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A UNIFORM-CONSISTENCY BARRIER ON FINITE-DIFFERENCE SCHEMES OF POSITIVE TYPE FOR CONVECTION-DIFFUSION EQUATIONS*

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Abstract. In this note the author shows a uniform-consistency barrier on finite-difference schemes of positive type for convection-diffusion equations; i.e., any difference scheme of positive type cannot approximate $Lu = -\varepsilon\Delta u + \vec{f}\cdot\nabla u + gu$ to $O(h^\alpha)$ $(\alpha > 1)$ accuracy uniformly in ε.

Key words. difference schemes of positive type, truncation error, uniform-difference schemes, convection-diffusion equations

AMS subject classifications. 65L05, 65N06

1. Introduction. Consider a discrete method for solving a differential equation $Lu = f$. Let L_h be a discrete approximation to L defined on a difference or a finite element mesh Ω_h depending on some meshwidth h. L_h is monotone if $L_h v \geq 0$ implies $v \geq 0$, where v is a function defined on Ω_h. Let w be a barrier function, i.e., a normalized function with $\max w(x_i) = 1$, such that $L_h w(x_i) \geq c > 0$, $\forall x_i \in \Omega_h$. If L_h is monotone, it is shown (see [1], for instance) that

$$\|L_h^{-1}\|_\infty \leq c^{-1},$$

where $\|L_h\|_\infty = \sup_{v\neq 0} \|L_h v\|_\infty / \|v\|_\infty$ and $\|v\|_\infty = \max_{x_i\in\Omega_h} |v(x_i)|$ is the maximum norm, and the discretization error $\|u - u_h\|_\infty$ is bounded by the truncation error $\|L_h u - f\|_\infty$ as follows:

$$\|u - u_h\|_\infty \leq c^{-1}\|L_h u - f\|_\infty.$$

In general, it is not easy to check if a discrete difference linear operator L_h is monotone. Consider a finite-difference scheme

$$L_k^h u = \left(2a_{0,\dots,0}T(0,\dots,0) - \sum_{i_1=-p_1}^{q_1} \cdots \sum_{i_k=-p_k}^{q_k} a_{i_1,\dots,i_k} T(i_1,\dots,i_k)\right) u \tag{1}$$

in k-dimensions for a uniform mesh, where u is a function defined on Ω_h, $T(\beta_1, \beta_2, \dots, \beta_k)$ is a translation operator defined by

$$T(\beta_1, \beta_2, \dots, \beta_k)u(x_1, x_2, \dots, x_k) = u(x_1 + \beta_1 h, x_2 + \beta_2 h, \dots, x_k + \beta_k h),$$

and the β_i's are integers. L_k^h is of positive type if

$$a_{i_1 i_2,\dots,i_k} \geq 0, \quad i_1 = -p_1, \dots, q_1, i_2 = -p_2, \dots, q_2, \dots, i_k = -p_k, \dots, q_k, \tag{2}$$

$$2a_{0,\dots,0} \geq \sum_{i_1=-p_1}^{q_1} \sum_{i_2=-p_2}^{q_2} \cdots \sum_{i_k=-p_k}^{q_k} a_{i_1 i_2,\dots,i_k}. \tag{3}$$

It is well known that L_k^h is automatically monotone if L_k^h is of positive type. Hence much attention has been paid to difference schemes of positive type [2], [4], [6]–[8]. In particular,

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difference schemes are useful for numerical methods for singular perturbation problems [2]–[8]. Unfortunately, in 1978, Kellogg and Tsan [6] showed that any three-point difference scheme of positive type cannot approximate $L(u) = -\varepsilon u'' + b(x)u' + g(x)u$ to $O(h^\alpha)$ ($\alpha > 1$) accuracy uniformly in ε. This implies that it is difficult to obtain a highly accurate approximation of the solution if any three-point difference scheme of positive type is used to solve a convection-diffusion equation $-\varepsilon u'' + b(x)u' + g(x)u = 0$.

The aim of this note is to show that any difference scheme of positive type cannot approximate $Lu = -\varepsilon \Delta u + \vec{f} \cdot \nabla u + gu$ to $O(h^\alpha)$ ($\alpha > 1$) accuracy uniformly in ε. It is shown first that any difference scheme of positive type cannot approximate $L(u) = -\varepsilon u'' + b(x)u' + g(x)u$ to $O(h^\alpha)$ ($\alpha > 1$) accuracy uniformly in ε, which generalizes the result given by Kellogg and Tsan [6]. We then extend the result to higher dimensions.

2. Main results. Let L_h be a discrete approximation to an operator L defined on a difference mesh Ω_h depending on some meshwidth h and let $\sigma(h)$ be a positive function of the meshwidth h. If there exists a positive constant C independent of h such that

$$|L_h v(x_i) - Lv(x_i)| \leq C\sigma(h), \quad \forall x_i \in \Omega_h, \tag{4}$$

where v is a smooth function, it is said that L_h approximates L to $O(\sigma(h))$ accuracy.

Let

$$L^h v_n = -\sum_{i=-p}^{-1} a_i(\varepsilon, h, n) v_{n+i} + a_0(\varepsilon, h, n) v_n - \sum_{i=1}^{q} a_i(\varepsilon, h, n) v_{n+i} \tag{5}$$

denote an approximation to the operator

$$Lu = -\varepsilon u'' + f(x)u' + g(x)u, \tag{6}$$

where p and q are nonnegative integers and $f(x)$ is not identically zero. First we prove that any difference scheme L^h of positive type cannot approximate L to $O(h^\alpha)$ ($\alpha > 1$) accuracy uniformly in ε.

THEOREM 2.1. *Suppose that L^h approximates L to $O(h^\alpha)$ ($\alpha > 1$) accuracy uniformly in ε. Then L^h is not of positive type.*

Proof. Denote $a_i = a_i(\varepsilon, h, n)$, $i = -p, \ldots, q$ for convenience in the proof. Under the assumption of the theorem, $L^h(x^k) = L(x^k) + \gamma_k$ for $k = 0, 1, 2$, where $|\gamma_k| \leq C_k h^\alpha$, C_k is independent of h and ε. Hence

$$-\sum_{i=-p}^{-1} a_i + a_0 - \sum_{i=1}^{q} a_i = g(x) + \gamma_0,$$

$$-\sum_{i=-p}^{-1} a_i(x+ih) + a_0 x - \sum_{i=1}^{q} a_i(x+ih) = f(x) + g(x)x + \gamma_1,$$

$$-\sum_{i=-p}^{-1} a_i(x+ih)^2 + a_0 x^2 - \sum_{i=1}^{q} a_i(x+ih)^2$$
$$= -2\varepsilon + 2f(x)x + g(x)x^2 + \gamma_2.$$

By direct computation, one obtains the following equations.

$$-\sum_{i=-p}^{-1} a_i + a_0 - \sum_{i=1}^{q} a_i = g(x) + \gamma_0,$$

$$\sum_{i=-p}^{-1} ia_i - \sum_{i=1}^{q} ia_i = (f(x) + \gamma_1 - x\gamma_0)h^{-1},$$

$$-\sum_{i=-p}^{-1} i^2 a_i - \sum_{i=1}^{q} i^2 a_i = (-2\varepsilon + \gamma_2 - 2x\gamma_1 + x^2\gamma_0)h^{-2}.$$

Adding the last two equations shows

$$\begin{aligned}(7)\qquad &\sum_{i=1}^{p} i(i-1)a_{-i} + \sum_{i=1}^{q} i(i+1)a_i \\ &= (2\varepsilon - \gamma_2 + 2x\gamma_1 - x^2\gamma_0)h^{-2} - (f(x) + \gamma_1 - x\gamma_0)h^{-1}.\end{aligned}$$

Since $f(x)$ is not identically zero, one can find a point x such that either $f(x) > 0$ or $f(x) < 0$. In the former case, since $|\gamma_k| \le C_k h^\alpha$, $k = 0, 1, 2$, (7) becomes strictly negative for ε sufficiently small, which implies L^h is not of positive type. In the latter case, doing a transformation $x \mapsto c - x$, we have similarly that

$$\begin{aligned}(8)\qquad &\sum_{i=1}^{p} i(i-1)a_{-i} + \sum_{i=1}^{q} i(i+1)a_i \\ &= (2\varepsilon - \gamma_2 + 2(c-x)\gamma_1 - (c-x)^2\gamma_0)h^{-2} + (f(c-x) - \gamma_1 + (c-x)\gamma_0)h^{-1}.\end{aligned}$$

The conclusion follows from the same argument. □

In fact, we can see from the proof that L^h cannot approximate L to $O(h^\alpha)$ ($\alpha > 1$) accuracy uniformly in ε if $a_i \ge 0$, $i = -p, \dots, -1, 1, \dots, q$. The result given by Kellogg and Tsan [6] is the case of $p = q = 1$.

Now we consider finite difference schemes in k-dimensions. Let L_k^h defined by (1) denote an approximation to the operator

$$(9)\qquad L_k u = -\varepsilon\Delta u + \vec{f} \cdot \nabla u + gu$$

in a difference mesh Ω_h with a uniform meshwidth h, where $\vec{f} = (f_1(x_1), \dots, f_k(x_k))^T$ is not identically zero. Since an ordinary differential equation (ODE) is a special case of a partial differential equation (PDE), if something cannot be done for ODEs, it cannot in general be done for PDEs. We claim that the same result of Theorem 2.1 holds in k-dimensions. In fact, assume that $f_1(x_1)$ is not identically zero without loss of generality and let

$$(10)\qquad c_0 = 2a_{0,\dots,0} - \sum_{i_2=-p_2}^{q_2} \cdots \sum_{i_k=-p_k}^{q_k} a_{0i_2,\dots,i_k},$$

$$(11)\qquad c_i = \sum_{i_2=-p_2}^{q_2} \cdots \sum_{i_k=-p_k}^{q_k} a_{ii_2,\dots,i_k}, \qquad i = -p_1, \dots, -1, 1, \dots, q_1.$$

If L_k^h is of positive type, one can see that

$$(12)\qquad c_i \ge 0, \quad i = -p_1, \dots, q_1, \quad 2c_0 \ge \sum_{-p_i}^{q_1} c_i.$$

Consider $L_k^h(x_1^j) = L_k(x_1^j) + \gamma_j'$, $j = 0, 1, 2$. The proof of Theorem 2.1 shows the following theorem.

THEOREM 2.2. *If L_k^h approximates L_k to $O(h^\alpha)$ ($\alpha > 1$) accuracy uniformly in ε, then L_k^h is not of positive type.*

Theorem 2.2 reveals a uniform-consistency barrier on finite-difference schemes of positive type for convection-diffusion equations.

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