Univalent functions with univalent sections

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1 Introduction

Let S denote the collection of univalent analytic functions f on the unit disc Δ, normalized such that $f(0) = 0$, and $f'(0) = 1$. Already in 1928 Szegö [6] proved that all the sections

$$z + \sum_{k=2}^{n} a_k z^k \qquad (n = 2, 3, 4, \ldots)$$

of the power series of f are univalent on the disc $\Delta_{\frac{1}{4}}$. For $n > 2$ the sections are actually univalent on a larger disc, and if f has some additional properties (e.g. if f is convex, or star-like) then some of these properties are inherited by the sections. For more details we refer to [1] p.243-246.

In this paper we shall investigate the collection $V \subset S$ consisting of those functions f for which all sections

$$z + \sum_{k=2}^{n} a_k z^k \qquad (n = 2, 3, 4. \ldots)$$

are univalent on the whole unit disc Δ. It is surprising to see how many "every-day" functions belong to V. Of course, V is a closed subset of the compact set S; therefore, V is also compact.

2 Elementary results

Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. For $n = 2, 3, 4, \ldots$ we denote

$$f_n(z) = z + \sum_{k=2}^{n} a_k z^k.$$

If $f \in V$, i.e. if $f_n \in S$ $(n = 2, 3, \ldots)$ then f'_n has no zeros on Δ. Thus the absolute value of the product of the zeros of

$$f'_n(z) = 1 + 2a_2 z + \cdots + n a_n z^{n-1}$$

is at least 1, i.e.

$$|a_n| \leq \frac{1}{n}.$$

This inequality is best possible: the function

$$b_n : z \to z + \frac{1}{n} z^n$$

belongs to V since for $z \neq w$ we have

$$\begin{aligned} \left| \tfrac{b_n(z) - b_n(w)}{z - w} \right| &= \left| 1 + \tfrac{1}{n} \left(z^{n-1} + z^{n-2} w + \cdots + w^{n-1} \right) \right| \\ &\geq 1 - \tfrac{1}{n} \left(|z|^{n-1} + \cdots + |w|^{n-1} \right) \\ &> 1 - \tfrac{1}{n}(1 + \cdots + 1) = 0. \end{aligned}$$

Note that all the functions b_n are star-like ($b_n \in S^*$) since

$$z \frac{b_n'(z)}{b_n(z)} = \frac{1 + z^{n-1}}{1 + \frac{1}{n} z^{n-1}}$$

maps Δ onto the disc with center $\frac{n}{n+1}$ and radius $\frac{n}{n+1}$, so

$$Re\, z \frac{b_n'(z)}{b_n(z)} > 0.$$

The necessary condition

$$|a_n| \leq \frac{1}{n}$$

is not sufficient. The function

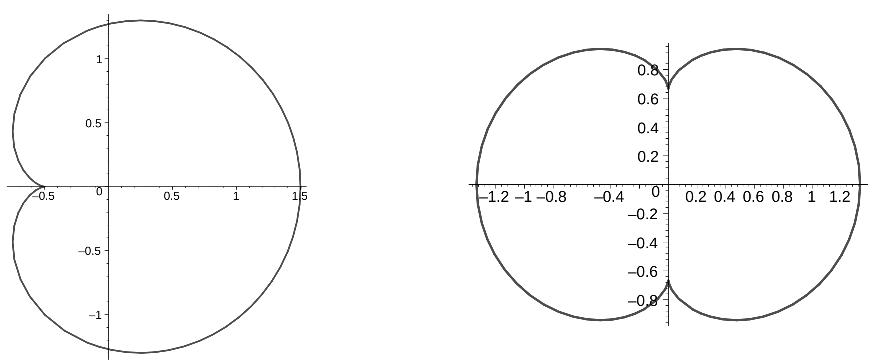
$$f : z \to z + \tfrac{1}{2} z^2 - \tfrac{1}{3} z^3$$

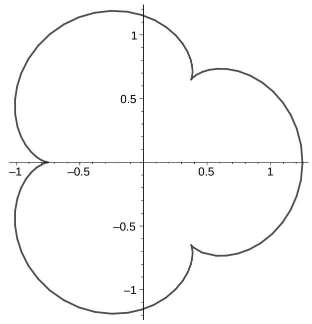
is not injective on Δ since

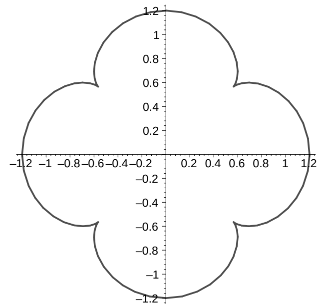
$$f' : z \to 1 + z - z^2$$

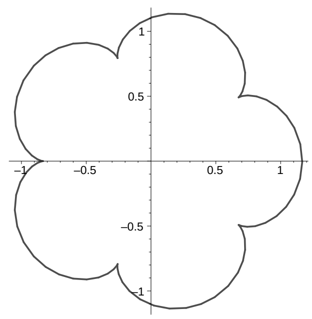
has a zero at $\frac{1}{2} - \frac{1}{2}\sqrt{5}$.

The illustration shows the image domains $b_n(\Delta)$ for $n = 2, 3, 4, 5$ and 6.







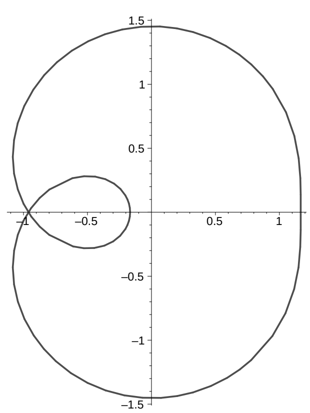


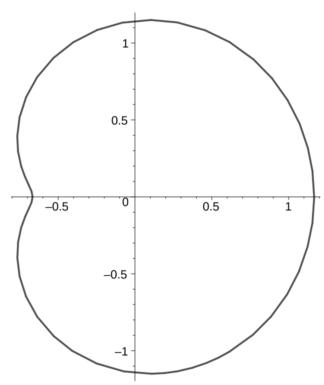
If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and if $|a_n| \le \frac{1}{n}$ then f is injective on $\Delta_{\frac{1}{2}}$; for $z, w \in \Delta_{\frac{1}{2}}$, $z \neq w$ we have

$$\left| \frac{z^n - w^n}{z - w} \right| < n \left(\tfrac{1}{2}\right)^{n-1}$$

hence

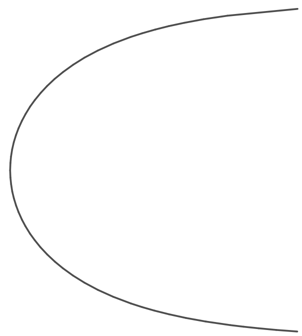
$$\left| \frac{f(z) - f(w)}{z - w} \right| = \left| 1 + \sum_{n=2}^{\infty} a_n \frac{z^n - w^n}{z - w} \right| > 1 - \sum_{2}^{\infty} \left(\tfrac{1}{2}\right)^{n-1} = 0.$$





By far the most important example is the function

$$l : z \to -\log(1-z) = \sum_{k=1}^{\infty} \frac{z^k}{k}.$$



Theorem 2.1 $l \in V$.

Proof. We show first that $l \in K$ (the collection of convex functions). A necessary and sufficient condition is $zl'(z) \in S^*$ and this is evident from

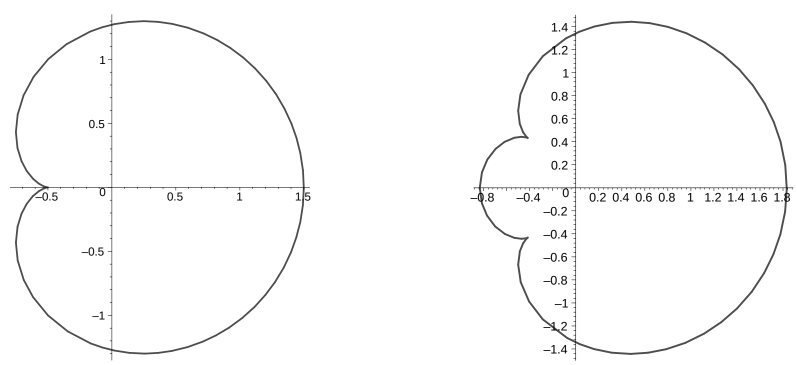
$$zl'(z) = \frac{z}{1-z}.$$

We even have $zl'(z) \in K$. The fact that $l \in K$ enables us to show that its section l_n are close-to-convex.

$$\frac{l_n'(z)}{l'(z)} = (1-z)\sum_{k=1}^{n} z^{k-1} = 1 - z^n,$$

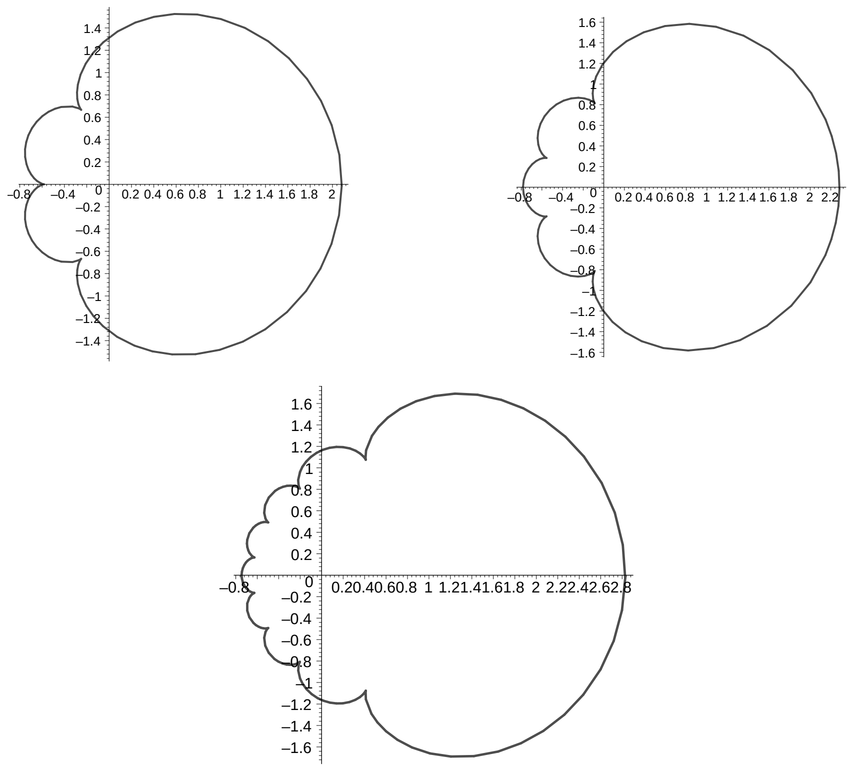
and since $Re(1-z^n) > 0$, we conclude that $l_n \in C \subset S$. ∎

The illustration shows the image domains $l_n(\Delta)$ for $n = 2, 3, 4, 5$ and 9.



Corollary 2.1 *If $f \in V$, then*

$$\begin{aligned} |f(z)| &\le l(|z|) = -\log(1-|z|), \\ |f'(z)| &\le l'(|z|) = \tfrac{1}{1-|z|}. \end{aligned}$$



The proof follows immediately from the fact that for every $f \in V$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, we already know that $|a_k| \leq \frac{1}{k}$. ■

Since $V \subset S$ we have of course the general distortion inequalities

$$\begin{aligned} |f(z)| &\geq \tfrac{|z|}{(1+|z|)^2}, \\ |f'(z)| &\geq \tfrac{1-|z|}{(1+|z|)^3}. \end{aligned}$$

For most examples we have much better lower bounds. A small improvement for all $f \in V$ can be obtained as a consequence of the following refinement of Koebe's $\frac{1}{4}$-theorem.

Theorem 2.2 *If $f \in V$, then $\Delta_{\frac{2}{5}} \subset f(\Delta)$.*

Proof. If $w \notin f(\Delta)$, then $\frac{wf}{w-f} \in S$. The power series for this function is

$$z + \left(a_2 + \frac{1}{w}\right) z^2 + \cdots$$

From $|a_2 + \frac{1}{w}| \leq 2$ and $|a_2| \leq \frac{1}{2}$ we obtain that

$$\left|\frac{1}{w}\right| \leq |a_2| + \left|a_2 + \frac{1}{w}\right| \leq \tfrac{5}{2},$$

i.e.

$$|w| \geq \frac{2}{5}.$$

This bound is not sharp. Equality would imply that f is a rotation of

$$\frac{z}{1+\frac{1}{2}z+z^2} = z - \tfrac{1}{2}z^2 - \tfrac{3}{4}z^2 + \cdots$$

but this function is not in V since $|a_3| = \frac{3}{4} > \frac{1}{3}$. ■

Application of this result to $z \to \frac{1}{r}f(rz)$ (with $0 < r \leq 1$) shows that

$$\left|\frac{1}{r}f(re^{it})\right| \geq \tfrac{2}{5},$$

i.e.

$$\left|f(re^{it})\right| \geq \tfrac{2}{5}r,$$

i.e.

$$|f(z)| \geq \tfrac{2}{5}|z|.$$

Thus we have:

$$|f(z)| \geq \begin{cases} \frac{|z|}{(1+|z|)^2} & |z| \leq \frac{1}{2}\sqrt{10}-1, \\ \frac{2}{5}|z| & |z| > \frac{1}{2}\sqrt{10}-1. \end{cases}$$

3 Close-to-convex functions

In order to give a unified presentation of some more examples of functions from V we list some auxiliary results about close-to-convex functions. We recall the definition: $f \in H(\Delta)$ is close-to-convex (notation: $f \in C$) if there exists $\vartheta \in \mathbb{R}$ and $g \in K$ such that for all $z \in \Delta$

$$Re\, e^{i\vartheta}\frac{f'(z)}{g'(z)} > 0.$$

This is equivalent to: there exists $\vartheta \in \mathbb{R}$ and $h \in S^*$ such that for all $z \in \Delta$

$$Re\, e^{i\vartheta} z\frac{f'(z)}{h(z)} > 0.$$

It is not difficult to show that

$$K \subset S^* \subset C \subset S,$$

and that all these inclusions are proper. [6], p.8.

Lemma 3.1 *Let* $f : z \to z + \sum_{k=2}^{\infty} a_k z^k \in H(\Delta)$. *In each of the following cases we have* $f \in C$.

a) $Re\, f'(z) > 0$,

b) $Re(1-z)f'(z) > 0$,

c) $Re(1-z^2)f'(z) > 0$,

d) $Re(1-z)^2 f'(z) > 0$.

Proof. a) Take $\vartheta = 0$, $g(z) = z$; b) Take $\vartheta = 0$, $g(z) = l(z) = -\log(1-z)$; c) Take $\vartheta = 0$, $h(z) = \frac{z}{1-z^2}$; d) Take $\vartheta = 0$, $h(z) = \frac{z}{(1-z)^2}$. ■

Note that condition c) is satisfied if $\sum_0^\infty |ka_k - (k+2)a_{k+2}| \leq 1$, and that condition d) is satisfied if

$$a_0 > 0,\ a_1 = 1 \ \text{ and } \ \sum_0^\infty |ka_k - 2(k+1)a_{k+1} + (k+2)a_{k+2}| \leq 1.$$

Lemma 3.2 *Let* $f(z) = z + \sum_{k=2}^\infty a_k z^k$ *and let* $\sum_{k=2}^\infty k|a_k| \leq 1$.
Then $f \in S^*$ *and* $|f'(z)| \geq 1 - |z|$.

Proof.

$$\begin{aligned} |zf'(z) - f(z)| &= \left|\sum_{k=2}^\infty (k-1)a_k z^k\right| \leq \sum_{k=2}^\infty (k-1)|a_k| \cdot |z|^k \\ &= \sum_{k=2}^\infty k|a_k| \cdot |z|^k - \sum_{k=2}^\infty |a_k| \cdot |z|^k \leq |z| \sum_{k=2}^\infty k|a_k| - \sum_{k=2}^\infty |a_k| \cdot |z|^k \\ &\leq |z| - \sum_{k=2}^\infty |a_k| \cdot |z|^k \leq |f(z)|. \end{aligned}$$

Thus $\left|z\frac{f'(z)}{f(z)} - 1\right| \leq 1$, and so $Re\, z\frac{f'(z)}{f(z)} > 0$, i.e. $f \in S^*$.
Furthermore

$$|f'(z)| = \left|1 + \sum_{k=2}^\infty ka_k z^{k-1}\right| \geq 1 - \sum_{k=2}^\infty k|a_k| \cdot |z|^{k-1} \geq 1 - |z| \sum_{k=2}^\infty k|a_k| \geq 1 - |z|.$$

■

Lemma 3.3 *Let* $f(z) = \sum_{k=1}^\infty a_k z^k$ *with* $a_1 = 1$ *and let*

$$\sum_{k=2}^\infty |ka_k - (k+1)a_{k+1}| \leq 1.$$

Then $f \in C$. *(This holds in particular if* $1 \geq 2a_2 \geq 3a_3 \geq \ldots \geq 0$*).*
Moreover, if $a_2 = \frac{1}{2}$ *then* $|f'(z)| \geq 1 - |z|$.

Proof. $(1-z)f'(z) = 1 + \sum\limits_{k=1}^{\infty} \left((k+1)a_{k+1} - ka_k\right) z^k.$
We deduce that

$$Re\,(1-z)f'(z) \geq 1 - \sum_{k=1}^{\infty} |(k+1)a_{k+1} - ka_k| \cdot |z|^k > 0,$$

so from lemma 3.1 we conclude that $f \in C$.
If moreover $a_2 = \frac{1}{2}$, then

$$(1-z)f'(z) = 1 + \sum_{k=2}^{\infty} \left((k+1)a_{k+1} - ka_k\right) z^k,$$

so we have

$$\begin{aligned} |1-z| \cdot |f'(z)| &\geq 1 - \sum_{k=2}^{\infty} |(k+1)a_{k+1} - ka_k| \cdot |z|^2 \geq 1 - |z|^2 \\ &= (1+|z|)(1-|z|) \geq |1-z| \cdot (1-|z|), \end{aligned}$$

and so $|f'(z)| \geq 1 - |z|$. ■

Lemma 3.4 *Let* $f(z) = \sum\limits_{k=1}^{\infty} a_{2k-1} z^{2k-1}$ *with* $a_1 = 1$ *and let*

$$\sum_{k=1}^{\infty} |(2k+1)a_{2k+1} - (2k-1)a_{2k-1}| \leq 1.$$

Then $f \in C$ *and* $|f'(z)| \geq 1 - |z|$. *(This holds in particular if* $1 \geq 3a_3 \geq 5a_5 \geq \ldots \geq 0$*).*

Proof. $(1-z^2)f'(z) = 1 + \sum\limits_{k=1}^{\infty} \left((2k+1)a_{2k+1} - (2k-1)a_{2k-1}\right) z^{2k}.$
We deduce that

$$Re(1-z^2)f'(z) \geq 1 - \sum_{k=1}^{\infty} |(2k+1)a_{2k+1} - (2k-1)a_{2k-1}| \cdot |z|^{2k} > 0,$$

so from lemma 3.1 we conclude that $f \in C$.
We also see that

$$\begin{aligned} |1-z^2| \cdot |f'(z)| &\geq 1 - \sum_{k=1}^{\infty} |(2k+1)a_{2k+1} - (2k-1)a_{2k-1}| \cdot |z|^2 \geq 1 - |z|^2 \\ &= (1+|z|)(1-|z|) \geq (1+|z|^2)(1-|z|) \geq |1-z^2| \cdot (1-|z|), \end{aligned}$$

and so $|f'(z)| \geq 1 - |z|$. ■

Lemma 3.5 *Let* $g \in K$*; then* $|g(z)| \geq |z|(1-|z|)$.

Proof. The distortion theorem for convex functions [5] p. 12 implies that

$$|g(z)| \geq \frac{|z|}{1+|z|}$$

and since $(1+|z|)(1-|z|) \leq 1$, the assertion follows. ■

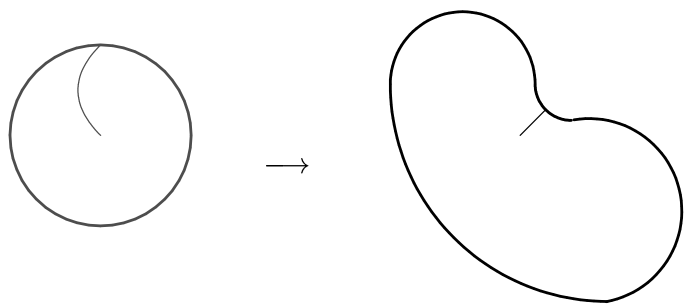
We finish this section with a lemma that gives a relation between the growth of a univalent function and its derivative.

Lemma 3.6 *Let $f \in S$ and let ρ be a continuous function on $[0,1]$ such that $|f'(z)| \geq \rho(|z|)$ for all $z \in \Delta$. Let R be the primitive of ρ for which $R(0)=0$. Then we have*

$$|f(z)| \geq R(|z|).$$

Proof. For a given $r \in (0,1)$, let z_0 be a point on Γ_r for which

$$|f(z_0)| = \min_{|z|=r}\{|f(z)|\}.$$



Let γ be the curve from 0 to z_0 that is mapped by f onto the segment $[0, f(z_0)]$. Then we have

$$f(z_0) = \int_\gamma f'(\zeta)d\zeta = \int_0^1 f'(\gamma(t)) \cdot \gamma'(t)dt = \int_0^1 (f(\gamma(t)))'\,dt.$$

Now $\arg\,(f(\gamma(t)))' = \arg f(z_0)$ independent of t.
Then

$$\begin{aligned} |f(z_0)| &= \textstyle\int_0^1 \left|(f(\gamma(t)))'\right| dt = \int_0^1 |f'(\gamma(t))| \cdot |\gamma'(t)| dt \\ &\geq \textstyle\int_0^1 \rho(|\gamma(t)|) \cdot |\gamma'(t)| dt \geq \left|\int_0^1 \rho(|\gamma(t)|)\gamma'(t)dt\right| \\ &= R(\gamma(1)) = R(|z_0|). \end{aligned}$$

■

4 Examples

We start with the remark that V is invariant under rotation and dilatation; more precisely, if $f \in V$ and $\zeta \in \overline{\Delta}\backslash\{0\}$ then

$$f_\zeta : z \to \frac{1}{\zeta} f(\zeta z) \in V.$$

Moreover, if we know that $|f'(z)| \geq 1-|z|$ (and hence by lemma 3.6 $|f(z)| \geq |z| - \frac{1}{2}|z|^2$) then we have

$$|f'_\zeta(z)| = |f'(\zeta z)| \geq 1 - |\zeta z| \geq 1 - |z|$$

and

$$|f_\zeta(z)| = \left|\frac{1}{\zeta}\right| \cdot |f(\zeta z)| \geq \left|\frac{1}{\zeta}\right| (|\zeta z| - \tfrac{1}{2}|\zeta z|^2) = |z| - \tfrac{1}{2}|\zeta|\,|z|^2 \geq |z| - \tfrac{1}{2}|z|^2.$$

1) We have already seen that for $n = 2, 3, 4, \ldots$ the function

$$b_n : z \to z + \frac{1}{n} z^n$$

belongs to V. We have also noticed that b_n is actually star-like and it is easy to see that

$$|b'_n(z)| \geq 1 - |z|,$$

and

$$|b_n(z)| \geq |z| - \tfrac{1}{2}|z|^2.$$

2) We have also seen that

$$l : z \to -\log(1-z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$$

belongs to V. We showed that $l \in K$ and its sections l_n are close-to-convex. It follows from lemma 3.3 that

$$|l'_n(z)| \geq 1 - |z| \quad \text{and} \quad |l'(z)| \geq 1 - |z|.$$

Therefore by lemma 3.6

$$|l_n(z)| \geq |z| - \tfrac{1}{2}|z|^2 \quad \text{and} \quad |l(z) \geq |z| - \tfrac{1}{2}|z|^2.$$

3) The function dilog is defined by

$$\text{dilog}(z) = \int_{[0,z]} -\frac{1}{\zeta} \log(1-\zeta) d\zeta = \sum_{k=1}^{\infty} \frac{z^k}{k^2};$$

dilog and its sections satisfy the conditions of lemma 3.3 so these functions are close-to-convex. Actually,

$$\text{dilog} = l * l$$

so by the Polya-Schoenberg theorem [5] p.27, dilog $\in K$. This can also be shown in the following elementary way

$$\text{dilog}(z) \in K \iff z(\text{dilog}(z))' \in S^*,$$

and this is obviously the case since

$$z(\text{dilog}(z))' = l(z) \in K.$$

From this relation follows that

$$|(\mathrm{dilog}(z))'| = |\frac{1}{z}l(z)| \geq 1 - \tfrac{1}{2}|z| \geq 1 - |z|,$$

and therefore by lemma 3.6

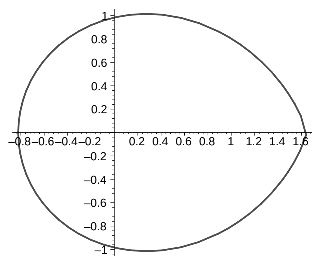
$$|\mathrm{dilog}(z)| \geq |z| - \tfrac{1}{4}|z|^2 \geq |z| - \tfrac{1}{2}|z|^2.$$

We shall show that the sections dilog_n satisfy similar inequalities:

$$|(\mathrm{dilog}_n(z))'| = |\frac{1}{z}l_n(z)| \geq 1 - \tfrac{1}{2}|z| \geq 1 - |z|.$$

Again by lemma 3.6

$$|\mathrm{dilog}_n(z)| \geq |z| - \tfrac{1}{4}|z|^2 \geq |z| - \tfrac{1}{2}|z|^2.$$



4) $g = -1 + \exp$ is the inverse of $l_{-1} : z \to -l(-z)$;

$$g(z) = \sum_{k=1}^{\infty} \frac{z^k}{k!}.$$

By lemma 3.3, g and the sections g_n are close-to-convex. Therefore $g \in V$. Also by lemma 3.3

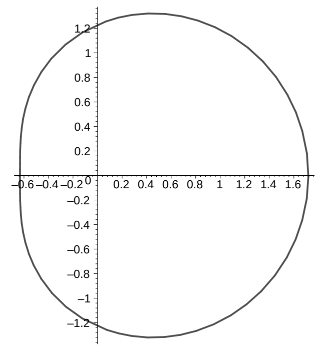
$$|g_n'(z)| \geq 1 - |z| \quad \text{and} \quad |g'(z)| \geq 1 - |z|,$$

and by lemma 3.6

$$|g_n(z)| \geq |z| - \tfrac{1}{2}|z|^2 \quad \text{and} \quad |g(z)| \geq |z| - \tfrac{1}{2}|z|^2.$$

The function g is even convex since

$$Re\left(1 + z\frac{g''(z)}{g'(z)}\right) = Re(1 + z) > 0.$$



5) In order to show that arctan $\in V$ we consider the rotated function

$$\arctan_i : z \to -i \arctan(iz) = \sum_{k=1}^{\infty} \frac{z^{2k-1}}{2k-1} = \tfrac{1}{2}(l(z) - l(-z)).$$

As a consequence of lemma 3.4 this function and the sections are close-to-convex, and this shows that arctan $\in V$. Lemma 3.4 also shows that

$$|\arctan_n'(z)| \geq 1 - |z|, \quad |\arctan'(z)| \geq 1 - |z|$$

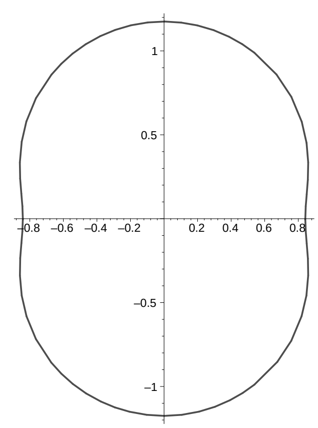
and by lemma 3.6

$$|\arctan_n(z)| \geq |z| - \tfrac{1}{2}|z|^2, \quad |\arctan(z)| \geq |z| - \tfrac{1}{2}|z|^2.$$

Since arctan maps Δ onto the vertical strip $\{z : -\frac{\pi}{4} < Rez < \frac{\pi}{4}\}$ we see that arctan $\in K$.

6) sin $\in V$. Application of lemma 3.2 shows that sin and all its sections are star-like. Moreover we obtain the lower bounds

$$\begin{aligned} |\sin_n'(z)| &\geq 1 - |z|, & |\sin'(z)| &\geq 1 - |z|, \\ |\sin_n(z)| &\geq |z| - \tfrac{1}{2}|z|^2, & |\sin(z)| &\geq |z| - \tfrac{1}{2}|z|^2. \end{aligned}$$



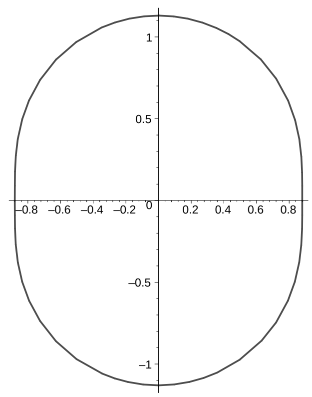
7) Another application of lemma 3.2 shows that $2J_1 \in V$.

$$J_1(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!} \left(\frac{z}{2}\right)^{2k+1}$$

so our assertion follows from

$$\begin{aligned} 2\sum_{k=1}^{\infty} \tfrac{2k+1}{2^{2k+1}k!(k+1)!} &\leq \sum_{k=1}^{\infty} \tfrac{1}{2^{2k-1}(k!)^2} = \tfrac{1}{2} + \sum_{k=2}^{\infty} \tfrac{1}{2^{2k-1}(k!)^2} \\ &\leq \tfrac{1}{2} + \tfrac{1}{8}\sum_{k=1}^{\infty} \tfrac{1}{(k!)^2} < \tfrac{1}{2} + \tfrac{1}{8}\sum_{k=2}^{\infty} \tfrac{1}{k!} < \tfrac{1}{8} + \tfrac{e}{8} < 1. \end{aligned}$$

Lemma 3.2 also provides us with the estimate $|2J_1'(z)| \geq 1 - |z|$, so by lemma 3.6 we have $|2J_1(z)| \geq |z| - \frac{1}{2}|z|^2$. We have of course similar estimates for the sections.



8) arcsin $\in V$. The power series

$$\arcsin(z) = \sum_{k=0}^{\infty} \frac{1.3.5\ldots(2k-1)}{2^k.k!} \cdot \frac{z^{2k+1}}{2k+1}$$

satisfies the conditions of lemma 3.4. Therefore arcsin and all the sections are close-to-convex so arcsin $\in V$. Furthermore we have

$$|\arcsin'(z)| \geq 1 - |z|, \quad |\arcsin(z)| \geq |z| - \tfrac{1}{2}|z|^2$$

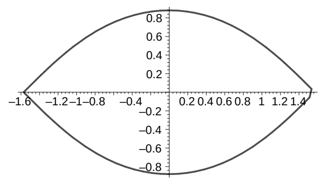
and similar estimates for the sections. Note that arcsin is convex:

$$1 + z\frac{\arcsin''(z)}{\arcsin'(z)} = \frac{1}{1-z^2}$$

has positive real part.
9) Let $0 < \beta \leq 1$ and let

$$f_\beta(z) = \frac{1}{\beta}\{1 - (1-z)^\beta\}.$$

$f_\beta(z) = z + \frac{1-\beta}{1} \cdot \frac{z^2}{2} + \frac{1-\beta}{1} \cdot \frac{2-\beta}{2} \cdot \frac{z^3}{3} + \frac{1-\beta}{1} \cdot \frac{2-\beta}{2} \cdot \frac{3-\beta}{3} \cdot \frac{z^4}{4} + \cdots$.
From lemma 3.3 we see that f_β and its sections belong to C.



Since $a_2 \neq \frac{1}{2}$ we cannot apply lemma 3.3 to obtain estimates for the derivative. Nevertheless we have

$$|f_\beta'(z)| = \frac{1}{|1-z|^{1-\beta}} \geq \frac{1}{(1+|z|)^{1-\beta}} = \frac{(1+|z|)^\beta}{1+|z|} \geq \frac{1}{1+|z|} \geq 1-|z|.$$

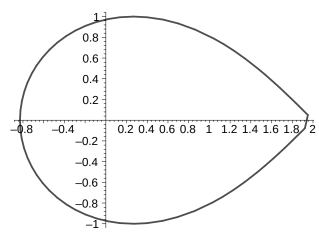
We are not able to prove such a bound for the sections.
Actually $f_\beta \in K$ since

$$1 + z\frac{f_\beta''(z)}{f_\beta'(z)} = \frac{1-\beta z}{1-z}$$

has positive real part. f_β is related to the function l of example 2 in several ways,

$$\lim_{\beta \downarrow 0} f_\beta = l,$$

and if we denote $l_\beta(z) = \frac{1}{\beta} l(\beta z)$, then $f_\beta = l_\beta^{-1} \circ l$.



10) $f : z \to ze^{\frac{1}{2}z}$ is star-like since

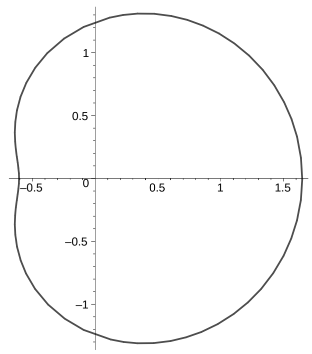
$$z\frac{f'(z)}{f(z)} = 1 + \tfrac{1}{2}z$$

has positive real part. From lemma 3.3 we deduce that the sections are close-to-convex, so $f \in V$. Also from lemma 3.3 we obtain

$$|f_n'(z)| \geq 1 - |z| \quad \text{and} \quad |f'(z)| \geq 1 - |z|,$$

and therefore from lemma 3.6

$$|f_n(z)| \geq |z| - \tfrac{1}{2}|z|^2 \quad \text{and} \quad |f(z)| \geq |z| - \tfrac{1}{2}|z|^2.$$



11) Let f be such that $zf'(z) \in K$. (This is the case with l and with dilog.) We shall show that $f \in V$.
Proof. $zf'(z) \in K \subset S^*$, hence $f \in K \subset S$. In example 2 we have seen that $l_n \in C$ so by the Polya-Schoenberg theorem [5] p.27

$$f_n = l_n * zf' \in C \subset S,$$

so $f \in V$. lemma 3.5 provides us with the lower bound

$$|f'(z)| \geq 1 - |z|.$$

Therefore

$$|f(z)| \geq |z| - \tfrac{1}{2}|z|^2.$$

The choice

$$zf'(z) = -1 + \sqrt{\frac{1+z}{1-z}}$$

leads to

$$f(z) = \arcsin(z) - \log \frac{2}{z^2}(1 - \sqrt{1-z^2}).$$

The choice

$$zf'(z) = 2(1 - \sqrt{1-z})$$

leads to

$$f(z) = 4(1 - \sqrt{1-z}) - 4 \log \frac{2}{z}(1 - \sqrt{1-z}).$$

At the end of this section we mention a general method to construct new examples from old ones. By the theorem of Polya-Schoenberg we know that the transformation T defined by

$$Tf(z) = l * f(z) = \int_{[0,z]} \frac{f(\zeta)}{\zeta} d\zeta$$

satisfies

$$T(K) \subset K, \;\; T(S^*) \subset S^*, \;\; T(C) \subset C.$$

Hence, if $f \in V$ and $f_n \in C$ for all n, then $Tf \in V$. Moreover if we know that $|f(z)| \geq |z| - \frac{1}{2}|z|^2$, then we have

$$|(Tf)'(z)| = \left|\frac{f(z)}{z}\right| \geq 1 - \tfrac{1}{2}|z| \geq 1 - |z|,$$

and so by lemma 3.6

$$|Tf(z)| \geq |z| - \tfrac{1}{4}|z|^2 \geq |z| - \tfrac{1}{2}|z|^2.$$

A similar transformation was described by Libera [2].

$$Lf(z) = \frac{2}{z}\int_{[0,z]} f(\zeta)d\zeta.$$

Libera showed that

$$L(K) \subset K, \;\; L(S^*) \subset S^*, \;\; L(C) \subset C.$$

Again, if $f \in V$ and $f_n \in C$ for all n, then $Lf \in V$. If we know that $|f(z)| \geq |z| - \frac{1}{2}|z|^2$ then by a generalization of lemma 3.6 to locally univalent functions we obtain

$$|Lf(z)| \geq |z| - \tfrac{1}{3}|z|^2 \geq |z| - \tfrac{1}{2}|z|^2.$$

Remark 4.1 Already in 1936 Robertson [3] showed that if $f \in V$ then all the Cesaro partial sums of the power series for f are univalent on the whole disc Δ.

5 Polynomials

All polynomials $f : z \to z + a_2z^2$ of degree 2 with $|a_2| \leq \frac{1}{2}$ belong to V and they obviously satisfy the inequality

$$|f'(z)| \geq 1 - |z|, \quad |f(z)| \geq |z| - \tfrac{1}{2}|z|^2.$$

In this paragraph we shall prove that for every polynomial in V of degree 3, we have similar inequalities. There is no loss of generality to assume that $a_3 > 0$, and by considering $-f(-z)$ we can replace a_2 by $-a_2$. We distinguish three cases.

Theorem 5.1 *If $f \in V$, $f(z) = z + a_2z^2 + a_3z^3$ with $a_3 = \frac{1}{3}$, then $f \in C$ and*

$$|f'(z)| \geq 1 - |z|, \quad |f(z)| \geq |z| - \tfrac{1}{2}|z|^2.$$

Proof.

$$\begin{aligned} f(z) &= z + a_2z^2 + \tfrac{1}{3}z^3, \\ f'(z) &= 1 + 2\alpha_2 z + z^2. \end{aligned}$$

f' is zerofree on Δ so the zeros of f' have modulus 1 and they are conjugate to each other. Thus

$$f'(z) = (z - e^{i\vartheta})(z - e^{-i\vartheta}),$$

and we deduce that $2a_2 = -(e^{i\vartheta} + e^{-i\vartheta})$, i.e. $a_2 = -\cos\vartheta \in \mathbb{R}$. Since $f \in V$ we have $|a_2| \leq \frac{1}{2}$, i.e.

$$|\cos\vartheta| \leq \tfrac{1}{2}.$$

By lemma 2 we have: $z \to z + a_2 z^2 \in S^*$ and this enables us to show that $f \in C$. Indeed,

$$z\frac{f'(z)}{z + a_2 z^2} = 1 + z\frac{z + a_2}{1 + a_2 z}$$

has positive real part since $\left|\frac{z+a_2}{1+a_2 z}\right| < 1$ for real $a_2 \in (-1, 1)$.
Consequently $f \in C$. To obtain the inequalities, we write $z = re^{it}$. An elementary computation leads to

$$\begin{aligned} |f'(z)|^2 - (1-|z|)^2 &= |z-e^{i\vartheta}|^2 \cdot |z-e^{-i\vartheta}|^2 - (1-|z|^2) \\ &= r^2(1-r)^2 + 4r^2(\cos\vartheta - \cos t)^2 + 2r(1-r)^2(1-2\cos\vartheta\cos t) \end{aligned}$$

and since $|\cos\vartheta| \leq \frac{1}{2}$ we have $1 - 2\cos\vartheta\cos t \geq 0$ and this shows that

$$|f'(z)| \geq 1 - |z|.$$

As before, lemma 3.6 implies that $|f(z) \geq |z| - \frac{1}{2}|z|^2$. ■

Theorem 5.2 *If $f \in V$, $f(z) = z + a_2 z^2 + a_3 z^2$ with $0 < a_3 < \frac{1}{3}$ and $a_2 \in \mathbb{R}$, then $f \in C$ and*

$$|f'(z)| \geq 1 - |z|, \quad |f(z)| \geq |z| - \tfrac{1}{2}|z|^2.$$

Proof. As in the previous theorem we have: $z \to z + a_2 z^2 \in S^*$, and

$$z\frac{f'(z)}{z + a_2 z^2} = 1 + z\frac{a_2 + 3a_3 z}{1 + a_2 z}.$$

From $-\frac{1}{2} \leq a_2 \leq \frac{1}{2}$, $0 < 3a_3 < 1$ we obtain that for $z \in \Delta$

$$\left|\frac{a_2 + 3a_3 z}{1 + a_2 z}\right| \leq 1.$$

Thus $Re\, \frac{zf'(z)}{z+a_2 z^2} > 0$ and $f \in C$.
Let $\rho = \frac{1}{\sqrt{3a_3}}$. The condition $a_2 \in \mathbb{R}$ is equivalent to:
there exists $\vartheta \in \mathbb{R}$ such that

$$f'(z) = \frac{1}{\rho^2}(z - \rho e^{i\vartheta})(z - \rho e^{-i\vartheta}).$$

The case $\rho \geq 2$ is relatively easy.

$$|f'(z)| = \left|1 - \frac{e^{-i\vartheta} z}{\rho}\right| \cdot \left|1 - \frac{e^{i\vartheta} z}{\rho}\right| \geq \left(1 - \frac{|z|}{\rho}\right)^2 \geq 1 - 2\frac{|z|}{\rho} \geq 1 - |z|.$$

Assume further that $1 < \rho < 2$. Normalize f so that

$$a_2 = -\frac{\cos\vartheta}{\rho} \leq 0.$$

From $|a_2| \leq \frac{1}{2}$ we deduce that

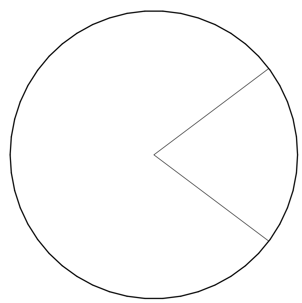
$$0 \leq \cos\vartheta \leq \frac{\rho}{2}.$$

To obtain the inequalities we write $z = re^{it}$. An elementary computation leads to

$$\begin{aligned}&\rho^4\left(|f'(z)|^2 - \left(1-\tfrac{|z|}{\rho}\right)^2\right)\\ &\quad = r^2(\rho-r)^2 + 4\rho^2r^2(\cos\vartheta - \cos t)^2 + 2\rho r(\rho-r)^2(1-2\cos\vartheta\cos t).\end{aligned}$$

If $1 - 2\cos\vartheta\cos t \geq 0$, i.e. if $\cos t \leq \frac{1}{2\cos\vartheta}$ and this is certainly the case if $\cos t \leq \frac{1}{\rho}$ we see that

$$|f'(z)| \geq 1 - \frac{|z|}{\rho} \geq 1 - |z|.$$



For other values of z we argue as follows:

$$|f'(z)| = \left|1 - 2\frac{z}{\rho}\cos\vartheta + \left(\frac{z}{\rho}\right)^2\right| \geq 1 - |z|\left|\frac{2\cos\vartheta}{\rho} - \frac{z}{\rho^2}\right|,$$

so we are done if

$$\left|\frac{2\cos\vartheta}{\rho} - \frac{z}{\rho^2}\right| \leq 1$$

i.e. if

$$|z - 2\rho\cos\vartheta| \leq \rho^2.$$

These points z lie in a disc D. We have $0 \in D$ because

$$2\rho\cos\vartheta \leq \rho^2.$$

Note that

$$\frac{1}{\rho} + i\sqrt{1 - \frac{1}{\rho^2}} \in D$$

$\Longleftrightarrow$

$$\left(\frac{1}{\rho} - 2\rho\cos\vartheta\right)^2 + 1 - \frac{1}{\rho^2} \le \rho^4$$

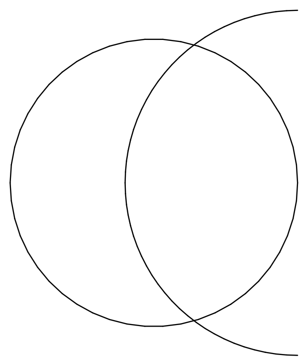
$\Longleftrightarrow$

$$4\rho^2\cos^2\vartheta - 4\cos\vartheta \le \rho^4 - 1.$$

We have

$$4\rho^2\cos^2\vartheta - 4\cos\vartheta \le 4(\rho^2-1)\cos\vartheta$$
$$= 4\tfrac{\rho^4-1}{\rho^2+1}\cos\vartheta \le \tfrac{2\cos\vartheta}{\rho}(\rho^4-1) \le \rho^4 - 1,$$

since $\frac{2\cos\vartheta}{\rho} \le 1$.



This shows that for all $z \in \Delta$ we have $|f'(z)| \ge 1 - |z|$. As before we also have $|f(z)| \ge |z| - \frac{1}{2}|z|^2$. ■

Theorem 5.3 *If $f \in V$, $f(z) = z + a_2z^2 + a_3z^3$ with $0 < a_3 < \frac{1}{3}$ and $a_2 \notin \mathbb{R}$, then*

$$|f'(z)| \ge 1 - |z|, \quad |f(z)| \ge |z| - \tfrac{1}{2}|z|^2.$$

Proof. Normalize f so that $Re\, a_2 \le 0$; denote the zeros of f' with $ae^{-i\vartheta}$ and $be^{i\vartheta}$ where $1 \le a < b$ and $0 < \vartheta \le \frac{\pi}{2}$.

$$f'(z) = \left(1 - \frac{e^{i\vartheta}z}{a}\right)\left(1 - \frac{e^{-i\vartheta}z}{b}\right) = 1 - \left(\frac{e^{i\vartheta}}{a} + \frac{e^{-i\vartheta}}{b}\right)z + \frac{1}{ab}z^2.$$

There is an easy case. If $\frac{1}{a} + \frac{1}{b} \le 1$, then

$$|f'(z)| \ge \left(1 - \frac{|z|}{a}\right)\left(1 - \frac{|z|}{b}\right) \ge 1 - \left(\frac{1}{a} + \frac{1}{b}\right)|z| \ge 1 - |z|.$$

From now on we assume that $\frac{1}{a} + \frac{1}{b} > 1$, and since $b > a \ge 1$, we even have

$$1 < \frac{1}{a} + \frac{1}{b} < 2.$$

Since $f \in V$, its second coefficient has modulus less then $\frac{1}{2}$ so

$$\left|\frac{e^{i\vartheta}}{a} + \frac{e^{-i\vartheta}}{b}\right| \le 1$$

i.e.

$$|ae^{-i\vartheta} + be^{i\vartheta}| \le ab.$$

(This condition can be written as $4\cos^2\vartheta \le ab - \frac{(a-b)^2}{ab}$.)

We divide Δ into three regions. For those z for which

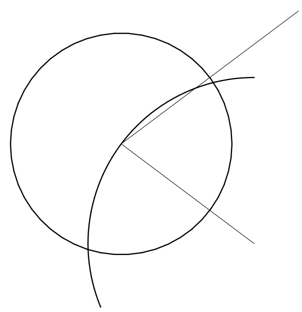
$$\left|1 - e^{i\vartheta}\frac{z}{a}\right| \ge 1$$

i.e.

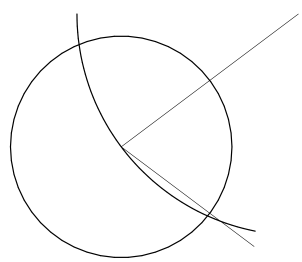
$$\left|z - e^{-i\vartheta}a\right| \ge a$$

we have

$$|f'(z)| \ge \left|1 - e^{-i\vartheta}\frac{z}{b}\right| \ge 1 - \frac{|z|}{b} \ge 1 - |z|.$$



A similar argument holds for those z with $\left|z - e^{i\vartheta}b\right| \ge b$.



We also have

$$f'(z) = 1 - \left(\frac{e^{i\vartheta}}{a} + \frac{e^{-i\vartheta}}{b}\right)z + \frac{1}{ab}z^2$$

and therefore

$$|f'(z)| \ge 1 - |z| \cdot \left|\frac{e^{i\vartheta}}{a} + \frac{e^{-i\vartheta}}{b} - \frac{z}{ab}\right|.$$

For those values of z for which

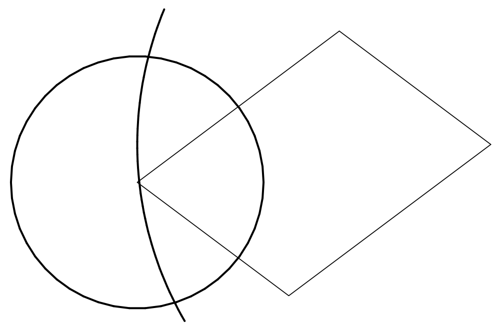
$$\left|\frac{e^{i\vartheta}}{a} + \frac{e^{-i\vartheta}}{b} - \frac{z}{ab}\right| \le 1$$

i.e.

$$|z - (ae^{-i\vartheta} + be^{i\vartheta})| \le ab$$

we have

$$|f'(z)| \geq 1 - |z|.$$



We shall show that these three regions cover Δ. We do this by proving that certain arcs of $\Gamma_{ab}(ae^{-i\vartheta})$ and $\Gamma_b(be^{i\vartheta})$ lie inside $\Delta_{ab}(ae^{-i\vartheta} + be^{i\vartheta})$.
We claim that for $t \in [\vartheta, \pi - \vartheta]$ we have

$$ae^{-i\vartheta} + ae^{it} \in \Delta_{ab}(ae^{-i\vartheta} + be^{i\vartheta}).$$

This is the case $\iff$

$$|ae^{it} - be^{i\vartheta}| \leq ab$$

$\iff$

$$a^2 + b^2 - 2ab\cos(t - \vartheta) \leq a^2b^2$$

$\iff$

$$\cos(t - \vartheta) \geq \frac{a^2 + b^2 - a^2b^2}{2ab}.$$

Now $\cos(t - \vartheta)$ is decreasing on $[\vartheta, \pi - \vartheta]$, so it suffices to show that

$$\cos(\pi - 2\vartheta) = -\cos 2\vartheta \geq \frac{a^2 + b^2 - a^2b^2}{2ab},$$

i.e.

$$\cos 2\vartheta \leq \frac{a^2b^2 - a^2 - b^2}{2ab}.$$

This is precisely the condition $4\cos^2\vartheta \leq ab - \frac{(a-b)^2}{ab}$. ■

6 Extreme points

As we mentioned before, the set V is a compact subset of the space $H(\Delta)$ of analytic functions on the unit disc. It is therefore of interest to study $\mathrm{Ext}(V)$, the set of extreme points of V. Our first result is a direct consequence of corollary 2.1.

Theorem 6.1 *$l \in \mathrm{Ext}(V)$.*

Proof. Choose $x \in (0,1)$. For every $f \in V$ we have

$$\operatorname{Re} f(x) \leq |f(x)| \leq -\log(1-x) = \operatorname{Re} l(x).$$

Moreover l is the only function for which there is equality, hence $l \in \operatorname{Ext}(V)$. ■

Of course, all the rotations l_ζ of l (with $|\zeta| = 1$)

$$l_\zeta : z \to \frac{1}{\zeta} l(\zeta z)$$

are also extreme points of V. Our next theorem says that these are not yet all.

Theorem 6.2 $\{l_\zeta : |\zeta| = 1\} \subsetneq$ *Ext*(V).

Proof. Consider the linear functional L on $H(\Delta)$ defined by

$$L(f) = \operatorname{Re}(20a_2 - 3a_3).$$

Then

$$L(l_{e^{it}}) = 10\cos t - \cos 2t \leq 9$$

while

$$L(z + \tfrac{1}{2}z^2) = 10.$$

This shows that $\operatorname{Ext}(V)$ contains more functions then only rotations of l. ■

The functions b_n of example 1) cannot be extreme points of V if $n > 2$ because they are convex combinations of sections of l_ζ. Take $\zeta = e^{\frac{2\pi i}{n-1}}$ and consider

$$l_n(z) = \sum_{k=1}^{n} \frac{1}{k} z^k.$$

$l_n \in V$ and

$$b_n(z) = \frac{1}{n-1} \sum_{m=1}^{n-1} \zeta^{-m} l_n(\zeta^m z).$$

Thus $b_n \notin \operatorname{Ext}(V)$.

Sometimes it is easy to show that certain classes of functions do not contain extreme points.

Theorem 6.3 *If $f \in V$ and if $f_n \in K$ for all n, then $f \notin \operatorname{Ext}(V)$.*

Proof. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$; $f_n(z) = z + \sum_{k=2}^{n} a_k z^k$.

By the results of Ruscheweyh [4] we see that all the functions

$$g(z) = z + \sum_{k=2}^{\infty} \left(a_k + \tfrac{1}{k.3^k}\right) z^k; \qquad g_n(z) = z + \sum_{k=2}^{n} \left(a_k + \tfrac{1}{k.3^k}\right) z^k,$$
$$h(z) = z + \sum_{k=2}^{\infty} \left(a_k - \tfrac{1}{k.3^k}\right) z^k; \qquad h_n(z) + z + \sum_{k=2}^{n} \left(a_k - \tfrac{1}{k.3^k}\right) z^k,$$

belong to S^*. Hence $g,h \in V$ and since $f = \frac{1}{2}g + \frac{1}{2}h$ we conclude that $f \notin \mathrm{Ext}(V)$. ■

In this paragraph we shall show that all sections l_n of l are extreme points of V. We do this by showing that l_n is not a convex combination of functions from V. Suppose on the contrary that for some $n \in \{2,3,4,\ldots\}$ l_n is a convex combination of functions φ and ψ from V. Then, since coefficients of functions from V satisfy $|a_n| \leq \frac{1}{m}$, we have

$$\begin{aligned} \varphi(z) &= z + \tfrac{1}{2}z^2 + \cdots + \tfrac{1}{n}z^n + \cdots \\ \psi(z) &= z + \tfrac{1}{2}z^2 + \cdots + \tfrac{1}{n}z^n + \cdots \end{aligned}$$

Let $k+1 > n$ be the smallest number for which φ (and hence ψ) has a nonzero coefficient. Since $\varphi, \psi \in V$ we have

$$\begin{aligned} z + \tfrac{1}{2}z^2 + \cdots + \tfrac{1}{n}z^n + a_{k+1}z^{k+1} \in S \\ z + \tfrac{1}{2}z^2 + \cdots + \tfrac{1}{n}z^n + b_{k+1}z^{k+1} \in S \end{aligned}$$

and there is an non-trivial convex combination of a_{k+1} and b_{k+1} equal to zero. We shall investigate for which complex number λ the polynomial

$$p : z \to z + \tfrac{1}{2}z^2 + \cdots + \frac{1}{n}z^n + \frac{1}{k+1}z^{k+1}$$

is not injective on Δ. p is certainly not injective if p' has zeros on Δ. We have the following chain of equivalent assertions

$$\begin{aligned}
& p' \text{ has zeros on } \Delta \\
\Longleftrightarrow \quad & 1 + z + z^2 + \cdots + z^{n-1} = -\lambda z^k \quad \text{has solutions on } \Delta \\
\Longleftrightarrow \quad & 1 + z + z^2 + \ldots + z^{n-1} = -\lambda z^k \quad \text{has solutions on } \Delta\backslash\{0\} \\
\Longleftrightarrow \quad & \tfrac{1}{z^k} + \tfrac{1}{z^{k-1}} + \cdots + \tfrac{1}{z^{k-n+1}} = -\lambda \quad \text{has solutions on } \Delta\backslash\{0\} \\
\Longleftrightarrow \quad & z^k + z^{k-1} + \cdots + z^{k-n+1} = -\lambda \quad \text{has solution on } \mathbb{C}\backslash\Delta \\
\Longleftrightarrow \quad & z^k + z^{k-1} + \cdots + z^{k-n+1} = -\lambda \quad \text{has less then } k \text{ solutions on } \overline{\Delta} \\
\Longleftrightarrow \quad & z^{k-n+1}(1 + z + \cdots + z^{n-1}) \quad \text{assumes the value } -\lambda \text{ less then } k \text{ times on } \overline{\Delta} \\
\Longleftrightarrow \quad & h(z) = z^{k-n+1}\tfrac{z^n-1}{z-1} \quad \text{assumes the value } -\lambda \text{ less then } k \text{ times on } \overline{\Delta}.
\end{aligned}$$

Thus, it is appropriate to study the curve $h \circ \Gamma$ and to find out for which λ we have $\mathrm{Ind}_{h\circ\Gamma}(-\lambda) < k$.
The curve $h \circ \Gamma$ intersects the real axis if for some value of $z \in \Gamma$ we have

$$h(z) = \overline{h(z)}$$

i.e.

$$z^{k-n+1}(1+z+\cdots+z^{n-1}) = \overline{z^{k-n+1}(1+z+\cdots+z^{n-1})}.$$

Multiply both sides with z^{k-n+1}

$$z^{2(k-n+1)}(1+z+\cdots+z^{n-1}) = \overline{1+z+\cdots+z^{n-1}} = 1+\frac{1}{z}+\cdots+\frac{1}{z^{n-1}}.$$

Multiply with z^{n-1}

$$z^{2k-n+1}(1+z+\cdots+z^{n-1}) = 1+z+\cdots+z^{n-1}$$

i.e.

$$(z^{2k-n+1}-1)\frac{z^n-1}{z-1} = 0.$$

Hence: real intersections occur if

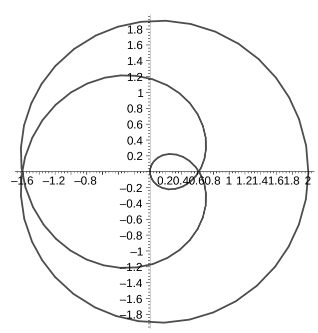
1) z is a $2k-n+1^{\text{th}}$ root of unity. If $z \neq 1$ then $h \circ \Gamma$ has self intersection.

2) z is a n^{th} root of unity. If $z \neq 1$ then $h \circ \Gamma$ passes through 0.

Similar computations show that $h \circ \Gamma$ intersects the imaginary axis if $z^{2k-n+1} = -1$ and of course if $\frac{z^n-1}{z-1} = 0$.
So $h \circ \Gamma$ passes through 0 at $z_\nu = e^{\frac{2\pi i}{n}\nu}$ $(\nu = 1, 2, \ldots n-1)$. In these points we have

$$h'(z_\nu) = -\frac{ni}{2\sin\frac{\pi\nu}{n}} e^{\frac{2\pi i}{n}(k-\frac{1}{2})\nu}$$

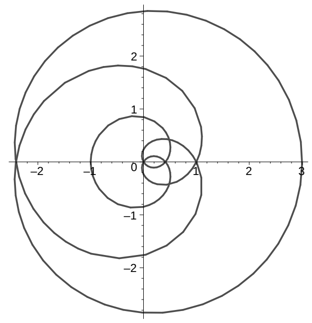
so $h \circ \Gamma$ has a tangent in the direction $\frac{\pi}{2} + \frac{2k-1}{n}\pi\nu$. This shows that $h \circ \Gamma$ has loops and these loops lie in a halfplane through the origin.



Only those values $-\lambda$ which lie outside these loops are assumed less then k times by h. Therefore we see that the polynomial p can only be univalent for those λ for which $-\lambda$ lie inside all the loops. These values certainly lie all in one halfplane through 0. Therefore φ and ψ cannot both be univalent, i.e. we have proved:

Theorem 6.4 *For all values of $n \in \{2, 3, \ldots\}$ we have*

$$l_n \in \operatorname{Ext}(V).$$



Of course the question arises whether V has more extreme points. A first step towards an answer is provided by the following: which functions are convex combinations of

$$z \to e^{-it} l_n(e^{it} z) \qquad (0 \le t \le 2\pi).$$

For $n = 2$ the answer is simple. Every polynomial $p : z \to z + a_2 z^2$ with $|a_2| \le \frac{1}{2}$ can be obtained in this way. If $|a_2| \ne \frac{1}{2}$ there are infinitely many such representations. In order to see which polynomials

$$p : z \to z + a_2 z^2 + \cdots + a_n z^n$$

can be written as

$$\int_0^{2\pi} e^{-it} l_n(e^{it} z) d\mu(t)$$

we look for probability measures μ on $[0, 2\pi]$ such that

$$\int_0^{2\pi} e^{(k-1)it} d\mu(t) = k a_k \quad (k = 2, \ldots n).$$

By the Riesz representation theorem there is a $1-1$ correspondence between probability measures μ and the positive linear functionals T on $C(\Gamma)$ for which $T(1) = 1$.

Lemma 6.1 *A linear functional T on $C(\Gamma)$ is positive $\iff$ T is bounded and $\|T\| = T(1)$.*

Proof. $\Longrightarrow$ If $\varphi \ge 0$, then $T(\varphi) \ge 0$. Therefore, for real valued functions φ we have $T(\varphi) \in \mathbb{R}$.
Since $\pm\varphi \le \|\varphi\|$ we have from positivity

$$0 \le T(\mp\varphi + \|\varphi\|) = \mp T(\varphi) + \|\varphi\| T(1)$$

hence

$$|T(\varphi)| \le T(1) \cdot \|\varphi\|.$$

If φ is complex valued, then we have for every $t \in [0, 2\pi]$

$$\text{Re } e^{it} T(\varphi) = \text{Re } T(e^{it}\varphi) = T(\text{Re}(e^{it}\varphi)) \le T(|e^{it}\varphi|) = T(|\varphi|) \le T(1)\|\varphi\|$$

so for a suitable choice of φ we obtain

$$|T(\varphi)| \le T(1)\|\varphi\|,$$

i.e. $\|T\| = T(1)$.

$\Longleftarrow$ Conversely, let $\|T\| = T(1)$. We have to show that for all functions $\varphi \geq 0$ we have $T(\varphi) \geq 0$. Let $\varphi \geq 0$ and let $a \in \mathbb{R}$. The functions

$$\varphi - \tfrac{1}{2}\|\varphi\| - ia$$

satisfy the inequality

$$\|\varphi - \tfrac{1}{2}\|\varphi\| - ia\| \leq \sqrt{\tfrac{1}{4}\|\varphi\|^2 + a^2} = |\tfrac{1}{2}\|\varphi\| + ia|.$$

Therefore

$$|T(\varphi) - T(1)(\tfrac{1}{2}\|\varphi\| + ia)| = |T(\varphi - \tfrac{1}{2}\|\varphi\| - ia)| \leq T(1) \cdot |\tfrac{1}{2}\|\varphi\| + ia|.$$

Apparently $T(\varphi)$ lies in every disc with center $T(1) \cdot (\frac{1}{2}\|\varphi\| + ia)$ and radius $T(1)|\frac{1}{2}\|\varphi\| + ia|$, hence

$$T(\varphi) \in [0, T(1) \cdot \|\varphi\|].$$

In particular $T(\varphi) \geq 0$. ■

Now assume that $a_2, a_3, \ldots, a_n \in \mathbb{C}$ are given. we want to find out if there exists a probability measure μ on $[0, 2\pi]$ such that

$$\int_0^{2\pi} e^{(k-1)t} d\mu(t) = a_k \qquad (k = 2, \ldots, n).$$

If the linear functional T defined on the set of polynomials of degree n by

$$T(\gamma_1 + \gamma_2 z + \cdots + \gamma_n z^n) = \gamma_1 + \gamma_2 a_2 + \cdots + \gamma_n a_n$$

has norm $T(1) = 1$, then T is positive. By Hahn-Banach T can be extended to a continuous linear functional on $C(\Gamma)$ with norm $1 = T(1)$ so the extension is positive, and by the theorem of Riesz T corresponds to a measure. Hence we arrive at the necessary and sufficient condition:

There exists a probability measure μ on $[0, 2\pi]$ such that $\int_0^{2\pi} e^{(k-1)t} d\mu(t) = a_k$ $(k = 2, \ldots, n)$

$$\Longleftrightarrow$$

For every sequence $\gamma_1, \gamma_2, \ldots, \gamma_n$ we have

$$|\gamma_1 + \gamma_2 a_2 + \cdots + \gamma_n a_n| \leq \max_{0 \leq t \leq 2\pi} |\gamma_1 + \gamma_2 e^{it} + \cdots + \gamma_n e^{int}|.$$

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