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The Stringer Bound in Case of Uniform Taintings*

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Abstract—The Stringer bound is a widely used nonparametric $100(1-\alpha)\%$ upper confidence bound for the fraction of errors in an accounting population. This bound has been found in practice to be rather conservative. In the present paper, we give recursive relations for obtaining the exact distribution of the Stringer bound in the case where the underlying distribution of the taintings is a uniform distribution on the interval [0,1], or a distribution with positive mass at zero and conditionally uniform on (0,1]. Based on these recurrence relations, we find a concrete counterexample which shows that the Stringer bound is not always conservative.

Keywords—Order statistics, Conservatism of a test, Linear combinations of order statistics, Stringer bound.

1. INTRODUCTION

In the N.R.C. report 'Statistical Models and Analysis in Auditing' (1988), reprinted in *Statistical Science* [1], an excellent presentation has been given on statistical issues and other statistical techniques in auditing. One of the issues which draws attention in this paper is the open question of the Stringer bound problem, which is about 30 years old. The Stringer bound is in fact a linear combination of order statistics of the underlying taintings, where the coefficients have a complicated structure as differences of solutions of certain equations, which cannot be solved explicitly. The coefficients cannot be generated by the help of a fixed score-generating function, so that the problem lies also out of the scope of the well-investigated cases of L-statistics.

The Stringer bound is a widely used $100(1-\alpha)\%$ upper confidence bound for the fraction of errors in an accounting population. Although the bound has been found in practice to be rather conservative, not even an intuitive explanation can be found in auditing literature. Moreover, no rigorous mathematical proof of the correctness of the Stringer bound as an upper confidence bound and also no counterexamples are available, see e.g., [2,3]. However, recently Pap and Van Zuijlen [4] showed that the Stringer bound is asymptotically not conservative for confidence levels $1-\alpha$, with α in the interval $(1/2,1)$ and proposed on the basis of an asymptotic analysis a modified Stringer bound which is asymptotically correct for every nominal confidence level α.

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In the present paper, we will study the distribution of the Stringer bound in a finite sample situation, where the underlying distribution of the taintings is a uniform distribution on the interval [0,1], or a distribution with positive mass at zero and conditionally uniform on (0,1]. This latter distribution plays an important role in auditing. We will present recursive relations for obtaining the exact distribution of the Stringer bound for a sample size $n+1$ from the distribution for a sample size n. These recurrence relations enable us to find in principle for every fixed sample size the distribution of the Stringer bound in the case of the above mentioned underlying distributions of the taintings.

Finally, we will use these recurrence relations in order to find with the aid of a computer program a concrete counterexample in the case of a finite sample size which shows that the Stringer bound is not always conservative. It turns out that the conservatism (and even the validity) of the Stringer bound breaks down for a confidence level below a half and for a sample size not lower than about 16.

For a description of the practical situation which leads to the initial mathematical model in this paper, we refer to [2,4–6] and also to an unpublished manuscript of Gill and Van Zuijlen [7].

2. THE CASE OF UNIFORM DISTRIBUTION

Let $U_1, U_2, \dots, U_n$ be independent random variables, uniformly distributed on the interval $[0,1]$. Let $U_{1:n} \le U_{2:n} \le \dots \le U_{n:n}$ be the ordered sample. For $\alpha \in (0,1)$, let $p_j^{(n)} \in [0,1]$, $j = 0, 1, \dots, n-1$ be defined by

$$\mathbb{P}\left\{U_{j+1:n} \le p_j^{(n)}\right\} = 1 - \alpha.$$

Evidently

$$\mathbb{P}\left\{U_{j+1:n} \le x\right\} = \sum_{k=j+1}^{n} \binom{n}{k} x^k (1-x)^{n-k},$$

which implies $0 < p_0^{(n)} < p_1^{(n)} < \dots < p_{n-1}^{(n)} < 1$. Moreover, let $U_{0:n} = 0$, $U_{n+1:n} = 1$, and $p_{-1}^{(n)} = 0$, $p_n^{(n)} = 1$. The Stringer bound (for the mean taint $\mu = 1/2$ of the variables U_k) is

$$\bar{\mu}_{ST}^{(n)} = \sum_{j=0}^{n} \left(p_j^{(n)} - p_{j-1}^{(n)}\right) U_{n-j+1:n} = \sum_{j=1}^{n+1} p_{n-j+1}^{(n)} \left(U_{j:n} - U_{j-1:n}\right).$$

We will investigate the probability

$$\mathbb{P}\left\{\bar{\mu}_{ST}^{(n)} \ge \frac{1}{2}\right\}.$$

Introduce the variables $V_j^{(n)} = U_{j:n} - U_{j-1:n}$, $j = 1, \dots, n+1$. (They are the so-called uniform spacings.) It is well known that the vector $(U_{1:n}, \dots, U_{n:n})$ is uniformly distributed on the set

$$\{z \in \mathbb{R}^n \mid 0 \le z_1 \le \dots \le z_n \le 1\},$$

with density function

$$f_{U_{1:n},\dots,U_{n:n}}(z_1, \dots, z_n) = \begin{cases} n! & \text{if } 0 \le z_1 \le \dots \le z_n \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

This implies that the density function of the vector $(V_1^{(n)}, \dots, V_n^{(n)})$ is

$$f_{V_1^{(n)},\dots,V_n^{(n)}}(z_1, \dots, z_n) = \begin{cases} n! & \text{if } z_j \ge 0,\ 1 \le j \le n;\ z_1 + \dots + z_n \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

so that the vector $(V_1^{(n)}, \dots, V_n^{(n)})$ is uniformly distributed on the set

$$\{z \in \mathbb{R}^n \mid z_j \geq 0,\ 1 \leq j \leq n;\ z_1 + \cdots + z_n \leq 1\}.$$

Consequently, for $z_j \geq 0$, $1 \leq j \leq n+1$, $z_1 + \cdots + z_{n+1} \leq 1$, we have

$$\begin{aligned}
&\mathbb{P}\left\{V_1^{(n)} \geq z_1, \dots, V_{n+1}^{(n)} \geq z_{n+1}\right\} \\
&\quad = \mathbb{P}\left\{V_1^{(n)} \geq z_1, \dots, V_n^{(n)} \geq z_n, V_1^{(n)} + \cdots + V_n^{(n)} \leq 1 - z_{n+1}\right\} \\
&\quad = (1 - z_1 - \cdots - z_{n+1})^n .
\end{aligned}$$

Hence, the vector $(V_1^{(n)}, \dots, V_{n+1}^{(n)})$ is uniformly distributed on the set

$$\mathcal{G} = \left\{z \in \mathbb{R}^{n+1} \mid z_j \geq 0,\ 1 \leq j \leq n+1;\ z_1 + \cdots + z_{n+1} = 1\right\},$$

which is a regular n-dimensional pyramid. The Stringer bound can be expressed as

$$\bar{\mu}_{ST}^{(n)} = \sum_{j=1}^{n+1} p_{n-j+1}^{(n)} V_j^{(n)},$$

and since the distribution of the vector $(V_1^{(n)}, \dots, V_{n+1}^{(n)})$ does not change after permutations of its coordinates, we obtain

$$\mathbb{P}\left\{\bar{\mu}_{ST}^{(n)} \geq \frac{1}{2}\right\} = \mathbb{P}\left\{\sum_{j=1}^{n+1} p_{j-1}^{(n)} V_j^{(n)} \geq \frac{1}{2}\right\}.$$

Definition 1. *The sequence of functions* $f_n : \{x \in \mathbb{R}^n \mid 0 < x_1 < \cdots < x_n \leq 1\} \to [0,1]$, $n = 2, 3, \dots$, *is defined as*

$$f_n(x_1, \dots, x_n) = \mathbb{P}\left\{\sum_{k=1}^{n} x_k V_k^{(n-1)} \geq \frac{1}{2}\right\}.$$

We have the relation

$$\mathbb{P}\left\{\bar{\mu}_{ST}^{(n)} \geq \frac{1}{2}\right\} = f_{n+1}\left(p_0^{(n)}, \dots, p_{n-1}^{(n)}, 1\right).$$

Proposition 1. *If* $x_j = 1/2$ *for some* $j \in \{1, \dots, n+1\}$, *then*

$$f_{n+1}\left(x_1, \dots, x_{j-1}, \frac{1}{2}, x_{j+1}, \dots, x_{n+1}\right) = f_n(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}). \tag{1}$$

Proof. The left hand side is equal to

$$\mathbb{P}\left\{\sum_{k \neq j} x_k V_k^{(n)} + \frac{1}{2} V_j^{(n)} \geq \frac{1}{2}\right\} = \mathbb{P}\left\{\sum_{k \neq j} x_k \widetilde{V}_k^{(n)} \geq \frac{1}{2}\right\},$$

where the joint distribution of the variables

$$\left(\widetilde{V}_k^{(n)} = \frac{V_k^{(n)}}{1 - V_j^{(n)}};\ k \neq j,\ 1 \leq k \leq n+1\right)$$

is the same as the distribution of $(V_k^{(n-1)}\,;\ 1 \le k \le n)$. (The distribution of the vector

$$\left(V_1^{(n)}, \dots, V_{j-1}^{(n)}, V_{j+1}^{(n)}, \dots, V_{n+1}^{(n)}\right)$$

under the condition $V_j^{(n)} = y$ is uniform on the set

$$\{z \in \mathbb{R}^n \mid z_k \ge 0,\ 1 \le k \le n;\ z_1 + \dots + z_n = 1 - y\}\,;$$

thus, the distribution of the vector

$$\left(\widetilde{V}_1^{(n)}, \dots, \widetilde{V}_{j-1}^{(n)}, \widetilde{V}_{j+1}^{(n)}, \dots, \widetilde{V}_{n+1}^{(n)}\right)$$

under the condition $V_j^{(n)} = y$ is uniform on the set

$$\{z \in \mathbb{R}^n \mid z_k \ge 0,\ 1 \le k \le n;\ z_1 + \dots + z_n = 1\}\,,$$

which does not depend on y; hence, its unconditional distribution is the same.) ∎

PROPOSITION 2. *If* $1/2 \in (x_j, x_{j+1})$ *for some* $j \in \{1, 2, \dots, n\}$, *then*

$$\begin{aligned} &f_{n+1}(x_1, \dots, x_{n+1}) \\ &\quad = y_{j+1}^{j} f_n(x_1, \dots, x_j, x_{j+2}, \dots, x_{n+1}) + y_j^{j+1} f_n(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}), \end{aligned} \tag{2}$$

where

$$y_j^i = \frac{x_i - (1/2)}{x_i - x_j}.$$

Remark that

$$y_j^i + y_i^j = 1.$$

PROOF. The statement (2) has the form

$$A = y_{j+1}^j B + y_j^{j+1} C$$

which can be written as

$$y_{j+1}^j A + y_j^{j+1} A = y_{j+1}^j B + y_j^{j+1} C.$$

Thus, we have to prove

$$\frac{A - B}{C - A} = \frac{y_j^{j+1}}{y_{j+1}^j}.$$

On the basis of Proposition 1 and the condition $(1/2) \in (x_j, x_{j+1})$, it is equivalent to

$$\frac{\mathbb{P}\left\{\sum_{k \ne j+1} x_k V_k^{(n)} + (1/2) V_{j+1}^{(n)} < (1/2) \le \sum_{k=1}^{n+1} x_k V_k^{(n)}\right\}}{\mathbb{P}\left\{\sum_{k=1}^{n+1} x_k V_k^{(n)} < (1/2) \le \sum_{k \ne j} x_k V_k^{(n)} + (1/2) V_j^{(n)}\right\}} = \frac{x_{j+1} - (1/2)}{(1/2) - x_j}.$$

Since the vector $(V_1^{(n)}, \dots, V_{n+1}^{(n)})$ is uniformly distributed in the n-dimensional pyramid

$$\mathcal{G} = \left\{z \in \mathbb{R}^{n+1} \mid 0 \le z_j,\ 1 \le j \le n+1;\ z_1 + \dots + z_{n+1} = 1\right\},$$

the above ratio is equal to the ratio of the volumes of the (nonregular) n-dimensional pyramids

$$\mathcal{G}_1 = \left\{ z \in \mathbb{R}^{n+1} \;\middle|\; \sum_{k \ne j+1} x_k z_k + \frac{1}{2} z_{j+1} < \frac{1}{2} \le \sum_{k=1}^{n+1} x_k z_k \right\} \bigcap \mathcal{G}$$

and

$$\mathcal{G}_2 = \left\{ z \in \mathbb{R}^{n+1} \;\middle|\; \sum_{k=1}^{n+1} x_k z_k < \frac{1}{2} \le \sum_{k \neq j} x_k z_k + \frac{1}{2} z_j \right\} \bigcap \mathcal{G}.$$

These pyramids have the same base (namely the set $\{z \in \mathbb{R}^{n+1} \mid \sum_{k=1}^{n+1} x_k z_k = (1/2)\} \bigcap \mathcal{G}$), and the vertex of $\mathcal{G}_1$, respectively, $\mathcal{G}_2$ is the point A_{j+1}, respectively, A_j, where A_k is the point whose k^{th} coordinate is 1 and the other coordinates are 0. The ratio of the heights of the pyramids $\mathcal{G}_1$ and $\mathcal{G}_2$ is equal to the ratio of the length of the sections $\overline{A_{j+1}C}$ and $\overline{CA_j}$, where the point C is the intersection of the line joining the points A_j and A_{j+1}, and the hyperplane $\{z \in \mathbb{R}^{n+1} \mid \sum_{k=1}^{n+1} x_k z_k = (1/2)\}$. This ratio is just

$$\frac{x_{j+1} - (1/2)}{(1/2) - x_j},$$

and the ratio of the volumes of $\mathcal{G}_1$ and $\mathcal{G}_2$ is the same. Thus, (2) is proved. ∎

REMARK 1. Clearly

$$f_2(x_1, x_2) = \begin{cases} 0, & \text{if } 0 < x_1 < x_2 \le \frac{1}{2}, \\ y_1^2, & \text{if } 0 < x_1 \le \frac{1}{2} < x_2 \le 1, \\ 1, & \text{if } \frac{1}{2} \le x_1 < x_2 \le 1, \end{cases} \tag{3}$$

since

$$f_2(x_1, x_2) = \mathbb{P}\left\{ x_1 V_1^{(1)} + x_2 V_2^{(1)} \ge \frac{1}{2} \right\} = \mathbb{P}\left\{ (x_2 - x_1) V_1^{(1)} \le x_2 - \frac{1}{2} \right\}.$$

Using (1) and the recursive equation (2), we obtain

$$f_3(x_1, x_2, x_3) = \begin{cases} 0, & \text{if } 0 < x_1 < x_2 < x_3 \le \frac{1}{2}, \\ y_1^3 y_2^3, & \text{if } 0 < x_1 < x_2 \le \frac{1}{2} < x_3 \le 1, \\ y_1^3 y_2^1 + y_1^2, & \text{if } 0 < x_1 \le \frac{1}{2} < x_2 < x_3 \le 1, \\ 1, & \text{if } \frac{1}{2} < x_1 < x_2 < x_3 \le 1, \end{cases}$$

and

$$f_4(x_1, x_2, x_3, x_4) = \begin{cases} 0, & \text{if } 0 < x_1 < x_2 < x_3 < x_4 \le \frac{1}{2}, \\ y_1^4 y_2^4 y_3^4, & \text{if } 0 < x_1 < x_2 < x_3 \le \frac{1}{2} < x_4 \le 1, \\ y_1^4 y_2^4 y_3^2 + \left(y_1^4 y_3^1 + y_1^3\right) y_2^3, & \text{if } 0 < x_1 < x_2 \le \frac{1}{2} < x_3 < x_4 \le 1, \\ \left(y_1^4 y_3^1 + y_1^3\right) y_2^1 + y_1^2 & \text{if } 0 < x_1 \le \frac{1}{2} < x_2 < x_3 < x_4 \le 1, \\ 1, & \text{if } \frac{1}{2} < x_1 < x_2 < x_3 < x_4 \le 1. \end{cases}$$

REMARK 2. For $n = 2, 3, \ldots$, $i = 1, 2, \ldots, n$, let $f_n(i)$ be the i^{th} row in the formula for f_n. By definition, we put $f_n(0) = 0$ and $f_n(n+1) = 1$. For example, we find from formula (3) that

$$\begin{aligned} f_2(1) &= y_1^2 \\ f_2(2) &= 1. \end{aligned}$$

For $j = 1, 2, \ldots, n$, denote by $f_n(i, j)$ the formula $f_n(i)$ where the indices $j, j+1, \ldots, n$ are replaced by $j+1, j+2, \ldots, n+1$, respectively. From Proposition 2, we have the following recursive procedure to generate $f_{n+1}(i)$:

$$f_{n+1}(i) = y_{n-i+2}^{n-i+1} f_n(i-1, n-i+2) + y_{n-i+1}^{n-i+2} f_n(i, n-i+1)$$

for $i = 1, \ldots, n+1$.

REMARK 3. One can prove easily the inequality $\mathbb{P}\left\{\bar{\mu}_{ST}^{(n)} \geq (1/2)\right\} = f_{n+1}(p_0^{(n)}, \dots, p_{n-1}^{(n)}, 1) \geq 1-\alpha$ for $n = 1, 2$. We shall show it for $n = 3$. We write p_j instead of $p_j^{(3)}$. We have $p_0 = 1 - \sqrt[3]{\alpha}$ and $p_2 = \sqrt[3]{1-\alpha}$.

First we consider the case $\alpha \in [7/8, 1)$. Then $0 < p_0 < p_1 < p_2 \leq 1/2$, thus

$$f_4(p_0, p_1, p_2, 1) = y_1^4 y_2^4 y_3^4 \geq \left(y_1^4\right)^3 = \frac{1}{8(1-p_0)^3} = \frac{1}{8\alpha} \geq \frac{1}{8} \geq 1 - \alpha.$$

In the case $\alpha \in [1/2, 7/8)$, we have $0 < p_0 < p_1 \leq (1/2) < p_2 < 1$ and

$$\begin{aligned} f_4(p_0, p_1, p_2, 1) &= y_1^4 y_2^4 y_3^2 + \left(y_1^4 y_3^1 + y_1^3\right) y_2^3 \geq \min\left\{y_1^4 y_2^4, y_1^4 y_3^1 + y_1^3\right\} \\ &\geq \min\left\{y_1^4 y_2^4, y_1^4, 1\right\} = y_1^4 y_2^4 = \frac{1}{4(1-p_0)(1-p_1)} = \frac{1}{4\alpha}\frac{(1-p_0)^2}{1-p_1} \geq 1-\alpha, \end{aligned}$$

since $\alpha = (1-p_0)^3 = (1-p_1)^3 + 3p_1(1-p_1)^2$ and $0 \leq p_1 \leq (1/2)$ imply

$$\frac{(1-p_0)^2}{1-p_1} = \frac{(1-p_1)(1+2p_1)}{1-p_0} \geq 1.$$

In the case $\alpha \in [1/8, 1/2)$, we have $0 < p_0 \leq (1/2) < p_1 < p_2 < 1$ and

$$\begin{aligned} f_4(p_0, p_1, p_2, 1) &= \left(y_1^4 y_3^1 + y_1^3\right) y_2^1 + y_1^2 = \left(\left(1 - y_4^1\right) y_3^1 + 1 - y_3^1\right) y_2^1 + 1 - y_2^1 \\ &= 1 - y_2^1 y_3^1 y_4^1 \geq 1 - y_4^1 \geq 1 - \alpha, \end{aligned}$$

since

$$y_4^1 = 1 - \frac{1}{2(1-p_0)} = 1 - \frac{1}{2\sqrt[3]{\alpha}} \leq \alpha.$$

In the case $\alpha \in (0, 1/8)$, we have $(1/2) < p_0 < p_1 < p_2 < 1$, and $f_4(p_0, p_1, p_2, 1) = 1$ trivially implies the inequality.

REMARK 4. Using the recursive equations as in Remarks 1 and 2, one can show that the inequality $\mathbb{P}\left\{\bar{\mu}_{ST}^{(n)} \geq (1/2)\right\} \geq 1 - \alpha$ does not hold for $n = 20$ and α near to 1. Figure 1 contains the cases $n = 1, 2, 5, 10, 20$ and the limit function as $n \to \infty$. This limit function $\alpha \mapsto \Phi(rz_{1-\alpha})$ with Φ denoting the standard normal distribution function, $z_{1-\alpha} = \Phi^{-1}(1-\alpha)$ and $r = \pi\sqrt{3}/4$ has been derived in [4].

3. THE CASE OF TAINTINGS WITH POSITIVE MASS AT ZERO

Now let $T_1, T_2, \dots, T_n$ be independent, identically distributed random variables in the interval $[0, 1]$ such that $\mathbb{P}\{T_k < x \mid T_k > 0\} = x$ for $x \in [0, 1]$. Let $\varrho = \mathbb{P}\{T > 0\}$. Now the mean taint is

$$\mu = \mathbb{E}(T) = \frac{\varrho}{2}.$$

Let $T_{1:n} \leq T_{2:n} \leq \cdots \leq T_{n:n}$ be the ordered sample. Moreover, let $T_{0:n} = 0$, $T_{n+1:n} = 1$. The Stringer bound is

$$\bar{\mu}_{ST}^{(n)} = \sum_{j=0}^{n} \left(p_j^{(n)} - p_{j-1}^{(n)}\right) T_{n-j+1:n}$$

and we shall investigate the probability

$$\mathbb{P}\left\{\bar{\mu}_{ST}^{(n)} \geq \frac{\varrho}{2}\right\}.$$

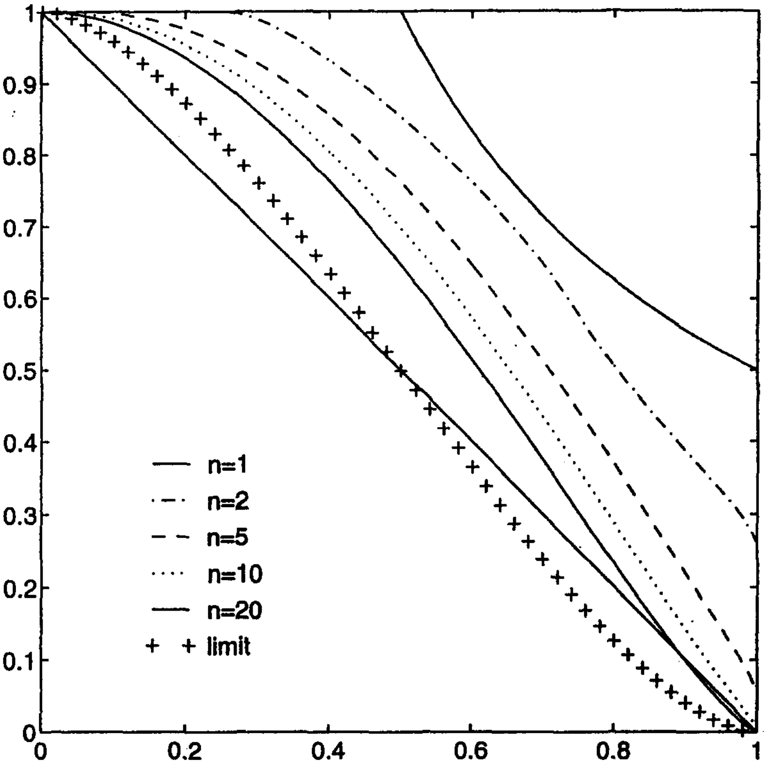

Figure 1. The value of $\mathbb{P}\left\{\bar{\mu}_{ST}^{(n)} \geq (1/2)\right\}$ as a function of α.

Let M_n be the number of the nonzero $T_1, \dots, T_n$. We have

$$\mathbb{P}\left\{\bar{\mu}_{ST}^{(n)} \geq \frac{\varrho}{2}\right\} = \sum_{k=0}^{n} \binom{n}{k} \varrho^k (1-\varrho)^{n-k} \mathbb{P}\left\{\bar{\mu}_{ST}^{(n)} \geq \frac{\varrho}{2} \;\middle|\; M_n = k\right\}.$$

For $k = 1, \dots, n$, one can show that the distribution of the vector $(T_{n-k+1:n}, \dots, T_{n:n})$ under the condition $M = k$ is the same as the (unconditional) distribution of the vector $(U_{1:k}, \dots, U_{k:k})$. Hence,

$$\mathbb{P}\left\{\bar{\mu}_{ST}^{(n)} \geq \frac{\varrho}{2} \;\middle|\; M_n = k\right\} = \mathbb{P}\left\{\sum_{j=0}^{k} \left(p_j^{(k)} - p_{j-1}^{(k)}\right) U_{k-j+1:k} \geq \frac{\varrho}{2}\right\}.$$

DEFINITION 2. *The sequence of functions* $g_n : \{x \in \mathbb{R}^n \mid 0 < x_1 < \cdots < x_n < 1\} \to [0,1]$, $n = 2, 3, \dots,$ *is defined as*

$$g_n(x_1, \dots, x_n) = \mathbb{P}\left\{\sum_{k=1}^{n} x_k V_k^{(n-1)} \geq \frac{\varrho}{2}\right\}.$$

As in Section 2, one can show the following two properties.

PROPOSITION 3. *If* $x_j = (\varrho/2)$ *for some* $j \in \{1, 2, \dots, n+1\}$, *then*

$$g_{n+1}\left(x_1, \dots, x_{j-1}, \frac{\varrho}{2}, x_{j+1}, \dots, x_{n+1}\right) = g_n(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}). \tag{4}$$

PROPOSITION 4. *If* $(\varrho/2) \in (x_j, x_{j+1})$ *for some* $j \in \{1, 2, \dots, n\}$, *then*

$$\begin{aligned} &g_{n+1}(x_1, \dots, x_{n+1}) \\ &\quad = u_{j+1}^{j} g_n(x_1, \dots, x_j, x_{j+2}, \dots, x_{n+1}) + u_j^{j+1} g_n(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}), \end{aligned} \tag{5}$$

where

$$u_j^i = \frac{x_i - (\varrho/2)}{x_i - x_j}.$$

Finally, we obtain

$$\mathbb{P}\left\{\bar{\mu}_{ST}^{(n)} \geq \frac{\varrho}{2}\right\} = \sum_{k=0}^{n} \binom{n}{k} \varrho^k (1-\varrho)^{n-k} g_{k+1}\left(p_0^{(n)}, \ldots, p_k^{(n)}\right),$$

where the function g_1 is defined by

$$g_1(x_1) = \begin{cases} 0, & \text{if } 0 < x_1 < \frac{\varrho}{2}, \\ 1, & \text{if } \frac{\varrho}{2} \leq x_1 < 1. \end{cases}$$

Using Propositions 3 and 4, we obtain

$$\mathbb{P}\left\{\bar{\mu}_{ST}^{(1)} \geq \frac{\varrho}{2}\right\} = \begin{cases} \varrho q_1^2, & \text{if } 0 < p_0^{(1)} \leq \frac{\varrho}{2}, \\ 1, & \text{if } \frac{\varrho}{2} < p_0^{(1)} < 1, \end{cases}$$

and

$$\mathbb{P}\left\{\bar{\mu}_{ST}^{(2)} \geq \frac{\varrho}{2}\right\} = \begin{cases} \varrho^2 q_1^3 q_2^3, & \text{if } 0 < p_0^{(2)} < p_1^{(2)} \leq \frac{\varrho}{2}, \\ \varrho^2\left(q_1^3 q_2^1 + q_1^2\right) + 2\varrho(1-\varrho) q_1^2, & \text{if } 0 < p_0^{(2)} \leq \frac{\varrho}{2} < p_1^{(2)} < 1, \\ 1, & \text{if } \frac{\varrho}{2} < p_0^{(2)} < p_1^{(2)} < 1, \end{cases}$$

where

$$q_j^i = \frac{p_{i-1}^{(n)} - (\varrho/2)}{p_{i-1}^{(n)} - p_{j-1}^{(n)}}.$$

Figure 2 contains the cases $n = 1, 2, 5, 10, 20$ and the limit function as $n \to \infty$ if $\varrho = (1/2)$.

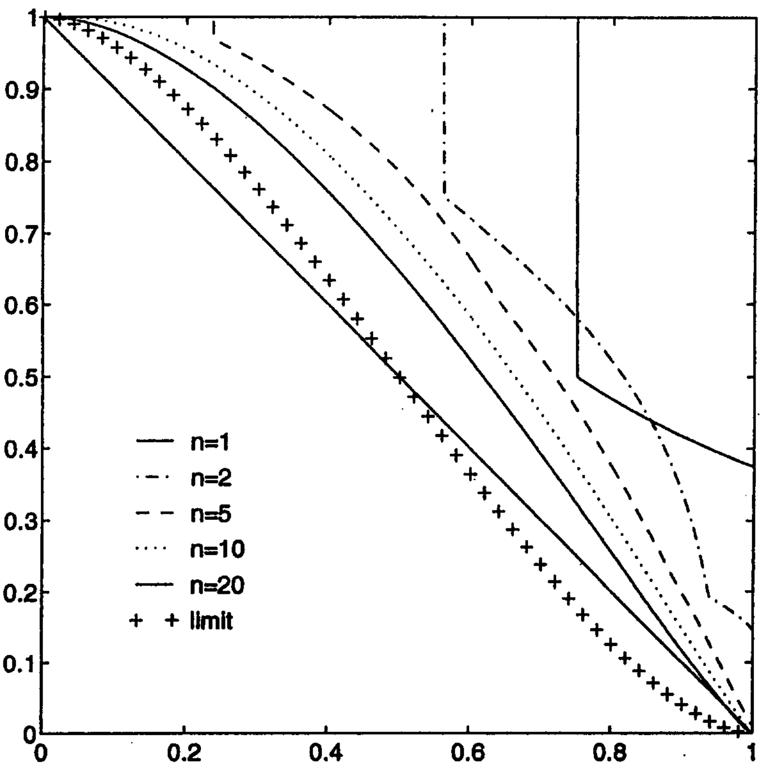

Figure 2. The value of $\mathbb{P}\left\{\bar{\mu}_{ST}^{(n)} \geq (1/2)\right\}$ as a function of α in the case $\varrho = (1/2)$.

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