Higher order Bose-Einstein correlations in $\pi^+p$ and $K^+p$ collisions at 250 GeV/$c$

EHS/NA22 Collaboration

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Abstract. Bose-Einstein correlations up to fourth order are presented at $\sqrt{s} = 22$ GeV. Genuine third-order correlations are observed. The experimental data are compared with predictions from a quantum statistical approach of radiation from a partially coherent source and with the FRITIOF model.

1 Introduction and formalism

The dynamical properties of complex systems can be described by two-, three- and more-particle correlation functions. In statistical physics, e.g. in the theory of gases, multiparticle correlations are the result of interactions between pairs, triplets, etc. of molecules [1]. In high energy strong interaction physics, the correlation characteristics of multihadron production reflect the complex (and not yet established) dynamical properties of the space-time evolution and hadronization of a quark-gluon system [2]. A specific case of short-range two- and multiparticle correlations, the Bose-Einstein (BE) correlation, has an interference origin due to the symmetry properties of the probability amplitude of two or more identical bosons (pions) (for reviews see e.g. [3-6]). The characteristics of BE correlations are determined by a number of factors: The space-time size and shape of the pion source, its non-static properties (the velocities of its movement and expansion), the existence of two or more sources of different size, the interrelation between chaotic and coherent sources etc.

In the most general case, the inclusive $q$-particle densities $\rho_q(1,...,q)$ (where the kinematical variables of the particles are abbreviated to their number $1,...,q$) are expressed in terms of the cluster expansion familiar from statistical physics [1,7,8]:

\begin{align}
\rho_2(1,2) &= C_2(1,2) + \rho_1(1)\rho_1(2), \\
\rho_3(1,2,3) &= C_3(1,2,3) + \sum_{(3)}\rho_1(1)\rho_2(2,3) \\
&\quad - 2\rho_1(1)\rho_1(2)\rho_1(3), \\
\rho_4(1,2,3,4) &= C_4(1,2,3,4) + \sum_{(4)}\rho_1(1)\rho_3(2,3,4) \\
&\quad + \sum_{(3)}\rho_2(1,2)\rho_2(3,4) \\
&\quad - 2\sum_{(6)}\rho_1(1)\rho_1(2)\rho_3(3,4) \\
&\quad + 6\rho_1(1)\rho_1(2)\rho_1(3)\rho_1(4),
\end{align}

etc, where the summations indicate that all possible permutations have to be taken. The number under the summation sign indicates the number of terms. The correlation functions or (factorial) cumulant functions $C_q(1,...,q)$ vanish whenever any one of their arguments becomes statistically independent of the others. They represent the genuine $q$-particle correlations, while the other terms in the expansions (1)-(3) reflect the "trivial" contributions from lower-order densities.

It is often convenient to use the normalized inclusive densities and correlations:
The normalization of the inclusive densities \( \rho_1, \rho_2, \rho_3, \) and \( \rho_4 \) for identical particles is defined from the condition that their integration over phase space results, respectively, in \( \langle n \rangle, \langle n(n-1) \rangle, \) and \( \langle n(n-1)(n-2) \rangle \) for the dipion ensemble with very small energy difference \( q_0 = E - 2 \mu r^2, \) where \( n \) is the particle multiplicity.

The normalized inclusive density for two identical pions is

\[
S_2 (1, 2) = 1 + K_2 (1, 2).
\]

In the limit of a completely chaotic and static pion source, \( K_2 (1, 2) \) reduces to the square of the Fourier transform \( F(p_1 - p_2, E_1 - E_2) \) of the space-time distribution of the source, \( K_2 (1, 2) = |F(1, 2)|^2, \) where \( p_i \) and \( E_i \) \((i = 1, 2)\) are the three-momentum and energy of pion \( i \), respectively.

Frequently, for convenience, the Lorentz-invariant collective variable

\[
Q_i^2 \equiv Q_i^2 \equiv -(P_1 - P_2) \cdot (P_1 + P_2)^2 - 4 M_\pi^2
\]

is introduced \((M_\pi \text{ is the pion mass})\) instead of the four momenta \( P_1, P_2 [9,10] \), and an empirical (Gaussian) parametrization is used for \(|F(Q_i^2)|^2\):

\[
K_2 (Q_i^2) = |F(Q_i^2)|^2 = \exp \left(-r^2 Q_i^2\right).
\]

The parameter \( r \) in (8) can be related to the pion source radius for a pion pair ensemble with very small energy difference \( q_0 = E_1 - E_2 \) or in the dipion rest frame, where \( q_0^2 = 0 \) [4,11]. If the pion source, observed in the dipion rest frame, has a spherically symmetric Gaussian form

\[
f(x) = \frac{1}{r_0 \sqrt{2 \pi}} \exp \left(-\frac{|x|^2}{2r_0^2}\right),
\]

then the parameter \( r \) is related to the source r.m.s. radius (averaged over the dipion ensemble).

In terms of the \( Q_{ij} \) variables and for the case of a completely chaotic source, the normalized inclusive three-pion density is \([12]\)

\[
S_3 (1, 2, 3) = 1 + |F(Q_{12}^2)|^2 + |F(Q_{13}^2)|^2 + |F(Q_{23}^2)|^2 + 2 \text{Re}\{F(Q_{12}^2) F(Q_{13}^2) F(Q_{23}^2)\}
\]

with

\[
K_3 (Q_{12}^2) = 2 \text{Re}\{F(Q_{12}^2) F(Q_{13}^2) F(Q_{23}^2)\}.
\]

In general, the genuine three-particle correlation \( K_3 (1, 2, 3) \) is not expressed completely in terms of the two-particle correlation function (8), but contains also new information on the phase of the Fourier transform of the source. To the extent that phase factors may be neglected, \( K_3 \) is related to \( K_2 \) via the expression

\[
K_3 (Q_{12}^2) = 2 \exp \left(-\frac{r^2}{2} Q_{12}^2\right) = 2 \sqrt{K_2 (Q_{12}^2)}
\]

with

\[
Q_{12}^2 = Q_{13}^2 = (P_1 + P_2 + P_3)^2 - 9 M_\pi^2 = Q_{12}^2 + Q_{13}^2 + Q_{23}^2.
\]

In a more general case, chaotic and coherent components may coexist in the pion source [11,13-16]. Although the coherent source of itself does not cause any BE correlation, superposition of chaotic and coherent radiation changes the interference pattern and the interrelation between correlations of different order, as between (8) and (12).

Pion radiation by a partially coherent source (with the chaoticity parameter \( p = (n_{cha})/(n) \), where \( n_{cha} \) denotes the chaotic fraction in the pion average multiplicity) can be described in the framework of quantum statistics, applying an approach analogous to that used in quantum optics. Usually, it is assumed that the coherent source is pointlike and the chaotic source has a Gaussian form, \( f(x) \sim \exp(-|x|^2/r^2) \), with \( r(r.m.s.) = r \sqrt{3} = r \sqrt{3/2} \) for the simplified case of a symmetric configuration in momentum space one has

\[
Q_{2q}^2 \equiv Q_{12}^2 = Q_{13}^2 = \ldots = Q_{(q-1)q}^2 = 2 Q_q^2/(q(q-1)) \quad (14)
\]

(for example, \( Q_2^2 = 1/3 Q_1^2 \), \( Q_3^2 = 4/6 Q_2^2 \)), where \( Q_2^2 \) and \( Q_3^2 \) are defined in (7) and (13),

\[
Q_2^2 = \left(\sum_{i=1}^{q} P_i\right)^2 - (M_\pi)^2
\]

and, in general,

\[
Q_q^2 = \left(\sum_{i=1}^{q} P_i\right)^2 - (q M_\pi)^2.
\]

The normalized two-, three- and four-pion inclusive densities are \([14-16]\):

\[
R_2 (Q_2^2) = 1 + 2p(1-p) \exp(-r^2 Q_2^2) + p^2 \exp(-2r^2 Q_2^2),
\]

\[
R_3 (Q_3^2) = 1 + 6p(1-p) \exp(-\frac{1}{3}r^2 Q_3^2) + 3p^2(3-2p) \exp(-\frac{2}{3}r^2 Q_3^2) + 2p^3 \exp(-r^2 Q_3^2),
\]

\[
R_4 (Q_4^2) = 1 + 12p(1-p) \exp(-\frac{1}{6}r^2 Q_4^2) + 6p^2(7-8p+2p^2) \exp(-\frac{1}{3}r^2 Q_4^2) + 4p^3(11-9p) \exp(-\frac{1}{2}r^2 Q_4^2) + 9p^4 \exp(-\frac{2}{3}r^2 Q_4^2).
\]

The normalized two- and three-pion correlation functions are:

\[
K_2 (Q_2^2) = 2p(1-p) \exp(-r^2 Q_2^2) + p^2 \exp(-2r^2 Q_2^2),
\]

\[
K_3 (Q_3^2) = 6p^2(1-p) \exp(-\frac{2}{3}r^2 Q_3^2) + 2p^3 \exp(-r^2 Q_3^2).
\]

In quantum optics, the two-particle correlation function is also parametrized by an exponential form as a function of
\[ Q_q = \sqrt{Q_q^2}. \] In this parametrization one gets for a symmetric configuration [16]:

\[ R_2(Q_2) = 1 + 2p(1 - p) \exp(-rQ_2) + p^2 \exp(-2rQ_2) \]  
\[ R_3(Q_3) = 1 + 6p(1 - p) \exp\left(-\frac{1}{\sqrt{3}}rQ_3\right) + 3p^2(3 - 2p) \exp\left(-\frac{2}{\sqrt{3}}rQ_3\right) + 2p^3 \exp\left(-\frac{3}{\sqrt{3}}rQ_3\right) \]  
\[ R_4(Q_4) = 1 + 12p(1 - p) \exp\left(-\frac{1}{\sqrt{6}}rQ_4\right) + 6p^2(7 - 8p + 2p^2) \exp\left(-\frac{2}{\sqrt{6}}rQ_4\right) + 4p^3(11 - 9p) \exp\left(-\frac{3}{\sqrt{6}}rQ_4\right) + 9p^4 \exp\left(-\frac{4}{\sqrt{6}}rQ_4\right). \]  

In (22)-(24), the parameter \( r \) is not directly related to the source radius, but characterizes the correlation range in \( Q \)-space. At \( p \to 1 \) (completely chaotic source) and symmetric configuration, (20) and (21) reduce to (8) and (12), respectively. At \( p < 1 \) (partially coherent source), the normalized correlation functions (20) and (21) in contrast with (8) and (12), now contain two exponential terms. The maximum values of the normalized densities (17) and (18) are smaller than, respectively, \( R_0(0) = 2 \) and \( R_3(0) = 6 \) expected for a completely chaotic source (cf. (17) and (18) at \( p = 1 \) with (6), (8) and (10)).

One should stress, that the above properties of a partially coherent source can be reproduced by a superposition of two completely chaotic sources with radii \( r_{Q_2}^0 \) and \( r_{Q_4}^0 \) accidentally related by \( r_{Q_2}^0/r_{Q_4}^0 \approx \sqrt{2} \) [4]. However, the predictions of quantum statistics are definite and contain, in a simplified case, only two free parameters \( p \) and \( r \) for the correlation of all orders. So, an experimental observation of higher order correlations allows, in principle, to establish quantum statistics, a basic approach in various physical fields (such as quantum mechanics and field theory, condensed matter physics, nuclear physics etc.), also in multiparticle production processes.

Apart from a few exceptions, correlations of three and more particles have experimentally been studied only during recent years [17-26]. While the evidence for the three-particle short-range rapidity correlations observed in [17-21] is inconclusive, significant genuine correlations are observed at small invariant mass of particle triplets [25] and at small \( Q^2 \) for three-, four- and five-particle systems [26]. Three-particle BE correlations are studied in [18,19,20,23,24] and are found to be consistent with (12). BE correlations measured in [24] up to fifth order have manifested some inconsistency with the expectation from quantum statistics: while the chaoticity parameter \( p \) is practically constant, the parameter \( r \) turns out to increase with increasing order \( q \).

The importance of the role of higher order BE correlations in multiparticle production and the scarcity of available data stimulate further experimental investigation in this field. In this work we present new experimental data on higher order (third and fourth order) BE correlations in \( (\pi^+ K^-)p \) collisions at 250 GeV/c from the NA22 experiment, performed at the CERN SPS with the help of the European Hybrid Spectrometer EHS. Earlier results of this experiment on two-particle BE correlations are published in [27-29].

Data sample and reconstruction procedure are described in short in the following section. In Sect. 3 we present the experimental data on the (genuine) BE correlations and the results of their analysis. Conclusions are summarized in Sect. 4.

2 Data sample and reconstruction procedure

The experimental set-up of the European Hybrid Spectrometer (EHS) is described in detail in [30]. In the NA22 experiment, EHS has been exposed to a positive meson enriched beam of 250 GeV/c momentum. The hadronic beam content has been 15.3% K\(^+\), 38.9% \( \pi^+ \) and 45.8% p. The K\(^+\) and \( \pi^+ \) components have been individually tagged and proton events have been vetoed.

The data reduction procedures are described in detail in [31,32]. A rapid cycling bubble chamber RCBC filled with hydrogen has been used as an active vertex detector. Tracks of secondary charged particles are reconstructed from hits in the wire and drift chambers of the spectrometer and from measurement in the RCBC, to an accuracy of \( (1 - 2.5\%) \) when reconstructed in the first lever arm and 1.5% when reconstructed in the full spectrometer. The resolution in \( Q^2 \) is estimated to be \( 8.10^{-4} \text{ GeV}^2 \) at \( Q^2 < 0.04 \text{ GeV}^2 \).

The event-selection criteria are described in detail in [25]. Accepted events are satisfactorily measured and reconstructed and contain at least two negative charged tracks with momentum error less than 4%. Each accepted track is required to lie in the region of Feynman variable \( |x| \rho < 0.5 \), in order to reduce possible correlations due to phase space restriction and biases due to violation of momentum and energy conservation. Single diffraction dissociation [32,33] is excluded. The number of accepted events is equal to 102568. For each event, a weight is introduced in order to normalize to the non-single-diffractive topological cross sections [31,33].

All negative particles are assumed to have pion mass. The contamination from other particles is estimated to be \( (7 \pm 3\%) \) [33].

3 Experimental results

3.1 The normalized higher-order densities

The normalized \( q \)-particle densities \( N_q(Q_q^2) \) are determined as

\[ R_q(Q_q^2) = N_q(Q_q^2)/N^\text{BG}_q(Q_q^2), \]  

where \( N_q(Q_q^2) \) is the number of \( q \)-particle combinations at given \( Q_q^2 \) and \( N^\text{BG}_q(Q_q^2) \) that for the reference (background) sample composed by combining tracks randomly chosen from different events of the same charged particle multiplicity.
The contribution from mixed events of a given multiplicity in $N^Bq(Q^2)$ is proportional to that of real events in $N_q(Q^2)$. The nominator and the denominator in (25) are normalized in is proportional to that of real events in $N_q(Q^2)$.

Here, we choose

The genuine three-particle correlation function $C_3(q^2; Q_3)$ and its normalized form $K_3(q^2; Q_3)$ are extracted by means of (2). The product $p_3 p_1 p_1$ in (2) is determined by combining three particles with a given $Q^2_3$ randomly chosen from different events with $n_{-} > 3$. The product $p_2 p_1$ is determined by combining three particles with a given $Q^2_3$, two of which are chosen from the same event and the other from another event with $n_{-} > 3$. The density $p_3$ is determined by combining three particles with a given $Q^2_3$ chosen from the same event.

Two methods of normalization are applied for the density function $p_3$ and the combinations $p_2 p_1$ and $p_1 p_1$. In method I, we use the total number of three-pion combinations in the interval $0 \leq Q^2_3 \leq (Q^2_{3})_{\text{max}}$ or $0 \leq Q_q \leq (Q_{q})_{\text{max}}$ as described above. In method II, we use the normalization described in the introduction: $(n(n-1)(n-2))$ for $p_3$, $(n)(n(n-1))$ for $p_2 p_1$, and $(n)^3$ for $p_1 p_1$. These two methods lead to very similar results.

In Figs. 1a,c,e and 2a,c,e, the measured ratios $R_2(q^2; Q^2)_q = N_q(q^2; Q^2)_q/N^B_q(q^2; Q^2)_q$ and $R_3(q^2; Q^2)_q = N_q(q^2; Q^2)_q/N^B_q(q^2; Q^2)_q$ are shown for $q=2,3,4$, respectively. Figures 1b,d,f and 2b,d,f present the same distributions corrected for Coulomb repulsion of the like-charge pions in the final state: each two-pion combination in $N^B_q(Q^2_q)$ or $Q^2_3$ is weighted by a factor (known as Gamov factor [34])

$$W_2 = G^{-1}(Q^2_q) = \frac{\exp(2\pi\eta) - 1}{2\pi\eta},$$

with $\eta = \alpha M_\pi/Q_2$. For a triplet and a quadruplet of pions containing, respectively, three and six pair combinations with variable $Q_{ij}$ a factor [19]

$$W_q = \prod_{i<j} G^{-1}(Q_{ij}), \quad (q = 3, 4)$$

is used.

In order to check the consistency of the $\pi^+$ and $K^+$ data, we have fitted the ratios $R_2(q^2; Q^2)_q$ and $R_3(q^2; Q^2)_q$ separately for pion and kaon induced reactions to the empirical dependence

$$R_q(q^2) = \gamma_q [1 + \lambda_q \exp(-r^2_{q} Q^2_{q})][1 + \delta_q Q^2_{q}],$$

where $\lambda_q$ characterizes the strength of the interference effects, $\gamma_q$ is a normalization coefficient and $\delta_m$ is introduced to account for a possible variation of $R_q(q^2)$ outside the interference peak. The fit results given in Tables 1 and 2 for $\pi^+$ and $K^+$ data, separately, are in agreement with each other. In Figs. 1 and 2 and in the following we, therefore, use the combined ($\pi^+/K^+$) data.

The extracted parameter value, $r_2 = 0.82 \pm 0.01$ fm, is consistent with the available data (see [27,28]). Also $r_3 = 0.51 \pm 0.02$ fm is in agreement with the results from pp-collisions at $\sqrt{s} = 26$ GeV [20], $e^+ e^-$ annihilation at $\sqrt{s} = 3 - 34$ GeV [18,19] and $\gamma \gamma$-collisions [19].
Table 1. The results of fitting the various data samples for \( q = 2 \) by function (28)

<table>
<thead>
<tr>
<th>Sample</th>
<th>( r ) (fm)</th>
<th>( \lambda )</th>
<th>( \gamma )</th>
<th>( \delta ) (GeV(^{-2}))</th>
<th>( \chi^2/NDF )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without Coulomb corrections</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \pi^+p )</td>
<td>0.82 ± 0.01</td>
<td>0.33 ± 0.02</td>
<td>0.957 ± 0.005</td>
<td>0.029 ± 0.006</td>
<td>93/96</td>
</tr>
<tr>
<td>( K^+p )</td>
<td>0.83 ± 0.02</td>
<td>0.33 ± 0.03</td>
<td>0.956 ± 0.009</td>
<td>0.029 ± 0.011</td>
<td>107/96</td>
</tr>
<tr>
<td>( (\pi^*/K^+)p )</td>
<td>0.82 ± 0.01</td>
<td>0.33 ± 0.02</td>
<td>0.953 ± 0.004</td>
<td>0.035 ± 0.006</td>
<td>115/96</td>
</tr>
<tr>
<td>With Coulomb corrections</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \pi^+p )</td>
<td>0.86 ± 0.01</td>
<td>0.38 ± 0.02</td>
<td>0.957 ± 0.005</td>
<td>0.024 ± 0.006</td>
<td>101/96</td>
</tr>
<tr>
<td>( K^+p )</td>
<td>0.87 ± 0.02</td>
<td>0.38 ± 0.03</td>
<td>0.957 ± 0.008</td>
<td>0.024 ± 0.010</td>
<td>110/96</td>
</tr>
<tr>
<td>( (\pi^*/K^+)p )</td>
<td>0.85 ± 0.01</td>
<td>0.38 ± 0.02</td>
<td>0.954 ± 0.004</td>
<td>0.030 ± 0.005</td>
<td>127/96</td>
</tr>
</tbody>
</table>

Table 2. The results of fitting the various data samples for \( q = 3 \) by function (28)

<table>
<thead>
<tr>
<th>Sample</th>
<th>( r ) (fm)</th>
<th>( \lambda )</th>
<th>( \gamma )</th>
<th>( \delta ) (GeV(^{-2}))</th>
<th>( \chi^2/NDF )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without Coulomb corrections</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \pi^+p )</td>
<td>0.52 ± 0.02</td>
<td>0.88 ± 0.07</td>
<td>0.982 ± 0.010</td>
<td>−0.016 ± 0.008</td>
<td>122/96</td>
</tr>
<tr>
<td>( K^+p )</td>
<td>0.51 ± 0.03</td>
<td>0.85 ± 0.11</td>
<td>0.960 ± 0.019</td>
<td>0.006 ± 0.015</td>
<td>114/96</td>
</tr>
<tr>
<td>( (\pi^*/K^+)p )</td>
<td>0.51 ± 0.02</td>
<td>0.86 ± 0.05</td>
<td>0.972 ± 0.009</td>
<td>−0.006 ± 0.007</td>
<td>143/96</td>
</tr>
<tr>
<td>With Coulomb corrections</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \pi^+p )</td>
<td>0.53 ± 0.02</td>
<td>1.00 ± 0.07</td>
<td>0.987 ± 0.010</td>
<td>−0.024 ± 0.010</td>
<td>126/96</td>
</tr>
<tr>
<td>( K^+p )</td>
<td>0.53 ± 0.03</td>
<td>1.02 ± 0.13</td>
<td>0.967 ± 0.018</td>
<td>−0.005 ± 0.018</td>
<td>115/96</td>
</tr>
<tr>
<td>( (\pi^*/K^+)p )</td>
<td>0.52 ± 0.02</td>
<td>0.99 ± 0.06</td>
<td>0.977 ± 0.009</td>
<td>−0.013 ± 0.009</td>
<td>149/96</td>
</tr>
</tbody>
</table>

In the following, the measured two-, three- and four-particle densities are analyzed in the framework of the quantum statistical approach of partially coherent source radiation. The data of Figs. 1 and 2 are fitted, respectively, by (17)-(19) multiplied by a background factor \( g(1 + \delta_0 Q^2) \), and (22)-(24) multiplied by \( g(1 + \delta_0 Q^2) \), where parameters \( \gamma_0 \) and \( \delta_0 \) have the same meaning as in (28). All parameters are fitted for every order \( q \), separately. The fit results are given in Tables 3 and 4 and shown as curves in Figs. 1 and 2.

For a verification, the FRITIOF-7 model [35] is used with two-particle BE parametrization according to JETSET 7.3 [36]. An exponential parametrization is used of the form \( R(Q) = 1 + \lambda \exp(-r Q) \) with parameter values \( \lambda = 0.30 \) and \( r = 0.7 \) fm. These values are lower than the values \( \lambda = 0.4 \) and \( r = 0.8 \) fm used in earlier NA22 publications, since the latter overestimate the higher order correlations. The generated events are subject to the same selection criteria as the real data. The MC results corresponding to the data in Tables 3 and 4 are given in Tables 5 and 6, respectively. No Coulomb repulsion is used in the MC and, therefore, no correction according to (27) is needed. The MC results are in reasonable agreement with the data and are indistinguishable from the experimental fits shown in Figs. 1 and 2.

3.2 The \( q \)-dependence of the radius \( r \)

In Fig. 3, the parameters \( r \) and \( p \) are presented as a function of \( q \), the order of the correlation. Also shown are data for \( pp \) collisions at \( \sqrt{s} = 630 \) and 900 GeV [24]. In these experiments a substantial increase of \( r \) with increasing \( q \) is observed. The NA22 data exhibit a similar trend but with smaller statistical significance. Moreover, the FRITIOF results, also plotted in Fig. 3, indicate a \( q \)-dependence quite similar to our data.

In the quantum-statistical model discussed in Sect. 1, the parameters \( r \) and \( p \) are supposed to be the same for all orders, in clear contradiction with the trend of the combined data on \( r \). This, however, does not necessarily invalidate the QS approach as such, in view of several simplifying assumptions underlying (17-19) and (22-24) such as: symmetric configuration of \( q \) particles, pointlike coherent source, stationary source, additivity (as opposed to multiplicativity) of coherent and chaotic components, etc. [13-15].

We have checked, whether the increase of the radius with increasing order is due to the symmetric pair approximation used in (17)-(19) and (22)-(24), while the experimental data contain also non-symmetric pairs. We studied the 3rd and 4th order correlations for "quasi-symmetric" triplets (requiring for each pair \( Q_i/Q_j = 1/3 \pm 1/6 \) and...
Table 3. The results of fitting the data sample by the functions (17) to (19) multiplied by $\gamma_q(1 + \delta Q^2)$

<table>
<thead>
<tr>
<th>Order $q$</th>
<th>$r$ (fm)</th>
<th>$p$</th>
<th>$\gamma$</th>
<th>$\delta$ (GeV$^{-2}$)</th>
<th>$\chi^2/NDF$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Without Coulomb corrections</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.80 ± 0.03</td>
<td>0.19 ± 0.01</td>
<td>0.953 ± 0.004</td>
<td>0.035 ± 0.006</td>
<td>113/96</td>
</tr>
<tr>
<td>3</td>
<td>0.86 ± 0.03</td>
<td>0.15 ± 0.01</td>
<td>0.969 ± 0.010</td>
<td>-0.004 ± 0.007</td>
<td>140/96</td>
</tr>
<tr>
<td>4</td>
<td>1.17 ± 0.07</td>
<td>0.19 ± 0.03</td>
<td>1.081 ± 0.020</td>
<td>-0.070 ± 0.011</td>
<td>106/95</td>
</tr>
<tr>
<td></td>
<td>With Coulomb corrections</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.83 ± 0.03</td>
<td>0.22 ± 0.01</td>
<td>0.954 ± 0.004</td>
<td>0.030 ± 0.005</td>
<td>123/96</td>
</tr>
<tr>
<td>3</td>
<td>0.87 ± 0.03</td>
<td>0.17 ± 0.01</td>
<td>0.974 ± 0.009</td>
<td>-0.013 ± 0.007</td>
<td>145/96</td>
</tr>
<tr>
<td>4</td>
<td>1.07 ± 0.07</td>
<td>0.18 ± 0.02</td>
<td>1.060 ± 0.024</td>
<td>-0.070 ± 0.013</td>
<td>98/95</td>
</tr>
</tbody>
</table>

Table 4. The results of fitting the data sample by the functions (22) to (24) multiplied by $\gamma_q(1 + \delta Q^2)$

<table>
<thead>
<tr>
<th>Order $q$</th>
<th>$r$ (fm)</th>
<th>$p$</th>
<th>$\gamma$</th>
<th>$\delta$ (GeV$^{-1}$)</th>
<th>$\chi^2/NDF$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Without Coulomb corrections</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.98 ± 0.08</td>
<td>0.56 ± 0.05</td>
<td>0.876 ± 0.018</td>
<td>0.122 ± 0.022</td>
<td>57/52</td>
</tr>
<tr>
<td>3</td>
<td>1.29 ± 0.11</td>
<td>0.49 ± 0.06</td>
<td>0.844 ± 0.044</td>
<td>0.102 ± 0.035</td>
<td>65/49</td>
</tr>
<tr>
<td>4</td>
<td>1.82 ± 0.13</td>
<td>0.55 ± 0.53</td>
<td>0.984 ± 0.074</td>
<td>-0.050 ± 0.051</td>
<td>47/44</td>
</tr>
<tr>
<td></td>
<td>With Coulomb corrections</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.95 ± 0.07</td>
<td>0.69 ± 0.08</td>
<td>0.879 ± 0.016</td>
<td>0.113 ± 0.020</td>
<td>68/52</td>
</tr>
<tr>
<td>3</td>
<td>1.25 ± 0.08</td>
<td>0.63 ± 0.08</td>
<td>0.851 ± 0.039</td>
<td>0.083 ± 0.025</td>
<td>65/49</td>
</tr>
<tr>
<td>4</td>
<td>1.70 ± 0.14</td>
<td>0.38 ± 0.42</td>
<td>0.982 ± 0.082</td>
<td>-0.063 ± 0.054</td>
<td>48/44</td>
</tr>
</tbody>
</table>

Table 5. The results of fitting the FRITIOF sample by the functions (17) to (19) multiplied by $\gamma_q(1 + \delta Q^2)$

<table>
<thead>
<tr>
<th>Order $q$</th>
<th>$r$ (fm)</th>
<th>$p$</th>
<th>$\gamma$</th>
<th>$\delta$ (GeV$^{-2}$)</th>
<th>$\chi^2/NDF$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.92 ± 0.02</td>
<td>0.17 ± 0.01</td>
<td>0.951 ± 0.002</td>
<td>0.060 ± 0.003</td>
<td>110/96</td>
</tr>
<tr>
<td>3</td>
<td>0.84 ± 0.02</td>
<td>0.13 ± 0.01</td>
<td>0.951 ± 0.006</td>
<td>0.021 ± 0.005</td>
<td>94/96</td>
</tr>
<tr>
<td>4</td>
<td>1.07 ± 0.04</td>
<td>0.17 ± 0.01</td>
<td>1.053 ± 0.014</td>
<td>-0.068 ± 0.007</td>
<td>188/95</td>
</tr>
</tbody>
</table>

Table 6. The results of fitting the FRITIOF sample by the functions (22) to (24) multiplied by $\gamma_q(1 + \delta Q^2)$

<table>
<thead>
<tr>
<th>Order $q$</th>
<th>$r$ (fm)</th>
<th>$p$</th>
<th>$\gamma$</th>
<th>$\delta$ (GeV$^{-1}$)</th>
<th>$\chi^2/NDF$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.08 ± 0.01</td>
<td>0.44 ± 0.03</td>
<td>0.866 ± 0.010</td>
<td>0.162 ± 0.013</td>
<td>78/52</td>
</tr>
<tr>
<td>3</td>
<td>1.18 ± 0.08</td>
<td>0.41 ± 0.02</td>
<td>0.772 ± 0.037</td>
<td>0.199 ± 0.043</td>
<td>31/50</td>
</tr>
<tr>
<td>4</td>
<td>1.33 ± 0.08</td>
<td>0.47 ± 0.11</td>
<td>0.729 ± 0.105</td>
<td>0.144 ± 0.109</td>
<td>75/44</td>
</tr>
</tbody>
</table>
quadruplets (requiring $Q_2^2/Q_1^2 = 1/6 \pm 1/6$). We find (with Coulomb corrections included) $r_3 = 0.97 \pm 0.07$ fm and $r_4 = 1.15 \pm 0.08$ fm for the Gaussian parametrizations (18) and (19) and $r_3 = 1.39 \pm 0.26$ fm and $r_4 = 1.76 \pm 0.18$ fm for the exponential parametrizations (23) and (24). So, the increase for "quasi-symmetric" combinations is not less pronounced than for the total sample (cf. Tables 3 and 4).

We have checked, furthermore, whether the increase of the radius with increasing order is caused by the fact that somewhat different samples are used to study correlations of different orders ($\geq 6$ prong events for $2^{nd}$ order, $\geq 8$ for $3^{rd}$ order and $\geq 10$ for $4^{th}$ order). We analyzed the $2^{nd}$ order correlations for $\geq 8$ and $\geq 10$ prong events and $3^{rd}$ order correlations for $\geq 10$ prong events and find results practically the same as those presented in Tables 3 and 4.

3.3 Genuine three-particle correlations

As mentioned in Sect. 2, our data allow to extract the normalized genuine three-particle correlation function $K_3(Q_3^2)$. The function $K_3(Q_3^2) + 1$ is shown in Fig. 4a after Coulomb correction. A non-zero $K_3$ is observed for $Q_3^2 < 0.2$(GeV/c)$^2$. We have checked that the effect is also present before Coulomb correction.

We now investigate whether the observed genuine three-particle correlation can be fully expressed in terms of the exponentially (29) 

$$F(Q^2) = \frac{1}{2} \left[ 1 + \gamma_1 + \gamma_2 \exp(-Q_2^2/Q_1^2) \right] (1 + \delta Q_2^2). \tag{29}$$

Fig. 3. The extracted parameters $r$ and $a$ of Gaussian (a,c) and exponential (b,d) parametrization as a function of the order $q$ of the correlation, for UA1 at 630 and 900 GeV [24] and for this experiment compared to FRITIOF with BE at 22 GeV.

We, therefore, fit the data of Fig. 4 by (29) and compare the resulting $r_2$ and $\lambda_2$ to the values given above. The fit results, practically the same for normalization method I and II described above, are presented in Table 7. Considering the large errors, the resulting parameters $r_2$ and $\lambda_2$ do not contradict those of the two-particle correlations and, therefore, do not allow to reveal new information on the phase of the Fourier transform $F(Q_2^2)$.

The parameter values used in FRITIOF provide "output" results consistent with those given in the last line of Table 1. The function $K_3(Q_3^2) + 1$ extracted from FRITIOF events is presented in Fig. 4b. It contains a noticeably smaller deviation from unity than observed for the real events in Fig. 4a. The fit by (29) leads to the very small value of $\lambda_3 = 0.07 \pm 0.03$, indicating that no significant "genuine" correlations are simulated in the three-particle density.

An alternative method is applied in [23]. Using the simplest parametrization for the two-particle correlation (cf. (28))
Table 7. The results of fitting the normalized three particle correlation function by (29)

<table>
<thead>
<tr>
<th>normalization method</th>
<th>$r_2$(fm)</th>
<th>$\lambda_2$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
<th>$\chi^2/NDF$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without Coulomb corrections</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>1.22*0.25-0.33</td>
<td>0.33 ± 0.15</td>
<td>1.000 ± 0.008</td>
<td>-0.002 ± 0.007</td>
<td>77/96</td>
</tr>
<tr>
<td>II</td>
<td>1.24*0.28-0.37</td>
<td>0.32 ± 0.15</td>
<td>1.007 ± 0.008</td>
<td>-0.001 ± 0.007</td>
<td>77/96</td>
</tr>
<tr>
<td>With Coulomb corrections</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>1.24*0.21-0.27</td>
<td>0.40 ± 0.15</td>
<td>0.999 ± 0.008</td>
<td>-0.001 ± 0.007</td>
<td>76/96</td>
</tr>
<tr>
<td>II</td>
<td>1.25*0.23-0.28</td>
<td>0.39 ± 0.15</td>
<td>1.004 ± 0.008</td>
<td>-0.001 ± 0.007</td>
<td>76/96</td>
</tr>
</tbody>
</table>

\[ K_2(Q_2^2) = \lambda_2 \exp(-r_2^2Q_2^2), \quad (30) \]

neglecting the phase factor of the Fourier transform $F(Q_{ij})$ and replacing the latter in (10) and (11) by $F(Q_{ij}) = \sqrt{K_2(Q_{ij}^2)}$, one obtains for the normalized three-particle density

\[ R_3(Q_{12}^2, Q_{13}^2, Q_{23}^2) = 1 + \lambda_2 \sum_{i<j}^3 \exp(-r_{ij}^2Q_{ij}^2) \]

\[ + 2\lambda_2^{3/2} \exp(-r_{ij}^2/2) \sum_{i<j}^3 Q_{ij}^2. \quad (31) \]

The normalized density $R_3(Q_3^2)$ or $Q_3$ can be recalculated by means of (25) after weighting the denominator by (31). The parameter values $r_2 = 0.82 \pm 0.03$ fm and $\lambda_2 = 0.34 \pm 0.03$, experimentally determined from the sample of events with at least three $\pi^-$-mesons, are close to the values presented in Table 1 for the total event sample.

In [23] this procedure has revealed small irregular deviations from the constant value $R_3(Q_3^2) = 1$ expected for the case, that the three-particle correlation is completely expressed in terms of two-particle correlations. Our data for the weighted densities $R_3(Q_3^2)$ and $R_3(Q_3)$ are presented in Fig. 5. They indeed indicate a small three-particle interference effect not described in terms of two-particle correlations.

At $q_0 < 0.3$ GeV, we extract the two-particle correlation parameters, $r_2 = 1.01 \pm 0.02$ fm and $\lambda_2 = 0.44 \pm 0.03$, larger than those obtained without applying a $q_0$-cut. These values are used for weighting by (31) the densities $R_3(Q_3^2)$ and $R_3(Q_3)$ for triplets, in which all three doublets simultaneously satisfy the restriction $q_0 < 0.3$ GeV (not shown). We again observe small, but statistically significant deviations from the expectation based on two-particle correlations.

4 Summary

A study of Bose-Einstein correlations up to fourth order has been performed in $(\pi^+/K^+)$-interactions at 250 GeV/c with the help of the EHS spectrometer. Genuine third-order correlations are observed which, except for small effects, can be described in terms of second-order correlations.

The data on second to fourth order correlations are compared with FRITIOF and satisfactory agreement is observed. The data are also analysed in the framework of a simplified optical model, based on the quantum statistics of a partially coherent radiation source. Our data are only marginally in agreement with this model.

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versity, Finland, and the University of Warsaw and Institute of Nuclear Problems, Poland for early contributions to this experiment. This work is part of the research program of the "Stichting voor Fundamenteel Onderzoek der Materie (FOM)", which is financially supported by the "Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO)". We further thank NWO for support of this project within the program for subsistence to the former Soviet Union (07-13-038).

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