \in D(G).$$

For all $\varphi \in D(G)$ the element $\tilde{\varphi} \in D(G)$ is defined by

$$\tilde{\varphi}(g) = \varphi(g^{-1}), \quad g \in G.$$

The symbol $*$ will denote convolution on $G$.

A Hilbert subspace of distributions is understood to be a Hilbert space which is continuously imbedded in $D'(G)$. It turns out (see Schwartz [14], [15]) that a Hilbert subspace $\mathcal{H}$ of $D'(G)$ is completely characterized by its reproducing kernel, being the linear map $H : D(G) \to \mathcal{H}$ such that:

$$\langle T, \varphi \rangle = (T, H\varphi)_\mathcal{H}, \quad T \in \mathcal{H}, \varphi \in D(G),$$

where $(\cdot, \cdot)_\mathcal{H}$ denotes the inner product on $\mathcal{H}$.

Conversely, every linear map $H : D(G) \to D'(G)$ such that $\langle H\varphi, \varphi \rangle \geq 0$ for all $\varphi \in D(G)$, is the reproducing kernel of a unique Hilbert subspace of $D'(G)$.

Let $\mathcal{H}$ be a Hilbert space of $D'(G)$ and $u : D'(G) \to D'(G)$ a continuous linear map. Denoting the adjoint of $u$ by $u^*$, the expression $uHu^*$ defines a kernel. The Hilbert subspace corresponding to this kernel is said to be the image of $\mathcal{H}$ under $u$ and is denoted by $u(\mathcal{H})$. 
In particular, if \( \lambda \) is a positive scalar then the image of \( \mathfrak{H} \) under the linear map \( T \rightarrow \sqrt{\lambda} T \) will be denoted by \( \lambda \mathfrak{H} \). So \( \lambda \mathfrak{H} = \mathfrak{H} \) as a linear space and \( (S, T) \mathfrak{H} = \frac{1}{\lambda} (S, T) \mathfrak{H} \) for all \( S, T \in \mathfrak{H} \).

If \( \mu(\mathfrak{H}) = \mathfrak{H} \) then \( \mathfrak{H} \) is said to be invariant under \( \mu \); this is the case if and only if \( \mu \) acts as a unitary operator on \( \mathfrak{H} \).

Given a non-zero element \( T \in D'(G) \), the one dimensional Hilbert subspace \([T]\) of \( D'(G) \) is understood to be:
\[
[T] = \{ \alpha T | \alpha \in \mathbb{C} \}
\]

where the inner product is defined by \( \langle \alpha T, \beta T \rangle = \alpha \bar{\beta} \).

A Hilbert subspace \( \mathfrak{H} \) of \( D'(G) \) is said to be bi-invariant if it is invariant under left and right translations on \( G \) and if these translations act as unitary operators on \( \mathfrak{H} \); an example is presented by \( L^2(G) \).

It can be proved (see Maurin [8], p. 196) that in case of a bi-invariant Hilbert subspace the reproducing kernel \( H \) is given by
\[
H \varphi = S * \varphi = \varphi * S, \quad \varphi \in D(G),
\]
where \( S \) is on \( G \) a central distribution of positive type.

Now, let \( \mathfrak{H} \) be a bi-invariant Hilbert subspace of \( D'(G) \) and let \( L \) and \( R \) be the left and right regular representation of \( G \) on \( D'(G) \). For every \( x, y \in G \), denote the restriction of the linear operator \( L_x R_y \) to \( \mathfrak{H} \) by \( B(x, y) \), thus defining a unitary representation of \( G \times G \) on \( \mathfrak{H} \). The representation \( B \) will be referred to as the bi-regular representation on \( \mathfrak{H} \). The space \( \mathfrak{H} \) is said to be minimal bi-invariant if \( B \) is irreducible.

Let \( U \) be an irreducible representation of \( G \) on a Hilbert space \( \mathfrak{K} \) and assume that for all \( \varphi \in D(G) \) the operator \( U(\varphi) \) is Hilbert-Schmidt. By a result of Dixmier and Malliavin (see [3]), every \( \varphi \in D(G) \) can be written as a sum of elements \( \alpha * \beta \) where \( \alpha, \beta \in D(G) \). It follows that the operators \( U(\varphi) \) are actually of trace class. Denote the distribution \( \varphi \rightarrow \text{tr} U(\varphi) \), the character of \( U \), by \( \mathcal{X} \). A kernel \( H_\varphi \) can now be defined by
\[
H_\varphi \varphi = \mathcal{X} * \varphi, \quad \varphi \in D(G),
\]
or, equivalently, by
\[
\langle H_\varphi, \psi \rangle = \text{tr} (\overline{U(\psi)} U(\varphi)), \quad \varphi, \psi \in D(G).
\]
The corresponding Hilbert subspace of \( D'(G) \) will be denoted by \( \mathfrak{K}_U \).

**Theorem 1.** The Hilbert subspace \( \mathfrak{K}_U \) is minimal bi-invariant and the bi-regular representation on \( \mathfrak{K}_U \) is equivalent to \( U \times U \).
Proof. Let \( \mathcal{R} \) be the conjugate space of \( \mathcal{K} \) and let \( \xi \to \bar{\xi} \) be the canonical anti-isomorphism from \( \mathcal{K} \) onto \( \mathcal{R} \). For all \( \xi, \eta \in \mathcal{K} \) the continuous map \( g \to (U(g)\eta, \xi) \) will be denoted by \( (U(\cdot)\eta, \xi) \). Define the linear map \( \Phi : \mathcal{K} \otimes \mathcal{R} \to D'(G) \) by

\[
\Phi(\xi \otimes \eta) = (U(\cdot)\eta, \xi).
\]

Now the condition that \( U(\phi) \) be Hilbert-Schmidt for all \( \phi \in D(G) \) is the same as saying that \( \Phi \) is continuous; its continuous extension to \( \mathcal{K} \otimes \mathcal{R} \) will be denoted by \( \tilde{\Phi} \). It is easily verified that

\[
L_x R_y \Phi = \Phi[U(x) \otimes U(y)], \quad x, y \in G.
\]

By irreducibility of \( \bar{U} \times U \) (see [7]) it follows that \( \tilde{\Phi} \) is injective.

Define \( \mathcal{H} = \tilde{\Phi}(\mathcal{K} \otimes \mathcal{R}) \) and transport (by means of \( \tilde{\Phi} \)) the Hilbert space structure of \( \mathcal{K} \otimes \mathcal{R} \) onto \( \mathcal{H} \). Now \( \mathcal{H} \) is a bi-invariant Hilbert subspace of \( D'(G) \) and the bi-regular representation on \( \mathcal{H} \) is equivalent to \( \bar{U} \times U \). This proves that \( \mathcal{H} \) is minimal bi-invariant.

Next we determine the reproducing kernel \( H \) of \( \mathcal{H} \) to prove that \( \mathcal{H} = \mathcal{H}_U \).

Choose an orthonormal base \( (\xi_i) \) in \( \mathcal{K} \). By construction of \( \mathcal{H} \), the functions \( (U(\cdot)\xi_i, \xi_j) \) form an orthonormal base in \( \mathcal{H} \).

Expanding \( H\phi \) in terms of this base one easily verifies that for all \( \phi, \psi \in D(G) \)

\[
\langle H\phi, \psi \rangle = \text{tr}(U(\psi)^* U(\phi)) = \langle H_U \phi, \psi \rangle.
\]

It follows that \( H = H_U \); consequently \( \mathcal{H} = \mathcal{H}_U \).

Following the lines of Godement in [4] and Klamer in [6], a converse to theorem 1 can be proved:

**Theorem 2.** If \( G \) is a unimodular Lie group of type I then to every minimal bi-invariant Hilbert subspace \( \mathcal{H} \) of \( D'(G) \) there corresponds (up to equivalence) a unique \( U \) such that \( \mathcal{H} = \lambda \mathcal{H}_U \).

**Proof.** Let \( \mathcal{H} \) be a minimal bi-invariant Hilbert subspace of \( D'(G) \) with reproducing kernel \( H \). Denote the dense linear subspace \( H(D(G)) \) of \( \mathcal{H} \) by \( \mathcal{H}_0 \) and equip this space with the inner product of \( \mathcal{H} \).

The space \( \mathcal{H}_0 \) can be turned into an irreducible Hilbert algebra (see Dixmier [2], Godement [4]) by defining a multiplication:

\[
H\phi \cdot H\psi = H(\phi \ast \psi) = (H\phi) \ast \psi = \phi \ast (H\psi)
\]

and an involution:

\[
H^* = H^*.
\]

The restriction of the left regular representation \( L \) to \( \mathcal{H} \) will be denoted by \( L|_\mathcal{H} \), the von Neumann algebra generated by \( L|_\mathcal{H} \) by \( \mathcal{L} \).
Now on the one hand one has on $\mathcal{L}$ a semi finite normal trace $\text{Tr}$ (the natural trace corresponding to the Hilbert algebra $\mathfrak{S}_0$) such that for all $\varphi, \psi \in D(G)$

$$\text{Tr}(L(\varphi)|_g^* L(\varphi)|_g) = (H\varphi, H\psi) = \langle H\varphi, \psi \rangle.$$ 

On the other hand, $\mathcal{L}$ being a factor of type I, there exists a closed $\mathcal{L}$-invariant subspace $\mathfrak{S}$ of $\mathfrak{S}$ such that the representation $U = L|_{\mathfrak{S}}$ is irreducible. Denoting the standard trace of an operator on $\mathfrak{S}$ by $\text{tr}$, a second semi finite normal trace on $\mathcal{L}$ can be defined by

$$A \to \text{tr}(A|_{\mathfrak{S}_0}) \quad (A \in \mathcal{L}_+).$$

Both traces on the factor $\mathcal{L}$ are necessarily proportional (see [2]).

It follows that for some $\lambda > 0$ one has for all $\varphi, \psi \in D(G)$

$$\langle H\varphi, \psi \rangle = \text{Tr}(L(\varphi)|_g^* L(\varphi)|_g) = \lambda \text{tr}(L(\varphi)|_g^* L(\varphi)|_g) = \lambda \text{tr}(U(\varphi)|_g^* U(\varphi)) = \lambda \langle U\varphi, \psi \rangle.$$ 

This proves that $H = \lambda H_G$, that is $\mathfrak{S}_0 = \lambda \mathfrak{S}_G$. Unicity of $U$ is left to the reader.

The structure of the space $\mathfrak{S}_G$ in case of a flat orbit

Let $G$ be a simply connected nilpotent Lie group, $\mathfrak{g}$ the associated Lie algebra and $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. It is well known that in this case the exponential map $\exp: \mathfrak{g} \to G$ is a global diffeomorphism. The inverse of this map will be denoted by $\log: G \to \mathfrak{g}$. By the exponential map a Lebesgue measure on $\mathfrak{g}$ is transformed into a (left and right) Haar measure on $G$. If $T \in D(G)$, a distribution $T \circ \exp$ on $\mathfrak{g}$ can be defined by

$$(T \circ \exp)(\varphi) = T(\varphi \circ \log), \quad \varphi \in D(G).$$

In this way the notion of composing a function on $G$ with the exponential map is generalized to distributions. The map $T \to T \circ \exp$ is an isomorphism from $D'(G)$ onto $D'(\mathfrak{g})$. The image of a Hilbert subspace $\mathfrak{S}$ under the map $T \to T \circ \exp$ is denoted by $\mathfrak{S} \circ \exp$, being a Hilbert subspace of $D'(\mathfrak{g})$. The Schwartz spaces of tempered distributions on $\mathfrak{g}$ and $\mathfrak{g}^*$ will be denoted by $S'(\mathfrak{g})$ and $S'(\mathfrak{g}^*)$, the Fourier transform from $S'(\mathfrak{g})$ onto $S'(\mathfrak{g}^*)$ by $\mathcal{F}$, and its inverse by $\mathcal{F}^{-1}$.

For a tempered Radon measure $\mu$ on $\mathfrak{g}^*$ the associated Hilbert subspace $A^2(\mu)$ of $S'(\mathfrak{g}^*)$ is defined:

$$A^2(\mu) = \{ f \mu | f \in L^2(\mu) \},$$

where the inner product is given by

$$(f \mu, g \mu) = \int f g \mu.$$ 

Kirillov proved in [5] that, in case of a nilpotent Lie group, there exists a one to one correspondence between irreducible representations of $G$ (up to equivalence) and orbits of
the co-adjoint action of $G$ on $\mathfrak{g}^*$. There exists a canonical invariant Radon measure $\mu_U$ on the orbit corresponding to $U$, such that Kirillov's famous trace formula is valid:

$$\text{tr} \, U(\varphi) = \int \mathcal{F} (\varphi \circ \exp) \, d\mu_U, \quad \varphi \in D(G)$$

(see Corwin and Greenleaf [1]; Kirillov [5]; Pukánszky [11]; Raïs [12]).

The following theorem is stated in the notations introduced above:

**Theorem 3.** Let $G$ be a simply connected nilpotent Lie group. If $U$ is an irreducible unitary representation corresponding to a flat orbit of the co-adjoint action, then

$$\mathcal{S}_U \circ \exp = \mathcal{F}A^2 (\mu_U).$$

**Proof.** Let $\mathcal{O}$ be a non-trivial flat orbit of dimension $m$ in $\mathfrak{g}^*$. Then $\mathcal{O} = \mathcal{V} + F$, where $\mathcal{V}$ is a linear subspace of $\mathfrak{g}^*$ and $F \in \mathfrak{g}^*$ such that $F \notin \mathcal{V}$. Now both $\mathcal{V}$ and the linear span $\mathcal{V} + \mathbb{R} F$ of $\mathcal{V}$ and $F$ are invariant under the co-adjoint action. Therefore, by Engel's theorem there exists a base $\{L_1, \ldots, L_n\}$ of $\mathfrak{g}^*$ such that

$$L_1, \ldots, L_m \in \mathcal{V},$$

$$L_{m+1} = F,$$

$$(\text{Ad} \, g)^* L_i \in \mathbb{R} L_1 + \cdots + \mathbb{R} L_{i-1} + L_i, \quad g \in G, \ i = 1, \ldots, n. ~ \text{Let} \ \{X_1, \ldots, X_n\} \ \text{be the co-base in} \ \mathfrak{g} \ \text{of} \ \{L_1, \ldots, L_n\}. \ \text{Then:}$$

$$\text{Ad} \, g \, X_i \in X_i + \mathbb{R} X_{i+1} + \cdots + \mathbb{R} X_n, \quad g \in G, \ i = 1, \ldots, n.$$

Through the bases $\{X_1, \ldots, X_n\}$ and $\{L_1, \ldots, L_n\}$ the spaces $\mathfrak{g}$ and $\mathfrak{g}^*$ are identified with $\mathbb{R}^m$. In this way the orbit $\mathcal{O}$ in $\mathfrak{g}^*$ can be described as:

$$\mathcal{O} = \mathbb{R}^m \times \{1\} \times \{0\} \times \cdots \times \{0\}.$$ 

Denoting the Lebesgue measure on $\mathbb{R}^m$ by $\lambda$ and the Dirac measure in the point $a$ by $\delta_a$, an invariant measure $\mu$ on $\mathcal{O}$ is defined:

$$\mu = \lambda \otimes \delta_1 \otimes \delta_0 \otimes \cdots \otimes \delta_0.$$ 

Now one has:

$$A^2 (\mu) = L^2 (\mathbb{R}^m) \otimes [\delta_1] \otimes [\delta_0] \otimes \cdots \otimes [\delta_0].$$

Consequently, writing $e^{i\tau}$ for the function $t \rightarrow e^{i\tau}$:

$$(*) \quad \mathcal{F}A^2 (\mu) = L^2 (\mathbb{R}^m) \otimes [e^{i\tau}] \otimes [1] \otimes \cdots \otimes [1].$$

The image of $\mathcal{F}A^2 (\mu)$ under the linear map $T \rightarrow T \circ \log$ from $D'(\mathfrak{g})$ onto $D'(G)$ will be denoted by $\mathcal{F}A^2 (\mu) \circ \log$ or, shortly, by $S_\mu$.

Modifying $\mu$, if necessary, by a scalar factor it turns out that $S_\mu = S_U$, where $U$ is the irreducible unitary representation corresponding to $\mathcal{O}$. 
To prove this, the operator $\mathcal{L}_A : g \to g$ is introduced:

$$\exp A \exp X = \exp \mathcal{L}_A(X).$$

The Baker-Campbell-Hausdorff formula implies that for all $A = (a_1, \ldots, a_n)$ and $X = (x_1, \ldots, x_n)$

$$\mathcal{L}_A(X) = (x_1 + a_1, x_2 + a_2, x_3 + a_3 + Q_1(a_1, a_2, x_1, x_2), \ldots$$

$$\ldots, x_n + a_n + Q_{n-2}(a_1, \ldots, a_{n-1}, x_1, \ldots, x_{n-1}))$$

where the $Q_i$ are polynomials.

By $(\ast)$ an element $f \in \mathcal{F}A^2(\mu)$ is of the form:

$$f(x_1, \ldots, x_n) = \varphi(x_1, \ldots, x_m)e^{ixm^{n+1}},$$

where $\varphi \in L^2(\mathbb{R}^m)$. But then:

$$(f \circ \mathcal{L}_A)(x_1, \ldots, x_n) = \tilde{\varphi}(x_1, \ldots, x_m)e^{ixm^{n+1}},$$

where $\tilde{\varphi} \in L^2(\mathbb{R}^m)$ with $\|\tilde{\varphi}\|_2 = \|\varphi\|_2$. It thus appears that $\mathcal{L}_A$ acts as a unitary operator on $\mathcal{F}A^2(\mu)$. The space $\mathcal{F}A^2(\mu)$ being invariant under the action of $\mathcal{L}_A$, it follows that $\mathcal{S}_\mu = \mathcal{F}A^2(\mu) \circ \log$ is a left invariant Hilbert subspace of $D'(G)$.

Next, the reproducing kernels of $\mathcal{S}_\mu$ and $\mathcal{S}_U$ are compared to prove that $\mathcal{S}_\mu = \mathcal{S}_U$. Noting that the reproducing kernel of $A^2(\mu)$ is presented by the map $\varphi \to \varphi \mu$, it is easily verified that the kernel of a Hilbert subspace $\mathcal{F}A^2(\mu)$ in $D'(G)$ is given by

$$\varphi \to \mathcal{F}A^2 \ast \varphi, \quad \varphi \in D(G),$$

where $\ast$ denotes the convolution on $G$.

Therefore the reproducing kernel $H_\mu$ of $\mathcal{S}_\mu$ can be described in the following way:

$$\langle H_\mu \varphi, \psi \rangle = \langle \mathcal{F}A^2 \ast (\varphi \circ \exp), \psi \circ \exp \rangle$$

for all $\varphi, \psi \in D(G)$.

Denoting the distribution $\varphi \to \text{tr} U(\varphi)$ on $G$ by $\mathcal{X}$, the reproducing kernel $H_U$ of $\mathcal{S}_U$ was defined as:

$$H_U \varphi = \mathcal{X} \ast \varphi, \quad \varphi \in D(G),$$

where $\ast$ denotes convolution on $G$.

Letting $\psi$ converge to the Dirac measure in the identity $e$ of $G$, and realizing that Kirillov's trace formula is a way of saying that $\mathcal{X} \circ \exp = \mathcal{F}A^2 \mu$ (where, if necessary, $\mu$ is modified by a scalar factor), one has:
Because both $\mathcal{H}_U$ and $\mathcal{H}_\mu$ are left invariant, this implies $(H_U \varphi)(g) = (H_\mu \varphi)(g)$ for all $\varphi \in D(G)$ and all $g \in G$. It follows that $H_U = H_\mu$, that is $\mathcal{H}_U = \mathcal{H}_\mu$. This proves the theorem.

A consequence of theorem 3 is that, in case of a flat orbit, the space $\mathcal{H}_U$ consists of locally square integrable functions. This is a fact of some importance in connection to the Fourier analysis on $G$, in which the spaces $\mathcal{H}_U$ can be seen as building blocks (see Thomas [16]). In theorem 3, as will be seen in the example given below, one can not drop the condition that the orbit be flat. An interesting connection to the work done by C. Moore and A. Wolf in [9] is expressed in the following theorem:

**Theorem 4.** Let $G$ be a simply connected nilpotent Lie group. If $G$ admits a square integrable irreducible unitary representation, then

$$\mathcal{H}_U \circ \exp = \mathcal{F}A^2(\mu_U)$$

for almost every $U$ with respect to the Plancherel measure.

**Proof.** The existence of a square integrable irreducible unitary representation is a guarantee that almost all orbits are flat (see Moore and Wolf [9]).

An example is now given of a three step nilpotent Lie group, where $\mathcal{H}_U \circ \exp = \mathcal{F}A^2(\mu_U)$ for almost every $U$ with respect to the Plancherel measure.

**Example.** Let $G$ be the Lie algebra generated by elements $X_1, X_2, X_3, X_4$ where the non-zero Lie brackets are given by:

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4.$$  

The corresponding simply connected Lie group $G$ is three step nilpotent.

An element $X \in g$ will be denoted by $X = (x_1, x_2, x_3, x_4)$, thus identifying $g$ with $\mathbb{R}^4$. Similarly, the dual $g^*$ of $g$ will be identified with $\mathbb{R}^4$. The element $\ell = (\ell_1, \ell_2, \ell_3, \ell_4)$ is understood to be the linear form $X \to \sum_i \ell_i x_i$. Now the co-adjoint action of $G$ on $g^*$ can be described by:

$$\text{Ad}(\exp X)^*(\ell) = \left( \ell_4 - x_2 \ell_3 - \left( x_3 + \frac{1}{2} x_1 x_2 \right) \ell_4, \ell_2 + x_1 \ell_3 + \frac{1}{2} x_1^2 \ell_4, \ell_3 + x_1 \ell_4, \ell_4 \right).$$

Let $\mathcal{O}$ be the orbit through the point $\ell = (0, 0, 0, 1)$.

The map $\Phi : \mathbb{R}^2 \to \mathcal{O}$, defined by:

$$\Phi(s, t) = \left( s, \frac{1}{2} t^2, t, 1 \right)$$
presents a diffeomorphism by which the Lebesgue measure on \( \mathbb{R}^2 \) is transformed into an invariant measure \( \mu \) on \( \mathcal{O} \).

The Hilbert subspace \( \mathcal{H} = \mathcal{F}A^2(\mu) \) in \( D'(g) \) has a reproducing kernel \( K \) defined by:
\[
\langle K\phi, \varphi \rangle = \langle \mathcal{F}\mu \ast \phi, \varphi \rangle = \langle \mu, |\mathcal{F}\phi|^2 \rangle.
\]

It will be pointed out that the Hilbert subspace \( \mathcal{H} \circ \log \) of \( D'(G) \) is not invariant under left translations, so it cannot be a space of type \( \mathfrak{S}_0 \).

As before, for all \( a \in g \) the operator \( \mathcal{L}_a : g \rightarrow g \) is introduced by:
\[
\exp A \exp X = \exp \mathcal{L}_a(X).
\]

(It turns out, as will be seen in the sequel, that the expression \( \langle K(\varphi \circ \mathcal{L}_a), \varphi \circ \mathcal{L}_a \rangle \) cannot be constant in \( a \). Consequently \( \mathcal{H} \circ \log \) is not invariant under left translations.) In case \( A = (a, 0, 0, 0) \) the operator \( \mathcal{L}_a \) will be denoted by \( \mathcal{L}_a \). By the Baker-Campbell-Hausdorff formula one has:
\[
\mathcal{L}_a(X) = \left( x_1 + a, x_2, x_3 + \frac{1}{2} ax_2, x_4 + \frac{1}{2} ax_3 + \frac{1}{12} a^2 x_2 - \frac{1}{12} ax_1 x_2 \right).
\]

Denoting \( \phi_a = \phi e^{-ix_1 x_2 x_3 x_4 / 12} \), one can derive that:
\[
\mathcal{F}(\varphi \circ \mathcal{L}_a)(\ell) = e^{ita} \mathcal{F}(\varphi_{id})(t) \left( \ell_2 - \frac{1}{2} a \ell_3 + \frac{1}{12} a^2 \ell_4, \ell_3 - \frac{1}{2} a \ell_4, \ell_4 \right).
\]

Next, the expression \( \langle K(\varphi \circ \mathcal{L}_a), \varphi \circ \mathcal{L}_a \rangle \) will be set into a form in which it is clear that it cannot be constant in \( a \):
\[
\langle K(\varphi \circ \mathcal{L}_a), \varphi \circ \mathcal{L}_a \rangle = \langle \mu, |\mathcal{F}(\varphi \circ \mathcal{L}_a)|^2 \rangle = \int \int |\mathcal{F}(\varphi \circ \mathcal{L}_a)(\ell)|^2 d\mu(\ell)
\]
\[
= \int \int |\mathcal{F}(\varphi \circ \mathcal{L}_a)\left( s, \frac{1}{2} t^2, t, 1 \right)|^2 ds dt
\]
\[
= \int \int |\mathcal{F}(\varphi_a)\left( s, \frac{1}{2} t^2 - \frac{1}{2} at + \frac{1}{12} a^2, t - \frac{1}{2} a, 1 \right)|^2 ds dt.
\]

If \( \varphi \) is of the form \( \varphi = \varphi_1 \circ \varphi_2 \circ \varphi_3 \circ \varphi_4 \) then (writing \( \hat{\varphi} \) for the Fourier transform of \( \varphi \)):
\[
\langle K(\varphi \circ \mathcal{L}_a), \varphi \circ \mathcal{L}_a \rangle
\]
\[
= \int \int |\mathcal{F}(\varphi_1 \circ \varphi_2 \varphi_3 \varphi_4 e^{-ix_1 x_2 / 12})(s, \frac{1}{2} t^2 - \frac{1}{2} at + \frac{1}{12} a^2)\hat{\varphi}_3\left( t - \frac{1}{2} a \right)|^2 |\hat{\varphi}_4(1)|^2 ds dt
\]
\[
= |\hat{\varphi}_4(1)|^2 \int |\hat{\varphi}_3(t)|^2 \left\{ \int |\mathcal{F}(\varphi_1 \circ \varphi_2 \varphi_3 \varphi_4 e^{-ix_1 x_2 / 12})(s, \frac{1}{2} t^2 - \frac{1}{24} a^2)|^2 ds \right\} dt.
\]
Now let $|\phi_3|^2$ converge to the Dirac measure in the origin. Then:

$$\lim \langle K(\phi \circ \Omega_a), \phi \circ \Omega_a \rangle = |\phi_4(1)|^2 \int \mathcal{F}(\phi_1 \otimes \phi_2 e^{-i s x_1 x_2/2}) \left( s, -\frac{1}{24} a^2 \right)^2 ds.$$ 

Examing the second order derivative with respect to $a$ in the point $a = 0$, one verifies that

$$\lim \langle K(\phi \circ \Omega_a), \phi \circ \Omega_a \rangle$$

can not be constant in $a$ for all $\phi_1$, $\phi_2$ and $\phi_4$.

If $C^\circ$ is replaced by an orbit through a point $\ell = (0, \alpha, 0, \beta)$, where $\beta \neq 0$, then the same arguments remain valid. It thus appears that $\mathcal{S}_U \circ \exp = \mathcal{F}A^2(\mu_U)$ for almost every $U$ in the Plancherel measure.

It is an interesting note that in this example the irreducible representations $U$ which do satisfy $\mathcal{S}_U \circ \exp = \mathcal{F}A^2(\mu_U)$ are exactly the ones corresponding to flat orbits.

**Problem.** If an irreducible unitary representation $U$ satisfies $\mathcal{S}_U \circ \exp = \mathcal{F}A^2(\mu_U)$, does it follow that the corresponding orbit is flat?

References


