Endomorphisms of the Plane Sending Linear Coordinates to Coordinates

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Let k be a field of characteristic zero. We show that an endomorphism of $k[X_1, X_2]$ which sends each linear coordinate to a coordinate is an automorphism of $k[X_1, X_2]$.

In [2] the authors raised the following question (referred to as Problem 1): let k be a field of characteristic zero and $A := k[X_1, \cdots, X_n]$ the polynomial ring over k. Is every k-endomorphism of A which sends each coordinate of A to a coordinate of A an automorphism of A? (A polynomial f of A is called a *coordinate* of A if there exist $F_2, \cdots, F_n$ in A such that $A = k[f, F_2, \cdots, F_n]$.) Problem 1 was answered affirmatively for $n = 2$ in [2]. The case $n \geq 3$ remains open for arbitrary k. However, as was observed by Derksen, a negative answer to Problem 1 for algebraically closed fields would give a counterexample to the Jacobian Conjecture. More explicitly, he shows that (see Lemma 2.4 in [2])if a k-endomorphism φ of A sends linear coordinates to coordinates, then $\det J\varphi(x) \neq 0$ for all $x \in k^n$, so $\det J\varphi \in k^*$ if k is algebraically closed! (A polynomial in A is called a *linear coordinate* if it is of the form $c_1X_1 + \cdots + c_nX_n$ for some c_i in k, not all zero). Based on Derksen's Lemma, Jelonek[3] gives a positive answer in any dimension to Problem 1 for algebraically closed fields! (He shows that the coordinate preservation property implies that φ is proper which together with $\det J\varphi \in k^*$ shows that φ is an automorphism of A.)

In [4], Mikhalev, Yu and Zolotykh, motivated by Derksen's Lemma, consider the following stronger version of Problem 1, referred to as Problem 2: is every endomorphism of A which sends every linear coordinate to a coordinate an automorphism? They show that if k is a non-algebraically closed field, then the answer to Problem 2 is negative for all $n \geq 3$. However the answer to Problem 2 for non-algebraically closed fields remained open for $n = 2$. (The algebraically closed case was already proved in [2].) The aim of this paper is

Preprint submitted to Elsevier Preprint *4 July 1997*

to fill this gap by giving an affirmative answer. More precisely we show that for every field k (char $k = 0$) an endomorphism of $k[X, Y]$ which sends each linear coordinate to a coordinate is an automorphism.

The proof of this result is based on a modified version of a result due to Rabier[5], where a useful characterization of a coordinate of $k[X, Y]$ is given. An easy proof of this modified version is given in Section 3.

1 The Main Result

Throughout this section k denotes a field of characteristic zero and $R := k[X, Y]$ the polynomial ring in two variables over k. The main result of this paper is the following.

Theorem 1.1 *Let φ be a k-endomorphism of R which sends each linear coordinate to a coordinate. Then φ is an automorphism of R.*

The proof of this theorem is based on the following proposition, a modified version of Theorem 2.1 in Rabier[5]. The proof will be given in Section 3 (see Proposition 3.1).

Proposition 1.2 *Let P be a coordinate of R with $n := \deg_Y P > 0$ and let $D := P_Y \partial_X - P_X \partial_Y$. Then $D^n(X) \in k$.*

Proof of Theorem 1.1

i) Let $P := \varphi(X)$. So P is a coordinate of R. Hence there exists an automorphism ψ of R with $\psi(X) = P$. Then $\psi^{-1}\varphi$ is an endomorphism of R sending X to X. Furthermore, if f is a linear coordinate of R, then $\psi(f)$ is a coordinate, whence $\psi^{-1}\varphi(f)$ is a coordinate too. Consequently, replacing φ by $\psi^{-1}\varphi$ we may assume that $\varphi(X) = X$.

(ii) So let $\varphi(X) = X$ and $g = \varphi(Y)$. The hypothesis implies that $g - cX = \varphi(Y - cX)$ is a coordinate of R for all $c \in k$. It follows easily that $n := \deg_Y g > 0$. Write $g = g_n Y^n + g_{n-1} Y^{n-1} + \cdots + g_1 Y + g_0$, with $g_i \in k[X]$ for all i. Since g is a coordinate, it follows from Corollary 1.4 of [1], that $g_n \in k^*$. Replacing Y by $Y - \frac{1}{n}\frac{g_{n-1}}{g_n}$, we may assume that $g_{n-1} = 0$. Repacing g by $g_n^{-1} g$, we may also assume that $g_n = 1$. In the remainder of this proof we will show that $n = 1$ (which concludes the proof, since if $n = 1$ then $\varphi = (X, g_1 Y + g_0)$ with $g_1 \in k^*$, so φ is an automorphism).

iii) So from now on we assume that $n \geq 2$, $g_n = 1$ and $g_{n-1} = 0$ and we will derive a contradiction. Since $g - cX$ is a coordinate of R for all $c \in k$, it follows from Proposition 1.2 that

$$(D + c\partial_Y)^n(X) \in k \tag{1.1}$$

for all $c \in k$ where $D := g_Y\partial_X - g_X\partial_Y$. Let T be a new variable and consider the polynomial

$$H(T) := (D + T\partial_Y)^n \in k[X, Y][T].$$

It follows from (1.1) and Lemma 1.3 below that $H(T) \in k[T]$. In particular, $A :=$ the T^{n-2}-coefficient of $H(T)$ belongs to k. From Proposition 2.1 of Section 2, we deduce that $g_X \in k$. Consequently $g = \lambda X + a(Y)$, for some $\lambda \in k$ and $a(Y) \in k[Y]$. However $g - cX$ is a coordinate of R for all $c \in k$. So, in particular, $a(Y) = g - \lambda X$ is a coordinate of R, a contradiction since $\deg a(Y) = n \geq 2$.

Lemma 1.3 *Let k be an infinite field and $f_0, \cdots, f_n \in k[X, Y]$. If $f(T) := \sum f_p T^p$ (where T is a new variable) is such that $f(c) \in k$ for all $c \in k$, then $f_p \in k$ for all k, i.e., $f(T) \in k[T]$.*

PROOF. Let $i, j \in \mathbf{N}$ with $i + j > 0$ and let $f_{p,i,j}$ be the coefficient of the monomial X^iY^j appearing in f_p. Then the hypothesis implies that $\sum f_{p,i,j}c^p = 0$ for all $c \in k$. Since k is infinite this implies that $f_{p,i,j} = 0$ for all p and all $i, j \geq 0$ with $i + j > 0$, i.e. $f_p \in k$ for all p.

2 A Technical Proposition

Throught this section we have the following notation: $g := Y^n + g_{n-2}Y^{n-2} + \cdots + g_1Y + g_0 \in k[X, Y]$ with $n \geq 2$ and $g_i \in k[X]$ for all i. Put $D := g_Y\partial_X - g_X\partial_Y$, $D(T) := D + T\partial_Y$, where T is a new variable and $\partial := \partial_Y$. Furthermore define $D_i := \partial^i(g_Y)\partial_X - \partial^i(g_X)\partial_Y$, for all $i \geq 0$. By Leibniz rule (and induction on k) one obtains

$$\partial^k D = \sum_{i=0}^{k} \binom{k}{i} D_i \partial^{k-i} \tag{2.1}$$

for all $k \geq 0$. Let $A :=$ the T^{n-2}-coefficient of $D(T)^n(X)$. Since $D(T)^n(X) = D(T)^{n-1}(g_Y)$, we obtain

$$A = \sum_{k=0}^{n-2} \partial^k D \partial^{n-k-2}(g_Y) = \sum_{k=0}^{n-2} \partial^k D \partial^{n-k-1}(g). \tag{2.2}$$

So by (2.1) and (2.2) we get

$$A=\sum_{k-0}^{n-2}\sum_{i=0}^{k}\binom{k}{i}D_i\partial^{n-i-1}(g). \tag{2.3}$$

Proposition 2.1 *If $A\in k$, then $g_X\in k$.*

PROOF. i) We first show that $g'_j=0$ for all $j\geq 1$. Suppose the contrary and let $j\geq 1$ be maximal such that $g'_j\neq 0$. We will show that the Y^j-coefficient of A is nonzero, contradicting that $A\in k$. Since $g_X=g'_jY^j+$ lower order Y-terms, one easily verifies that $\deg_Y D_i\partial^{n-i-1}(g)\leq j$, for all $0\leq i\leq n-2$. By (2.3), $\deg_Y A\leq j$. So to show that the Y^j-coefficient of A is nonzero it suffices to show that $\partial^jA\neq 0$. From (2.2) and (2.1) we deduce

$$\begin{aligned}\partial^jA&=\sum_{k=0}^{n-2}\partial^{j+k}D\partial^{n-k-1}(g)\\&=\sum_{k=0}^{n-2}(\sum_{i=0}^{j+k}\binom{j+k}{i}D_i\partial^{j+k-i}\partial^{n-k-1}(g)\\&=\sum_{k=0}^{n-2}\sum_{i=0}^{j+k}\binom{j+k}{i}D_i\partial^{j+n-i-1}(g).\end{aligned}$$

Furthermore, it is left to the reader to verify (using again that $g_X=g'_jY^j+$ lower order Y-terms) that

$$D_i(\partial^{j+n-i-1}(g))=0 \tag{2.4}$$

for $i\neq j$, and

$$D_j(\partial^{n-1}(g))=-j!n!g'_j. \tag{2.5}$$

Consequently, $\partial^jA=\sum_{k=0}^{n-2}\binom{j+k}{j}(-j!)n!g'_j\neq 0$, as desired.

ii) By i) we get $g=a(Y)+g_0(X)$ where $a(Y)=Y^n+$ lower order Y-terms. Consequently, $D=a'(Y)\partial_X-g'_0(X)\partial_Y$ and by (2.2)

$$\begin{aligned}A&=\sum_{k=0}^{n-2}\partial^kDa^{(n-k-1)}(Y)\\&=\sum_{k=0}^{n-2}\partial^k(-g'_0(X)a^{(n-k)}(Y))\end{aligned}$$

$$= -g_0'(X)(n-1)a^{(n)}(Y)$$
$$= -(n-1)(n!)g_0'(X).$$

Since $n \geq 2$ and $A \in k$, this implies that $g_0' \in k$, whence $g_X \in k$.

3 A Modification of A Result of Rabier

In [5], Rabier proved the following result: let $P \in R := k[X_1, X_2]$ with $\deg P = n \geq 1$ and put $D_P := P_{X_2}\partial_{X_1} - P_{X_1}\partial_{X_2}$. If P is a coordinate of R, then $(D_P)^n(X_1)$ and $(D_P)^n(X_2)$ belong to k and not both are zero. Furthermore it is shown in [5] that the converse is true as well. In the proof of Theorem 1.1 we need the following modified version of the result mentioned above.

Proposition 3.1 *Let P be a coordinate of R with* $\deg_{X_2} P = n \geq 1$. *Then* $(D_P)^n(X_1) \in k^*$.

PROOF. Put $F_1 := P$ and let (F_1, F_2) be an automorphism of R with $\det J(F_1, F_2) = 1$. Then $D_P = -\frac{d}{dF_2}$ and if (G_1, G_2) denotes the inverse of (F_1, F_2) then $X_1 = G_1(F_1, F_2)$. So

$$(D_P)^n(X_1) = (-1)^n (\frac{d}{dF_2})^n G_1(F_1, F_2) \tag{3.1}$$

By (1.6) and (1.8) of [1] we have $n = \deg_{X_2} F_1 = \deg_{Y_2} G_1(Y_1, Y_2) = \deg_{F_2}$ $G_1(F_1, F_2)$ or

$$n = \deg_{F_2} G_1(F_1, F_2) \tag{3.2}$$

¿From (3.1) we deduce that $(D_P)^n(X_1) = (-1)^n n! c$, where c is the coefficient of F_2^n in $G_1(F_1, F_2)$. By Corollary 1.4 [1], $c \in k^*$. So the desired result follows from (3.2).

Acknowledgement

The second author likes to thank the Department of Mathematical Sciences of Oakland University for the warm hospitality during his visit when this work was initiated.

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