Polynomial Maps and a Conjecture of Samuelson

Arno van den Essen
Department of Mathematics
Catholic University Nijmegen
Toernooiveld, Nijmegen, The Netherlands

and

T. Parthasarathy
Indian Statistical Institute
Delhi Centre
New Delhi-110016, India

Submitted by Robert M. Guralnick

ABSTRACT

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-mapping. It was conjectured by Samuelson that if the upper left-hand principal minors of the Jacobian of F do not vanish on $\mathbb{R}^n$, then F is injective. However, in 1965 Gale and Nikaido gave a simple counterexample to the case $n = 2$. In this paper we show that the Samuelson conjecture is true for polynomial mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$. Furthermore, we give a precise description of such maps.

INTRODUCTION

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map. Then the local inverse-function theorem asserts that F is locally invertible in a neighborhood of a point $p \in \mathbb{R}^n$ if and only if $\det(JF)(p) \neq 0$. It is much more difficult to decide if a C^1-map $F : \mathbb{R}^n \to \mathbb{R}^n$ is globally invertible. Suppose F is (globally) invertible. Then certainly $\det(JF)(p) \neq 0$ for all $p \in \mathbb{R}^n$. Conversely one can ask: if the Jacobian of F does not vanish on $\mathbb{R}^n$, is F globally invertible? Already for $n = 1$ we have a counterexample, namely $F(x) = e^x$. A criterion for global invertibility of C^1-functions is not known (as far as we know); however, several sufficient conditions for global injectivity are known in the literature

655 Avenue of the Americas, New York, NY 10010 0024-3795/92/$5.00

[4, 6, 7]. Most of these results were initiated by the following conjecture, due to Samuelson [9]: if all minors $\det[(\partial F_i/\partial x_j)(p))_{1 \leqslant i,j \leqslant r}]$ do not vanish for all $p \in \mathbb{R}^n$ and all $1 \leqslant r \leqslant n$, then F is globally injective. The following counterexample was given by Gale and Nikaido in [4]: take $F = (f, g)$ with $f = e^{2x} - y^2 + 3$, $g = 4e^{2x}y - y^3$. Then $\partial_x f > 0$ and $\det JF > 0$ on $\mathbb{R}^2$. However, F is not injective, since $F(0, \pm 2) = (0, 0)$.

Polynomial mappings behave much better. First there is the following result, due to Bialynicki-Birula and Rosenlicht [2], asserting that every injective polynomial map from $\mathbb{R}^n$ to $\mathbb{R}^n$ is necessarily surjective. The inverse of such a mapping is in general not a polynomial mapping [for example take $F(x) = x + x^3$]. One can wonder if the Samuelson conjecture holds for polynomial mappings; we don't know the answer. In fact the Samuelson conjecture for polynomial mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$ is a special case of the so-called real-Jacobian conjecture (see [5]), which asserts that even the condition $\det JF(p) \neq 0$ for all $p \in \mathbb{R}^n$ is sufficient to prove that F is globally injective.

Even better behaved are the polynomial mappings $F : \mathbb{C}^n \to \mathbb{C}^n$. Here the Bialynicki-Birula–Rosenlicht result asserts that an injective polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$ is necessarily invertible with an inverse which is also a polynomial mapping.

The aim of this note is to show that the Samuelson conjecture is true for polynomial mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$. In fact we show that a polynomial mapping from $\mathbb{C}^n$ to $\mathbb{C}^n$ satisfying the hypothesis of the Samuelson conjecture is a product of n elementary polynomial mappings and a (linear) diagonal map (see Theorem 2.1). Of course the Samuelson conjecture for polynomial mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$ is just a very special case of the famous Jacobian conjecture, which asserts that the nonvanishing of $\det JF$ on $\mathbb{C}^n$ implies by itself the injectivity (and hence the invertibility) of F. The Jacobian conjecture is still open for all $n \geqslant 2$ (see [1, 3, 8] for more details).

1. NOTATIONS AND PRELIMINARIES

Let A be a commutative ring. By $A[X] := A[X_1, \ldots, X_n]$ we denote the polynomial ring in n variables $X_1, \ldots, X_n$ with coefficients in A. On $A[X]$ we have the usual A-derivations $\partial/\partial X_j$ for all $1 \leqslant j \leqslant n$. Let $F = (F_1, \ldots, F_n) : A^n \to A^n$ be a polynomial map, i.e., each F_i belongs to $A[X]$. The Jacobian matrix of F, defined by the $n \times n$ matrix $(\partial F_i/\partial X_j)_{1 \leqslant i,j \leqslant n}$, will be denoted by either JF or $(JF)(X)$ or $J(F_1, \ldots, F_n; X_1, \ldots, X_n)$. Furthermore, for each $1 \leqslant p \leqslant n$ we denote by $|JF|_p$ or $|JF|_p(X)$ the determinant of the $p \times p$ submatrix $(\partial F_i/\partial X_j)_{1 \leqslant i,j \leqslant p}$ of JF. If $G =$

$(G_1, \ldots, G_n)$ is another polynomial map, the Jacobian matrix of the composed map $F \circ G$ is given by the chain rule, i.e.

$$J(F \circ G) = (JF)(G) \cdot JG. \tag{1.1}$$

Let $1 \leqslant i \leqslant n$. A polynomial map $E = (E_1, \ldots, E_n)\colon A^n \to A^n$ is called *elementary* (of type i) if $E_j = X_j$ for all $j \neq i$ and $E_i - X_i$ does not contain X_i.

LEMMA 1.2. *Let $F = (F_1, \ldots, F_n)\colon A^n \to A^n$ be a polynomial map, and E an elementary polynomial map of type 1. Then $|JF \circ E|_p = |JF|_p(E)$ for all $1 \leqslant p \leqslant n$.*

Proof. Let $1 \leqslant p \leqslant n$. If $p = n$, the result follows from (1.1), since $|JE|_n = 1$.

The case $p < n$ we reduce to the case $p = n$. Therefore define $\tilde{F} := (F_1, \ldots, F_p, X_{p+1}, \ldots, X_n)$. Observe that $(F_1(E), \ldots, F_p(E), X_{p+1}, \ldots, X_n) = \tilde{F} \circ E$, since E is of type 1. So we get

$$|J(F \circ E)|_p = |J(F_1(E), \ldots, F_p(E), X_{p+1}, \ldots, X_n)| = |J(\tilde{F} \circ E)|_n \text{ [by (1.1)]}$$

$$= |J\tilde{F}|_n(E) \cdot |JE|_n = |JF|_p(E),$$

since $|J\tilde{F}|_n = |JF|_p$ and $|JE|_n = 1$. ■

2. A POLYNOMIAL VERSION OF THE SAMUELSON CONJECTURE

Throughout this section A is a commutative ring of characteristic zero, i.e., $na \neq 0$ for all $a \in A \setminus \{0\}$ and all $n \in \mathbb{Z} \setminus \{0\}$. The set of units of A we denote by A^*.

THEOREM 2.1. *Let $F = (F_1, \ldots, F_n)\colon A^n \to A^n$ be a polynomial map such that $|JF|_i \in A^*$ for all $1 \leqslant i \leqslant n$. Then $F = D \circ E_{(n)} \circ \cdots \circ E_{(1)}$, where each $E_{(p)}$ is an elementary polynomial map of type p, and D is the diagonal map defined by $D = (d_1 X_1, \ldots, d_n X_n)$, where $d_1 = |JF|_1$ and $d_i = |JF|_i |JF|_{i-1}^{-1}$ for all $2 \leqslant i \leqslant n$.*

Proof.

(i) Put $c_i := |JF|_i$. So $c_i \in A^*$ for all $1 \leqslant i \leqslant n$. Let D' be the diagonal map $(c_1^{-1} X_1, c_2^{-1} c_1 X_2, \ldots, c_n^{-1} c_{n-1} X_n)$. One readily verifies that $|JD' \circ F|_i = 1$ for all $1 \leqslant i \leqslant n$. So replacing F by $D' \circ F$, we may assume that $|JF|_i = 1$ for all i.

(ii) From $|JF|_1 = 1$ we get $F_1 = X_1 + f_1$ for some $f_1 \in A[X_2, \ldots, X_n]$. Put $E_{(1)} = (X_1 + f_1, X_2, \ldots, X_n)$, we define $G := F \circ E_{(1)}^{-1}$. Then $G = (G_1, \ldots, G_n)$ with $G_1 = X_1$. Furthermore $|JG|_p = 1$ for all $1 \leqslant p \leqslant n$ by Lemma 1.2.

(iii) Consider $G_2, \ldots, G_n$ as elements of $A'[X_2, \ldots, X_n]$, where $A' := A[X_1]$, and define the polynomial map $G^* := (G_2, \ldots, G_n)$ from A'^{n-1} to A'^{n-1} (in the variables $X_2, \ldots, X_n$). Now observe that

$$\det J(G_2, \ldots, G_p; X_2, \ldots, X_p) = |JG|_p \qquad \text{for all} \quad 2 \leqslant p \leqslant n$$

(since $G_1 = X_1$). By (ii) we know that $|JG|_p = 1$; hence $\det J(G_2, \ldots, G_p; X_2, \ldots, X_p) = 1$ for all $2 \leqslant p \leqslant n$. By induction on the number of variables it follows that $(G_2, \ldots, G_n) = E^*_{(n)} \circ \cdots \circ E^*_{(2)}$, where $E^*_{(p)} = (X_2, \ldots, X_{p-1}, X_p + f_p, X_{p+1}, \ldots, X_n)$ with $f_p \in A'[X_2, \ldots, \hat{X}_p, \ldots, X_n]$ for all $2 \leqslant p \leqslant n$. Since $G = (X_1, G_2, \ldots, G_n)$, it follows that $G = E_{(n)} \circ \cdots \circ E_{(2)}$, where $E_{(p)} = (X_1, E^*_{(p)}) = (X_1, X_2, \ldots, X_{p-1}, X_p + f_p, X_{p+1}, \ldots, X_n)$. Together with $G = F \circ E_{(1)}^{-1}$ and (i), this completes the proof. ■

COROLLARY 2.2 (Polynomial version of the Samuelson conjecture). *If $F : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map such that $|JF|_i(p) \neq 0$ for all $p \in \mathbb{C}^n$ and all $1 \leqslant i \leqslant n$, then F is invertible (hence injective).*

Proof. Since each nonconstant polynomial in $\mathbb{C}[X]$ has a zero in $\mathbb{C}^n$, it follows that $|JF|_i \in \mathbb{C}^*$ for all $1 \leqslant i \leqslant n$. Then apply Theorem 2.1. ■

The second author thanks the University of Nijmegen for hospitality during his stay.

REFERENCES

1 H. Bass, E. H. Connell, and D. Wright, The Jacobian conjecture: Reduction of degree and formal expension of the inverse, *Bull. Amer. Math. Soc.* 7:287–329 (1982).

2 A. Bialynicki-Birula and M. Rosenlicht, Injective morphisms of real algebraic varieties, *Proc. Amer. Math. Soc.* 13:200–203 (1962).

3 A. van den Essen, Polynomial maps and the Jacobian conjecture, in *Proceedings of the 1990 Computational Algebra Seminar*, CWI Tract 84, 1991, pp. 29–44.

4 D. Gale and H. Nikaido, The Jacobian matrix and global univalence of mappings, *Math. Ann.* 159:81–93 (1965).

5 K. Kurdyka and K. Rusek, Polynomial-rational bijections of $\mathbb{R}^n$, *Proc. Amer. Math. Soc.* 102:804–808 (1988).
6 C. Olech, T. Parthasarathy, and G. Ravindran, Almost N-matrices and their applications to the linear complementarity problem, *Linear Algebra Appl.* 145:107–125 (1990).
7 T. Parthasarathy, *On Global Univalence Theorems*, Lecture Notes in Math. 977, Springer-Verlag, New York, 1983.
8 K. Rusek, Polynomial Automorphisms, Preprint 456, Polish Acad. of Science, 1989.
9 P. A. Samuelson, *Prices of Factors and Goods in General Equilibrium*, *Rev. Econom. Stud.* 21, 1953.

Received 7 October 1991; final manuscript accepted 21 November 1991