Moduli spaces of configurations

een wetenschappelijke proeve op het gebied van de
Natuurwetenschappen, Wiskunde en Informatica

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Introduction

This thesis is about moduli of sequences of linear subspaces of projective spaces. We call such a sequence of linear subspaces a *configuration*. The easiest example of a linear subspace of a projective space is a point in the projective space. Thus the easiest example of a configuration is a sequence of points in a projective space, say an element of $(\mathbb{P}^n)^N$. On the configuration space $(\mathbb{P}^n)^N$ acts the special linear group $SL(n+1)$ and we are interested in the orbits of this action.

Of course, we can consider the collection of these orbits just as a set, but it would be nicer if we had the following:

1. an algebraic variety X,
2. a morphism of algebraic varieties $\phi : (\mathbb{P}^n)^N \to X$, such that
3. ϕ is surjective, and
4. for $x \in X$ the fibres $\phi^{-1}(x)$ are orbits of the action of $SL(n+1)$.

This is the idea of a *moduli space*. The variety X would parametrize all possible configurations of N points in $\mathbb{P}^n$ upto linear equivalence.

In general moduli spaces are varieties or schemes parametrizing some class of objects upto some type of equivalence. They are a central subject in algebraic geometry. In general, and in our case, it is not possible to construct a moduli space that parametrizes *all* the equivalence classes of the given objects. Let's make this clearer in an example. Four points on a projective line have a cross ratio. If $[x_1, y_1], \dots, [x_4, y_4] \in \mathbb{P}^1$ are the points, the cross ratio is the following point on $\mathbb{P}^1$:

$$\left[\det\begin{pmatrix} x_1 & y_1 \\ x_3 & y_3 \end{pmatrix} \cdot \det\begin{pmatrix} x_2 & y_2 \\ x_4 & y_4 \end{pmatrix}, \det\begin{pmatrix} x_1 & y_1 \\ x_4 & y_4 \end{pmatrix} \cdot \det\begin{pmatrix} x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}\right].$$

On the locus $U \subset (\mathbb{P}^1)^4$ where the four points are distinct we have: $P, Q \in U$ are in the same orbit if and only if P and Q have the same cross ratio. However, the rational map $\phi : (\mathbb{P}^1)^4 \dashrightarrow \mathbb{P}^1$ is not defined if three of the four points coincide.

David Mumford, in his famous book *Geometric Invariant Theory*, [MFK94], often referred to as *GIT*[1], invented a method to determine quotients for the action of a reductive group G on an algebraic variety X. In his method the *invariants* of the action are very important. If we choose a line bundle $\mathcal{L}$ on X such that the action of G extends to $\mathcal{L}$, then we say that $f \in \Gamma(X, \mathcal{L}^{\otimes k})$ is an invariant of degree k if it satisfies

[1] Although *moduli stacks* perhaps are even more popular nowadays, in this thesis we study our quotients with GIT, which is considered both classical and alive.

$f(g \cdot x) = f(x)$ for $x \in X$ and $g \in G$. The quotient map $\phi : X \dashrightarrow X//G$ is given by invariants. It isn't defined on the points where all invariants vanish. These points are called *unstable*. Points that are not unstable are called *semi-stable*. In the cross ratio example the configurations where (at least) three points coincide are unstable and the other configurations are semi-stable.

Though the quotient map $\phi : X \dashrightarrow X//G$ is defined on the semi-stable locus, the quotient $X//G$ isn't an orbit space everywhere. In our example consider the set all configurations of $(P_1, \ldots, P_4) \in (\mathbb{P}^1)^4$, such that $P_1 \neq P_2 = P_3 \neq P_4$. Obviously this set consists of two orbits: the orbit where $P_1 = P_4$ and the orbit where $P_1 \neq P_4$. Both orbits are mapped by ϕ to $[1, 0] \in \mathbb{P}^1$. In Mumford's theory there is a subset U of the set of semi-stable points such that $\phi|_U : U \to X//G$ maps different orbits to different points. The points in this subset U are called *stable*. The stable configurations of $(\mathbb{P}^1)^4$ are the configurations where the four points are distinct.

There are no concepts in Geometric Invariant Theory, nor in this thesis, more important than semi-stability and stability. Instead of configurations of only points on a line we investigate GIT for the action of $SL(n+1)$ on a sequence of linear subspaces of $\mathbb{P}^n$, i.e. the variety on which the special linear group acts is:

$$\mathcal{C} := \prod_{i=1}^{N} \mathrm{Gr}(d_i + 1, n + 1).$$

Here $\mathrm{Gr}(d_i+1, n+1)$ denotes the *Grassmannian variety*, whose elements parametrize the d_i+1-dimensional linear subspaces of a $n+1$-dimensional vectorspace. An element of $\mathcal{C}$ is a configuration. As I have explained, in order to obtain quotients with nice properties, one has to exclude unstable configurations, which are configurations that are in some sense "too degenerate". Mumford had already given a useful criterion that checks, in a fixed configuration space $\mathcal{C}$, whether an element of $\mathcal{C}$ is unstable or semi-stable, or even stable. One could also vary the configuration spaces, i.e. vary the number of subspaces N, their dimensions d_i and the dimension of the ambient projective space n. It is not clear a priori that a configuration space $\mathcal{C}$ contains any stable element. Thus a natural purpose in our research has been to give criteria on the configuration spaces for the existence of stable (resp. semi-stable) elements. We obtained some results in this direction.

The other principal theme of our investigations is an involution on sequences of points in projective spaces, which goes by the name of *Gale duality*. Given a sequence of N points in $\mathbb{P}^n$ that is sufficiently general, one can choose homogeneous coordinates such that it is represented by the rows of a matrix

$$\begin{pmatrix} I_{n+1} \\ A \end{pmatrix}.$$

Then its Gale dual is the sequence of N points in $\mathbb{P}^{N-n-2}$ represented by the rows of the matrix

$$\begin{pmatrix} A^T \\ I_{N-n-1} \end{pmatrix}.$$

It was known that this association of points has remarkable geometric interpretations, on which is elaborated in an article by Eisenbud and Popescu, *The projective geometry of the Gale transform*, [EP00]. The association of points sets can be generalized to an association of sequences of linear subspaces as follows. Choose on each of the subspaces linearly independent points spanning the subspace. Consider the set of all these points. Take its Gale dual. Form a new sequence of subspaces by taking the spans of appropiate subsets of the Gale dual set of points. One can check that this construction is independent of the choice of the points spanning the subspaces. It gives an association between elements of the configuration spaces

$$\prod_{i=1}^{N} \mathrm{Gr}(d_i+1, n+1) \text{ and } \prod_{i=1}^{N} \mathrm{Gr}(d_i+1, \tilde{n}+1),$$

where

$$\tilde{n} = \sum_{i=1}^{N}(d_i+1) - n - 2.$$

Eisenbud and Popescu, and we, wondered if this generalization has any geometric meaning. In this thesis we investigate this question and we were able to give a down-to-earth geometric construction for the generalized Gale dual.

The Gale dual is defined upto linear transformation, so it is natural to ask if there exists an isomorphism of the moduli spaces

$$\prod_{i=1}^{N} \mathrm{Gr}(d_i+1, n+1)//SL(n+1) \cong \prod_{i=1}^{N} \mathrm{Gr}(d_i+1, \tilde{n}+1)//SL(\tilde{n}+1).$$

(Note that we abuse notation: the quotient can only be taken of the subset of the semistable points of the configuration spaces.) In the case of sequences of points this question had already been answered affirmatively. We showed that such an isomorphism also exists in the general case.

Results and overview of this thesis

Let's now give a more detailed outline of the contents of any of the four chapters of this thesis. Chapter 1 is a short introduction to GIT intended for the beginner. It is a collection of well known definitions and theorems without proofs. It is explained what we mean by a good quotient, that GIT is a way to produce good quotients for the action of a reductive group on an algebraic variety and how to construct these quotients by considering rings of invariants. At the end of the chapter some attention is given to *linearizations* of the group action. A linearization is, loosely said, a choice of the embedding inside projective space of the variety on which the reductive group acts.

Chapter 2 introduces the configuration spaces of sequences of linear subspaces. We repeat the Mumford Criterion for (semi-)stability. In some sense this criterion measures the level of degeneracy of a configuration. It states that a sequence of linear subspaces is unstable if the intersection with arbitrary other linear subspaces is "too big". For example, a sequence of five lines in $\mathbb{P}^4$ is unstable if there exists a line intersecting these five lines. With the Mumford Criterion stability can be checked very well, even in a difficult configuration space, though computing the corresponding moduli space is in general nearly impossible, except for some small cases. Dolgachev and Ortland studied the case of configurations of points extensively in *Point sets in projective spaces and Theta functions*, [DO88]. We dedicated the rest of the chapter to some other computable examples of moduli spaces of configurations of points and hyperplanes. A beauty is the moduli space of configurations of three points and three lines in the plane, which is the toric variety in $\mathbb{P}^5$ given by the single equation

$$XYZ = UVW.$$

We study the natural birational map to the moduli space of six points in the plane, which is the double cover of $\mathbb{P}^4$ branched along the *Igusa quartic* (a famous quartic threefold).

Chapter 3 deals with Gale duality. We give several equivalent definitions of the generalized Gale transform. Necessary for the statement that Gale duality gives an isomorphism of the corresponding moduli spaces is the statement that a configuration is stable (resp. semi-stable) if and only if its Gale dual is stable (resp. semi-stable). We were able to prove this for the standard linearization, before we could prove the desired isomorphism of moduli spaces. The key to the isomorphism is the *Gelfand-MacPherson correspondence*, which establishes a connection between certain linear subspaces upto a torus action and sequences of points upto a linear action. This can be generalized, replacing the torus action by an action of certain block matrices and sequences of points by sequences of linear subspaces. Then, combining this correspondence with the usual duality in projective spaces leads to the sought generalized

Gale duality isomorphism. This was discovered and proven independently by Yi Hu, [Hu05]. Another important result in this chapter is that we found a geometric way to construct the Gale dual of a sequence of linear subspaces that involves nothing more than taking spans, intersections and projections in vectorspaces. At the end of the chapter we elaborate on *quivers*. Our aim is to state the *Domokos-Zubkov* theorem, which gives all semi-invariants of quivers. A special case of this theorem gives all invariants of configuration spaces, though it doesn't determine their invariant rings.

Chapter 4 is mainly about the question: which configuration spaces have stable (resp. semi-stable) elements? First we realized that for a fixed configuration space $\mathcal{C}$ there is an algorithm to check whether or not there exist stable (resp. semi-stable) elements. We describe this algorithm, which uses some Schubert calculus. Then we concentrated on the special case of configuration spaces where all linear subspaces have the same dimension, i.e. on spaces of the form

$$\mathrm{Gr}(d+1, n+1)^N.$$

We could prove the statement that for each $d, n \in \mathbb{N}$ there exists $N_0 \in \mathbb{N}$ (resp. $N_1 \in \mathbb{N}$) such that $\mathrm{Gr}(d+1, n+1)^N$ has stable (resp. semi-stable) elements if and only if $N \geq N_0$ (resp. $N \geq N_1$). The next step in our research has been to determine the values of the function N_0. Because of the numerous isomorphisms between our moduli spaces given by Gale duality and the usual duality in projective spaces, it is possible to estimate the values of N_0 quite precisely.

Some attention is paid to the quotients of four medials. That is the case that $N = 4$ and $n = 2d+1$. These spaces have the special feature that there do not exist any stable elements, but nevertheless a good quotient exists.

At the end of the chapter, and of the thesis, we consider the concept of a *stable resolution*. A stable resolution of a variety X on which a reductive group G acts is, roughly speaking, a surjective equivariant morphism of varieties $Y \to X$ which resolves the strictly semi-stable locus. Strictly semi-stable elements are semi-stable elements that are not stable. They are exactly those points of X that prevent the GIT quotient $X//G$ from being an orbit space. We give an idea about how to resolve the strictly semi-stable elements of configuration spaces. For point sets the stable resolution is some kind of generalization of a Fulton-MacPherson compactification.

Notations and conventions

In the whole thesis we will work over an arbitrary algebraically closed field k. Wherever we require more properties of k, we will say it. In the first chapter we will often talk about a variety without specifying a particular one. By the word *variety* we mean an *abstract variety*, i.e. an integral, separated scheme of finite type over $\mathrm{Spec}(k)$, see [Har77], page 105. In the other chapters we will talk about specific varieties, such as (subvarieties of) $\mathbb{P}^n$. All are defined over k.

For a projective variety $X \subset \mathbb{P}^n$ we denote by $X^* \subset k^{n+1}$ the *affine cone* over X, i.e.

$$X^* := \{x^* \in k^{n+1} \mid \text{the class of } x^* \text{ is an element of } X\}.$$

Given a quasi-projective variety X and an invertible sheaf $\mathcal{L}$ on X, we will sometimes consider $\mathcal{L}$ as a line bundle. We speak of the projection $\pi : \mathcal{L} \to X$, which locally is the product of X and the affine line $\mathbb{A}^1$. But, at other times, we consider line bundles as invertible sheaves. For the fibre of an invertible sheaf $\mathcal{L}$ on X, considered as line bundle, we use the notation

$$\mathcal{L}_x := \pi^{-1}(x).$$

We define the dimension of the empty set to be -1.

1 Short introduction to Geometric Invariant Theory

Though there are several good introductionary books on this subject, I write here my own introduction to Geometric Invariant Theory. For the expert it will not contain any surprises. For the non-expert it will explain the most important concepts in not too many pages.

The interested reader finds more details in [Dol03], [MFK94] and [New78] (books on which this chapter is based).

1.1 Group actions on algebraic varieties

The main purpose of Geometric Invariant Theory is as follows. Given a variety (or scheme) X and a group G, acting on X, one wants to construct a quotient of this action. In the category of sets one just takes X/G to be the set of orbits. In the category of varieties (or schemes) it is far more difficult. In general there will be no quotient which is an orbit space. This is easy to see. Suppose that X/G has the structure of a variety, such that the map $\pi : X \to X/G$ is a morphism. Then in particular each orbit has to be closed, because π is continuous.

EXAMPLE 1.1.1 The action of $GL(n)$ on k^n has two orbits and one of them is not closed.

This trivial example is typical: almost never all orbits of the actions considered are closed. However, we will see that "in most cases" there exists an open set $U \subset X$ such that U/G has the structure of algebraic variety and $U \to U/G$ is a morphism (so in particular U is a union of closed orbits). The meaning of "in most cases" is not very clear. In fact part of this thesis is about the question in *which* cases such an open set U exists for a certain class of varieties.

Let us now start with our first definitions.

DEFINITION 1.1.2 An *algebraic group* is a group G, which is at the same time a variety, such that the maps $\mu : G \times G \to G$ (multiplication) and $\iota : G \to G$ (taking inverse) are morphisms of varieties.

DEFINITION 1.1.3 An *algebraic group action* on a variety X is a group action $\sigma : G \times X \to X$, where G is an algebraic group and σ is a morphism of varieties.

One easy example of an algebraic group action we have already seen, is the action of $GL(n)$ on k^n. Given a homomorphism of algebraic groups $G \to GL(n)$, we get an action of G on k^n.

DEFINITIONS 1.1.4 A *rational representation* is homomorphism of algebraic groups $G \to GL(n)$. A *linear action* is an action of G on k^n via a rational representation. A *linear algebraic group* is an algebraic group isomorphic to a closed subgroup of $GL(n)$.

Important examples of linear algebraic groups are: $SL(n)$ (a special linear group), $\mathbb{G}_m := GL(1)$ (the multiplicative group) and $T^n := \mathbb{G}_m^n$ (a torus group).

The following definition gives us the minimal requirements for what we should call a quotient of an algebraic group action.

DEFINITION 1.1.5 Let $\sigma : G \times X \to X$ be an algebraic group action. A *categorical quotient* of X by G is a variety Y with a morphism $\Phi : X \to Y$ such that

1. the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \Phi} \\ X & \xrightarrow{\Phi} & Y \end{array}$$

   commutes ("Φ is constant on orbits"), and

2. if $\Psi : X \to Z$ is any morphism that is constant on orbits, then there exists a unique morphism $\Xi : Y \to Z$ with $\Psi = \Xi \circ \Phi$.

Note that if a categorical quotient exists, it is unique.

In order to construct categorical quotients in general we first restrict our attention to a special case. Suppose X is an affine variety, write $X = \mathrm{Spec}(A)$, where A is a finitely generated k-algebra. For every $g \in G$ there is an automorphism $\sigma_g : X \to X$, defined by

$$\sigma_g(x) := \sigma(g, x).$$

These automorphisms of X correspond to k-algebra automorphisms $\sigma_g^* : A \to A$.

DEFINITION 1.1.6 Let G be an algebraic group and R a k-algebra. A *rational action* of G on R is a map

$$h : R \times G \to R$$

with the properties

1. $h(f, gg') = h(h(f, g), g')$ and $h(f, 1) = f$ for all $f \in R$, $g, g' \in G$,

2. the map $f \mapsto h(f, g)$ is a k-algebra automorphism of R for all $g \in G$, and

3. every element of R is contained in a finite-dimensional subspace which is invariant under G and on which G acts by a rational representation.

DEFINITION 1.1.7 Let G be an algebraic group, R a k-algebra and $h : R \times G \to R$ a rational action. Then we call

$$R^G := \{f \in R \mid h(f, g) = f \text{ for all } g \in G\}$$

the subalgebra of invariant elements.

PROPOSITION 1.1.8 *Let $\sigma : G \times X \to X$ be an algebraic group action on an affine variety $X = \mathrm{Spec}(A)$. Then the induced map*

$$\begin{aligned} h : A \times G &\to A \\ (f, g) &\mapsto \sigma_g^*(f) \end{aligned}$$

is a rational action of G on A.

PROOF See the definition of rational action above. The first two properties are trivial to check. For the third property: see [New78], lemma 3.1. □

In the situation of this proposition, we can ask for a candidate for a categorical quotient Y. Suppose it exists and is affine, write $Y = \mathrm{Spec}(B)$. We can identify A with morphisms $\Psi : X \to \mathbb{A}^1$ to the affine line. The definition of categorical quotient tells us that such a morphism factors through Y if and only if it is constant on orbits. Algebraically this means that $B = A^G$.

So, if Y is to be affine, A^G has to be finitely generated. In general, given a rational action of an algebraic group G on a finitely generated k-algebra R, the subalgebra of invariants R^G is not finitely generated. See [Nag58]. This is the famous counterexample of Nagata against Hilbert's fourteenth problem.

However, we will see that in important cases the algebra of invariants is finitely generated. What we need is the concept of a reductive group. One needs to take care over the terminology, because different authors use the several concepts associated with the word "reductive" in different ways. The terminology we use coincides with the terminology used in [Dol03] and [New78].

DEFINITION 1.1.9 A linear algebraic group G is *linearly reductive* (resp. *geometrically reductive*) if for every linear action of G on k^n and every non-zero invariant vector $v \in k^n$ there exists a G-invariant function $f \in k[x_1, \ldots, x_n]$ with $\deg(f) = 1$ (resp. $\deg(f) \geq 1$) such that $f(v) \neq 0$.

The definition of a reductive algebraic group is somewhat more technical. Each linear algebraic group G has a unique maximal connected normal solvable subgroup. This is called the *radical* of G. Now G is *reductive* if its radical is isomorphic to a torus group.

Let us give immediately the main theorem (for our purposes) about reductive algebraic groups.

THEOREM 1.1.10 (NAGATA) *Let G be a geometrically reductive group acting rationally on a finitely generated k-algebra R. Then R^G is finitely generated.*

PROOF There are several places where a proof is given. We refer to [Nag64], [New78], ch. 3, par. 2 or [Dol03], par. 3.4. □

Here are some facts about reductive algebraic groups. Every geometrically reductive group is reductive, see [NM64]. If $\mathrm{char}(k) = 0$, every reductive group is linearly reductive. This fact is contributed to Weyl. For a long time it was conjectured (by Mumford), but not proven, that, in arbitrary characteristic, every reductive group is geometrically reductive. A proof was given in 1974 by Haboush, see [Hab75].

In his famous book Geometric Invariant Theory, often referred to as GIT, Mumford shows how quotients can be constructed for the actions of geometrically reductive groups. Haboush' theorem is important, because it extends GIT to arbitary characteristic.

For our purposes it is only important that all groups we use are reductive algebraic groups.

THEOREM 1.1.11 *$GL(n)$, $SL(n)$, $\mathbb{G}_m$ and products of these groups are reductive algebraic groups over $\mathbb{C}$ (over any field).*

PROOF See textbooks, such as [New78] (page 50) and [Dol03] (page 42). □

In fact Hilbert has proven already in [Hil90] that the ring of invariants $A^{SL(n)}$, where A is a finitely generated k-algebra, is finitely generated.

The aim of this paragraph is a theorem about the categorical quotient of an affine variety for the action of a reductive group. Before we give this theorem, we give some more definitions. There are more properties one would like to have when one has constructed a categorical quotient.

DEFINITION 1.1.12 Let $\sigma : G \times X \to X$ be an algebraic group action. A *good quotient* of X by G is a variety Y and a morphism $\Phi : X \to Y$ such that

1. Y (together with Φ) is a categorical quotient,
2. for any subset $U \subset Y$, the inverse image $\Phi^{-1}(U)$ is open if and only if U is open,
3. for any open subset $U \subset Y$, the homomorphism $\Phi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\Phi^{-1}(U))$ is an isomorphism onto $\mathcal{O}_X(\Phi^{-1}(U))^G$, and
4. Φ is surjective.

If we are in the situation of the definition above, it follows that

1. if W is a closed invariant subset of X, then $\Phi(W)$ is closed in Y, and
2. if W_1 and W_2 are closed invariant subsets of X with $W_1 \cap W_2 = \emptyset$, then $\Phi(W_1) \cap \Phi(W_2) = \emptyset$.

In fact one can rephrase the definition above using these properties of closed invariant subsets.

DEFINITION 1.1.13 Let $\sigma : G \times X \to X$ be an algebraic group action. A *geometric quotient* of X by G is a variety Y and a morphism $\Phi : X \to Y$ such that

1. Y (together with Φ) is a good quotient, and
2. the image of the map $\Psi : G \times X \to X \times X$ given by

$$(g, x) \mapsto (\sigma(g, x), x)$$

is $X \times_Y X$.

A geometric quotient is the best we can hope for, because it is even an orbit space for the action of G on X.

Recall we were looking for a categorical quotient for the action of an algebraic group on an affine variety. The following theorem says that, if the group is reductive, our candidate for an affine categorical quotient is in fact a very good candidate.

THEOREM 1.1.14 *Let G be a reductive group acting on an affine variety $X = \mathrm{Spec}(A)$. Then $Y = \mathrm{Spec}(A^G)$, together with the map $\phi : X \to Y$, is a good quotient of X by G.*

PROOF See [New78], ch. 3, par. 3. □

Usually we will work in this thesis with projective varieties, and not with affine varieties. In the next paragraph we will look more closely at reductive group actions on projective varieties.

1.2 GIT quotients

Until now we required our actions on affine varieties to be linear, i.e. the groups act via a rational representation. This means, loosely said, that one considers the actions as happening inside a k^n. The analogue of this, when we consider actions on quasi-projective varieties, is the concept of linearization with reference to an invertible sheaf.

DEFINITION 1.2.1 Let $\sigma : G \times X \to X$ be an algebraic group action on a quasi-projective variety X, and let $\mathcal{L}$ be a line bundle on X, with projection map: $\pi : \mathcal{L} \to X$. A *linearization* of this action is an action $\beta : G \times \mathcal{L} \to \mathcal{L}$ such that

1. the diagram

$$\begin{array}{ccc} G \times \mathcal{L} & \xrightarrow{\beta} & \mathcal{L} \\ \downarrow{\scriptstyle \mathrm{id}\times\pi} & & \downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow[\sigma]{} & X \end{array}$$

   commutes, and

2. for all $X \in X$ and all $g \in G$ the map $\mathcal{L}_x \to \mathcal{L}_{\sigma(g,x)}$ given by

$$y \mapsto \beta(g, y)$$

   is linear.

We call a *G-linearized line bundle* over X a pair of a line bundle $\mathcal{L}$ and its linearization β. A morphism of G-linearized line bundles is a G-equivariant morphism of line bundles. Thus we can speak of isomorphism classes of G-linearized line bundles on X and one can show (see [Dol03], chap. 7) the set of isomorphism classes of G-linearized line bundles on X has an abelian group structure. We denote this group by $\mathrm{Pic}^G(X)$, and we have a natural homomorphism

$$T : \mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$$

which is forgetting the linearization. This homomorphism is not necessarily surjective. For an elaboration on the existence of linearizations, see §1.3. We quietly assume at this moment that linearizing the action is not a big problem. Then we can go on and define the following concepts.

DEFINITION 1.2.2 Let X be a quasi-projective variety, acted upon by G, a reductive algebraic group. Let this action be linearized with respect to $\mathcal{L}$, a line bundle on X. Let $x \in X$.

1. x is called *semi-stable* with respect to $\mathcal{L}$ if there exists $m \geq 0$ and $s \in \Gamma(X, \mathcal{L}^m)^G$ such that $X_s = \{y \in X \mid s(y) \neq 0\}$ is affine and contains x,
2. x is called *stable* with respect to $\mathcal{L}$ if there exists $m \geq 0$ and $s \in \Gamma(X, \mathcal{L}^m)^G$ such that $X_s = \{y \in X \mid s(y) \neq 0\}$ is affine and contains x, G_x is finite and all orbits of G in X_s are closed, and
3. x is called *unstable* with respect to $\mathcal{L}$ if x is not semi-stable.

Note the obvious fact that every stable point (with respect to $\mathcal{L}$) is stable (with respect to $\mathcal{L}$). Here are some notations we use.

$$\begin{array}{lcl} X^{ss}(\mathcal{L}) & := & \text{locus of semi} - \text{stable points} \\ X^{s}(\mathcal{L}) & := & \text{locus of stable points} \\ X^{us}(\mathcal{L}) & := & \text{locus of unstable points} \\ X^{sss}(\mathcal{L}) & := & X^{ss}(\mathcal{L}) \setminus X^{s}(\mathcal{L}) \end{array}$$

Often we will omit the line bundle in question and write X^s, etcetera. Elements of X^{sss} are called *strictly semi-stable* points, the non-stable semi-stable points.

REMARK 1.2.3 In the literature, for example in [MFK94] itself, also exists the notion of a *properly stable* point. This is exactly a point that we call a stable point. In the same literature the notion of a *stable* point is used, by which is meant a semi-stable points such that nearby orbits are closed (i.e. their definition of stable only drops the condition that the stabilizer is finite).

REMARK 1.2.4 See the definition of semi-stable above. If the sheaf $\mathcal{L}$ is ample the set X_s is affine automatically. So in that case we only have to find an invariant section (of a power) of $\mathcal{L}$ which is non-zero on x. This is important for us because all sheaves we will consider, are ample sheaves.

As shows the following theorem, the semi-stable locus is an open subset of X over which a good quotient exists.

THEOREM 1.2.5 (MUMFORD) *Let G be a reductive group acting on a quasi-projective variety X. Let the action be linearized with respect to an ample line bundle $\mathcal{L}$. Then there exists a good quotient*

$$\pi : X^{ss}(\mathcal{L}) \to X^{ss}(\mathcal{L})//G$$

There exists an open set $U \subset X^{ss}(\mathcal{L})//G$ such that $X^s(\mathcal{L}) = \pi^{-1}(U)$ and the restriction of π to $X^s(\mathcal{L})$ is a geometric quotient of $X^s(\mathcal{L})$ by G. Moreover: $X^{ss}(\mathcal{L})//G$ is a quasi-projective variety.

There exists a converse of the theorem above, saying that under some (weak) conditions subsets $U \subset X$ for which a categorical quotient $U//G$ exist, are of the form $U = X^{ss}(\mathcal{L})$ for some linearization with respect to an ample line bundle $\mathcal{L}$. We won't eleborate on this. See [MFK94], page 41.

REMARK 1.2.6 The stable locus may be empty, but this will not always mean that there doesn't exist an open subset U inside the semi-stable locus such that $U \to U//G$ is a geometric quotient. In order to have an orbit space it isn't necessary that the stabiliser of a point $x \in U$ is finite, but that its dimension is constant in a neighbourhood of x. We define:

$$X^{reg} := \{x \in X \mid \dim(G_x) \text{ is constant in a neigbourhood of } x\}.$$

Cf. Remark 1.2.3 above. A point $x \in X$ is stable in the sense of [MFK94] if and only if $x \in X^{ss} \cap X^{reg}$. The set X^{reg} is a disjoint union of open subsets $X_i \subset X^{reg}$ with $i = 0, \ldots, \dim(G)$, where X_i consists of all points $x \in X^{reg}$ with $\dim(G_x) = i$. It is clear that $X_0 \cap X^{ss} = X^s$ (stable in our sense). Theorem 1.2.5 above remains true if one interprets the word stable as stable in the sense of [MFK94]. Thus it can happen that $X^s = \emptyset$, but nevertheless the GIT quotient $X^{ss}//G$ has the property that it is almost everywhere an orbit space. We will encounter an example of this situation when we are going to look at the quotient of configurations of 4 lines in projective 3-space. See Example 4.2.3 and paragraph 4.2.2.

REMARK 1.2.7 The set X^{reg} from the previous remark is the maximal subset $U \subset X$ such that a geometric quotient $U \to U//G$ exists. If for some subset $V \subset X$ a geometric quotient $V \to V//G$ exists, then $V \subset X^{reg}$. See [Dol03], §6.3.

The following corollary we will use in practice.

COROLLARY 1.2.8 *If in the previous theorem X is projective and $\mathcal{L}$ very ample, we have*

$$X^{ss}(\mathcal{L})//G \cong \mathrm{Proj}(R^G)$$

where

$$R = \oplus_{k \geq 0} \Gamma(X, \mathcal{L}^k).$$

Thus X is a projective variety.

REMARK 1.2.9 If in the situation of the corollary above $X \subset \mathbb{P}^n$, $\mathcal{L} := \mathcal{O}_X(1)$ and the action is by the group $G = SL(n+1)$ and it is linearized with respect to this

$\mathcal{L}$, the unstable points are precisely the points on which all invariant functions vanish. We define:

$$\mathcal{N} := X^{us}(\mathcal{L}) = \{x \in X \mid s(x) = 0 \text{ for all } s \in R^G\}.$$

The idea to consider this set $\mathcal{N}$ goes back to Hilbert. It is called the *nullcone* and its elements are called *nullforms*. Nullforms can't be distinguished by invariant functions. In fact if we consider the special case that R^G is generated by generators $s_0, \ldots, s_k$ of the same degree, then the rational map $X \dashrightarrow \mathbb{P}^k$ given by

$$x \mapsto (s_0(x), \ldots, s_k(x))$$

is the quotient map (when restricted to the semi-stable locus). The nullcone is the locus where this map isn't defined.

In most cases it turns out to be very difficult to find explicit invariants. Nevertheless we have a useful tool to determine which points are (semi-)stable. This is the so called one-parameter criterion or numerical criterion for stability. The idea is as follows. Let G be a reductive algebraic group acting on a projective variety $X \subset \mathbb{P}^n$ via a homomorphism $G \to GL(n+1)$. In other words: the action is linearized with respect to the invertible sheaf $\mathcal{L}$ corresponding to the embedding $X \subset \mathbb{P}^n$. We can consider the induced action of G on the affine cone $X^* \subset k^{n+1}$. Let $x^* \in X^*$ be a point whose class is $x \in X$. Then another way to say whether or not x is unstable is given by this equivalence (which is not difficult to prove):

$$x \in \mathcal{N} \iff 0 \in \overline{Gx^*}.$$

We could also check this for subgroups of G. If $0 \in \overline{Hx^*}$ for any subgroup $H < G$, x is unstable, because $\overline{Hx^*} \subset \overline{Gx^*}$. In fact the one-parameter criterion will say it is sufficient to check this only for the so-called one-parameter subgroups of G.

DEFINITION 1.2.10 If G is an algebraic group, a *one-parameter subgroup* of G is a non-trivial homomorphism of algebraic groups $\mathbb{G}_m \to G$ and we denote the set of one-parameter subgroups of G by $PS(G)$.

A one-parameter subgroup λ of G can be viewed as an action of $\mathbb{G}_m$ on X, or as an action on X^*. It is a fact (not proven here) that we can choose coordinates such that the action on X^* is given by

$$\lambda(t) \cdot x^* = (t^{a_0} x_0, \ldots, t^{a_m} x_m)$$

for certain $a_0, \ldots, a_m \in \mathbb{Z}$. Now consider the map

$$\phi_x^* : \mathbb{A}^1 \setminus \{0\} \to k^{n+1}, \;\; t \mapsto \lambda(t) \cdot x^*.$$

If this map can be extended to a map $\mathbb{A}^1 \to k^{n+1}$ by sending the origin to the origin then it is clear that 0 is in the closure of the orbit of x^* of the one-parameter subgroup of G, so that x is unstable. Using the diagonal form of the action we see that 0 is in this closure if and only if all a_i for which $x_i \neq 0$ are strictly positive. This observation leads to the following definition.

DEFINITION 1.2.11

$$\mu^{\mathcal{L}}(x, \lambda) := \min_i \{a_i \mid x_i \neq 0\}.$$

One can show that the function $\mu^{\mathcal{L}}$ doesn't depend on the diagonalization of the one-parameter action. We can use this function to check unstability. If a one-parameter subgroup λ of G and a point $x \in X$ satisfy $\mu^{\mathcal{L}}(x, \lambda) > 0$, then x is unstable. We will stop now to elaborate on an explanation of the numerical criterion and state it.

THEOREM 1.2.12 *Let G be an action of a reductive group on a projective variety X, linearized with reference to an ample line bundle $\mathcal{L}$. Let $x \in X$. Then:*

$$\begin{array}{lcll} x \in X^{ss}(\mathcal{L}) & \iff & \mu^{\mathcal{L}}(x, \lambda) \leq 0 & \text{for all } \lambda \in PS(G), \\ x \in X^{s}(\mathcal{L}) & \iff & \mu^{\mathcal{L}}(x, \lambda) < 0 & \text{for all } \lambda \in PS(G). \end{array}$$

PROOF See textbooks such as [Dol03] and [New78] for more information. The criterion is due to Mumford and Hilbert. □

1.3 Linearization of the action

The reader probably hoped, in the previous paragraph, that an action of an algebraic group G on X can always be linearized with reference to some line bundle $\mathcal{L}$. This is certainly true in the example we consider in this thesis. In fact we will see in a moment that often more than one linearization is possible, with fixed invertible sheaf $\mathcal{L}$. The quotient can change together with a change of linearization. Whenever no confusion arises as to which linearization we choose, we just write

$$X^{ss}(\mathcal{L})//G$$

for the GIT quotient. It will happen, however, that we do consider our quotients with respect to different linearization. When this is the case, we denote by

$$X^{ss}_{\beta}(\mathcal{L})$$

the semi-stable locus for the linearization $\beta : G \times \mathcal{L} \to \mathcal{L}$, and by

$$X^{ss}_{\beta}(\mathcal{L})//G$$

its GIT quotient.

We give some theorems saying how many linearizations are possible for a given action and invertible sheaf.

DEFINITION 1.3.1

$$\chi(G) := \mathrm{Hom}(G, \mathbb{G}_m) = \{\text{homomorphisms of algebraic groups } G \to \mathbb{G}_m\}$$

This group $\chi(G)$ is called the *group of rational characters* of G.

Recall that $T : \mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$ is the homomorphism that forgets the linearization. The dimension of the kernel of T is a measure of the amount of linearizations a given invertible sheaf on X allows. In the theorem below $p_1 : G \times X \to G$ denotes the first projection.

THEOREM 1.3.2 *If* $\mathcal{O}(G \times X)^* = p_1^{-1}(\mathcal{O}(G)^*)$ *then*

$$\ker(T) \cong \chi(G).$$

The condition $\mathcal{O}(G \times X)^* = p_1^{-1}(\mathcal{O}(G)^*)$ may look strange, but in fact it is a corollary of a more general theorem (which we don't give, because we only need the given statement). The condition is fulfilled for example if X is just affine space or if X is connected and proper over k, because in those cases $\mathcal{O}(X)^* = k^*$.

The following is the main theorem about the existence and the amount of linearizations.

THEOREM 1.3.3 *Let G be a connected, affine algebraic group acting on a normal variety* X *then we have an exact sequence of groups*

$$0 \to \ker(T) \to \mathrm{Pic}^G(X) \xrightarrow{T} \mathrm{Pic}(X) \to \mathrm{Pic}(G)$$

PROOF See [Dol03], §7.2. It is not relevant here to explain the map $\mathrm{Pic}(X) \to \mathrm{Pic}(G)$. $\square$

REMARK 1.3.4 Since it can happen that there are many possible linearizations, a priori there are many different quotients for the same action. It turns out however, that in the case of line bundles giving projective geometric quotients, there are only finitely many quotients and all these are birational to one another. See an article of Dolgachev and Hu, [DH98]. We will not focus in this thesis on all possible linearizations and their resulting quotients, but almost always take a standard linearization. Cf. Remark 2.1.6 and Example 2.3.3.

2 Moduli spaces of sequences of linear subspaces

Let n and N be natural numbers satisfying $n, N \geq 1$. From now on we consider the action of $SL(n+1)$ on sequences of N linear subspaces of $\mathbb{P}^n$. For a natural number d with $0 \leq d \leq n$ we denote by

$$\mathbb{G}(d,n) := Gr(d+1, n+1)$$

the Grassmannian of $d+1$-dimensional subspaces of a $n+1$-dimensional vectorspace over k. Let

$$\underline{d} := (d_1, \ldots, d_N)$$

be a sequence of natural numbers satisfying $0 \leq d_1, \ldots, d_N \leq n$. The main object of this thesis is:

$$\mathcal{C}(\underline{d}, n) := \prod_{i=1}^{N} \mathbb{G}(d_i, n).$$

To the vector $\underline{d}$ we will refer as a *sequence of dimensions*. The product $\mathcal{C}(\underline{d}, n)$ is a *configuration space* of sequences of linear subspaces. An element of it we will call a *configuration*. Whenever there can be no confusion about the dimension n of the ambient projective space and the sequence of dimensions $\underline{d}$ we will omit these data from the notation and just write $\mathcal{C}$ for the configuration space.

REMARK 2.0.5 Note that we allow the dimensions d_i of the subspaces to be equal to the dimension of the ambient space n. Of course the "Grassmannian" $\mathbb{G}(n,n) \cong \mathrm{Spec}(k)$ isn't interesting at all, but allowing it is useful when we consider Gale transforms. Cf. Remark 3.1.3 and Example 3.4.3.

We are interested in the GIT quotients

$$\mathcal{C}^{ss}//SL(n+1).$$

A first question we can ask ourselves is which configurations are stable (resp. semistable) in $\mathcal{C}$.

2.1 The Mumford Criterion for stability

In fact there is a criterion for stability of configurations in $\mathcal{C}$. Before we can state it we have to investigate the action of $SL(n+1)$ on $\mathcal{C}$ more thoroughly. We need to know what are the possible linearizations of this action. See §1.3 about this subject. Consider the exact sequence

$$0 \to \chi(SL(n+1)) \to \mathrm{Pic}^{SL(n+1)}(\mathcal{C}) \xrightarrow{T} \mathrm{Pic}(\mathcal{C}) \to \mathrm{Pic}(SL(n+1)).$$

Recall that $T : \mathrm{Pic}^{SL(n+1)}(\mathcal{C}) \to \mathrm{Pic}(\mathcal{C})$ is the natural map, which forgets the linearization.

PROPOSITION 2.1.1

$$\mathrm{Pic}(SL(n+1)) \cong 0$$

PROOF

$$SL(n+1) \cong S \setminus H,$$

where S is the hypersurface of $\mathbb{P}^{n^2}$ given by the equation

$$\det((x_{ij})_{i,j\in\{1,\ldots,n\}}) - x_{00}^n = 0,$$

and H the hyperplane given by $x_{00} = 0$. There is an exact sequence ([Har77], ch. 2, prop. 6.5)

$$\mathbb{Z} \to \mathrm{Pic}(S) \to \mathrm{Pic}(S \setminus H) \to 0.$$

The first map is given by $1 \mapsto [H]$ and $\mathrm{Pic}(S) \cong \mathbb{Z} \cdot [H]$ (Lefschetz theorem), so the proposition follows, because the first map is a surjection. □

PROPOSITION 2.1.2

$$\chi(SL(n+1)) \cong 0$$

PROOF The only rational characters of $GL(n+1)$ are of the form $A \mapsto \det(A)^g$ with $g \in \mathbb{Z}$ (see [DC71], ch. 2, §4), i.e. $\chi(GL(n+1)) \cong \mathbb{Z}$. From this it follows that $\chi(SL(n+1)) = 0$. □

Thus it follows that every invertible sheaf on $\mathcal{C}$ admits a unique linearization for the action of $SL(n+1)$. What remains now is the question which invertible sheaves can exist on $\mathcal{C}$.

PROPOSITION 2.1.3 *For d and n natural numbers with $0 \leq d < n$*

$$\mathrm{Pic}(\mathbb{G}(d,n)) \cong \mathbb{Z}.$$

PROOF $\mathbb{G}(d,n) \subset \mathbb{P}^{\binom{n+1}{d+1}-1}$ by the Plückerembedding. The part of $\mathbb{G}(d,n)$ where one of the Plückercoordinates is non-zero, is isomorphic to $\mathbb{A}^{(n-d)(d+1)}$. Any $\mathcal{L} \in \mathrm{Pic}(\mathbb{G}(d,n))$ is zero, when restricted to $\mathbb{A}^{(n-d)(d+1)}$, so is itself isomorphic to a line bundle associated with a divisor which is a multiple of a hyperplane section. □

DEFINITION 2.1.4 We denote by

$$\underline{k} := (k_1, \ldots, k_N) \in \mathbb{Z}^N$$

a sequence of integers, which we call *weights*.

With each $\underline{k} \in \mathbb{Z}^N$ we associate an invertible sheaf on $\mathcal{C}$. Let $p_i : \mathcal{C} \to \mathbb{G}(d_i, n)$ denote the i-th projection.

DEFINITION 2.1.5 Let $n, N \in \mathbb{N}$ satisfy $n, N \geq 1$. Let $\underline{d} = (d_1, \ldots, d_N) \in \mathbb{Z}^N$ satisfy $0 \leq d_1, \ldots, d_N < n$. Let $\underline{k} := (k_1, \ldots, k_N) \in \mathbb{Z}^N$. The following is an invertible sheaf on $\mathcal{C}(\underline{d}, n)$:

$$\mathcal{L}_{\underline{k}} := \bigotimes_{i=1}^{N} p_i^*(\mathcal{O}_{\mathbb{G}(d_i,n)}(1)^{\otimes k_i}).$$

For a Grassmannian variety we have that $H^1(X, \mathcal{O}_X) = 0$ and $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$. Thus it follows from the Künneth formula, that Grassmannians X and Y satisfy $\mathrm{Pic}(X \times Y) \cong \mathrm{Pic}(X) \times \mathrm{Pic}(Y)$, so it follows that $\mathrm{Pic}(\mathcal{C}) \cong \mathbb{Z}^N$ and every invertible sheaf on $\mathcal{C}$ is isomorphic to $\mathcal{L}_{\underline{k}}$ for a certain $\underline{k} \in \mathbb{Z}^N$.

The line bundle $\mathcal{L}_{\underline{k}}$ is very ample if and only if all k_i satisfy $k_i > 0$. If this is the case we refer to $\underline{k}$ as a *sequence of positive weights*. The projective embedding of $\mathcal{C}$ corresponding to such a $\mathcal{L}_{\underline{k}}$ is a composition of a product of Plücker embeddings, then a product of k_i-uple Veronese embeddings and finally the Segre embedding.

REMARK 2.1.6 Usually we will consider the action of $SL(n+1)$ with respect to the sheaf $\mathcal{L}_{(1,\ldots,1)}$, i.e. we use the sequence of weights $\underline{k} = (1, \ldots, 1)$. We call this the *standard linearization*.

REMARK 2.1.7 There are cases in which one better takes another linearization than the standard one. See Example 2.3.3.

The following corollary summarizes what we have seen.

COROLLARY 2.1.8

$$\mathrm{Pic}^{SL(n+1)}(\mathcal{C}) = \{\mathcal{L}_{\underline{k}} \mid \underline{k} \in \mathbb{Z}^N\}$$

We give a criterion for stability of configurations of sequences of linear subspaces.

THEOREM 2.1.9 (MUMFORD CRITERION) *Let $\underline{d}$, $\underline{k}$ and $\mathcal{C}$ be a as above. We consider the action of $SL(n+1)$ on $\mathcal{C}$ linearized by the line bundle $\mathcal{L}_{\underline{k}}$. A configuration $(V_1, \ldots, V_N) \in \mathcal{C}$ is semi-stable for this action and linearization if and only if for each proper linear subspace $V \subset \mathbb{P}^n$:*

$$\frac{\sum_{i=1}^N k_i(\dim(V_i \cap V) + 1)}{\sum_{i=1}^N k_i(d_i + 1)} \leq \frac{\dim(V) + 1}{n+1}$$

The configuration $(V_1, \ldots, V_N)$ is stable if and only if we have a strict inequality in the equation above.

PROOF (See [Dol03], th. 11.1, [MFK94], prop. 4.3 and [New78], th. 4.17.) We repeat a proof here. It makes use of the 1-parameter criterion for stability, see theorem 1.2.12. Every 1-parameter subgroup of $SL(n+1)$ is conjugate to one of the form

$$\lambda(t) = \mathrm{diag}(t^{a_0}, \ldots, t^{a_n})$$

(where $t \in \mathbb{G}_m$, the $a_i \in \mathbb{Z}$, and $\sum a_i = 0$; one can also assume that $a_0 \geq \ldots \geq a_n$). It is sufficient to check the 1-parameter criterion for these special λ. Let $x := (V_1, \ldots, V_N)$ be a configuration. What is $\mu^{\mathcal{L}_{\underline{k}}}(x, \lambda)$?

Choose linearly independent elements of $\mathbb{P}^n$, denoted by $e_0, \ldots, e_n$ and denote for $i = 0 \ldots n$ by E_i the span of $e_0, \ldots, e_i$. For an arbitrary element $V \in \mathbb{G}(d, n)$ and a natural number j with $0 \leq j \leq d$ there is a unique integer ν_j with

$$\dim(V \cap E_{\nu_j}) = j \text{ and } \dim(V \cap E_{\nu_j - 1}) = j - 1.$$

Thus ones sees that V is the rowspace of a matrix of the form

$$\begin{pmatrix} a_{00} & \ldots & a_{0\nu_0} & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ a_{10} & \ldots & \ldots & a_{1\nu_1} & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{d0} & \ldots & \ldots & \ldots & \ldots & \ldots & a_{d\nu_d} & 0 & \ldots & 0 \end{pmatrix}.$$

Here $a_{j\nu_j} \neq 0$ for each j. Interesting for us is that we can see something about the Plückercoordinates of V. We see that $p_{i_0 \ldots i_d}(V) = 0$ if $i_j > \nu_j$ for some j and we see that $p_{i_{\nu_0} \ldots i_{\nu_d}}(V) \neq 0$.

The projective coordinates of x in the embedding defined by $\mathcal{L}_{\underline{k}}$ are products of N monomials of degree d_i in the Plückercoordinates of the V_i (see the remark above). For λ as above we have

$$p_{i_0 \dots i_d}(\lambda(t) \cdot V) = t^{a_{i_0} + \dots + a_{i_d}} p_{i_0 \dots i_d}(V).$$

Now, looking at the definition of $\mu^{\mathcal{L}_{\underline{k}}}(x, \lambda)$, which is the minimum of the exponents of t in a diagonal action (for the coordinates $\neq 0$), one sees that

$$\mu^{\mathcal{L}_{\underline{k}}}(x, \lambda) = \sum_{i=1}^{N} k_i (\sum_{j=0}^{d_i} a_{\nu_j^i}),$$

where the superscript ^{i} in ν_j^i denotes ν_j for the subspace V_i of $\mathbb{P}^n$. We rewrite this equation and use that $\dim(V_i \cap E_j) - \dim(V_i \cap E_{j-1})$ is either 0 or 1 (it is 1 if and only if $j = \nu_j^i$):

$$\mu^{\mathcal{L}_{\underline{k}}}(x, \lambda) = \sum_{i=1}^{N} k_i (\sum_{j=0}^{n} a_j (\dim(V_i \cap E_j) - \dim(V_i \cap E_{j-1}))) =$$

$$\sum_{i=1}^{N} k_i ((d_i + 1) a_n + \sum_{j=0}^{n-1} (\dim(V_i \cap E_j) + 1)(a_j - a_{j+1}) =$$

$$a_n \sum_{i=1}^{N} k_i (d_i + 1) + \sum_{j=0}^{n-1} (\sum_{i=1}^{N} k_i (\dim(V_i \cap E_j) + 1)(a_j - a_{j+1})).$$

The last formula is a linear function of the a_i. The a_i satisfy the conditions $\sum a_i = 0$ and $a_0 \geq \dots \geq a_n$. The points

$$a_0 = \dots = a_s = n - s, a_{s+1} = \dots a_n = -(s+1),$$

(for $0 \leq s \leq n-1$) span the 1-faces of this convex cone. The above linear function is negative if and only if it is negative for its extreme values. Thus we get

$$\mu^{\mathcal{L}_{\underline{k}}}(x, \lambda) < 0$$

if and only if

$$-\sum_{i=1}^{N} k_i (d_i + 1)(s+1) + (n+1)(\sum_{i=1}^{N} k_i (\dim(V_i \cap E_s) + 1)) < 0$$

for each s with $0 \leq s \leq n-1$. Now note that for each proper linear subspace $V \subset \mathbb{P}^n$ with $\dim(V) = s$ we could have chosen the coordinates $e_0, \ldots, e_n$ in such a way that $V = E_s$. Also, of course, the $<$ can be replaced by $\leq$. Thus the theorem follows. $\square$

2.2 Point sets in projective spaces

Sequences of points are the first example one looks at. An excellent and well known book on this subject is [DO88] and as always [Dol03] is useful too.

Here we consider sequences of points and hyperplanes at the same time. More precise, we consider the action of $SL(n+1)$ on a product $X := (\mathbb{P}^n)^{N_1} \times ((\mathbb{P}^n)^*)^{N_2}$ (with respect to the standard linearization). Let V denote the underlying vector space of $\mathbb{P}^n$, i.e. $\mathbb{P}^n := \mathbb{P}(V)$ where $V \cong k^{n+1}$. Let $\tilde{X}$ denote the product $\tilde{X} := V^{N_1} \times (V^*)^{N_2}$. An element of X can be represented by an element of $\tilde{X}$, i.e. a sequence $(x_1, \ldots, x_{N_1}, x_1^*, \ldots, x_{N_2}^*)$, where the $x_i \in V$ and the $x_i^* \in V^*$. We introduce a short notation for some expressions in these variables. If $0 < i_1, i_2, \ldots, i_{n+1} \leq N_1$ are natural numbers, we define:

$$[i_1 i_2 \ldots i_{n+1}] := \det(x_{i_1}, x_{i_2}, \ldots, x_{i_{n+1}})$$

and similarly for the dual vectorspaces: if $0 < j_1, j_2, \ldots, j_{n+1} \leq N_2$, we define:

$$[j_1 j_2 \ldots j_{n+1}]' := \det(x_{j_1}^*, x_{j_2}^*, \ldots, x_{j_{n+1}}^*).$$

These expressions can be considered as functions in $k[\tilde{X}]$ and are called *brackets* or *bracket functions*. They are obviously invariant functions. Their importance is that they generate the ring of invariants $k[\tilde{X}]^{SL(n+1)}$ together with another class of invariant functions on $\tilde{X}$, which we introduce now. For $i \in \{1, \ldots, N_1\}$ and $j \in \{1, \ldots, N_2\}$ we define:

$$(ij) := \langle x_j^*, x_i \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes nothing else than the pairing $V \times V^* \to k$ (on the i-th point of V^{N_1} and the j-th point of $(V^*)^{N_2}$).

THEOREM 2.2.1 (FIRST FUNDAMENTAL THEOREM OF INVARIANT THEORY) *The ring of invariants $k[\tilde{X}]^{SL(n+1)}$ is generated by all brackets*

$$[i_1 i_2 \ldots i_{n+1}],\ [j_1 j_2 \ldots j_{n+1}]',\ \textit{and}\ (ij),$$

with $0 < i, i_1, i_2, \ldots, i_{n+1} \leq N_1$ *and* $0 < j, j_1, j_2, \ldots, j_{n+1} \leq N_2$.

PROOF A proof can be found in [Wey46] for the case $\mathrm{char}(k) = 0$, but in fact the result is true in any characteristic, see [DP76]. □

The First Fundamental Theorem of Invariant Theory is the key to solve the moduli problem for sequences of points. To show how this works we give here a short account of the theory of moduli spaces of point sets (which is well known). Later we will investigate some examples in which both points and hypersurfaces occur. The idea is to form homogeneous products of the basic invariants given by the theorem. A convenient way to describe these homogeneous products is the use of *tableaux*. The following definitions can be found for example in [Fu97]. Some of these definitions we will also need, when we are going to do Schubert Calculus (in §4.1.1).

DEFINITION 2.2.2 A *tableau* (often called *Young tableau*) is a matrix

$$\tau := \begin{pmatrix} \tau_{11} & \tau_{12} & \cdots & \tau_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ \tau_{m1} & \tau_{m2} & \cdots & \tau_{mn} \end{pmatrix}$$

such that the entries τ_{ij} are the natural numbers in $\{1, \ldots, S\}$ with $S \mid mn$. Say that $St = mn$, then furthermore it is required that each natural number appearing in the matrix occurs exactly t times.

In the situation of this definition we say that τ is a tableau on $\{1, \ldots S\}$ of *shape* $m \times n$.

DEFINITION 2.2.3 A tableau is called a *standard tableau* if it is increasing in each row and weakly increasing in each column, i.e.

$$\begin{array}{lll} \tau_{ij} & < & \tau_{i,j+1} \text{ for } i \in \{1, \ldots, m\} \text{ and } j \in \{1, \ldots, n-1\} \\ \tau_{ij} & \leq & \tau_{i+1,j} \text{ for } i \in \{1, \ldots, m-1\} \text{ and } j \in \{1, \ldots, n\}. \end{array}$$

Tableaux help us to describe products of brackets in an orderly and convenient way.

DEFINITION 2.2.4 Let τ be a tableau as in Definition 2.2.2. The *monomial* belonging to the tableau is defined by

$$m(\tau) := \prod_{k=1}^{m} [\tau_{k1}\tau_{k2} \ldots \tau_{kn}].$$

A monomial belonging to a standard tableau is called a *standard monomial*.

Now if V is an n-dimensional k-vectorspace and τ is a tableau, one checks easily that $m(\tau)$ is a homogeneous invariant inside $k[V^S]$ (for the action of $SL(n)$) of degree m. The following theorem is of the utmost importance.

THEOREM 2.2.5 *Let n and N be natural numbers, and consider the action of $SL(n+1)$ on $(\mathbb{P}^n)^N$ with respect to the standard linearization (see Remark 2.1.6). The homogeneous coordinate ring of the quotient*

$$(\mathbb{P}^n)^N//SL(n+1)$$

is in degree t generated by the standard tableaux with shape $m \times (n+1)$. (Recall that t satisfies: $Nt = m(n+1)$.)

PROOF See again one of Dolgachev's books. After one has shown that the ring of invariants is generated by monomials, one gives an algorithm to write each monomial as linear combination of standard monomials. This is called the *straightening algorithm.* □

Using this theorem one finds some nice examples, which we treat here shortly (see [Dol03] and [DO88] for details).

EXAMPLE 2.2.6 (CROSS RATIO) In the case of 4 points on a projective line we have these two standard tableaux in degree 1:

$$t_0 := \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ and } t_1 := \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

corresponding to the standard monomials

$$m(t_0) : (([x_i, y_i])_{i=1\ldots 4} \in (\mathbb{P}^1)^4 \mapsto \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \cdot \det \begin{pmatrix} x_3 & y_3 \\ x_4 & y_4 \end{pmatrix}, \text{ and}$$
$$m(t_1) : (([x_i, y_i])_{i=1\ldots 4} \in (\mathbb{P}^1)^4 \mapsto \det \begin{pmatrix} x_1 & y_1 \\ x_3 & y_3 \end{pmatrix} \cdot \det \begin{pmatrix} x_2 & y_2 \\ x_4 & y_4 \end{pmatrix}$$

One can also write down the standard tableaux in higher dimensions and thus find standard monomials of higher degree (than 1). Then one easily shows that all these higher degree monomials are products of $m(t_0)$ and $m(t_1)$. This is left to the reader (or can be found in the cited books).

EXAMPLE 2.2.7 (SEGRE CUBIC) The moduli space of 6 points on a line is another famous example. It turns out to be the Segre cubic threefold. This threefold can be given as the zero set in $\mathbb{P}^5$ of the following two equations:

$$\sum_{i=0}^{5} X_i = 0 \text{ and } \sum_{i=0}^{5} X_i^3 = 0.$$

See [DO88], page 14 for the calculation. It starts with writing down the standard monomials in degree 1 and 2 and showing that these generate all standard monomials. It ends with deriving relations between these invariants.

2.3 Points and lines in the plane

We elaborate on a nice example.

PROPOSITION 2.3.1 *The moduli space of sequences of 3 points and 3 lines in $\mathbb{P}^2$ is the toric variety lying in $\mathbb{P}^5$ given by the single equation $X_0X_3X_4 = X_1X_2X_5$.*

PROOF By the First Fundamental Theorem of Invariant Theory 2.2.1 we know exactly the (inhomogeneous) invariants for the action of $SL(3)$ on $(\mathbb{P}^2)^3 \times ((\mathbb{P}^2)^*)^3$. If we denote the points by p_1, p_2, p_3 and the lines by l_1, l_2, l_3, the basic invariants of homogeneous multi-degree $(1, 1, \ldots, 1)$ are:

$$\begin{array}{lcl} W & := & [123][123]' \\ X_0 & := & (11)(22)(33) \\ X_1 & := & (11)(23)(32) \\ X_2 & := & (12)(21)(33) \\ X_3 & := & (12)(23)(31) \\ X_4 & := & (13)(21)(32) \\ X_5 & := & (13)(22)(31) \end{array}$$

The first basic invariant is linearly dependent on the others:

$$W = X_0 + X_3 + X_4 - X_1 - X_2 - X_5$$

as one checks by hand (formula for a 3×3 determinant). One also checks easily that the X_i satisfy the following relation:

$$X_0X_3X_4 = X_1X_2X_5.$$

I claim that all higher multi-degree homogeneous invariants are (linear combinations of) products of the X_i. This is seen by solving some linear equations. A basic homogeneous invariant of multi-degree $(k, k, \ldots, k)$ has the form:

$$\prod_{(i,j)\in\{1,2,3\}^2} (ij)^{a_{ij}},$$

where the a_{ij} are natural numbers satisfying

$$\sum_{i=1}^{3} a_{ij} = k \text{ for each } j$$

and

$$\sum_{j=1}^{3} a_{ij} = k \text{ for each } i.$$

These are exactly the equations meaning that the matrix $(a_{ij})_{(i,j)\in\{1,2,3\}^2}$ is a semi-magic square of weight k. Such a semi-magic square has 4 independent equations: the sum of the entries of the first row equals the sum of the entries of each of the other two rows and furthermore it equals the sum of the entries of two of the columns (the last equation is redundant). So the solution space is 5-dimensional. The 6 permutation matrices span a 5-dimensional space of solutions. Each of them corresponds to one of the 6 basic invariants $X_0, X_1, \ldots, X_5$. □

The magic square example above can be generalized.

REMARK 2.3.2 Consider sequences of l points and l hyperplanes in $\mathbb{P}^{l-1}$. The basic invariants of multi-degree $\underline{k} = (1, \ldots, 1)$ are:

$$\begin{array}{lll} W & := & [12\ldots l][12\ldots l]' \\ X_0 & := & \\ \vdots & \vdots & \text{invariants corresponding} \\ \vdots & \vdots & \text{to the permutations of} \\ \vdots & \vdots & \text{the set } \{1,\ldots,p\} \\ X_{l!-1} & := & \end{array}$$

Because of the formula for the determinant the invariant W is a linear combination of $X_0, \ldots, X_{l!-1}$.

It is a fact (proven by König in 1916) that each semi-magic square of weight k is a linear combination with positive integer coefficients of k permutation matrices. This result implies that multi-degree $(k,\ldots,k)$ invariants for l points and l lines in $\mathbb{P}^{l-1}$ depend on the basic ones (as in example 2.3.1 above). There are lots of dependencies between the basic invariants. The dimension of the moduli space of l points and l lines in $\mathbb{P}^{l-1}$ is $2l(l-1)-(l^2-1)$, which equals l^2-2l+1. This corresponds to the fact that an $l\times l$ semi-magic square has l^2 unknowns and $2(l-1)$ independent equations to be solved.

The following example shows that one not always takes the standard linearization.

EXAMPLE 2.3.3 Take: $n=2$, $N=3$ and $\underline{d}=(0,0,1)$. Thus we consider 2 points and 1 line in the projective plane. We denote the points by p_1,p_2 and the line by l_1. Then these are the basic invariants according to the First Fundamental Theorem of Invariant Theory:

$$(11) \text{ and } (21).$$

One sees immediately that it is impossible to make homogeneous invariants out of these with respect to the standard linearization. This corresponds to the fact that there aren't any semi-stable configurations (as one checks using the Mumford Criterion).

Instead of taking the standard linearization one can give the sequence of weights $\underline{k}=(1,1,2)$ to the points and the line. We see that:

$$\mathcal{C}(\underline{d},n)^{ss}(\mathcal{L}_{\underline{k}})//SL(3)\cong \mathrm{Proj}(k[(11)(21)])\cong \text{a point}.$$

This is what one expects. Now one finds that a configuration is semi-stable if and only if none of the two points is on the line. There are no stable points, because all stabilizers have positive dimension. Note that the two points may coincide. On the locus where the two points coincide the stabilizer is bigger. On the locus where the two points do not coincide, the dimension of the stabilizer is constant and the quotient is geometric. So this example also illustrates Remark 1.2.6.

We give two additional examples of moduli spaces of points and lines in the projective plane.

EXAMPLE 2.3.4 Take $n=2$, $N=5$ and $\underline{d}=(0,0,0,0,1)$. Thus we consider sequences of 4 points and 1 line. Denote as always the points by $p_1,\ldots,p_4$ and the line by l_1. The basic homogeneous multi-degree $(1,1,1,1,1)$ invariants are:

$$[123](41'),\ [124](31'),\ [134](21'),\ [234](11'),$$

and one sees easily that the generating higher multi-degree invariants are products of these. Also one shows that:

$$[123](41') = [124](31') + [134](21') + [234](11').$$

So the moduli space is a $\mathbb{P}^2$. Mumfords criterion shows which configurations are stable (resp. semi-stable). One finds that a configuration is stable if the points $p_1, \dots p_4$ lie in general position and at most one of them lies on l_1. A configuration is semi-stable if and only if not all points $p_1, \dots p_4$ are collinear and at most 2 of the points are on l_1. The strictly semi-stable locus of the configuration space maps to 4 lines in general position in the moduli space.

EXAMPLE 2.3.5 Take $n = 2$, $N = 5$ and $\underline{d} = (0, 0, 0, 1, 1)$. Thus we consider sequences of 3 points and 2 lines. Denote as always the points by p_1, p_2, p_3 and the lines by l_1, l_2. This time there are no degree 1 or degree 2 homogeneous invariants. Basic multi-degree 3 invariants are for example:

$$[123](11')(12')(21')(22')(31')(32'),\ [123](11')^2(22')^2(31')(32')$$

One can represent them by 3x2 matrices. The entry at (i,j) gives the exponent of (ij'). Then

$$\begin{matrix}1&1\\1&1\\1&1\end{matrix}\quad\begin{matrix}1&1\\2&0\\0&2\end{matrix}\quad\begin{matrix}1&1\\0&2\\2&0\end{matrix}\quad\begin{matrix}2&0\\1&1\\0&2\end{matrix}\quad\begin{matrix}0&2\\1&1\\2&0\end{matrix}\quad\begin{matrix}2&0\\0&2\\1&1\end{matrix}\quad\begin{matrix}0&2\\2&0\\1&1\end{matrix}$$

represent the 7 basic multi-degree 3 invariants. Call them (in the same order as above) $x_1, \dots, x_7$. There exist the following relations between these invariants:

$$\begin{aligned}
x_1^2 &= x_2x_3\\
x_1^2 &= x_4x_5\\
x_1^2 &= x_6x_7\\
x_1x_2 &= x_4x_7\\
x_1x_3 &= x_5x_6\\
x_1x_4 &= x_2x_6\\
x_1x_5 &= x_3x_7\\
x_1x_6 &= x_3x_4\\
x_1x_7 &= x_2x_5.
\end{aligned}$$

The variety given by these equations is the closure in $\mathbb{P}^6$ of the torus

$$\{[1, x_2, x_2^{-1}, x_4, x_4^{-1}, x_4x_2^{-1}, x_2, x_4^{-1}] \mid x_2, x_4 \in \mathbb{C}^*\}.$$

This closure is the GIT quotient of sequences of 3 points and 2 lines in the projective plane. One proves this by showing that all higher degree homogeneous invariants are products of $x_1, \ldots, x_7$. Every multi-degree 3k invariant can be represented by a 3x2 matrix of natural numbers such that the sum of the entries of each row is 2k and the sum of the entries of each column is 3k. Thus

$$\begin{array}{cc} a & 2k-a \\ c & 2k-c \\ 3k-a-c & a+c-k \end{array}$$

represents the invariant

$$(11')^a(12')^{2k-a}(21')^c(22')^{2k-c}(31')^{3k-a-c}(32')^{a+c-k}[123]^k.$$

Here $a, c \in \{0, \ldots 2k\}$ and $k \leq a+c \leq 3k$. Such a matrix is a positive integral sum of the basic matrices. This fact is proven by solving a system of linear equations and showing that one can choose a solution with positive integers (this is a little bit of work; one has to look at some cases).

2.4 A map between quotients

Yet another example, which is well known, is the moduli space of 6 points in the plane. We summarize the result. Details can be found in [DO88], pages 17-19.

EXAMPLE 2.4.1 Take $n = 2$, $N = 6$, $\underline{d} = (0, \ldots, 0)$ and $\underline{k} = (1, \ldots, 1)$. The GIT quotient of 6 points in the plane with the standard linearization we denote by $\mathcal{D}$. It turns out that $\mathcal{D}$ is the double cover of $\mathbb{P}^4$ branched along the *Igusa quartic*. The Igusa quartic is a quartic threefold. It can be given as a hypersurface in $\mathbb{P}^4$ by the equation

$$\begin{aligned} F_4(Y_0, \ldots, Y_4) := \quad & (-Y_2Y_3 + Y_1Y_4 + Y_0Y_1 + Y_0Y_4 - Y_0Y_2 - Y_0Y_3 - Y_0^2)^2 \\ & -4Y_0Y_1Y_4(-Y_0 + Y_1 - Y_2 - Y_3 + Y_4) \end{aligned}$$

and its singular locus consists of 15 lines. Thus $\mathcal{D} \subset \mathbb{P}(1^5, 2)$. If $\mathbb{P}(1^5, 2)$ has coordinates $Y_0, \ldots, Y_5$ (and Y_5 has weight 2), then $\mathcal{D}$ is given by the equation

$$Y_5^2 = F_4(Y_0, \ldots, Y_4).$$

The quotient map $\mathcal{C}(\underline{d}, n)^{ss}(\mathcal{L}_{\underline{k}}) \to \mathcal{D}$ is given by

$$\begin{array}{rcl} Y_0 & := & [123][456] \\ Y_1 & := & [124][356] \\ Y_2 & := & [125][346] \\ Y_3 & := & [134][256] \\ Y_4 & := & [135][246] \\ Y_5 & := & [123][145][246][356] - [124][135][236][456] \end{array} .$$

Now recall the moduli space of sequences of 3 points and 3 lines in the plane (see proposition 2.3.1). Call this moduli space $\mathcal{T}$. Obviously there is an equivariant birational map

$$\psi : \mathcal{C}((0,0,0,1,1,1),2) \to \mathcal{C}((0,0,0,0,0,0),2)$$

sending 3 points and 3 lines $(p_1, p_2, p_3, l_1, l_2, l_3)$ to the 6 points $(l_1 \cap l_2, l_2 \cap l_3, l_3 \cap l_1, p_1, p_2, p_3)$. A calculation in the computer algebra package Maple shows that this map extends to a birational map on the quotients

$$\phi : \mathcal{T} \to \mathcal{D}$$

given by

$$\begin{array}{rcl} Y_0 & := & X_3 - X_2 + X_0 - X_5 + X_4 - X_1 \\ Y_1 & := & X_4 - X_2 \\ Y_2 & := & X_5 - X_0 \\ Y_3 & := & X_1 - X_0 \\ Y_4 & := & X_3 - X_2 \\ Y_5 & := & X_4X_0 + X_3X_0 - X_5X_1 + X_3X_4 - X_1X_2 - X_5X_2. \end{array}$$

The calculation in Maple goes as follows. One takes a generic configuration of 3 points and 3 lines. Without loss of generality one assumes that the lines are given by:

$$Z_0 = 0,\ Z_1 = 0,\ \text{and } Z_2 = 0,$$

(the Z_i are the coordinates of $\mathbb{P}^2$ of course) and the points have coordinates

$$[111],\ [1ab],\ \text{and } [1cd]$$

with some conditions on a, b, c and d in order to let the configuration be in general position. We denote by

$$\begin{array}{l} Q_{\mathcal{T}} : \mathcal{C}((0,0,0,1,1,1),2) \to \mathcal{T} \text{ and} \\ Q_{\mathcal{D}} : \mathcal{C}((0,0,0,0,0,0),2) \to \mathcal{D} \end{array}$$

the quotient maps and we would like it if

$$\phi \circ Q_{\mathcal{T}} = Q_{\mathcal{D}} \circ \psi.$$

One calculates the images of $Q_{\mathcal{T}}$ and $Q_{\mathcal{D}} \circ \psi$ of our generic point, one sees what has to be the map ϕ and one checks that the guessed map is the right map.

Further calculations (in Maple or by hand) yield the following properties of the map ϕ.

1. The only point where the map ϕ isn't defined is the point with homogeneous coordinates $P := [111111] \in \mathcal{T} \subset \mathbb{P}^5$. P is the image of strictly semi-stable points in the configuration space corresponding to configurations satisfying:
   (a) the three points coincide, or
   (b) the three lines coincide, or
   (c) two of the three points coincide and the three lines pass through a point, or
   (d) two of the three lines coincide and the three points lie on a line.
2. Denote by $\pi : \mathcal{D} \to \mathbb{P}^4$ the double covering map. The composition
   $$\pi \circ \phi : \mathcal{T} \setminus \{P\} \to \mathbb{P}^4$$
   is projection from the point P. Since $\deg(\mathcal{T}) = 3$ this projection is 2:1 generically (which is nice, because π is 2:1 of course). Note that $Y_0 = X_3 - X_2 + X_0 - X_5 + X_4 - X_1 = 0$ is the tangent plane to $\mathcal{T}$ at $[111111]$. The map ϕ is 1:1 on
   $$U := \{\text{locus where } Y_0 \neq 0\}.$$
   (Also on $\mathcal{T} \setminus U$ the map ϕ is generically 1:1.)
3. The 9 singular lines of $\mathcal{T}$ are mapped to 9 of the 15 singular lines of $\mathcal{D}$. The other 6 singular lines of $\mathcal{D}$ have planes as fibres.

3 Gale Duality (association) and invariants of sequences of linear subspaces

In this chapter we will investigate a generalization of Gale duality. An excellent overview of Gale duality of points sets is [EP00]. We will give all definitions and theorems that we need.

3.1 Generalized Gale Duality

In this paragraph we will give various equivalent definitions of the generalized Gale transform. We start with the most symmetric and abstract one.

DEFINITION 3.1.1 Let V, W and $L_1, \dots L_N$ be vectorspaces of dimensions $\dim(V) := r$, $\dim(W) := s$ and $\dim(L_i) := e_i$, such that $0 \leq e_i \leq \min(r, s)$ for all i and

$$\sum_{i=1}^{N} e_i = r + s.$$

Suppose we have injective linear maps $j_i : L_i \to V$ and surjective linear maps: $p_i : W \to L_i$. Also suppose we have an exact sequence

$$0 \to W \xrightarrow{\psi} \bigoplus_{i=1}^{N} L_i \xrightarrow{\phi} V \to 0$$

and the maps ϕ and ψ are given by

$$\begin{array}{lcll} \psi(x) & = & (p_1(x), \dots, p_N(x)) & \text{for } x \in W, \text{ and} \\ \phi(y_1, \dots, y_N) & = & \sum_{k=1}^{N} i_k(y_k) & \text{for } (y_1, \dots, y_N) \in \bigoplus_{k=1}^{N} L_k. \end{array}$$

In this situation we can dualize everything. We obtain an exact sequence

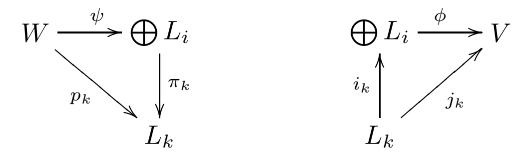
$$0 \to V^* \xrightarrow{\phi^*} \bigoplus_{i=1}^{N} L_i^* \xrightarrow{\psi^*} W^* \to 0,$$

with injective linear maps $p_k^* : L_k^* \to W^*$ and surjective linear maps $j_k^* : V^* \to L_i^*$. The maps ψ^* and ϕ^* are given by

$$\begin{array}{lcll} \phi^*(x) & = & (j_1^*(x), \dots, j_N^*(x)) & \text{for } x \in V^*, \text{ and} \\ \psi^*(y_1, \dots, y_N) & = & \sum_{k=1}^{N} p_k^*(y_k) & \text{for } (y_1, \dots, y_N) \in \bigoplus_{k=1}^{N} L_k^*. \end{array}$$

We call the set of linear subspaces $p_1^*(L_1^*), \dots, p_N^*(L_N^*)$ lying in W^* the *Gale dual* of the set of linear subspaces $j_1(L_1), \dots, j_N(L_N)$ lying in V.

REMARK 3.1.2 Another way of describing the maps ϕ and ψ is saying that for each $k \in \{1, \ldots, N\}$ the following two diagrams commute:



Here $i_k : L_k \to \bigoplus L_i$ denotes the natural inclusion and $\pi_k : \bigoplus L_i \to L_k$ denotes the natural projection. Then in the dual situation the two diagrams

commute for each $k \in \{1, \ldots, N\}$. Thus we see that the maps ϕ and ψ in the dual situation change to the maps ψ^* and ϕ^* given by the same equations.

REMARK 3.1.3 In the definition of Gale transform we allow the dimensions of the subspaces to be zero or maximal, i.e. equal to the dimension of one of the two ambient spaces. Having a maximal subspace is considered in Example 3.4.3. Our geometric construction of the Gale transform (see §3.3) also allows $\{0\}$ as a subspace (but it will only add a $\{0\}$ in the Gale dual).

Suppose we are given a finite-dimensional vector space V and a sequence of linear subspaces $V_1, \ldots, V_N$. Does a Gale dual exist and how do we find it? According to the definition we have to consider an exact sequence

$$0 \to \ker(\phi) \to \bigoplus_{i=1}^{N} V_i \xrightarrow{\phi} V \to 0.$$

In particular the subspaces have to span V. Furthermore we have to require that the maps

$$p_j : \ker(\phi) \to \bigoplus_{i=1}^{N} V_i \to V_j$$

are surjective.

PROPOSITION 3.1.4 *A Gale dual of a set $V_1, \ldots, V_N$ of subspaces of a vectorspace V exists if and only if each $N-1$ of the subspaces span V. The Gale dual is unique upto choice of coordinates. Taking the Gale dual is an involution on sequences of linear subspaces, compatible with possible permutations.*

PROOF It is easy to see that a map p_j is surjective if and only if

$$\mathrm{span}(V_1, \ldots, V_{j-1}, V_{j+1}, \ldots, V_N) \supset V_j,$$

which is the same as saying that

$$\mathrm{span}(V_1, \ldots, V_{j-1}, V_{j+1}, \ldots, V_N) = \mathrm{span}(V_1, \ldots, V_N) = V.$$

This proves the first statement. The second statement is clear from the definition (but had to be remarked somewhere). It is also clear that changing the order of a sequence of linear subspaces just changes the order of the subspaces of the Gale dual. □

Since our purpose is to investigate this duality for configurations (i.e. lying in projective space), the following definition is not surprising.

DEFINITION 3.1.5 Let $x := (V_1, \ldots, V_N) \in \mathcal{C}(\underline{d}, n)$ be a configuration. Suppose each $N-1$ of the N subspaces $V_1, \ldots, V_N \subset \mathbb{P}^n$ span $\mathbb{P}^n$. Now consider x as a sequence of linear subspaces of k^{n+1}. Take its Gale dual and consider it as a sequence of projective linear subspaces of a projective space. This is by definition the *Gale dual* of x. It is an element of $\mathcal{C}(\underline{d}, m)$, where $m = \sum(d_i + 1) - n - 2$.

REMARK 3.1.6 The *Gale transform* is the birational, equivariant map

$$\mathcal{C}(\underline{d}, n) \dashrightarrow \mathcal{C}(\underline{d}, \sum(d_i + 1) - n - 2),$$

which is taking the Gale dual.

REMARK 3.1.7 From Proposition 3.1.4 it is clear that a Gale dual can only exist if $\sum_{i=1}^N (d_i + 1) > n + 1$. If this condition is satisfied, it follows that configurations that are semi-stable with respect to the standard linearization admit a Gale dual. Suppose $x := (V_1, \ldots, V_N) \in \mathcal{C}(\underline{d}, n)$ does not admit a Gale dual, then according to Proposition 3.1.4 there exists a hyperplane H that contains $N-1$ of the subspaces V_i. If x is semi-stable, then

$$\frac{(\sum_{i=1}^N (d_i + 1)) - 1}{\sum_{i=1}^N (d_i + 1)} \leq \frac{\sum_{i=1}^N (\dim(V_i \cap H) + 1)}{\sum_{i=1}^N (d_i + 1)} \leq \frac{n}{n+1}$$

(using Mumford's Criterion 2.1.9 in the second inequality). Then $\sum_{i=1}^N (d_i + 1) \leq n + 1$, contrary to our assumption. So x is not semi-stable.

Gale duality is often referred to as *association*. Castelnuovo used this name in [Cas89], where he called two point sets of $2n + 2$ points in $\mathbb{P}^n$ that are Gale dual "gruppi associati di punti". We say that two configurations are *associated* if they are each other's Gale dual.

Association was first defined only in the case of points lying in projective space, i.e. $\underline{d} = (0, \ldots, 0)$. Choosing coordinates, one can represent a sequence of N points in $\mathbb{P}^n$ by a matrix X of size $N \times (n + 1)$. Then a Gale dual is a set of N points lying in $\mathbb{P}^{N-n-2}$ and can be represented by a matrix Y of size $N \times (N - n - 1)$. Now the definition easily translates to: the two sets of points represented by X and Y are associated if and only if there exists an invertible diagonal matrix Γ of size $N \times N$ such that

$$Y^T \cdot \Gamma \cdot X = 0.$$

The diagonal matrix Γ appears to avoid dependence on the choices of homogeneous coordinates.

REMARK 3.1.8 Yet another (though similar) way to define association of point sets: given a configuration of N points lying in $\mathbb{P}^n$, such that a Gale dual exists (see remark 3.1.4). Choose homogeneous coordinates and an ordering of the points such that the points can be represented as the rows of the matrix

$$\begin{pmatrix} I_{n+1} \\ A \end{pmatrix},$$

where I_{n+1} is the $(n + 1) \times (n + 1)$ identity matrix and A is a matrix of size $(N - n - 1) \times (n + 1)$. The Gale dual of this set of points, is the set of points in $\mathbb{P}^{N-n-2}$ represented by the rows of the matrix

$$\begin{pmatrix} A^T \\ I_{N-n-1} \end{pmatrix}.$$

This definition is indeed equivalent to the previous one, as shows:

$$\begin{pmatrix} A & I_{N-n-1} \end{pmatrix} \begin{pmatrix} -I_{n+1} & 0 \\ 0 & I_{N-n-1} \end{pmatrix} \begin{pmatrix} I_{n+1} \\ A \end{pmatrix} = 0.$$

At first sight these algebraic definitions do not seem to have any geometric meaning at all. To suggest that geometry really plays an important part, we give some examples, before we go on.

EXAMPLE 3.1.9 A set of 6 points in $\mathbb{P}^2$ is Gale dual to itself if and only if the 6 points lie on a conic. This follows immediately from [EP00], th. 7.1. This theorem implies that a point set $\Gamma \subset \mathbb{P}^n$ is self-associated if and only if $\#\Gamma = 2n+2$ and each $p \in \Gamma$ satisfies: $\Gamma \setminus \{p\}$ imposes the same number of conditions on quadrics as Γ does.

EXAMPLE 3.1.10 The Gale transform of a set of N points in $\mathbb{P}^1$ is the corresponding set of N points on the rational normal curve in $\mathbb{P}^{N-3}$, which is the image of a Veronese embedding. Conversely, through N points in $\mathbb{P}^{N-3}$ in general position goes a unique rational normal curve. The Gale dual of this point set is the set of points on $\mathbb{P}^1$ corresponding to the points on the rational normal curve. See [DO88], ch. 3, §2, prop. 2 for a proof.

The article [EP00] contains many examples and theorems about the geometry of the Gale transform in the case of sequences of points. We will investigate the geometric meaning of Gale duality in the case of configurations of linear subspaces in the next paragraph.

An important observation was made by Dolgachev and Ortland ([DO88], ch. 3, §1, th. 1). They showed that association of point sets, being defined upto projective equivalence, can in fact be extended to GIT quotients, i.e. there exists an isomorphism

$$\mathcal{C}((0,\ldots,0),n)^{ss}//SL(n+1) \cong \mathcal{C}((0,\ldots,0),m)^{ss}//SL(m+1),$$

where $m = N - n - 2$ and N is the number of points.

We will prove an analogous theorem for the case of arbitrary configurations. In our efforts to prove such a theorem, we first proved the following proposition (only for the standard linearization), which is necessary, but is not used in the eventual proof of theorem 3.1.13.

PROPOSITION 3.1.11 *Let $x := (V_1, \ldots, V_N) \in \mathcal{C}(\underline{d}, n)$ be a configuration that is semi-stable with respect to the standard linearization. Then its Gale dual is semi-stable and x is stable if and only if its Gale dual is stable.*

PROOF We use the Mumford Criterion for stability of configurations, see th. 2.1.9. We consider the subspaces V_i as linear subspaces of a vectorspace V and the Gale dual subspaces V_i^* as subspaces of W, as in Definition 3.1.1. Suppose x violates this criterion, i.e. is not stable. This means there exists a proper linear subspace $L \subset V$ such that

$$\frac{\sum_{i=1}^N \dim(V_i \cap L)}{\sum_{i=1}^N \dim(V_i)} \geq \frac{\dim(L)}{\dim(V)}$$

We assume that the spaces $V_i \cap L$ span L, because if they don't, we may replace L by $\operatorname{span}(V_1 \cap L, \ldots, V_N \cap L)$ and this space satisfies surely the same inequality as L. Thus we can make the following commutative diagram.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & W & \longrightarrow & \bigoplus_{i=1}^N V_i & \longrightarrow & V & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \bigoplus_{i=1}^N (V_i \cap L) & \longrightarrow & L & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

By diagram chasing or using the serpent's lemma one sees that this diagram can be extended to this big commutative diagram of vectorspaces and linear maps. The rows and columns are exact sequences.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M & \longrightarrow & A = \bigoplus_{i=1}^N A_i & \longrightarrow & B & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & W & \longrightarrow & \bigoplus_{i=1}^N V_i & \longrightarrow & V & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K & \longrightarrow & \bigoplus_{i=1}^N (V_i \cap L) & \longrightarrow & L & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

If we dualize this diagram, M^* can be considered as a subspace of W^* and plays the rôle of L as subspace of V. We will show that

$$\frac{\sum_{i=1}^N \dim(V_i^* \cap M^*)}{\sum_{i=1}^N \dim(V_i^*)} \geq \frac{\dim(M^*)}{\dim(W^*)},$$

i.e. the Gale dual of x is not stable. To prove this we need to know a little bit more about the vectorspaces A_i in the diagram above. The maps $V_i \to A_i$ are surjective, so the A_i^*

can be considered as subspaces of the V_i^*. Also, because the maps $W \to \bigoplus V_j \to V_i$ are surjective, the maps $M \to \bigoplus A_j \to A_i$ are surjective, so the A_i^* can be considered as subspaces of M^*. So we have: $A_i^* \subset V_i^* \cap M^*$ for all i. One would hope (and maybe expect) that $A_i^* \cong V_i^* \cap M^*$. Then the situation would be completely symmetric, but this isn't true in general (we will comment on this later). We show now that M^* satisfies the inequality above. Because

$$\dim(A) = \sum_{i=1}^{N} \dim(A_i) \leq \sum_{i=1}^{N} \dim(M^* \cap V_i^*)$$

it is sufficient to prove

$$\frac{\dim(A)}{\sum_{i=1}^{N} \dim(V_i)} \geq \frac{\dim(M)}{\dim(W)}$$

Because $\dim(M) = \dim(A) - \dim(B)$ and $\dim(W) = \sum_{i=1}^{N} \dim(V_i) - \dim(V)$ we have

$$\begin{array}{rcl}
\dfrac{\dim(A)}{\sum_{i=1}^{N} \dim(V_i)} & \geq & \dfrac{\dim(M)}{\dim(W)} \\
& \Longleftrightarrow & \\
\dim(A)(\sum_{i=1}^{N} \dim(V_i) - \dim(V)) & \geq & (\dim(A) - \dim(B))(\sum_{i=1}^{N} \dim(V_i)) \\
& \Longleftrightarrow & \\
\dim(A)\dim(V) & \leq & \dim(B)(\sum_{i=1}^{N} \dim(V_i)) \\
& \Longleftrightarrow & \\
1 - \dfrac{\dim(A)}{\sum_{i=1}^{N} \dim(V_i)} & \geq & 1 - \dfrac{\dim(B)}{\dim(V)} \\
& \Longleftrightarrow & \\
\dfrac{\sum_{i=1}^{N} \dim(V_i \cap L)}{\sum_{i=1}^{N} \dim(V_i)} & \geq & \dfrac{\dim(L)}{\dim(V)}
\end{array}$$

This last equivalence because $\sum_{i=1}^{N} \dim(V_i \cap L) = \sum_{i=1}^{N} \dim(V_i) - \dim(A)$ and $\dim(L) = \dim(V) - \dim(B)$. $\Box$

Now we will prove the GIT version of generalized Gale duality. Essentially the proof is an elaboration on the following observation. Let, as always, $n, N \in \mathbb{N}$ and let $\underline{d} = (d_1, \ldots, d_N) \in \mathbb{Z}^N$ satisfy $0 \leq d_1, \ldots, d_N < n$. Consider matrices of size

$(n+1)\times(\sum d_i+1)$. Such a matrix M can be seen as the concatenation of N matrices D_i of sizes $(n+1)\times(d_i+1)$:

$$M := (M_1,\ldots,M_N)$$

If we assume that all M_i have maximal rank, M can be considered as representative of a sequence of linear subspaces lying in $\mathbb{P}^n$ of dimensions d_i. On the other hand, if we assume that the matrix M itself has maximal rank, it can be considered as a linear subspace of dimension n of $\mathbb{P}^{\sum(d_i+1)-1}$. Let U be the set of matrices of size $(n+1)\times(\sum d_i+1)$, such that both conditions on the rank are fulfilled (this is equivalent to: the subspaces span $\mathbb{P}^n$). There's an action on U from the left of the group $GL(n+1)$. The quotient of this action is a Grassmannian: $\mathbb{G}(n,\sum(d_i+1)-1)$. We can also consider an action on U from the right by the group $H < GL(\sum(d_i+1))$, consisting of blockmatrices with (invertible) blocks of sizes $d_1+1,\ldots,d_N+1$. The quotient of this action is $\prod\mathbb{G}(d_i,n)$. Because multiplication of matrices is associative, these two actions commute, so at least set-theoretically we have a bijection

$$GL(n+1)\backslash\prod_{i=1}^{N}G(d_i+1,n+1)\cong G(n+1,\sum(d_i+1)-1)/H$$

The idea to associate configurations of points to elements of a Grassmannian comes from [GM82]. It is known as the *Gelfand-Macpherson correspondence*. Kapranov extended this correspondence to the level of Chow quotients and GIT quotients in [Kap92] (for the case of point sets: $d_i=0$ for all i).

We extend the Gelfand-MacPherson correspondence to the level of GIT quotients for arbitrary configurations. Let $\underline{d}$ be a sequence of dimensions. The actions we have to consider, are the action of $SL(n+1)$ on $\mathcal{C}(\underline{d},n)=\prod\mathbb{G}(d_i,n)$ and the action of

$$H_1 := \left\{ D\in SL(\sum(d_i+1)) \;\middle|\; \begin{array}{l}\text{matrices in blockform with}\\ \text{blocks } D_1,\ldots,D_N \text{ satisfying}\\ D_i\in GL(d_i+1)\text{ for all } i\end{array}\right\}$$

on $\mathbb{G}(n,\sum(d_i+1)-1)$.

In §2.1 we have seen that every invertible sheaf on $\mathcal{C}$ admits a unique linearization for the action of $SL(n+1)$. For a sequence of positive weights $\underline{k}$ and associated very ample line bundle $\mathcal{L}_{\underline{k}}$, denote this unique linearization by $\lambda_{\underline{k}}$.

In order to classify the possible linearizations for the action of H_1 we consider the exact sequence (of theorem 1.3.3)

$$0\to\chi(H_1)\to\mathrm{Pic}^{H_1}(\mathbb{G}(n,\sum(d_i+1)-1))\xrightarrow{T}\mathrm{Pic}(\mathbb{G}(n,\sum(d_i+1)-1))\to\mathrm{Pic}(H_1).$$

PROPOSITION 3.1.12 *We have the following isomorphisms for the groups in the exact sequence above:*

$$\chi(H_1) \cong \mathbb{Z}^N/(1,\ldots,1)\cdot\mathbb{Z},\ \mathrm{Pic}(\mathbb{G}(n,\sum(d_i+1)-1)) \cong \mathbb{Z},\ \mathrm{Pic}(H_1) \cong 0.$$

PROOF That $\mathrm{Pic}(\mathbb{G}(n,\sum(d_i+1)-1)) \cong \mathbb{Z}$ we already knew (Proposition 2.1.3). That $\chi(H_1) \cong \mathbb{Z}^N/(1,\ldots,1)$ isn't difficult if one knows that every rational character of $GL(n+1)$ is of the form $A \mapsto \det(A)^g$ with $g \in \mathbb{Z}$. To an element $\underline{k} \in \mathbb{Z}^N$, upto multiples of $(1,\ldots,1)$, corresponds the character $\chi_{\underline{k}} : H_1 \to k^*$ given by $\chi_{\underline{k}}(D_1,\ldots,D_N) = \det(D_1)^{k_1}\ldots\det(D_N)^{k_N}$. That $\mathrm{Pic}(H_1) \cong 0$ follows from the fact that $H_1 \cong S \setminus H$, where S is a hypersurface in $\mathbb{P}^{\sum(d_i+1)^2}$ and H is a hyperplane section (the proof is completely analogous to the proof of Proposition 2.1.1). □

On $\mathbb{G}(n,\sum(d_i+1)-1)$ we will only consider the invertible sheaf $\mathcal{O}(1)$. From the proposition it follows that for each $\underline{k} \in \mathbb{Z}^N$ (upto multiples of $(1,\ldots,1)$), there exists a corresponding linearization of the action of H_1. We denote this corresponding linearization by $\mu_{\underline{k}}$.

We can state our generalized Gelfand-MacPherson theorem.

THEOREM 3.1.13 *For* $\underline{k} := (k_1,\ldots,k_N) \in \mathbb{Z}^N$ *with all* $k_i > 0$ *there is an isomorphism of GIT-quotients*

$$\mathbb{G}(n,\sum(d_i+1)-1)^{ss}_{\mu_{\underline{k}}}(\mathcal{O}(1))//H_1 \cong \prod_{i=1}^{N}\mathbb{G}(d_i,n)^{ss}_{\lambda_{\underline{k}}}(\mathcal{L}_{\underline{k}})//SL(n+1).$$

PROOF We determine the homogeneous rings of invariants of both spaces and use Theorem 1.2.8.

Denote by $R(n,\sum(d_i+1)-1)$ the homogeneous coordinate ring of $\mathbb{G}(n,\sum(d_i+1)-1) = G(n+1,\sum d_i+1)$ in the Plücker embedding. One can identify it with the ring of polynomials Φ on matrices M of size $(n+1)\times(\sum d_i+1)$, such that all M and all $g \in SL(n+1)$ satisfy: $\Phi(g\cdot M) = M$. Because of the linearization $\mu_{\underline{k}}$ we use, the degree e part of the homogeneous ring of invariants is:

$$R(n,\sum(d_i+1)-1)^{H_1}_e = \left\{\begin{array}{l} \text{polynomials } \Phi \text{ on matrices} \\ M \in \mathrm{Mat}(n+1,\sum d_i+1), \\ \text{satisfying :} \\ \Phi(g\cdot M) = M \text{ and} \\ \chi_{\underline{k}}(D)^e\cdot\Phi(M) = \Phi(M\cdot D) \\ \text{for all } g \in SL(n+1),\ D \in H_1 \\ \text{and } M \in \mathrm{Mat}(n+1,\sum d_i+1) \end{array}\right\}.$$

Given $\underline{k} := (k_1, \dots k_n) \in \mathbb{Z}^N$ with all $k_i > 0$, we have a very ample line bundle $\mathcal{L}_{\underline{k}}$ on $\mathcal{C}(\underline{d}, n)$. This line bundle corresponds to the composition of the Plücker embedding of each factor $\mathbb{G}(d_i, n)$ and the k_i-uple embeddings. We denote by $S(\underline{d}, n)$ the homogeneous coordinate ring of $\mathcal{C}(\underline{d}, n)$ in this embedding. The degree e homogeneous component of $S(\underline{d}, n)$ can be identified with the set of polynomials Φ on N matrices $M := (M_1, \dots, M_N)$ of sizes $(n+1) \times (d_i + 1)$ such that

$$\Phi(M_1 \cdot D_1, \dots, M_N \cdot D_N) = \chi_{\underline{k}}(D)^e \Phi(M_1, \dots, M_N).$$

Here $D_1, \dots D_N$ are the blocks of an element $D \in H_1$. For the degree e homogeneous invariant part we have the additional requirement that

$$\Phi(g \cdot M) = \Phi(M) \text{ for all } g \in SL(n+1).$$

Thus we see that in fact

$$S(\underline{d}, n)^G_e = R(n, \sum(d_i + 1) - 1)^{H_1}_e,$$

which gives the desired isomorphism, because both varieties are the projective spectrum of the same graded ring. □

As an easy corollary we get the generalized Gale duality on the level of GIT quotients.

COROLLARY 3.1.14 *Let $\underline{k} \in \mathbb{Z}^N$ be a sequence of weights. Let $\underline{d} \in \mathbb{Z}^N_{\geq 0}$ be a sequence of dimensions, and let n and $\tilde{n} := \sum(d_i + 1) - n - 2$ be natural numbers. There is an isomorphism of GIT quotients*

$$\mathcal{C}(\underline{d}, n)^{ss}//SL(n+1) \cong \mathcal{C}(\underline{d}, \tilde{n})^{ss}//SL(\tilde{n}+1)$$

(both quotients with respect to the linearizations associated to $\underline{k}$.)

PROOF Apply the Gelfand-MacPherson isomorphism, then apply the usual duality of Grassmannians, then apply the Gelfand-MacPherson isomorphism again, i.e.

$$\begin{gathered}\mathcal{C}(\underline{d}, n)^{ss}//SL(n+1) \cong \\ \mathbb{G}(n, \sum(d_i+1) - 1)^{ss}//H_1 \cong \\ \mathbb{G}(\tilde{n}, \sum(d_i+1) - 1)^{ss}//H_1 \cong \\ \mathcal{C}(\underline{d}, \tilde{n})^{ss}//SL(\tilde{n}+1). \ \square\end{gathered}$$

The generalized Gelfand-MacPherson correspondence and its application to generalized Gale duality was discovered and proven independently by Yi Hu [Hu05].

3.2 Working with generalized Gale Duality

A more convenient formulation of Definition 3.1.1 goes as follows. Represent the injections $j_k : L_k \to V$ by $d_i \times r$ matrices A_i, and the surjections $p_k : W \to L_k$ by $s \times d_i$ matrices B_i. In fact this is just choosing coordinates. Then in these coordinates the map ϕ is given by the matrix

$$A := (A_1^t \dots A_N^t)$$

and the map ψ by the matrix

$$B := \begin{pmatrix} B_1 \\ \vdots \\ B_N \end{pmatrix}$$

and these are Gale dual if the product of these matrices is 0. Because of the dependency on the choice of coordinates a block matrix Γ appears:

REMARK 3.2.1 Let n, $\tilde{n}$ and $\underline{d} = (d_1, \dots, d_N)$ be natural numbers such that $\tilde{n} = \sum(d_i + 1) - n - 2$. The configurations $(V_1, \dots, V_N) \in \mathcal{C}(\underline{d}, n)$ and $(W_1, \dots, W_N) \in \mathcal{C}(\underline{d}, \tilde{n})$ are Gale dual to each other if and only if there exist matrices A_i of sizes $(d_i + 1) \times (n + 1)$ and matrices B_i of sizes $(d_i + 1) \times (\tilde{n} + 1)$ (for $i = 1, \dots, N$) and a block matrix

$$\Gamma := \begin{pmatrix} \Gamma_1 & 0 & \dots & 0 \\ 0 & \Gamma_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \Gamma_N \end{pmatrix}$$

(with the blocks $\Gamma_i \in GL(d_i + 1)$), such that

$$(A_1^t \dots A_N^t) \cdot \Gamma \cdot \begin{pmatrix} B_1 \\ \vdots \\ B_N \end{pmatrix} = 0.$$

REMARK 3.2.2 The following is a practical way to calculate a generalized Gale transform. Given a configuration $(V_1, \dots, V_N) \in \mathcal{C}(\underline{d}, n)$ take on each subspace V_i exactly $d_i + 1$ points $P_1^i, \dots P_{d_i+1}^i$ spanning the subspace. Take the Gale dual of the sequence of all these points, i.e. the Gale dual of

$$(P_1^1, P_2^1, \dots, P_{d_1+1}^1, P_1^2, \dots, P_{d_2+1}^2, \dots, P_{d_N+1}^N).$$

This is done most easily by using remark 3.1.8. We get a sequence of points

$$(Q_1^1, Q_2^1, \dots, Q_{d_1+1}^1, Q_1^2, \dots, Q_{d_2+1}^2, \dots, Q_{d_N+1}^N)$$

in $\mathbb{P}^{\tilde{n}}$ (where $\tilde{n} = \sum(d_i + 1) - n - 2$). Then take the configuration $(W_1, \dots, W_N) \in \mathcal{C}(\underline{d}, \tilde{n})$ obtained by taking the spans

$$W_i := \langle Q_1^i, \dots Q_{d_i+1}^i \rangle$$

of this Gale dual sequence of points. This is the Gale dual of our original configuration. This is the way Eisenbud and Popescu suggested the generalized Gale transform in their article [EP00].

3.3 The geometric interpretation

Consider the following construction. Let $x := (V_1, \dots, V_N) \in \mathcal{C}(\underline{d}, n)$ a configuration. Let $D = \sum_{i=1}^N (d_i + 1) - 1$. It is possible to choose subspaces V, W and $W_1, \dots, W_N$ of $\mathbb{P}^D$, such that $V \cong \mathbb{P}^n$, $W \cong \mathbb{P}^{D-n-1}$ and $W_i \cong \mathbb{P}^{d_i}$ for $i \in \{1, \dots, N\}$, and such that

$$\mathbb{P}^D = \langle V, W \rangle = \langle W_1, \dots, W_N \rangle.$$

Denote by $\pi : \mathbb{P}^D \dashrightarrow V$ the projection with centre W onto $V \cong \mathbb{P}^n$. It is possible to choose V, W and $W_1, \dots, W_N$ in such a way that

$$\pi(W_i) = V_i \text{ for all } i \text{ (as sets)}.$$

Thus $x \in \mathcal{C}(\underline{d}, n)$ is seen as the projection of the configuration $W_1, \dots, W_N$ in $\mathbb{P}^D$. On the other hand, consider for each $i \in \{1, \dots, N\}$ the join

$$H_i := \langle W_1, \dots, \hat{W}_i, \dots, W_N \rangle \subset \mathbb{P}^D.$$

It is a subspace of codimension $d_i + 1$. Then

$$(W \cap H_1, \dots, W \cap H_N)$$

is a configuration of subspaces of $W \cong \mathbb{P}^{D-n-1}$ with codimensions $d_1 + 1, \dots d_N + 1$.

We show now why this construction is in fact the Gale transform. Choose a basis $e_0, \dots, e_D$ of $\mathbb{P}^D$. Choose $V := \langle e_{D-n}, \dots, e_D \rangle$ and $W := \langle e_0, \dots, e_{D-n-1} \rangle$. Let the subspaces $V_i \subset V$ be the rowspaces of matrices A_i of size $(d_i + 1) \times (n + 1)$ (with respect to the basis $e_{D-n}, \ldots, e_D$ of V). Extend the matrices A_i to matrices B_i of size $(d_i+1) \times D$ as follows:

$$B_i := (C_i \mid A_i),$$

where C_i has size $(d_i+1) \times (D-n-1)$ and such that the matrix

$$A := \begin{pmatrix} B_1 \\ \vdots \\ B_N \end{pmatrix}$$

has rank D. This is possible if the subspaces $V_1, \ldots, V_N$ span V. The rowspace of a matrix B_i is the subspace W_i of the construction. Indeed

$$\pi(W_i) = V_i \text{ for all } i,$$

because projecting from W onto V is wiping out the first $D-n$ entries of a vector.

We can describe the whole situation with the following diagram.

$$W \xrightarrow{\begin{pmatrix} I_{D-n-1} \\ 0 \end{pmatrix}} (\oplus W_i)_e \xrightarrow{(A^t)^{-1}} (\oplus W_i)_a \xrightarrow{A^t} (\oplus W_i)_e \xrightarrow{\begin{pmatrix} 0 & I_{n+1} \end{pmatrix}} V$$

In this diagram the subscript $_e$ in $\mathbb{P}^D_e$ denotes that we consider $\mathbb{P}^D$ with respect to the basis $e_0, \ldots, e_D$ and the subscript $_a$ in $\mathbb{P}^D_a$ denotes that we consider $\mathbb{P}^D$ with respect to the basis given by the rows of the matrix A. Above the arrows in the diagram are given the matrices associated with the linear maps with respect to their bases.

Recall Definition 3.1.1. It is clear that we are exactly in the situation that is given there. We have:

$$\phi = \begin{pmatrix} 0 & I_{n+1} \end{pmatrix} \circ A^t,$$

$$\psi = (A^t)^{-1} \circ \begin{pmatrix} I_{D-n-1} \\ 0 \end{pmatrix}.$$

As in Remark 3.1.2 the maps j_k and p_k are the compositions of ϕ and ψ with the maps $i_k : W_k \to \oplus W_i$ and $\pi_k : \oplus W_i \to W_k$ given by the matrices:

$$i_k : \begin{pmatrix} 0 & \ldots & I_{d_i+1} & \ldots & 0 \end{pmatrix}, \; \pi_k : \begin{pmatrix} 0 \\ \vdots \\ I_{d_i+1} \\ \vdots \\ 0 \end{pmatrix}.$$

We see that the maps ϕ and ψ can also be given by the explicit formulas of Definition 3.1.1. Thus the sequence of subspaces $(W_1^*, \ldots, W_N^*)$ of W^* is the Gale dual of the sequence of subspaces $(V_1, \ldots, V_N)$ of V.

On the other hand, we had formed for $k \in \{1, \ldots, N\}$ the subspace $W \cap H_k$ of W. These satisfy:

$$v \in W \cap H_k \iff p_k(v) = 0.$$

In fact the subspaces $W \cap H_k$ are:

$$W \cap H_k = \{v \in W \mid f(v) = 0 \text{ for all } f \in W_k^*\}.$$

Thus we have established a geometric construction of Gale duality.

3.4 Examples of the generalized Gale transform

The problem with generalized Gale duality is that it is often trivial. Let us take as a first example: 4 points and 1 line lying in the plane. Take a configuration in general position. The Gale dual will also be a configuration of 4 points and 1 line in the plane. Without loss of generality we may assume that the points have homogeneous coordinates [100], [010], [001] and [111] and there are $a, b \in \mathbb{C}$ such that the line is represented by

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix},$$

such that $ab(1-a-b) \neq 0$. (This inequality prevents that one of the 4 points lies on the line.) The points $[10a]$ and $[01b]$ span the line. Now we apply one of the definitions of the generalized Gale transform. We take the Gale dual of 6 points (the last two spanning the line), and change coordinates (which is an easy computation).

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & a \\ 0 & 1 & b \end{bmatrix} \xrightarrow{\text{Gale transform}} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & a & b \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{change coordinates}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ \frac{b-1}{a} & -1 & -1 \\ -1 & \frac{a-1}{b} & -1 \end{bmatrix}$$

It is easy to check that the Gale dual line, represented by

$$\begin{bmatrix} \frac{b-1}{a} & -1 & -1 \\ -1 & \frac{a-1}{b} & -1 \end{bmatrix}$$

is the same line as original line. (Both are given by the equation $X_2 = aX_0 + bX_2$.) So, by computation we have seen now that taking the Gale dual of 4 points and 1 line in the plane is a trivial operation.

We also could have done this without computation by using geometric properties of the Gale transform. Recall example 3.1.9 which says that 6 points in $\mathbb{P}^2$ are self-associated if and only if they lie on a conic. Given 4 points and a line in $\mathbb{P}^2$, in order to calculate the Gale dual we may choose *any* 2 points spanning the line, so by choosing these 2 points in such a way that together with the 4 points they lie on a conic, one obtains the result. This observation (that "it is easy to lay 6 points on a conic") gives rise to the more general idea that self-association of sets of linear subspaces is often trivial, unless all the subspaces are points. We will specify the meaning of "often" here. The key to our idea is a theorem in [DO88], which we state after the following definition.

DEFINITION 3.4.1 Let $\Gamma \subset \mathbb{P}^n$ be a finite set of points. We say that Γ *fails to impose independent conditions on quadrics* if there exists $p \in \Gamma$ such that the dimension of the linear system of quadrics going through Γ equals the dimension of the linear sytem of quadrics going through $\Gamma \setminus \{p\}$.

In this case we say that Γ *fails by 1 to impose independent conditions on quadrics* if moreover there doesn't exist $q \in \Gamma$ (with $q \neq p$) such that the dimension of the linear sytem of quadrics going through $\Gamma \setminus \{p, q\}$ equals the dimension of the linear system of quadrics going through $\Gamma \setminus \{p\}$.

THEOREM 3.4.2 *Let $P := (p_1, \ldots, p_{2n+2}) \in (\mathbb{P}^n)^{2n+2}$ be stable and such that $p_i \neq p_j$ for $i \neq j$, then P is self-associated if and only if the set $\{p_1, \ldots, p_{2n+2}\}$ fails by 1 to impose independent conditions on quadrics.*

PROOF The proof of this can be found in [DO88], page 46. □

Now we consider the more general situation of configurations

$$(V_1, \ldots, V_N) \in \mathbb{G}(d, n)^N,$$

where N, d and n are chosen such that the Gale transform doesn't change the dimension of the ambient space, i.e. such that $N(d + 1) = 2(n + 1)$. In order to prove that this Gale transform is trivial for $d \neq 0$, we only would have to show that we can choose on each V_i the $d + 1$ points spanning it in such a way that the total point set we use to calculate its Gale dual fails by 1 to impose independent conditions on quadrics.

EXAMPLE 3.4.3 Cf. Remarks 2.0.5 and 3.1.3. It can be interesting to study Gale transforms of configurations where one of the subspaces is equal to the whole projective space. In the case of configurations on a line these turn out to be nothing else than projections. Let $(V_1, \ldots, V_N, V) \in \mathcal{C}(\underline{d}, n)$ a configuration, such that $\underline{d} = (d_1, \ldots, d_N, n)$ and $d_1, \ldots, d_N < n$. In order to calculate its Gale dual we choose points spanning the subspaces. In general we can choose coordinates such that the matrix representing all these points has the form

$$\begin{pmatrix} I_{n+1} \\ A \\ I_{n+1} \end{pmatrix}.$$

The Gale dual sequence of points in $\mathbb{P}^{2n+1}$ is given by

$$\begin{pmatrix} & I_{2n+1} & \\ A^t & & I_{n+1} \end{pmatrix}.$$

We see that the "subspace" V of $\mathbb{P}^n$ corresponds to a subspace of $V' \subset \mathbb{P}^{2n+1}$ spanned by the rows of the matrix

$$\begin{pmatrix} A^t & I_{n+1} \end{pmatrix}.$$

Projection from V' onto $\mathbb{P}^n$, for example the the $\mathbb{P}^n$ inside $\mathbb{P}^{2n+1}$ defined by putting the last $r+1$ coordinates zero, yields the configuration spanned by the points in the matrix

$$\begin{pmatrix} I_{n+1} \\ A^t \end{pmatrix}.$$

In the case that $n = 1$ we consider four points with cross ratio (say) $a \neq 0, 1, \infty$ and a line in $\mathbb{P}^1$. After a coordinate change these four points are represented by $(1,0)$, $(0,1)$, $(1,1)$ and $(1,a)$. In this case the matrix A, as in the above, is symmetric:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & a \end{pmatrix}.$$

The moduli space of configurations of 4 points and a line in $\mathbb{P}^3$ is $\mathbb{P}^1$. The quotient map is projecting the 4 points from the line onto a line and taking the cross ratio.

3.5 Invariants of arbitrary sequences of linear subspaces.

As we have seen in some examples (in §2.3) it is quite possible to understand quotients of configuration spaces of only points and hyperplanes. However, in general the quotients $\mathcal{C}(\underline{d}, n)//SL(n+1)$ are not easy to calculate, for one needs to understand completely their invariant rings. In the case of points and hyperplanes we have the fundamental theorem of invariant theory (Theorem 2.2.1), which is very convenient, because it gives all invariants. In the general case of sequences of linear subspaces of arbitrary dimensions we also have a theorem giving the invariants. It is a corollary of a theorem giving all semi-invariants of quivers in terms of determinants. This theorem is proven in [DZ01] and in [DW00].

3.5.1 The theorem of Domokos-Zubkov First we give the relevant definitions and some facts (without proofs) in order to be able to understand the theorem. These definitions and facts can be found in the mentioned articles and also in any text about quivers intended for the beginner. See for example [Der01].

A *quiver* Q is a directed graph (Q_0, Q_1), where Q_0 is the set of vertices and Q_1 is the set of arrows. We denote by $h : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ the maps that add to each arrow his head and his tail respectively.

A *representation* V of a quiver Q is a set

$$\{V(x) \mid x \in Q_0\}$$

of finite dimensional k-vectorspaces together with a set

$$\{V(a) : V(ta) \to V(ha) \mid a \in Q_1\}$$

of k-linear maps.

The *dimension vector* $\alpha : Q_0 \to \mathbb{N}$ of a representation V is defined by

$$\alpha(x) := \dim V(x) \text{ for } x \in Q_0.$$

By choosing bases we can identify each vectorspace $V(x)$ with $k^{\alpha(x)}$, i.e. we view V as an element of

$$\mathrm{Rep}(Q, \alpha) := \bigoplus_{a \in Q_1} \mathrm{Hom}(k^{\alpha(ta)}, k^{\alpha(ha)}).$$

This is called the *representation space* of the quiver Q with reference to the dimension vector α. Define

$$GL(\alpha) := \prod_{x \in Q_0} GL(\alpha(x)).$$

$GL(\alpha)$ acts on $\mathrm{Rep}(Q, \alpha)$ as follows. If

$$\phi = \{\phi(x) \in GL(\alpha(x)) \mid x \in Q_0\} \in GL(\alpha)$$

and $V \in \mathrm{Rep}(Q, \alpha)$ then for $a \in Q_1$

$$(\phi \cdot V)(a) := \phi(ha)V(a)\phi(ta)^{-1}.$$

By definition V and $W \in \mathrm{Rep}(Q, \alpha)$ are *isomorphic* if and only if they lie in the same $GL(\alpha)$-orbit. Because we have an action of $GL(\alpha)$ on $\mathrm{Rep}(Q, \alpha)$ we also have a *dual action* on the coordinate ring $k[\mathrm{Rep}(Q, \alpha)]$, as follows. If $g \in GL(\alpha)$ and $f \in k[\mathrm{Rep}(Q, \alpha)]$ then for $A \in \mathrm{Rep}(Q, \alpha)$

$$g \cdot f(A) = f(g^{-1}A)$$

(confer Definition 1.1.6 of rational action). The *ring of invariants* is defined as

$$I(Q, \alpha) := k[\mathrm{Rep}(Q, \alpha)]^{GL(\alpha)}.$$

However, this ring of invariants turns out to be trivial in a lot of cases (see [Der01]), for example in *our* case. More interesting is the following. Define

$$SL(\alpha) := \prod_{x \in Q_0} SL(\alpha(x)).$$

If $\chi : GL(\alpha) \to k^*$ is a character, then χ has the form:

$$\{\phi(x) \in GL(\alpha(x)) \mid x \in Q_0\} \in GL(\alpha) \mapsto \prod_{x \in Q_0} \det(\phi(x))^{\sigma(x)}.$$

The map $\sigma : Q_0 \to \mathbb{Z}$ is called the *weight*. The *ring of semi-invariants* is defined as

$$SI(Q, \alpha) := k[\mathrm{Rep}(Q, \alpha)]^{SL(\alpha)}.$$

If a semi-invariant $f \in SI(Q, \alpha)$ satisfies for all $g \in GL(\alpha)$:

$$g \cdot f = \prod_{x \in Q_0} \det(g(x))^{\sigma(x)} \cdot f \text{ for some weight } \sigma,$$

then we say that $f \in SI(Q, \alpha)_\sigma$ or that f is a semi-invariant of weight σ.

Before we can state the theorem we have to give only one definition. A *path* p in the quiver Q is a (finite) sequence of elements $a_1, \dots, a_s$ of Q_1, such that $h(a_i) = t(a_{i+1})$ for $i \in \{1, 2, \dots, s-1\}$. We call $h(p) := h(a_s)$ and $t(p) := t(a_1)$ the head and the tail of the path respectively. For a representation V of the quiver $V(p) : V(t(p)) \to V(h(p))$ denotes the homomorphism $V(a_s)V(a_{s-1}) \dots V(a_1)$.

THEOREM 3.5.1 (DOMOKOS-ZUBKOV) *Let Q be a quiver and β be an arbitrary dimension vector. The ring $SI(Q, \beta)$ is spanned (k-linearly) by semi-invariants of the form*

$$W \in \mathrm{Rep}(Q, \beta) \mapsto \det(g)$$

where

$$g : \oplus_{i=1}^{n} W(x_i) \to \oplus_{j=1}^{m} W(y_j)$$

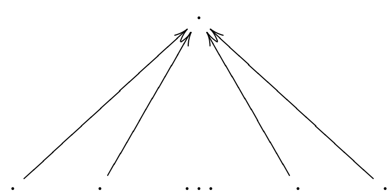
with $x_1, \ldots, x_n, y_1, \ldots y_m \in Q_0$ *such that* $\sum \beta(x_i) = \sum \beta(y_j)$ *and where*

$$g = \det \begin{pmatrix} g_{11} & g_{21} & \cdots & g_{m1} \\ g_{12} & g_{22} & \cdots & g_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ g_{1n} & g_{2n} & \cdots & g_{mn} \end{pmatrix}$$

with g_{ij} *a linear combination of* $W(p)$ *where* p *is a path with* $t(p) = x_i$ *and* $h(p) = y_i$*.*

REMARK 3.5.2 This formulation of the theorem can be found in [Der01]. There it is proven for an algebraicly closed field of characteristic 0. In [DW00] it only has to be an algebraicly closed field. In [DZ01] it is proven for an arbitary infinite field. Before the theorem was proven in general, it was known in special cases, for example in the case that the quiver has no loops. We only need a very special case of the theorem.

3.5.2 Domokos-Zubkov applied to our case Let $N \in \mathbb{N}$. Consider the following quiver.



We denote this quiver by $Q = (Q_0, Q_1)$ with $Q_0 := \{1, \ldots, N+1\}$ and $Q_1 := \{a_1, \ldots, a_N\}$. (We give the number $N+1$ to the upper point, the numbers $1, \ldots, N$ to the other N points, and we denote by a_i the arrow from point i to point $N+1$.) Let us consider this quiver with dimension vector $\alpha := (d_1, \ldots, d_N, n)$, such that $0 < d_i < n$ for each $i \in \{1, \ldots, N\}$.

Of course this quiver is strongly related to sequences of linear subspaces with dimensions $d_1, \ldots, d_N$ respectively of a vectorspace of dimension n. We will now show the connection between the ring of semi-invariants of the quiver with given dimension vector and the homogeneous invariant ring of the moduli space of sequences of linear subspaces (with respect to the same sequence of dimensions).

Choose a sequence of weights $\sigma := (\sigma_1, \ldots, \sigma_N, \Sigma)$ for the quiver Q. We consider the semi-invariants with weight σ. Let's write out some definitions.

$$SI(Q,\alpha)_\sigma = \{f \in k[\mathrm{Rep}(Q,\alpha)] \mid g \cdot f = \prod_{x \in Q_0} \det(g(x))^{\sigma(x)} \cdot f \text{ for } g \in GL(\alpha)\}$$

We write $(M_1, \ldots, M_N)$ for an element of $\mathrm{Rep}(Q,\alpha)$ and for an element $g \in GL(\alpha)$ we write $g := (A_1, \ldots, A_N, G)$ with the $A_i \in GL(d_i)$ and $G \in GL(n)$. In the following we view these elements as matrices. Then:

$$f \in SI(Q,\alpha)_\sigma \iff$$

$$f \text{ is a function on matrices } (M_1, \ldots, M_N) \text{ such that}$$

$$g \cdot f(M_1, \ldots, M_N) := f(g^{-1}(M_1, \ldots, M_N)) := f(G^{-1}M_1A_1, \ldots, G^{-1}M_NA_N) =$$
$$\prod_{i=1}^{N} \det(A_i)^{\sigma_i} \cdot \det(G)^{\Sigma} \cdot f(M_1, \ldots, M_N)$$

Look now at the description of the homogeneous ring of invariants in the proof of theorem 3.1.13. We see that

$$SI(Q,\alpha)_\sigma = R_1,$$

where R_1 is the degree 1 part of the homogeneous ring of invariants for the action of $SL(n)$ on $\mathbb{G}(\underline{d}, n-1)$ where $\underline{d} = (d_1 - 1, \ldots, d_N - 1)$ with respect to the sheaf $\mathcal{L}_{\underline{k}}$ with $\underline{k} = (\sigma_1, \ldots, \sigma_N)$.

Though Domokos-Zubkov gives us all invariants, it doesn't say anything about the relations between the invariants. This remains difficult. For the special case of 5 or 6 lines in $\mathbb{P}^3$ it is done in the thesis of Vazzana, [Vaz99].

REMARK 3.5.3 There are more results of the theory of quivers that could be applied to the case of configurations. One important remark (made by Harm Derksen) is that Gale duality can be seen as a case of the Bernstein-Gelfand-Ponomarev (BGP) reflection functors for quivers. In the special case of the quiver Q above it turns out to be as follows. From Q one gets a new quiver Q' by reversing the N arrows. Then a BGP reflection functor gives a map

$$\mathcal{F} : \mathrm{Rep}(Q,\alpha) \to \mathrm{Rep}(Q',\alpha'),$$

where $\alpha' = (d_1, \ldots, d_N, \sum d_i - n)$ is the Gale dual dimension vector. One can show that applying two times (to an arbitrary quiver) a BGP reflection functor gives the same representation (see for example [Der01], Lecture 5, Theorem 4). There's a theorem of Kac about the rings of semi-invariants of the representation space and a reflected representation space: they are isomorphic (see [Der01], Lecture 7, Theorem 4).

4 When do stable configurations exist?

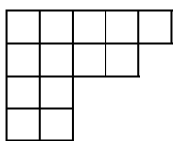
For an action of a reductive algebraic group G on a projective variety X, which is linearized with reference to an ample line bundle $\mathcal{L}$, we have the open subsets $X^{ss}(\mathcal{L}) \subset X^s(\mathcal{L}) \subset X$. A categorical quotient $X^{ss}(\mathcal{L})//G$ and a geometric quotient $X^s(\mathcal{L})//G$ exist. However the sets $X^{ss}(\mathcal{L})$ and $X^s(\mathcal{L})$ can be empty, which is a bad situation. In this chapter we investigate the possible emptyness of the stable and semi-stable loci of configurations of linear subspaces and we try to give exact criteria for the existence of (semi-)stable orbits.

4.1 Algorithm

Let $\underline{d} := (d_1, \ldots, d_N)$ be a sequence of dimensions, $\underline{k} := (k_1, \ldots, k_N)$ a sequence of weights. We consider the action of $SL(n+1)$ on $\mathcal{C} := \prod_{i=1}^{N} \mathbb{G}(d_i, n)$, linearized with reference to the sheaf $\mathcal{L}_{\underline{k}}$. See chapter 2 for definitions. An important observation is that for such a fixed configuration space $\mathcal{C}$ there exists an algorithm, which checks the existence of stable (resp. semi-stable) configurations.

4.1.1 Schubert calculus The algorithm uses some facts about Grassmannian varieties known as Schubert calculus. I will state these facts here without proofs. For more information one can read for example [GH78], ch. 1, §5, [Bri04] or [Fu97]. In this brief summary I follow the latter.

Let $\mathrm{Gr}(d, n)$ be the Grassmannian variety of d-dimensional linear subspaces of an n-dimensional vectorspace V. A *Young diagram* λ is a non-increasing finite sequence of positive integers, written as $\lambda = (\lambda_1, \ldots, \lambda_m)$. We think of it as a collection of boxes like this one:



So λ_1 is the (maximal) number of columns of λ and m the number of rows. The number of boxes of λ is denoted by $|\lambda| := \sum_{i=1}^{m} \lambda_i$. It is convenient to define $\lambda_i := 0$ for $i > m$. A *complete flag* F of V is a sequence

$$F = (V_1 \subset V_2 \subset \ldots \subset V_n),$$

where the V_i are linear subspaces with $\dim(V_i) = i$. Given a complete flag F and a Young diagram λ with at most d rows and $n - d$ columns, we define

$$\sigma_\lambda(F) := \{L \in \mathrm{Gr}(d, n) \mid \dim(L \cap V_{n-d+i-\lambda_i}) \geq i \text{ for } 1 \leq i \leq d\}.$$

Such a set is called a *Schubert variety*. It is an irreducible closed subvariety of $\mathrm{Gr}(d,n)$ of codimension $|\lambda|$. We denote by $[\sigma_\lambda]$ the class of $\sigma_\lambda(F)$ in the cohomology group $H^{2|\lambda|}(\mathrm{Gr}(d,n))$. This class is independent of the chosen complete flag. Furthermore, the classes $[\sigma_\lambda]$ give a basis over $\mathbb{Z}$ for the cohomology ring of $\mathrm{Gr}(d,n)$. The product of such classes is given by the formula

$$[\sigma_\lambda] \cdot [\sigma_\mu] = \sum_\nu c^\nu_{\lambda,\mu}[\sigma_\nu].$$

Here the product is taken over all Young diagrams ν with at most d rows and $n-d$ columns. The coefficients $c^\nu_{\lambda,\mu}$ are the *Littlewood-Richardson numbers*. These numbers also arise in other contexts. I will summarize here some properties of these numbers. Again the proofs can be found in textbooks about Schubert calculus, such as [Fu97].

For each triple of Young diagrams (λ,μ,ν) a Littlewood-Richardson number $c^\nu_{\lambda,\mu} \in \mathbb{N}$ is defined. As these numbers appear as coefficients for intersections in a cohomology ring, the following property does not surprise: $c^\nu_{\lambda,\mu} = 0$ if not $|\lambda| + |\mu| = |\nu|$.

We introduce a notation: if a is a natural number, then $[\sigma_a]$ denotes the Schubert class of the tableau (a), i.e. the tableau consisting of one row of length a. A Schubert class of the form $[\sigma_a]$ is called a *special Schubert class*.

Pieri's rule is a simple combinatorial rule to compute a product of a Schubert cycle with a special Schubert cycle. Suppose λ is special with 1 row of length a.

$$\lambda := \underbrace{\boxed{}\boxed{}\;\cdots\;\boxed{}\boxed{}}_{a}$$

Then Pieri's formula says that for any tableau μ

$$[\sigma_\lambda] \cdot [\sigma_\mu] = \sum_\nu [\sigma_\nu],$$

where the sum is taken over all tableaux ν that are obtained from μ by adding a boxes to μ, without adding two boxes in one row of μ.

Giambelli's rule expresses a Schubert class as a determinant of special Schubert classes. Suppose λ has d rows, then

$$[\sigma_\lambda] = \det(\sigma_{\lambda_i+j-i})_{1\le i,j\le d}.$$

Note that Pieri's en Giambelli's formula together give an algorithm to express a product of Schubert cycles as a sum of Schubert cycles. There exist more combinatorial rules for these computations. An algorithm to compute Littlewood-Richardson coefficients has been implemented by A. Buch.[1]

[1]He calls it 'Littlewood-Richardson calculator' and it is freely available at http://home.imf.au.dk/abuch/lrcalc/.

Note further that these rules are universal in the sense that they calculate Littlewood-Richardson numbers without paying attention at the actual cohomology ring we are doing the computation in. If we are working in $\mathrm{Gr}(d, n)$, i.e. only diagrams with at most d rows and $n - d$ columns represent Schubert cycles, then we have the *duality theorem*.

$$[\sigma_\lambda] \cdot [\sigma_\mu] = \begin{cases} 1 \text{ if } \lambda_i + \mu_{d+1-i} = n - d \text{ for all } 1 \leq i \leq n - d \\ 0 \text{ if } \lambda_i + \mu_{d+1-i} > n - d \text{ for any } i. \end{cases}$$

4.1.2 Outline of the algorithm We can now easily explain the algorithm. It consists of the following steps.

According to Mumford's Criterion, $\mathcal{C}^s(\mathcal{L}_{\underline{k}}) = \emptyset$ means the following statement. For every configuration $(V_1, \ldots, V_N) \in \mathcal{C}$, there exists a proper linear subspace $V \subset \mathbb{P}^n$ such that

$$\frac{\sum_{i=1}^N k_i(\dim(V_i \cap V) + 1)}{\sum_{i=1}^N k_i(d_i + 1)} \geq \frac{\dim(V) + 1}{n + 1}.$$

This statement can be checked using Schubert calculus in the Grassmannians $\mathbb{G}(l, n) = \mathrm{Gr}(l + 1, n + 1)$ for $l = 0, \ldots, n - 1$.

So let l be fixed. We define a *violation* of the Mumford Criterion to be a sequence of integers $(a_1, \ldots, a_N)$, satisfying $0 \leq a_i \leq \max(d_i, l)$ for all i, such that for each $(V_1, \ldots, V_N) \in \mathcal{C}$ there exists $L \in \mathbb{G}(l, n)$ with $\dim(L \cap V_i) \geq a_i$ for $i = 1, \ldots, N$ and

$$\sum_{i=1}^N k_i(a_i + 1) \geq (\frac{l + 1}{n + 1}) \sum_{i=1}^N k_i(d_i + 1).$$

Since the right hand side of this inequality is just a fixed rational number, the k_i are fixed, and the a_i satisfy $0 \leq a_i \leq \max(d_i, l)$, it is easy to determine all $(a_1, \ldots, a_N)$ that satisfy the inequality above. Such a sequence $(a_1, \ldots, a_N)$ we call a *potential violation*.

Let l and a potential violation $(a_1, \ldots, a_N)$ be fixed. For a given $(V_1, \ldots, V_n) \in \mathcal{C}$ we define the Schubert cycle

$$\sigma(a_i, i) := \{L \in \mathbb{G}(l, n) \mid \dim(L \cap V_i) \geq a_i\},$$

and we denote by $[\sigma(a_i, d_i)]$ its class in $H^*(\mathbb{G}(l, n), \mathbb{Z})$. Whether or not for each $(V_1, \ldots, V_N) \in \mathcal{C}$ there exists $L \in \mathbb{G}(l, n)$ with $\dim(L \cap V_i) \geq a_i$, depends on

$$[\sigma(a_1, 1)] \cdot [\sigma(a_2, 2)] \cdot \ldots \cdot [\sigma(a_N, N)] \in H^*(\mathbb{G}(l, n), \mathbb{Z}).$$

This intersection product being non-zero is equivalent to the statement that for general $(V_1, \ldots, V_N) \in \mathcal{C}$ there exists $L \in \mathbb{G}(l, n)$ with $\dim(L \cap V_i) \geq a_i$ for all

$i \in \{1, \ldots, N\}$, which is equivalent to the statement that for *every* $(V_1, \ldots, V_N) \in \mathcal{C}$ there exists $L \in \mathbb{G}(l, n)$ with $\dim(L \cap V_i) \geq a_i$ for all $i \in \{1, \ldots, N\}$.

In a lot of cases one sees that this intersection product is trivial by counting dimensions. If

$$\sum_{i=1}^{N} \operatorname{codim}(\sigma(a_i, i), \mathbb{G}(l, n)) > \dim(\mathbb{G}(l, n)),$$

then certainly this product is zero. Note that in the notation of Schubert cycles as above

$$[\sigma(a_i, i)] = [\sigma_\lambda] \in H^*(\mathrm{Gr}(l+1, n+1), \mathbb{Z}),$$

where λ is the rectangular diagram with $n - l + a_i - d_i$ columns and $a_i + 1$ rows. So the intersection product is zero if

$$\sum_{i=1}^{N} (a_i + 1)(n - l + a_i - d_i) > (l + 1)(n - l).$$

In general it is not clear whether or not the intersection product is trivial, in case the above inequality is not satisfied. We will quickly see some examples where we can check this (non-)triviality without doing an explicit calculation. In other cases (it seems) one really has to calculate the intersection product by computing Littlewood-Richardson numbers. As commented above this can be done for example using Pieri's en Giambelli's formulas (or just using the implementation by Anders Buch).

REMARK 4.1.1 Note that the algorithm has to do *many* operations, i.e. for each l from 0 to $n-1$ something has to be checked for each weighted partition (with bounded summands) of each natural number greater than or equal to

$$(\frac{l+1}{n+1}) \sum_{i=1}^{N} k_i(d_i + 1).$$

4.1.3 More Schubert calculus Note that all Schubert cycles appearing in our algorithm are of a very particular form: their Young diagrams are rectangular. A naive idea (in which the author believed for some time) is that all intersections of Schubert cycles with rectangular Young diagrams are non-trivial, unless, of course, the codimension of the intersection is bigger than the dimension of the Grassmannian variety in which the intersection takes place. This is not true, even in the case that all intersecting Schubert cycles are equal.

Let us investigate this more thoroughly. Let $d, n \in \mathbb{N}^*$ such that $d < n$. We consider intersections in the Grassmannian $\mathrm{Gr}(d, n)$. Let $s, l \in \mathbb{N}^*$ such that $s < n - d$

and $l < d$, and denote by (s^l) the partition $(s, \ldots, s)$ (with l times an s). In this situation it would be convenient if the following were true.

FALSE STATEMENT 4.1.2 *If $N \in \mathbb{N}^*$ is such that $Nsl \leq d(n-d)$, then*

$$\sigma_{(s^l)}^N \neq 0.$$

If this statement were true we would not need Schubert calculus in our algorithm in case we have such an intersection.

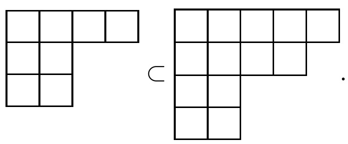
COUNTER EXAMPLE 4.1.3 Inside $\mathrm{Gr}(4,7)$ we have

$$\sigma_{(2,2,2)}^2 = 0,$$

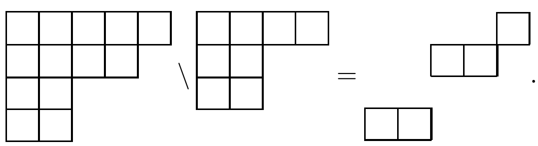
as is easily seen with the duality theorem. (The opposite Schubert variety of $\sigma_{(2,2,2)}$ inside $\mathrm{Gr}(4,7)$ is $\sigma_{(3,1,1,1)}$.)

All hopes for a theorem as general as statement 4.1.2 are shattered, but in even more special cases we can avoid calculations nevertheless. Thereto we need to delve a little bit deeper into Schubert calculus and Littlewood Richardson coefficients. The definitions and proposition below come from [Fu97].

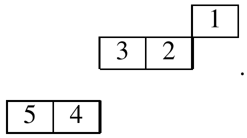
DEFINITION 4.1.4 Let λ^1, λ^2 be Young diagrams. If $\lambda_i^1 \leq \lambda_i^2$ for all i, we say that λ^2 *contains* λ^1. Notation: $\lambda^1 \subset \lambda^2$. Example:



DEFINITION 4.1.5 Let λ^1, λ^2 be Young diagrams, such that λ^2 contains λ^1. We denote by $\lambda^2 \setminus \lambda^1$ the collection of boxes one gets by removing from λ^2 the boxes of λ^1. Example:



DEFINITION 4.1.6 Let λ^1, λ^2 be Young diagrams, such that λ^2 contains λ^1. The *reverse numbering* of the boxes of $\lambda^2 \setminus \lambda^1$ is a numbering of the boxes of $\lambda^2 \setminus \lambda^1$ from the right to the left in each row, and from the top to the bottom. Example:



REMARK 4.1.7 Another property of Littlewood-Richardson coefficients can be formulated now:

$$c^{\nu}_{\lambda,\mu} \neq 0 \Rightarrow \lambda \subset \nu \text{ or } \mu \subset \nu.$$

See again [Fu97]. Recall definition 2.2.2 of a tableau. An equivalent formulation of this definition says that a tableau is a *filling* of a rectangular Young diagram (satisfying the property of Definition 2.2.2). If μ is a rectangular Young diagram of size $m \times n$ and we fill it with elements of $\{1, \ldots, S\}$, each element occuring t times, the resulting tableau τ is called a tableau on $\{1, \ldots, S\}$ with *shape* μ.

Now we can formulate a proposition ([Fu97], page 68, Prop. 4).

PROPOSITION 4.1.8 *Let μ, λ, ν be Young diagrams. The Littlewood Richardson number $c^{\nu}_{\lambda,\mu}$ is the number of standard tableaux U on the shape μ that satisfy the following properties:*

1. *if $k-1$ and k appear in the same row of the reverse numbering of $\nu \setminus \lambda$, then k occurs weakly above and strictly right of $k-1$ in U, and*
2. *if k appears in the box directly below j in the reverse numbering of $\nu \setminus \lambda$, then k occurs strictly below and weakly left of j in U.*

This allows us to prove a weaker version of Statement 4.1.2.

PROPOSITION 4.1.9 *Let $d, n \in \mathbb{N}^*$ such that $d < n$. Let $s, l \in \mathbb{N}^*$ such that $s < n-d$ and $l < d$. Let $N \in \mathbb{N}^*$ such that $Nsl \leq d(n-d)$. If additionally $s \mid n-d$, then*

$$\sigma^N_{(s^l)} \neq 0.$$

PROOF For $i \in \{1, \ldots, N\}$ we define a Young diagram λ_i as follows. Write $(isl) = a(ds) + b$, with $0 \leq b < ds$. Then $s \mid b$, say $b = cs$. Now define

$$\lambda_i := (\underbrace{(a+1)s, \ldots, (a+1)s}_{\text{length}=c}, \underbrace{as, \ldots, as}_{\text{length}=d-c}).$$

(One could also say one gets λ_i by putting an $l \times s$-rectangle i times inside an $d \times (n-d)$-rectangle in a certain way.) We use induction on i to prove that for each i the coefficient of σ_{λ_i} in $\sigma_{s^l}^i$ is non-trivial. For $i = 1$ it is trivially true, because $\lambda_1 = s^l$. Suppose now $i > 1$. Because by the induction hypothesis the coefficient of λ_{i_1} inside $\sigma_{s^l}^{i-1}$ is non-trivial, it is sufficient to prove that for each i the coefficient of σ_{λ_i} in $\sigma_{\lambda_{i-1}} \cdot \sigma_{s^l}$ is non-zero. We apply the proposition above with $\mu = (s^l)$, $\nu = \lambda_i$ and $\lambda = \lambda_{i-1}$. The coefficient $c^\nu_{\lambda,\mu}$ is the number of standard tableaux on the shape μ satisfying certain properties. For our purpose it is only important that $c^\nu_{\lambda,\mu} > 0$, so it suffices to show that there is at least one standard tableau on μ satisfying these properties. It is not difficult to see that one can take the standard tableau

$U :=$

1	2	…	s-1	s
s+1	s+2	…	2s-1	2s
⋮	⋮		⋮	⋮
(l-2)s+1	(l-2)s+2	…	(l-1)s-1	(l-1)s
(l-1)s+1	(l-1)s+2	…	ls-1	ls

on μ. There are two possibilities for the reverse numbering of $\nu \setminus \lambda$. It can be like this:

s	s-1	…	2	1
2s	2s-1	…	s+2	s+1
⋮	⋮		⋮	⋮
(l-1)s	(l-1)s-1	…	(l-2)s+2	(l-2)s+1
ls	ls-1	…	(l-1)s+2	(l-1)s+1

and the other possibility is that it has the following shape.

s	s-1	…	2	1
⋮	⋮		⋮	⋮
		…		

⋮ ⋮

		…		
⋮	⋮		⋮	⋮
ls	ls-1	…	(l-1)s+2	(l-1)s+1

It is trivial to check that in both cases U satisfies the conditions of proposition 4.1.8. $\square$

This proposition sometimes simplifies our algorithm, because we have to compute intersection products inside Grassmannians $\mathrm{Gr}(d, n)$. If all the subspaces have the same dimension these products can have the form

$$\sigma^N_{(s^l)}$$

where $Nsl \leq d(n-d)$. Whenever we also have that $s \mid n-d$ we can apply Proposition 4.1.9 and see immediately that the product is non-zero.

4.1.4 Example of the algorithm We apply the algorithm in the case of four lines in projective 3-space. (We take the linearization where all lines have the same weight 1.) Denote configurations of four lines by (l_1, l_2, l_3, l_4). Note that Mumford's Criterion in this case says that (l_1, l_2, l_3, l_4) is stable if and only if

$$\sum_{i=1}^{4} (\dim(l_i \cap V) + 1) < 2 \cdot (\dim(V) + 1)$$

for all proper linear subspaces $V \subset \mathbb{P}^3$. The algorithm simply checks this for $\dim(V) = 0, 1, 2$ respectively.

- $\dim(V) = 0$: we have to check if it is possible to configurate (l_1, l_2, l_3, l_4) in such a way that there does not exist a point V such that

  $$\sum_{i=1}^{4} (\dim(l_i \cap V) + 1) \geq 2,$$

  i.e. (l_1, l_2, l_3, l_4) don't have intersections. This is trivially possible, but let's see it by using the algorithm. The possible violations of the Mumford Criterion are all sequences (a_1, a_2, a_3, a_4), where each a_i is either -1 or 0, and satisfying

  $$a_1 + a_2 + a_3 + a_4 \geq 2,$$

  i.e. at least two of the a_i are 0. For example the sequence $(-1, -1, 0, 0)$ is a possible violation. The relevant Schubert cycle is:

  $$\sigma(0, 1) = \{V \in \mathbb{P}^3 \mid V \text{ lies on } l_i\} = \sigma_{\square\square}.$$

  Because $\dim(\mathbb{P}^3) = 3$ and $\mathrm{codim}(\sigma(0)) = 2$, and $2 \cdot 2 > 3$ the intersection product $\sigma(0)^2$ inside $H^*(\mathbb{P}^3, \mathbb{Z})$ is trivial. This means that the possible violation $(-1, -1, 0, 0)$ is not a real violation of the Mumford Criterion. The other possible violations are also checked quickly.

- $\dim(V) = 1$: is it possible to configurate our four lines in such a way that there does not exist a line V with

$$\sum_{i=1}^{4} (\dim(l_i \cap V) + 1) \geq 4.$$

  Possible violations are for example $(0, 0, 0, 0)$, $(1, 1, -1, -1)$ and $(1, 0, 0, -1)$. In the latter case for example the question is: can we configurate our four lines such that there does not exist a line V with $\dim(V \cap l_1) = 1$, and $\dim(V \cap l_2) = \dim(V \cap l_3) = 0$, i.e. such that the lines l_2 and l_3 do not intersect the line l_1. Of course we can! The only interesting possible violation is $(0, 0, 0, 0)$. Then the question is: does there exist a line V intersecting all four lines? The relevant Schubert cycle is

$$\sigma(0, 1) = \{V \in \mathbb{G}(1, 3) \mid V \cap l_i \neq \emptyset\} = \sigma_{\square}.$$

  The case is small enough to see quickly that $\sigma(1)^4 = 2$, but we can also apply proposition 4.1.9 with $n = 4$, $d = 2$ and $s = l = 1$. Thus we see the intersection product $\sigma(0, 1)^4 \neq 0$. This proves that the Mumford Criterion is violated for every possible configuration of lines (l_1, l_2, l_3, l_4).

- $\dim(V) = 2$: is irrelevant now, because we already know there do not exist stable points.

REMARK 4.1.10 Though not necessary in the small case above, we were able to use Proposition 4.1.9 to determine that a certain product of Schubert cycles is non-zero. It is likely that one can also avoid calculations in other situations than the situation of this proposition and see a priori that a certain product of Schubert cycles is non-zero.

4.2 Stability of sequences of subspaces of equal dimension

DEFINITION 4.2.1 Let $\underline{d} = (d_1, \ldots, d_N)$ be a sequence of dimensions. Consider the action of $SL(n+1)$ on $\mathcal{C}(\underline{d}, n)$. It is convenient to give a name to the GIT quotient:

$$\mathcal{Q}(\underline{d}, n) := \mathcal{C}(\underline{d}, n)^{ss} // SL(n + 1).$$

We quietely assumed here that the quotient is taken with reference to the sequence of weights $\underline{k} = (1, \ldots, 1)$. If there can be confusion about the sequence of weights $\underline{k}$ we are using, we write $\mathcal{Q}_{\underline{k}}(\underline{d}, n)$ for the quotient for the action linearized with reference to the sheaf $\mathcal{L}_{\underline{k}}$.

DEFINITION 4.2.2 We define the *virtual dimension* of $\mathcal{Q}(\underline{d}, n)$ to be the difference of the dimension of the configuration space $\mathcal{C}(\underline{d}, n)$ and the group $SL(n+1)$. Notation:

$$\operatorname{vdim}(\mathcal{Q}(\underline{d}, n)) := \sum_{i=1}^{N} (d_i + 1)(n - d_i) - (n+1)^2 + 1.$$

It is clear there can be no stable points in $\mathcal{C}(\underline{d}, n)$ if $\operatorname{vdim}(\mathcal{Q}(\underline{d}, n))$ is negative. The example below shows that this is not the only case it occurs.

EXAMPLE 4.2.3 We consider configurations of 4 lines in projective 3-space, i.e. $N = 4$, $n = 3$, $\underline{d} = (1,1,1,1)$, so

$$\mathcal{C}(\underline{d}, n) = \mathbb{G}(1,3)^4,$$

and

$$\operatorname{vdim}(\mathcal{Q}(\underline{d}, n)) = 4 \cdot \dim(\mathbb{G}(1,3)) - \dim(SL(4)) = 4 \cdot 4 - 15 = 1.$$

One checks (using Mumford's Criterion for stability) that a configuration in $\mathcal{C}(\underline{d}, n)$ can only be stable if there does not exist a line intersecting all 4 lines. However, it is easy to prove that such a line always exists. (We have already seen this for example in paragraph 4.1.4.) Hence $\mathcal{C}(\underline{d}, n)^s = \emptyset$. One also easily checks that configurations of 4 lines in general position are semi-stable.

We will return to this example. In fact we will give a class of examples where no stable points exist and show how one forms quotients nevertheless. Cf. Remark 1.2.6.

Now we try to find out for what values of $(\underline{d}, n)$ the space of configurations $\mathcal{C}(\underline{d}, n)$ has a stable point (or a semi-stable point). We restrict our attention to the case that the dimension of all subspaces is the same. An important observation is that non-existence of stable points can only occur if one considers small numbers of linear subspaces.

THEOREM 4.2.4 *Let d and n be fixed natural numbers satisfying $0 \leq d < n$. Consider for $N = 1, 2, 3, \ldots$ the sequence of dimensions $\underline{d}_N = (\underbrace{d, \ldots, d}_{\text{length}=N})$. There is a $N_0 \in \mathbb{N}$ (depending on d and n) such that*

$$\mathcal{C}(\underline{d}_N, n)^s \neq \emptyset \text{ for } N \geq N_0.$$

The proof has two simple ingredients. First we will show that existence of (semi-)stable points in $\mathcal{C}(\underline{d}_N, n)$ guarantees existence of (semi-)stable points in $\mathcal{C}(\underline{d}_{N+1}, n)$. Then we will give an upper bound on M with

$$\mathcal{C}(\underline{d}_M, n)^{ss} = \emptyset.$$

From there it isn't difficult to see that there also exists an upper bound for the non-existence of stable points.

LEMMA 4.2.5 *Let $\sigma : G \times X \to X$ be an action of a reductive algebraic group on a quasi-projective variety X, linearized with reference to an ample invertible sheaf $\mathcal{L}$. Let $N \in \mathbb{N}$. Suppose there exist stable points in X^N. (This is with respect to the induced action of G and line bundle $\prod p_i^*(\mathcal{L})$.) Then there exist stable points in X^{N+1}.*

PROOF If I is a subset of $\{1, \dots, N+1\}$, then we denote by p_I the projection $X^{N+1} \to X^I$. Choose $x \in X^{N+1}$ such that $p_I(x)$ is stable for every I with $|I| = N$. This is possible because the sets $(X^I)^s$ are open and X is irreducible. It means that there exist invariant sections σ_I of $(\prod_{i \in I} p_i^*(\mathcal{L}))^{r_I}$ for some positive integers r_I, such that $\sigma_I(p_I(x)) \neq 0$ and G_y is finite for all $y \in X^I_{\sigma_I}$. By taking powers of the σ_I we may assume they are sections of $(\prod_{i \in I} p_i^*(\mathcal{L}))^r$ for one and the same r (which is the least common multiple of the r_I).
Now define

$$\tau := \prod_{|I|=N} p_I^*(\sigma_I)$$

This is a section of $(\prod_{i=1}^{N+1} p_i^*(\mathcal{L}))^{Nr}$. Note further that

$$\tau(x) = \prod_{|I|=N} \sigma_I(p_I(x)) \neq 0$$

and τ is invariant because the σ_I are invariant. Because $\mathcal{L}$ is an ample invertible sheaf, $(\prod_{i=1}^{N+1} p_i^*(\mathcal{L}))^{Nr}$ is also ample and invertible, so X_τ is affine, being the complement of an ample divisor. Finally G_y is a finite group for all $y \in X_\tau^{N+1}$, because $G_y \subset G_{p_I(y)}$ which is finite because $p_I(y) \in X^I_{\sigma_I}$. So x is a stable point of X^{N+1}. □

REMARK 4.2.6 Of course the same holds if we replace stable in Lemma 4.2.5 by semi-stable. This is clear from the proof. If for some action of G on X, and for a certain N, $(X^N)^{ss} \neq \emptyset$, then it follows that $(X^{N+1})^{ss} \neq \emptyset$.

REMARK 4.2.7 It has to be remarked somewhere that a configuration $(V_1, \dots, V_N) \in \mathcal{C}(\underline{d}, n)$ can never be semi-stable if

$$W := \operatorname{span}(V_1, \dots, V_N) \neq \mathbb{P}^n.$$

This one checks trivially with the Mumford Criterion (see Theorem 2.1.9). The proper subspace $W \subset \mathbb{P}^n$ satisfies

$$1 = \frac{\sum_{i=1}^{N} k_i(\dim(V_i \cap W) + 1)}{\sum_{i=1}^{N} k_i(d_i + 1)} > \frac{\dim(W) + 1}{n + 1}.$$

Hence no semi-stable points exist if $\sum(d_i + 1) < n + 1$.

The proof of the following lemma is inspired by the theorem of Domokos-Zubkov (Theorem 3.5.1). It determines an upper bound on N with $\mathcal{C}(\underline{d}_N, n)^{ss} = \emptyset$.

LEMMA 4.2.8 *Let d and n be fixed natural numbers satisfying $0 \leq d < n$. Consider for $N = 1, 2, 3, \ldots$ the sequence of dimensions $\underline{d}_N = (\underbrace{d, \ldots, d}_{\text{length}=N})$. There is a $N_1 \in \mathbb{N}$ (depending on d and n) such that*

$$\mathcal{C}(\underline{d}_N, n)^{ss} \neq \emptyset \text{ for } N \geq N_1.$$

PROOF We take $N_1 = (n+1)(d+1)$ and give a non-trivial invariant for sequences of N_1 subspaces. According to Remark 1.2.4 this is sufficient.

Let $(L_1, \ldots, L_{N_1})$ denote a sequence of N_1 subspaces. Represent the L_i by $(d + 1) \times (n + 1)$-matrices A_i and consider the following determinant:

$$\sigma(L_1, \ldots, L_{N_1}) := \det \begin{pmatrix} A_1 & \ldots & A_{d(n+1)+1} \\ \vdots & & \vdots \\ A_{n+1} & \ldots & A_{(d+1)(n+1)} \end{pmatrix}$$

Clearly σ is an invariant: if $g \in SL(n + 1)$, then

$$g \cdot \sigma(L_1, \ldots, L_{N_1}) = \det \begin{pmatrix} g^{-1} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & g^{-1} \end{pmatrix} \cdot \begin{pmatrix} A_1 & \ldots & A_{d(n+1)+1} \\ \vdots & & \vdots \\ A_{n+1} & \ldots & A_{(d+1)(n+1)} \end{pmatrix},$$

so $g \cdot \sigma(L_1, \ldots, L_{N_1}) = \sigma(L_1, \ldots, L_{N_1})$. This invariant σ does not vanish everywhere. This is clear for example because in the set of matrices of size $(n + 1)(d + 1)$ the property of being non-singular and representing N_1 subspaces is a closed property. □

REMARK 4.2.9 It is clear from the proof above that one could easily take as an upper bound for the non-existence of semi-stable points:

$$N_2 = \frac{\mathrm{lcm}(d+1, n+1)}{\gcd(d+1, n+1)}.$$

This is the number of rectangles of size $(d+1) \times (n+1)$ one needs to form a square. In the case that $d+1 \mid n+1$ this is the smallest lower bound, because N_2 is the number of $\mathbb{P}^d$'s necessary to span $\mathbb{P}^n$ (cf. Remark 4.2.7). In general N_2 is certainly not the smallest lower bound. For example in the case of lines in $\mathbb{P}^4$ (i.e. $d = 1$, $n = 4$ and $N_2 = 5$) a number of 4 subspaces is sufficient (see the following example).

EXAMPLE 4.2.10 Consider configurations of lines in $\mathbb{P}^4$. Three lines do not suffice to have semi-stable configurations, because

$$\mathrm{vdim}(\mathcal{Q}((1,1,1),4)) = 3 \cdot 6 - 24 < 0.$$

The expected dimension of the moduli space of four lines in $\mathbb{P}^4$ is 0. Indeed, some checking of the Mumford Criterion (the algorithm) shows that it is possible to have stable configurations. Four lines in general position span $\binom{4}{2} = 6$ projective three-spaces, and the configuration is stable if and only if these three-spaces are in general position. All stable configurations of four lines are mapped to one point; they are all in the same $SL(5)$-orbit. If $l_1, \ldots, l_4$ are the lines and $W_{\{i,j\}}$ are the three-spaces $W_{\{i,j\}} := \langle l_i, l_j \rangle$, one easily recovers the lines from the three-spaces by

$$l_i = W_{\{i,j\}} \cap W_{\{i,k\}} \cap W_{\{i,l\}},$$

where i, j, k and l are such that $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

The proof of Theorem 4.2.4 has become simple.

PROOF (OF THEOREM 4.2.4) Lemma 4.2.8 asserts that for big N configurations of N subspaces can be semi-stable. From the definition of stable and semi-stable it follows that we only have to show that that for big N semi-stable configurations of N subspaces can have finite stabilisers. Because the function on $\mathcal{C}(\underline{d}_N, n)$

$$(V_1, \ldots, V_N) \mapsto \mathrm{Stab}_{(V_1, \ldots, V_N)}(SL(n+1))$$

is upper semi-continuous, we only have to construct for some N a configuration of N subspaces, that has trivial stabiliser. (Which is easy.) □

REMARK 4.2.11 Note that in this paragraph we only considered (semi-)stability with respect to the standard sequence of weights $\underline{k} = (1, \ldots, 1)$. We will not study (semi-)stability with respect to unusual sequences of weights (see only Example 2.3.3).

4.2.1 When can elements of configuration spaces be stable? This question was important in our research for some time. Below we state it again, slightly more precise.

QUESTION 4.2.12 Let d and n be fixed natural numbers satisfying $0 \leq d < n$. Consider for $N = 1, 2, 3, \ldots$ the sequence of dimensions $\underline{d}_N = (\underbrace{d, \ldots, d}_{\text{length}=N})$. What is the minimal number $N_0 \in \mathbb{N}$ of Theorem 4.2.4 such that

$$\mathcal{C}(\underline{d}_N, n)^s \neq \emptyset \text{ for } N \geq N_0.$$

Because the number N_0 obviously depends on d and n we consider it sometimes as a function to $\mathbb{N}$ and use the notation $N_0(d, n)$.

Given d, n and N as above, we would like to have simple and quick way to determine $N_0(d, n)$. There is a simple way. Apply the algorithm of paragraph 4.1 for $N = 1, 2, \ldots$ etc. and wait until it tells you there exist stable points. However, the calculations become rather lengthy even when d and n are small (cf. Remark 4.1.1).

Here's a trick to determine $N_0(d, n)$ for infinitely many pairs (d, n). If for a pair (d, n) a number of N subspaces suffices to have stable points, it means that $\mathcal{Q}(\underline{d}_N, n)$ is a geometric quotient. Recall Gale duality:

$$\mathcal{Q}(\underline{d}_N, n) \cong \mathcal{Q}(\underline{d}_N, N(d+1) - n - 2).$$

Because its virtual dimension equals its actual dimension:

$$\begin{aligned} &\mathrm{vdim}(\mathcal{Q}(\underline{d}_N, N(d+1) - n - 2)) = \mathrm{vdim}(\mathcal{Q}(\underline{d}_N, n)) = \\ &\dim(\mathcal{Q}(\underline{d}_N, n)) = \dim(\mathcal{Q}(\underline{d}_N, N(d+1) - n - 2)), \end{aligned}$$

it follows that $\mathcal{Q}(\underline{d}_N, N(d+1) - n - 2)$ is also a geometric quotient, i.e. stable points exist in $\mathcal{C}(\underline{d}_N, N(d+1) - n - 2)$.

PROPOSITION 4.2.13 *If $N \geq N_0(d, n)$, then*

$$N_0(d, N(d+1) - n - 2) \leq N$$

PROOF Clear from the above. □

If, for some small pair (d, n), we know $N_0(d, n)$ we get immediately an upper-bound for $N_0(d, N_0(d, n)(d + 1) - n - 2)$. There is also an obvious lower bound for the latter, given by the fact that $\mathrm{vdim}(\mathcal{Q}(\underline{d}_{N_0(d,n)}, N_0(d, n)(d + 1) - n - 2) \geq 0$. Then we determine $N_0(d, N_0(d, n)(d + 1) - n - 2)$ by applying the algorithm for the numbers of subspaces in between the upper and lower bound.

The usual duality in projective spaces gives us another simple rule to determine values of the map N_0:

$$N_0(d, n) = N_0(n - d - 1, n).$$

Producing a table Combining Gale duality and this rule allows us to determine $N_0(d, n)$ in a lot of cases. In order to get nicer formulas we change our variables:

$$\begin{array}{ccc} \delta & := & d + 1 \\ \gamma & := & n - d \\ M_0(\delta, \gamma) & := & N_0(d, n) - 2 \end{array}$$

Note that δ and γ are natural numbers greater than 1.

RULES 4.2.14 In terms of δ, γ and M_0 we have the following translation of the rules we have already seen to determine the values of M_0.

1. If $M \geq M_0(\delta, \gamma)$, then
$$M_0(\delta, M\delta - \gamma) \leq M.$$
2. $M_0(\delta, \gamma) = M_0(\gamma, \delta)$
3. The fact that the virtual dimension of a geometric quotient is greater than or equal to zero gives us:
$$M_0(\delta, \gamma) \geq \frac{\delta^2 + \gamma^2 - 1}{\delta\gamma}$$
as one easily checks.

We are now able to make a table, which is symmetric in its diagonal. At place (δ, γ) we put $M_0(\delta, \gamma)$. We start to fill some places. For example:

- The case $\delta = 1$. One needs $n + 2$ points in $\mathbb{P}^n$ to make a stable configuration: $M_0(1, \gamma) = \gamma$.
- The diagonal $\delta = \gamma$, i.e. $\mathbb{P}^d$'s in $\mathbb{P}^{2d+1}$. In the next paragraph we will elaborate on this case. It will turn out that $M_0(\delta, \delta) = 3$ for all δ. See Remark 4.2.20.

- The case $\gamma = \delta \pm 1$, i.e. $\mathbb{P}^d$'s in $\mathbb{P}^{2d+2}$. Using duality and Gale duality one sees:

$$\begin{aligned}&\mathcal{Q}((d,d,d,d),2d+2) \cong \\ &\mathcal{Q}((d+1,d+1,d+1,d+1),2d+2) \cong \\ &\mathcal{Q}((d+1,d+1,d+1,d+1),2d+4).\end{aligned}$$

  It follows that $\mathcal{Q}((d,d,d,d),2d+2) \cong \mathcal{Q}((1,1,1,1),4)$ for every $d \geq 1$, so $M_0(\delta,\delta+1) = M_0(\delta,\delta-1) = 2$ for all δ, because one needs at least 4 lines in $\mathbb{P}^4$ to have stable configurations. See example $4.2.10$.

After having filled some positions in the table, we can fill more positions using the rules 4.2.14. For example:

$$M_0(2,2a-1) = a \text{ for } a \geq 2.$$

(PROOF The virtual dimension rule tells us:

$$M_0(2,2a-1) \geq \frac{2^2+(2a-1)^2-1}{2(2a-1)} > a-1.$$

We have already seen that $M_0(1,2) = M_0(2,1) = 2$, so for $a \geq 2$ (according to the first rule of 4.2.14)

$$M_0(2,2a-1) \leq a.)$$

In an analogous manner one can fill many other places of the table. It appears that:

- Except for the diagonal (which is the case of $\mathbb{P}^d$'s in $\mathbb{P}^{2d+1}$) on all places the number of subspaces required is the minimal numbers of subspaces needed to make the virtual dimension positive.

- Places "far away" from the diagonal can always be filled by applying the rules. In order to fill places close to the diagonal we sometimes are obliged to use the algorithm.

Here's our table for $M_0(\gamma,\delta)$. The empty places close to the diagonal have to be calculated using the algorithm. However, the algorithm (implemented in the computer algebra package Maple) took too much time to give an answer (more than a day).

16	9	6	5	4		3	3						3	2	3
15	8	6	4	4		3	3					3	2	3	2
14	8	5	4	4	3	3					3	2	3	2	3
13	7	5	4	3	3	3				3	2	3	2	3	
12	7	5	4	3	3				3	2	3	2	3		
11	6	4	4	3	3			3	2	3	2	3			
10	6	4	3	3			3	2	3	2	3				
9	5	4	3	3		3	2	3	2	3					
8	5	3	3	3	3	2	3	2	3					3	3
7	4	3	3	3	2	3	2	3				3	3	3	3
6	4	3	3	2	3	2	3			3	3	3	3		
5	3	3	2	3	2	3	3	3	3	3	3	3	4	4	4
4	3	2	3	2	3	3	3	3	3	4	4	4	4	4	5
3	2	3	2	3	3	3	3	4	4	4	5	5	5	6	6
2	3	2	3	3	4	4	5	5	6	6	7	7	8	8	9
γ/δ	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

4.2.2 The quotient in the case of four medials

DEFINITION 4.2.15 Let V be a vectorspace of even dimension and $L \subset V$ a linear subspace. We call L a *medial*, if $\dim(V) = 2 \cdot \dim(L)$. A pair of medials (L_1, L_2) can obviously span the vectorspace V. If this is the case, the pair (L_1, L_2) is said to be *in general position* and we write

$$V = L_1 \oplus L_2.$$

For our purposes we consider medials as linear projective spaces in stead of affine linear spaces. Thus we let d be a natural number and look at a projective space V of dimension $n = 2d + 1$ and N subspaces $(L_1, \dots, L_N)$ of dimension d. If $N \geq 2$ there exist semi-stable points. According to Remark 4.2.6 we have to check this for $N = 2$. It is easily seen that if the pair (L_1, L_2) is in general position, we have equality in Mumford's criterion, i.e. (L_1, L_2) is semi-stable, but not stable.

We have seen two examples of medials already. Of course there is the case of 4 points on a line, where one forms a GIT quotient by taking the cross ratio of the 4 points. We also looked at 4 lines in projective 3-space (Example 4.2.3), but here we couldn't take the GIT quotient, because there are no stable configurations. One reason for the cases of 4 medials to be interesting is that they are exactly the cases in which the virtual dimension is 1.

PROPOSITION 4.2.16 *If N, d and n be positive integers with $d < n$ and $\underline{d}_N = (\underbrace{d, \dots, d}_{\text{length}=N})$ is a sequence of dimensions, then*

$$\operatorname{vdim}(\mathcal{Q}(\underline{d}_N, n)) = 1 \iff N = 4, \text{ and } n = 2d + 1.$$

PROOF Immediately from the definition of virtual dimension it follows that

$$\operatorname{vdim}(\mathcal{Q}(\underline{d}_N, n)) = 1 \iff N = \frac{(n+1)^2}{(n-d)(d+1)}.$$

Now the substitutions $d_1 := n - d$, and $d_2 := d + 1$ make things more obvious. We seek integer solutions (N, d_1, d_2) for the equality

$$N = \frac{(d_1 + d_2)^2}{d_1 d_2},$$

or in other words we seek integers (d_1, d_2) such that

$$d_1 d_2 \mid d_1^2 + d_2^2.$$

Note that all solutions are multiples of (d_1, d_2) satisfying $\gcd(d_1, d_2) = 1$. Assuming (d_1, d_2) is such a minimal solution we see for all primes p

$$p|d_1 \iff p|d_1^2 \text{ and } p|d_1^2 + d_2^2 \iff p|d_2^2 \text{ and } p|d_1^2 + d_2^2 \iff p|d_2,$$

so $d_1 = d_2 = 1$. This means that the only possibility is $d_1 = d_2$, i.e. $n = 2d + 1$ and $N = \frac{(2d+2)^2}{(d+1)^2} = 4$. □

The example of configurations of 4 lines in projective 3-space turns out to be a crucial example for the general case of 4 medials. Recall that the obstruction for stability in the case of 4 lines is the existence of a line (in general exactly 2 lines) intersecting these 4 lines. The following (well known) proposition generalizes this.

PROPOSITION 4.2.17 *Let $d \in \mathbb{N}$. The number of lines intersecting 4 linear subspaces in general position of dimension d of a projective space of dimension $2d + 1$ is $d + 1$.*

PROOF This is a calculation in $\operatorname{Gr}(2, 2d + 2)$. See §4.1.1 for Schubert calculus. Let V_{d+1} be a subspace of (affine) dimension $d + 1$. We have to consider the special Schubert class

$$[\sigma_d] = \text{the class of } \{L \in \operatorname{Gr}(2, 2d + 2) \mid \dim(L \cap V_{d+1}) \geq 1\}.$$

The number of lines intersecting 4 subspaces in general position is

$$[\sigma_d]^4 = ([\sigma_d]^2)^2 = (\sum_{i=0}^{d} \sigma_{(2d-i,i)})^2 = d+1.$$

(Use Pieri's rule and the duality theorem.) □

COROLLARY 4.2.18 *Let $d \in \mathbb{N}$ satisfy $d \geq 1$, let $\underline{d}$ be the sequence of dimensions $\underline{d} := (d,d,d,d)$. Then:*

$$\mathcal{C}(\underline{d}, 2d+1)^s = \emptyset.$$

PROOF If $(V_1, \ldots, V_4) \in \mathcal{C}(\underline{d}, 2d+1)$ and L is a line intersecting $V_1, \ldots V_4$, then we have

$$\frac{\sum_{i=1}^4 \dim(V_i \cap L) + 1}{\sum_{i=1}^4 (d+1)} = \frac{4}{4(d+1)} = \frac{2}{2d+2} = \frac{\dim(L)+1}{2d+2},$$

i.e. an equality in the criterion of Mumford. □

A more instructive way to see the instability in $\mathcal{C}(\underline{d}, 2d+1)$ is to recall the definition of stability (Definition 1.2.2). A configuration can only be stable if it's stabilizer in the special linear group is finite. The existence of the lines prevents this.

PROPOSITION 4.2.19 *Let $d \in \mathbb{N}$ and let δ be the sequence of dimensions $\underline{d} := (d,d,d,d)$. A general element of $\mathcal{C}(\underline{d}, 2d+1)$ has stabiliser inside $SL(2d+2)$ isomorphic to $(\mathbb{C}^*)^d$.*

PROOF Let $(V_1, \ldots, V_4) \in \mathcal{C}(\underline{d}, 2d+1)$ be a general element. Let $l_1, \ldots, l_{d+1}$ be the lines intersecting $V_1, \ldots, V_4$. Because of the assumed general position $l_1, \ldots, l_{d+1}$ span $\mathbb{P}^{2d+1}$. So we can choose a basis $e_1, e_2, \ldots, e_{2d+2}$ of the underlying affine space $\mathbb{C}^{2d+2}$ of $\mathbb{P}^{2d+1}$ such that the lines are given by $l_i := \langle e_{2i-1}, e_{2i} \rangle$. The configuration $(V_1, \ldots, V_4)$ is stabilised if and only if $l_1, \ldots, l_{d+1}$ are stabilised, so

$$\begin{gathered}\mathrm{Stab}(V_1, \ldots, V_4) = \\ \{\mathrm{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_{d+1}, \lambda_{d+1}) \mid \lambda_1 \cdot \lambda_2 \cdots \lambda_{d+1} = 1\},\end{gathered}$$

which is isomorphic to $(\mathbb{C}^*)^d$. □

In spite of the existence of the isotropy subgroups the GIT quotient is an orbit space on an open subset of all configurations, because on an open subset of all configurations the dimensions of the stabiliser is constant. See Remark 1.2.6.

Taking the quotient has a nice geometric interpretation here. The key to this is the existence of the $d+1$ lines. The basis construction is as follows. Given 4 medials $(V_1,\dots,V_4) \in \mathcal{C}(\underline{d}, 2d+1)$ in general position, take the $d+1$ lines through $(V_1,\dots,V_4)$. On each line we have the 4 intersection points with $(V_1,\dots,V_4)$. If on each line no 3 of these 4 points coincide, we get $d+1$ cross ratios. This gives an equivariant map from an open part $U \subset \mathcal{C}(\underline{d}, 2d+1)$ to $(\mathbb{P}^1)^{d+1}/S_{d+1} \cong \mathbb{P}^{d+1}$, and this turns out to be the quotient.

REMARK 4.2.20

$$\mathcal{C}((d,d,d,d,d), 2d+1)^s \neq \emptyset,$$

so $N_0(d, 2d+1) = 5$. Because semi-stable configurations exist the only thing we have to do is to find a configuration with trivial stabiliser (because $x \mapsto \mathrm{Stab}_x(SL(n+1))$ is upper semi-continuous). See the proof of Proposition 4.2.19. Without loss of generality we can assume that exactly the diagonal matrices of the form

$$\{\mathrm{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \dots, \lambda_{d+1}, \lambda_{d+1}) \mid \lambda_1 \cdot \lambda_2 \cdots \lambda_{d+1} = 1\}$$

stabilize the first four-tuple $(V_1,\dots,V_4)$ of $\mathbb{P}^d$'s. We choose the fifth subspace V_5 such that the free coefficients $\lambda_1,\dots,\lambda_{d+1}$ are killed, for example take V_5 to be the subspace spanned by the vectors $e_{2i-1} + e_{2i+2}$ for $i \in \{1,\dots,d\}$ and an arbitrary other vector. One sees that an element of $SL(n+1)$ stabilizing $(V_1,\dots,V_5)$ must be the identity.

REMARK 4.2.21 A complete description of the homogeneous invariant ring of the configuration space of 4 subspaces of a projective space is given in [HH96].

4.3 Stable resolutions of configuration spaces

In the whole thesis we were focussed on the existence of semi-stable points or the even better stable points. Until now we have only tried to solve this problem by adding subspaces to our configuration spaces. We have seen that by adding enough subspaces we always can obtain that stable points exist in the configuration space (Theorem 4.2.4) and we have given some indication of the meaning of "enough".

What we don't solve in this way is the possible existence of strictly semi-stable points. Recall that a strictly semi-stable point is a point that is neither unstable nor stable. These are exactly the points that prevent the GIT-quotient from being an orbit space. In this paragraph we discuss a method to get rid of these strictly semi-stable points by changing slightly the configuration space. The relevant concept is called a stable resolution.

DEFINITION 4.3.1 Let X be a quasi-projective variety, acted upon by G, a reductive algebraic group. Let this action be linearized with reference to $\mathcal{L}$, a line bundle on X. Suppose $X^s \neq \emptyset$. A *stable resolution* of the G-action on X is an equivariant map $\pi : Y \to X$, together with a linearization of the G-action on Y with reference to an ample line bundle $\mathcal{M}$ on Y, such that

1. $Y^{ss} = Y^s$, i.e. Y doesn't contain any strictly semi-stable points,
2. $\pi^{-1}(X^s) \subset Y^s$, and
3. π is an isomorphism over X^s.

REMARK 4.3.2 Of course the notions of stability and semi-stability in the definition above are with reference to the line bundles $\mathcal{L}$ and $\mathcal{M}$. As always we abuse notation.

In [Kir71] Kirwan gave a canonical procedure for obtaining stable resolutions, in the case that the ground field $k = \mathbb{C}$. Though this construction is a sequence of blowing ups, it is somewhat abstract.

4.3.1 Results of Reichstein More useful for us than Kirwan's construction is the article [Rei89]. Zinovy Reichstein gives here some results concerning the question: what happens to stable (resp. semi-stable) points under equivariant maps and especially equivariant blowing ups? His work is valid over any algebraicly closed field (not just the complex numbers). We state here those results that we'll use later on to construct stable resoultions of our configuration spaces. Thereto we fix some terminology. From now on

- G is a reductive algebraic group,
- X and Y are projective varieties. G acts on these varieties,
- the action on X is linearized by an ample line bundle $\mathcal{L}$, the action of Y is linearized by an ample line bundle $\mathcal{M}$, and
- $\pi : Y \to X$ is a G-equivariant morphism.

A first question could be which line bundle on Y we should take.

PROPOSITION 4.3.3 *For e large the linebundle*

$$K_e := \pi^* \mathcal{L}^e \otimes \mathcal{M}$$

is ample on Y and has a G-linearization. Furthermore Y^{us}, Y^{ss} and Y^{ps} are independent of e for large e.

THEOREM 4.3.4 *The notions of stability in X (with respect to $\mathcal{L}$) and Y (with reference to K_e for large e) are related in the following way.*

1. *If $p = \pi(q)$ is unstable in X, then q is unstable in Y.*
2. *If $p = \pi(q)$ is stable in X, then q is stable in Y.*

PROOF This is Theorem 2.1 of [Rei89]. □

The theorem above shows that stable points (resp. unstable points) remain stable (resp. unstable) under equivariant maps. From this theorem it is not clear what happens to the strictly semi-stable points. In the case that $\pi : Y \to X$ is a blowing-up Reichstein has some additional results.

DEFINITION 4.3.5 If we denote by

$$\alpha : X^{ss} \to X//G$$

the quotient map, then for a subset $C \subset X^{ss}$ we define

$$\tilde{C} := \alpha^{-1}\alpha(C).$$

For an arbitrary subset $C \subset X$ we define $\tilde{C}$ to be the closure in X of $\alpha^{-1}\alpha(C \cap X^{ss})$.

Note that if C is closed $\tilde{C}$ is a closed invariant subset of X^{ss}.

From now on the map $\pi : Y \to X$ denotes an equivariant blowing up. To be more precise: let $\mathcal{I}$ be a G-invariant sheaf of ideals and let $\pi : Y \to X$ the blowing up of $\mathcal{I}$. We denote by D the center of this blowing up and by E the exceptional divisor $\pi^{-1}(E)$. Because $\mathcal{L}$ is an ample linebundle on X there's r such that $\mathcal{L}^r$ is very ample.

PROPOSITION 4.3.6 *For large t the line bundle*

$$\mathcal{M} := \mathcal{O}(-E) \otimes \pi^* \mathcal{L}^{rt}$$

is very ample on Y and the action of G on Y can be linearized with reference to $\mathcal{M}$.

Still following [Rei89] we linearize now the action on the blowing up Y with reference to the sheaf K_e as above and compare stability in X and Y.

THEOREM 4.3.7 *Suppose $D \subset X^{ss}$. Consider $q \in Y$, such that $p = \pi(q)$ is semi-stable, and $p \notin D$. Then we have the following.*

1. *q is unstable if and only if $p \in \tilde{D}$.*
2. *q is not stable if and only if $p \in X^{sss}$.*

Proof: This is Theorem 2.3 of [Rei89]. □

REMARK 4.3.8 Note that the statements here seem to differ somewhat form the statements in [Rei89]. One difference is that Reichstein uses the terminology of "properly stable" (see Remark 1.2.3). The other difference is that Reichstein is unprecise in his definition of $\tilde{C}$: he defines it for an arbitrary set $C \subset X$ as $\alpha^{-1}\alpha(C)$.

To be able to say what happens to the points of X of the blow up locus, one needs to impose some smoothness conditions on X and D. For a closed subset $Z \subset X$ we denote the strict transform by Z', i.e. Z' is by definition the closure of $\pi^{-1}(Z \setminus D)$ in Y.

THEOREM 4.3.9 *Consider $q \in Y$, such that $p = \pi(q)$ is semi-stable. Assume that X and D are smooth at every point of $\alpha^{-1}(\alpha(p))$. Then we have the following.*

1. *q is unstable if and only if q lies in the strict transform $\tilde{D}'$ of $\tilde{D}$.*
2. *q is not stable if and only if q lies in the strict transform $(X^{sss})'$ of X^{sss}.*

Inspired by these theorems Reichstein gives a statement about obtaining a stable resolution by a sequence of blowing ups (and a resolution of singularities of X^{sss}). In the next section we will start to investigate our own blowing up processes in order to obtain a stable resolution of configuration spaces.

4.3.2 Compactifications of configuration spaces as stable resolutions The Theorems 4.3.7 and 4.3.9 of Reichstein give rise to the following idea. If we could blow up a locus D, such that $\tilde{D} = X^{sss}$, then according to these theorems we would have that every point in the blowing up that is not stable, is in fact unstable. In other words: we would have obtained a stable resolution. The problem is that some smoothness conditions on the locus D have to be fulfilled.

The situation of points on a line We first restrict ourselves to the case of configuration spaces of N points on a line. By the Mumford Criterion

$$\begin{array}{lcl} x \in (\mathbb{P}^1)^N \text{ is stable} & \Longleftrightarrow & \text{no } \frac{1}{2}N \text{ points coincide} \\ x \in (\mathbb{P}^1)^N \text{ is semi-stable} & \Longleftrightarrow & \text{at most } \frac{1}{2}N \text{ points coincide} \end{array}$$

If N is an odd number, we have that $((\mathbb{P}^1)^N)^{ss} = ((\mathbb{P}^1)^N)^s$. If N is an even number there do exist strictly semi-stable points. These are exactly the configurations where $\frac{1}{2}N$ points come together. In the moduli space several of these strictly semi-stable orbits are mapped to the same point. If $S \subset \{1, \ldots, N\}$, we denote by $\Delta_S \subset (\mathbb{P}^1)^N$ the corresponding diagonal. To each partition $\{1, \ldots, N\} = S_1 \cup S_2$ into two sets of the same size $\frac{1}{2}N$ corresponds one point in the moduli space. Configurations where all points with a label from one of the two sets of the partition come together, i.e. all points of $\Delta_{S_1} \cup \Delta_{S_2}$, are mapped to this single point. The minimal orbit corresponding to such a point is the orbit $\Delta_{S_1} \cap \Delta_{S_2}$.

Fulton-MacPherson compactification A well known compactification of configuration spaces was described by Fulton and MacPherson in [FM94]. If X is a nonsingular variety, the configuration space of N distinct (labeled) points is defined as

$$F(X, N) := X^N \setminus \Delta.$$

The *Fulton-MacPherson compactification* $FM(X, N)$ is a compactication of $F(X, N)$ with the property that it can be obtained by a sequence of blowing ups:

$$X_s \to X_{s-1} \to \ldots \to X_1 \to X_0$$

starting with $X_0 = X^N$ and resulting in $X_s = FM(X, N)$. The number of blowing ups $s = 2^N - N - 1$. In each step a locus corresponding to some diagonal is blown up, in the following order: $\Delta_{\{1,2\}}$, $\Delta_{\{1,2,3\}}$, $\Delta_{\{2,3\}}$, $\Delta_{\{1,3\}}$, $\Delta_{\{1,2,3,4\}}$, $\Delta_{\{2,3,4\}}$, $\Delta_{\{1,3,4\}}$, $\Delta_{\{1,2,4\}}$, $\Delta_{\{1,2,3,4,5\}}$, etcetera. In this way one gets a compactification with many nice properties. For details: see the article.

For our purposes this construction is interesting because in the case of points on a line some strictly semi-stable loci are blown up. However, $FM(\mathbb{P}^1, 4)$, the first case one considers, is not a stable resolution of $(\mathbb{P}^1)^4$. Let's analyze what happens. The first blowing up in the process corresponds to the diagonal $\Delta_{\{1,2\}}$. Because this is a variety of codimension 1, nothing is blown up. The next step is blowing up the diagonal $\Delta_{\{1,2,3\}}$. Because $\Delta_{\{1,2,3\}} \cap X^{ss} = \emptyset$, we have $\widetilde{\Delta_{\{1,2,3\}}} = \emptyset$, which yields that strictly semi-stable points remain unstable. Also in the other steps of the process only unstable loci are blown up, which doesn't help us at all. This example has shown us that surely we have to blow up a locus of strictly semi-stable points.

Ulyanov's compactification of configuration spaces Another compactification of configuration spaces was described by Ulyanov in [Uly02]. We will quickly see that this compactification is a stable resolution in the case of point sets on a line. First we describe shortly Ulyanov's construction. Again: for details: see the article. He uses *polydiagonals*. A polydiagonal is nothing else than an intersection of diagonals. Suppose we are going to construct the *Ulyanov compactification* of $F(X, N)$. Give it the name: $U(X, N)$. The polydiagonals of X^N correspond bijectively to the partitions of $\{1, \ldots, N\}$. For example: $\{1, \ldots, N-2\} \cup \{N-1, N\}$ corresponds to the polydiagonal $\Delta_{\{1,\ldots,N-2\}} \cap \Delta_{\{N-1,N\}}$. Ulyanov's compactification is also obtained by a sequence of blowing ups

$$X_{N-1} \to X_{N-2} \to \ldots \to X_1 \to X_0,$$

again starting with $X_0 = X^N$ and resulting in $X_N = U(X, N)$. This time the number of steps in the process is exactly $N - 1$. Each step is a blowing up of (a proper transform) of a union of polydiagonals. To be precise the map $X_i \to X_{i-1}$ is the blowing up of the union of all polydiagonals corresponding to partitions of $\{1, \ldots, N\}$ into exactly i subsets. An important property of this construction is that each X_i is smooth (if X_0 is smooth). The loci that are blown up in each step have become disjoint by a previous step in the process.

PROPOSITION 4.3.10

$$U(\mathbb{P}^1, N) \to (\mathbb{P}^1)^N$$

is a stable resolution of the action of $SL(2)$ *on* $(\mathbb{P}^1)^N$.

PROOF If N is odd, the statement is trivially true, because there are no strictly semi-stable points. If N is even, recall that the strictly semi-stable locus is

$$\bigcup \Delta_A,$$

where the union is taken over all partitions $\{1, \ldots, N\} = A$ such that $\#A = \frac{1}{2}N$. Let's now analyze what happens in the blowing up process. The first step is the blowing up of the small diagonal $\Delta_{\{1,\ldots,N\}}$. Because $\Delta_{\{1,\ldots,N\}} \subset X^{us}$ this doesn't have any influence on the strictly semi-stable points. However, the loci that will be blown up in the second step, have become disjoint. In the second step all proper transforms of polydiagonals corresponding to partitions into two subsets are blown up. Most of these again consist of unstable points, i.e. the polydiagonals $\Delta_A \cap \Delta_B$ with $\#A, \#B \neq \frac{1}{2}N$. However, the moment the polydiagonals $\Delta_A \cap \Delta_B$ with $\#A = \#B = \frac{1}{2}N$ are blown up, something does happen. This union of polydiagonals is part of $((\mathbb{P}^1)^N)^{sss}$.

Moreover, it consists exactly of the strictly semi-stable points with biggest stabilizer, i.e. it is the smallest set $D \subset (\mathbb{P}^1)^N$ with $\tilde{D} = ((\mathbb{P}^1)^N)^{sss}$. Now all conditions in Theorem 4.3.9 are satisfied: D is smooth (because of the blowing up in step 1) and the variety that is blown up in step 2 is smooth (because all steps in Ulyanov's process are smooth). It is clear that Theorem 4.3.9 then says: a point is not stable if and only if it is unstable. In the other blowing ups of Ulyanov's construction nothing changes (with respect to the variety being a stable resolution). □

REMARK 4.3.11 From the previous proposition it is clear that we have blown up too much, as we had blown up not enough in the case of the Fulton-MacPherson compactification. What we have to blow up minimally here is also clear:

1. first the small diagonal, in order to make the polydiagonals we are going to blow up in step 2 disjoint, and
2. then the polydiagonals which are the strictly semi-stable points with minimal orbit.

4.3.3 A compactification of a general configuration space In general we want to construct a stable resolution of

$$\mathcal{C} := \mathcal{C}(\underline{d}, n) := \prod_{i=1}^{N} \mathbb{G}(d_i, n).$$

We will show here a construction which is an analog of Ulyanov's compactification. Recall Mumford's Criterion (Theorem 2.1.9). It gives precise conditions for configurations $(L_1, \ldots, L_N) \in \mathcal{C}$ in terms of the intersection with another linear subspace, determining their stability. Inspired by the Mumford Criterion we define some subvarieties of $\mathcal{C}$.

DEFINITION 4.3.12 If $\sigma := (s_1, \ldots, s_N) \in \mathbb{Z}^N$ with $-1 \leq s_i \leq d_i$ for all i, and $l \in \{0, \ldots, n-1\}$, we define:

$$W_{\sigma,l} := \{(L_1, \ldots, L_N) \in \mathcal{C} \mid \text{there exists } V \text{ with } \dim(V) = l \text{ and } \dim(V \cap L_i) \geq s_i \text{ for all } i\}.$$

Let $D := \dim(\mathcal{C}) - 2$. Consider a sequence of blowing-ups:

$$Y_D \xrightarrow{\pi_D} Y_{D-1} \longrightarrow \ldots \longrightarrow Y_1 \xrightarrow{\pi_1} Y_0 \xrightarrow{\pi_0} \mathcal{C},$$

such that π_i is the blow-up (of a proper transform of) the union of those $W_{\sigma,l}$ satisfying $\dim(W_{\sigma,l}) = i$.

We repeat here a definition and a lemma from [Uly02].

DEFINITION 4.3.13 Two smooth subvarieties of a smooth algebraic variety W are said to intersect cleanly if not $U \subset V$ or $V \subset U$, their scheme-theoretic intersection is smooth and the tangent bundles satisfy $T(U \cap V) = TU \cap TV$.

LEMMA 4.3.14 *Let W be a smooth algebraic variety and let U, V be smooth subvarieties of W intersecting cleanly. Then:*

1. *the proper transforms of U and V in $\mathrm{Bl}_{U\cap V} W$ are disjoint, and*
2. *if Z is a smooth subvariety of $U \cap V$, then the proper transforms of U and V in $\mathrm{Bl}_Z W$ intersect cleanly.*

Now here's a strategy to obtain a stable resolution of $\mathcal{C}$.

1. In the Definition 4.3.12 in very step of the blowing up process a disjoint union of $W_{\sigma,l}$ is blown up. These unions are disjoint, because possible intersections of $W_{\sigma,l}$ are themselves $W_{\sigma,l}$ of smaller dimension. Because on the lemma on clean intersections above, the $W_{\sigma,l}$ become disjoint in this way, if
2. the varieties $W_{\sigma,l}$ intersect cleanly. Such an intersection is itself of the form $W_{\sigma,l}$. The problem is that these varieties aren't smooth in general.
3. The solution to obtain a stable resolution is as follows. First resolve the singularities of the varieties $W_{\sigma,l}$ and then apply the blowing up process of Definition 4.3.12.

Bibliography

[Bri04] M. Brion. *Lectures on the geometry of flag varieties.* Available at: arXiv:math.AG/0410240, 2004.

[Cas89] G. Castelnuovo. *Su certi gruppi associati di punti.* Ren. di Circ. Matem. Palermo 3, 179-192, 1889.

[DP76] C. de Concini, C. Procesi. *A characteristic free approach to invariant theory.* Advances in Math. 21, no. 3, 330-354, 1976.

[Der01] H. Derksen. *Quiver representations.* Notes from his lectures. Available at: http://www.math.lsa.umich.edu/hderksen, 2001

[DW00] H. Derksen, J. Weyman. *Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients.* J. Amer. Math. Soc. 13, no. 3, 467-479 (electronic), 2000.

[DC71] J.A. Dieudonné, J.B. Carrell. *Invariant Theory, Old and New.* Academic Press, 1971.

[DH98] I.V. Dolgachev, Y. Hu. *Variation of geometric invariant theory quotients.* Inst. Hautes Études Sci. Publ. Math. 87, 5-56, 1998.

[DO88] I.V. Dolgachev, D. Ortland. *Point sets in projective spaces and Theta functions.* (Volume 165 of Astérisque.) Société Mathématique de France, 1988.

[Dol03] I.V. Dolgachev. *Lectures on Invariant Theory.* (Volume 296 of London Mathematical Lecture note series.) Cambridge University Press, 2003.

[DZ01] M. Domokos, A.N. Zubkov. *Semi-invariants of quivers as determinants.* Transform. Groups 6, no. 1, 9-24, 2001.

[EP00] D. Eisenbud, S. Popescu. *The Projective Geometry of the Gale Transform.* Journal of algebra 230 (1), 127-173, 2000.

[FM94] W. Fulton, R.M. MacPherson. *A compactification of configuration spaces.* Ann. Math. 139, 183-225, 1994.

[Fu97] W. Fulton. *Young tableaux.* (Volume 35 of London Mathematical Society student texts.) Cambridge University Press, 1997.

[GM82] I.M. Gelfand, R.M. MacPherson. *Geometry in Grassmannians and a generalization of the dilogarithm.* Adv. in Math. 44, 279-312, 1982.

[GH78] P. Griffiths, J. Harris. *Principles of Algebraic Geometry.* Wiley-Interscience, 1978.

[Hab75] W.J. Haboush. *Reductive groups are geometrically reductive.* Ann. of Math. 102, 67-83, 1975.

[Har77] R. Hartshorne. *Algebraic Geometry.* (Volume 52 of Graduate Texts in Mathematics.) Springer-Verlag, 1977.

[HH96] R. Howe, R. Huang. *Projective invariants of four subspaces.* Adv. in Math. 118, 295-336, 1996.

[Hil90] D. Hilbert. *Über die Theorie der algebraischen Formen.* Math. Ann. 36, 473-534, 1890.

[Hu05] Y. Hu. *Stable Configurations of Linear Subspaces and Quotient Coherent Sheaves.* Quarterly Journal of Pure and Applied Mathematics, Volume 1, Number 1, 2005.

[Kap92] M.M. Kapranov. *Chow quotients of Grassmanninas I.*, In: I.M. Gelfand Seminar, Adv. in Soviet Math. 16, 29-111, 1993.

[Kir71] F.C. Kirwan. *Partial desingularisations of quotients of non-singular varieties and their Betti numbers.* Ann. Math. 122, 41-85, 1971.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric Invariant Theory.* (Volume 34 of Ergebnisse der Mathematik und ihrer Grenzgebiete.) 3rd enlarged edition, Springer-Verlag, 1994.

[Nag58] M. Nagata. *On the fourteenth problem of Hilbert.* Proc. International Congress of Math., 459-462, Edinburg, 1958.

[Nag64] M. Nagata. *Invariants of a group in an affine ring.* J. Math. Kyoto Univ. 3 369-377, 1963/1964.

[NM64] M. Nagata, and T. Miyata. *Note on semi-reductive groups.* J. Math. Kyoto Univ. 3 379-382, 1963/1964.

[New78] P.E. Newstead. *Introduction to Moduli Problems and Orbit Spaces.* (Volume 51 of Tata Institute Lecture Notes on Mathematics.) Springer-Verlag, 1978.

[Rei89] Z. Reichstein. *Stability and equivariant maps.* Invent. Math. 96, 349-383, 1989.

[Uly02] A.P. Ulyanov. *Polydiagonal compactification of configuration spaces.* J. Algebraic Geom. 11, 129-159, 2002.

[Vaz99] D. Vazzana. *Projections and invariants of lines in projective space.* Univ. Michigan Ph. D. thesis, 1999.

[Wey46] C.H.H. Weyl. *The classical groups: their invariants and representations.* Princeton Univ. Press, 1946.

Introductie tot het onderwerp

Dit proefschrift gaat over moduli van rijtjes lineaire deelruimten van projectieve ruimten. Zo'n rijtje lineaire deelruimten noemen we een *configuratie*. Het eenvoudigste voorbeeld van een lineaire deelruimte van een projectieve ruimte is een punt in de projectieve ruimte. Zo is het eenvoudigste voorbeeld van een configuratie een rijtje punten in de projectieve ruimte, zeg een element van $(\mathbb{P}^n)^N$. De speciale lineaire groep $SL(n+1)$ werkt op op de configuratieruimte $(\mathbb{P}^n)^N$ en wij zijn geïnteresseerd in de banen van deze actie.

Natuurlijk kunnen we de slechts de verzameling van deze banen beschouwen, maar liever zouden we het volgende hebben:

1. een algebraïsche variëteit X,
2. een morfisme van algebraïsche variëteiten $\phi : (\mathbb{P}^n)^N \to X$, zodat
3. ϕ surjectief is, en
4. voor elke $x \in X$ de vezel $\phi^{-1}(x)$ een baan is voor de actie van $SL(n+1)$.

Dit is het idee van een *moduliruimte*. Bovenstaande variëteit X zou alle mogelijke configuraties van N punten in $\mathbb{P}^n$ parametrizeren tot op lineaire equivalentie.

Moduliruimten zijn variëteiten of schema's die een zekere klasse van objecten parametrizeren tot op een zekere equivalentierelatie. Ze zijn erg belangrijk in de algebraïsche meetkunde. In het algemeen, en ook in ons geval, is het niet mogelijk om moduliruimten te construeren die *alle* equivalentieklassen van de objecten parametrizeren. Laten we dit duidelijk maken met een voorbeeld. Vier punten op een projectieve lijn hebben een dubbelverhouding. Als $[x_1, y_1], \dots, [x_4, y_4] \in \mathbb{P}^1$ de punten zijn, dan is de dubbelverhouding het volgende punt van $\mathbb{P}^1$:

$$\left[\det\begin{pmatrix} x_1 & y_1 \\ x_3 & y_3 \end{pmatrix} \cdot \det\begin{pmatrix} x_2 & y_2 \\ x_4 & y_4 \end{pmatrix}, \det\begin{pmatrix} x_1 & y_1 \\ x_4 & y_4 \end{pmatrix} \cdot \det\begin{pmatrix} x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}\right].$$

Voor elementen uit de de deelverzameling $U \subset (\mathbb{P}^1)^4$ waar de vier punten verschillend zijn hebben we: $P, Q \in U$ zitten in dezelfde baan dan en slechts dan als ze dezelfde dubbelverhouding hebben. Als echter drie van de vier punten samenvallen, is de rationale afbeelding $\phi : (\mathbb{P}^1)^4 \dashrightarrow \mathbb{P}^1$ niet gedefinieerd.

David Mumford heeft een methode uitgevonden om quotiënten te bepalen voor de actie van een reductieve groep G op een algebraïsche variëteit X. Deze staat beschreven in zijn beroemde boek *Geometric Invariant Theory*[2], [MFK94], vaak *GIT*

[2]Tegenwoordig zijn wellicht de zogenaamde *moduli stacks* populairder, maar in dit proefschrift houden we het bij de nog springlevende GIT.

genoemd. In zijn methode spelen *invarianten* een erg belangrijke rol. Als we een lijnenbundel $\mathcal{L}$ op X kiezen zodat de actie van G kan uitgebreid worden tot $\mathcal{L}$, dan zeggen we dat $f \in \Gamma(X, \mathcal{L}^{\otimes k})$ een invariant van graad k is als $f(g \cdot x) = f(x)$ voor alle $x \in X$ en $g \in G$. De quotiëntafbeelding $\phi : X \dashrightarrow X//G$ wordt gegeven door invarianten en is dus niet gedefinieerd in punten waar alle invarianten nul zijn. Deze punten heten *onstabiel*. Punten die niet onstabiel zijn heten *semi-stabiel*. In het voorbeeld van de dubbelverhouding zijn de configuraties van (tenminste) drie samenvallende punten onstabiel en de andere configuraties zijn semi-stabiel.

Hoewel er een quotiëntafbeelding $\phi : X \dashrightarrow X//G$ op de semi-stabiele locus gedefinieerd is, is het quotient $X//G$ niet overal een banenruimte. Beschouw in ons voorbeeld de verzameling van alle configuraties $(P_1, \ldots, P_4) \in (\mathbb{P}^1)^4$, zo dat $P_1 \neq P_2 = P_3 \neq P_4$. Het is duidelijk dat deze verzameling uit twee banen bestaat: de baan waar $P_1 = P_4$ en de baan waar $P_1 \neq P_4$, maar beide banen worden door ϕ op het punt $[1, 0] \in \mathbb{P}^1$ afgebeeld. In de theorie van Mumford is er een deelverzameling U van de semi-stabiele punten zo dat $\phi|_U : U \to X//G$ verschillende banen op verschillende punten afbeeldt. De punten in deze deelverzameling U heten *stabiel*. Zo zijn de stabiele configuraties van $(\mathbb{P}^1)^4$ de configuraties waar de vier punten verschillend zijn.

Er bestaan geen concepten in de Geometrische Invarianten Theorie (of in dit proefschrift) die belangrijker zijn dan stabiliteit en semi-stabiliteit. In plaats van puntconfiguraties doen we GIT voor de actie van $SL(n+1)$ op rijtjes lineaire deelruimten van $\mathbb{P}^n$, m.a.w. de variëteit waarop de speciale lineaire groep werkt is:

$$\mathcal{C} := \prod_{i=1}^{N} \mathrm{Gr}(d_i + 1, n + 1).$$

Hier bedoelen we met $\mathrm{Gr}(d_i + 1, n + 1)$ de Grassmannse variëteit, waarvan de elementen de $d_i + 1$-dimensionale lineaire deelruimten van een $n + 1$-dimensionale vectorruimte parametrizeren. Een element van $\mathcal{C}$ is een configuratie. Zoals is uitgelegd, moet men om quotiënten met goede eigenschappen te krijgen de onstabiele configuraties uitsluiten. Men zou kunnen zeggen dat de onstabiele configuraties in zekere zin ”te gedegenereerd” zijn. Mumford heeft in GIT al een handig criterium gegeven dat, in een vaste configuratieruimte $\mathcal{C}$, nagaat of een element van $\mathcal{C}$ onstabiel is of semi-stabiel, of zelfs stabiel. Wat men ook kan doen, is de configuratieruimten variëren. Men kan het aantal deelruimten N variëren, de dimensies d_i van de deelruimten en de dimensie n van de projectieve ruimte. Het is niet van tevoren duidelijk of zo'n configuratieruimte $\mathcal{C}$ een stabiel element bevat. Zo werd een natuurlijk doel in ons onderzoek om criteria op de configuratieruimten te geven voor het bestaan van stabiele (resp. semi-stabiele) punten. In deze richting hebben we enkele resultaten gehaald.

Het andere belangrijk thema in ons onderzoek is een involutie op rijtjes punten in projectieve ruimten genaamd *Gale dualiteit*. Gegeven een rijtje van N punten in $\mathbb{P}^n$ in voldoende algemene positie kan men homogene coördinaten kiezen zodat het gerepresenteerd wordt door de rijen van de matrix

$$\begin{pmatrix} I_{n+1} \\ A \end{pmatrix}.$$

De Gale duale van dit rijtje is dan gedefinieerd als het rijtje punten dat wordt gerepresenteerd door het de rijen van

$$\begin{pmatrix} A^T \\ I_{N-n-1} \end{pmatrix}.$$

Het was al bekend dat deze associatie bijzondere meetkundige eigenschappen heeft. Daarover schrijven Eisenbud en Popescu in het artikel *The projective geometry of the Gale transform*, [EP00]. Deze associatie van rijtjes punten kan asl volgt worden gegeneralizeerd tot een associatie van rijtjes lineaire deelruimten. Kies op ieder van de deelruimten lineair onafhankelijke punten die hem opspannen en beschouw het rijtje van al deze punten. Neem daarvan de Gale duale en maak dan een nieuw rijtje lineaire deelruimten door de opspansels te nemen van de geschikte deelverzamelingen van de Gale duale puntverzameling. Men kan nagaan dat deze constructie niet afhangt van de keuze van de punten die de deelruimten opspannen. Zo krijgt men dus een associatie tussen elementen van deze twee configuratieruimten:

$$\prod_{i=1}^{N} \mathrm{Gr}(d_i+1, n+1) \text{ and } \prod_{i=1}^{N} \mathrm{Gr}(d_i+1, \tilde{n}+1),$$

met

$$\tilde{n} = \sum_{i=1}^{N} (d_i+1) - n - 2.$$

Eisenbud en Popescu, en wij, vroegen ons af of deze generalizatie de een of andere meetkundige betekenis heeft. In dit proefschrift onderzoeken we deze vraag en we konden een eenvoudige meetkundige constructie geven van de de gegeneralizeerde Gale duale.

De Gale duale is gedefinieerd tot op lineaire transformaties. Daarom is het een natuurlijke vraag of er een isomorfisme bestaat tussen de moduliruimtes

$$\prod_{i=1}^{N} \mathrm{Gr}(d_i+1, n+1)//SL(n+1) \cong \prod_{i=1}^{N} \mathrm{Gr}(d_i+1, \tilde{n}+1)//SL(\tilde{n}+1).$$

(Let op: we zijn hier onzorgvuldig met onze notatie, omdat het quotiënt alleen maar genomen kan worden van de deelverzameling van de semi-stabiele punten van de configuratieruimtes.) In het geval van rijtjes van punten was deze vraag al bevestigend beantwoord. Wij hebben laten zien dat zo'n isomorfisme ook in het algemeen bestaat.

Samenvatting van dit proefschrift

Nu geven we in meer detail een samenvatting van de inhoud van elk van de vier hoofdstukken van dit proefschrift. Hoofdstuk 1 is een beknopte introductie van GIT bedoeld voor beginners. Het is een opsomming van bekende definities en stellingen zonder bewijzen. We leggen uit wat er bedoeld wordt met een goed quotiënt en dat GIT een manier is om goede quotïenten te maken voor acties van reductieve groepen op algebraïsche variëteiten. We leggen uit hoe je deze quotiënten construeert door invariantenringen te bekijken. Aan het eind van het hoofdstuk besteden we een beetje aandacht aan *linearizaties* van de groepsacties. Losjes gezegd is een linearizatie een keuze van de inbedding in de projectieve ruimte van de variëteit waarop de groep werkt.

In hoofdstuk 2 voeren we de configuratieruimtes van lineaire deelruimtes in. We herhalen (het al bekende) criterium van Mumford om (semi-)stabiliteit te testen. In zekere zin meet dit criterium de mate van gedegenereerdheid van een configuratie. Het zegt dat een rijtje van lineaire deelruimten onstabiel is als de doorsnijding met willekeurige andere lineaire deelruimten "te groot" is. Zo is bijvoorbeeld een rijtje van vijf lijnen in $\mathbb{P}^4$ onstabiel als er een lijn bestaat die deze vijf lijnen doorsnijdt. Met behulp van het criterium van Mumford kan men heel goed stabiliteit nagaan, zelfs in ingewikkelde configuratieruimten, terwijl het berekenen van de corresponderende moduli ruimte in het algemeen bijna onmogelijk is (behalve in enkele kleine gevallen). Dolgachev en Ortland hebben een uitgebreide studie gemaakt van het geval van puntconfiguraties in *Point sets in projective spaces and Theta functions*, [DO88]. De rest van het hoofdstuk hebben wij besteed aan enkele berekenbare voorbeelden van moduli ruimten can configuraties van punten en hypervlakken. De moduli ruimte van configuraties van drie punten en drie lijnen in het vlak is erg mooi. Het is de torische variëteit in $\mathbb{P}^5$ die wordt gegeven door de vergelijking

$$XYZ = UVW.$$

We bekijken de natuurlijke birationale afbeelding naar de moduli ruimte van zes punten in het vlak, de dubbele overdekking van $\mathbb{P}^4$ vertakt langs de (bekende) *Igusa quartic*.

Hoofdstuk 3 gaat over Gale dualiteit. We geven een aantal equivalente definities van de gegeneralizeerde Gale transformatie. Een nodige voorwaarde voor de bewering dat Gale dualiteit een isomorfisme induceert tussen de corresponderende moduli ruimten is de bewering dat een configuratie stabiel (resp. semi-stabiel) is dan en slechts dan als zijn Gale duale stabiel (resp. semi-stabiel) is. We konden deze bewering bewijzen voor de standaard linearizatie, voordat we het gewenste isomorfisme van moduli ruimten konden bewijzen. De sleutel tot het bewijzen van het isomorfisme is de *Gelfand-MacPherson correspondentie*, die een verband legt tussen zekere lineaire deelruimten tot op de actie van een torus en rijtjes punten tot op lineaire actie. Dit kan gegeneralizeerd worden, waarbij de torus actie wordt vervangen door een actie van bepaalde blokmatrices en de rijtjes punten door rijtjes lineaire deelruimten. Door deze correspondentie te combineren met de gewone dualiteit in projective ruimten krijgen we het gegeneralizeerde Gale dualiteit isomorfisme. Dit werd onafhankelijk van ons ook ontdekt en bewezen door Yi Hu, [Hu05]. Een ander belangrijk resultaat in dit hoofdstuk is een meetkundige constructie voor de gegeneralizeerde Gale dualiteit, waar niks anders bij komt kijken dan het nemen van opspansels, doorsnijdingen en projecties in vectorruimten. Aan het eind van het hoofdstuk gaan we in op *quivers*. We sturen aan op het formuleren van de stelling van *Domokos-Zubkov*, die alle semi-invarianten van quivers geeft. Een speciaal geval van deze stelling geeft wel alle invarianten van configuratieruimten, maar bepaalt niet de structuur van hun invariantenringen.

Hoofdstuk 4 gaat vooral over de vraag welke configuratieruimten stabiele (resp. semi-stabiele) elementen hebben. Het eerste dat we ons realizeerden was dat er voor een vaste configuratieruimte $\mathcal{C}$ een algoritme bestaat om te controleren of hij stabiele (resp. semi-stabiele) elementen bevat. We beschrijven dit algoritme, dat gebruik maakt van Schubert calculus. Daarna bekeken we vooral het speciale geval van configuraties waarvan alle deelruimten dezelfde dimensie hebben, i.e. ruimten van de vorm

$$\mathrm{Gr}(d+1, n+1)^N.$$

We konden bewijzen dat er voor iedere $d, n \in \mathbb{N}$ er een $N_0 \in \mathbb{N}$ (resp. een $N_1 \in \mathbb{N}$) bestaat zodat $\mathrm{Gr}(d+1, n+1)^N$ stabiele (resp. semi-stabiele) punten bevat dan en slechts dan als $N \geq N_0$ (resp $N \geq N_1$). Daarna was het doel van onze onderzoekingen het bepalen van de waarden van de functie N_0. Door de vele isomorfismes tussen onze moduli ruimten (Gale dualiteit en de gebruikelijke dualiteit in projectieve ruimten), is het mogelijk om de waarden van N_0 tamelijk goed te schatten.

Ook schenken we enige aandacht aan quotiënten van vier medialen. Dat is het geval dat $N = 4$ en $n = 2d + 1$. Deze ruimten hebben de speciale eigenschap dat er geen stabiele punten in bestaan, terwijl ze wel een goed quotiënt hebben.

Aan het einde van het hoofdstuk, en van het proefschrift, bekijken we *stabiele resoluties*. Een stabiele resolutie van een variëteit X met daarop een werking van een reductieve groep G is, losjes gezegd, een surjectief equivariant morfisme $Y \to X$ dat de zuiver semi-stabiele locus opheft. Zuiver semi-stabiele elementen zijn semi-stabiele elementen die niet stabiel zijn. Dit zijn precies de elementen van X die verhinderen dat $X//G$ een banenruimte is. We schetsen hoe je de strict semi-stabiele elementen van configuratieruimten zou moeten opheffen. Voor het geval van verzamelingen van punten is de stabiele resolutie een generalizatie van een Fulton-MacPherson compactificatie.

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Curriculum Vitae

Op 25 juli 1976 werd ik, Erik Reuvers, geboren in Eindhoven als eerste kind van Henny Reuvers en Marian Hendriks. Ik doorliep de lagere school tot in de vijfde klas in Eindhoven. In 1987 verhuisde ik met mijn familie naar Maastricht, waar ik het laatste gedeelte van de lagere school doorliep. Op mijn middelbare school, het St.-Maartencollege te Maastricht, vond ik elk vak interessant en wiskunde leek me het leukst. Daarom verhuisde ik in 1994 naar Nijmegen om wiskunde te studeren. Ook tijdens mijn studie vond ik elk vak interessant en de afstudeerrichting meetkunde trok mijn aandacht het meest. Ik volgde bij prof. dr. Jozef Steenbrink enkele meetkunde vakken en schreef bij hem mijn afstudeerscriptie "Een compactificatie van configuratieruimten". In 2000 studeerde ik cum laude af en aansluitend begon ik mijn promotieonderzoek met als begeleider wederom Jozef Steenbrink. Het resultaat van dit onderzoek is vastgelegd in het proefschrift dat u in uw hand heeft.

Dankwoord

Bij deze gelegenheid wil ik eenieder bedanken die aan mijn geluk heeft bijgedragen door zijn aanwezigheid in mijn leven. Zonder jullie had ik geen proefschrift kunnen schrijven. In het bijzonder bedank ik mijn begeleider Jozef Steenbrink voor alle hulp, voor zijn aanstekelijk enthousiasme, zijn geduld en vertrouwen. Ik dank mijn collega's Orsola en Martijn voor hun prettige aanwezigheid bij meetkundige gelegenheden. Martijn wil ik ook bedanken voor het regelmatig bij mij binnenlopen om over meetkunde of andere interessante dingen te praten. Ik dank Trees en Willy voor hun interesse en hulp bij tal van zaken. Ik dank mijn kamergenoten Danny, Paul, Michiel en Pim ieder voor hun eigen manier van aanwezig zijn, wat altijd een prettige ervaring was. Dennis, Frans, Jan-Willem, Joost, Matthijs, Paul, Pieter, Stefan en Stan bedank ik voor de fijne tijd die we op allerlei manieren samen hebben doorgebracht. Veel dank voor jullie onmisbare vriendschap en steun! Ik dank mijn vader en mijn moeder voor alles wat ze me hebben gegeven. Mijn vader dank ik voor de leuke puzzels die hij me vanaf mijn vijfde levensjaar zo nu en dan voorschotelt. Michiel, Frank en Suzanne: bedankt dat jullie er zijn! Por fín, la persona más importante eres tú, Claudia. Gracias. Eres mi vida.