On a robust and scalable linear elasticity solver based on a saddle point formulation.

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Abstract

This paper presents a newly developed iterative algorithm for solving problems of linear isotropic elasticity discretized by means of mixed finite elements. It continues work started in [2], [3], [8], [9] and [20].

The proposed method uses a pressure Schur complement approach to solve a saddle–point system arising in the mixed formulation. As an inner solver for the displacement field variables it uses an extension to the robust black–box multilevel procedure suggested in [9]. The proposed method works on a hierarchical sequence of finite element meshes to solve the problem with an arithmetic cost, nearly proportional to the dimension of the arising algebraic system. The coarsest mesh in the above sequence of meshes can consist of almost arbitrary triangular patches, which allows in practice to capture the solution even using a moderate number of successive refinement steps.

The rate of convergence of the algorithm is bounded uniformly with respect to the problem coefficients, namely the Young's modulus E and the Poisson ratio ν. This makes it possible to apply the method for a broad class of engineering problems.

KEY WORDS Linear elasticity, regularized Stokes equations, mixed finite elements, multilevel iterative methods, optimal order preconditioning, parameter–free solvers.

1 Introduction

Numerical modeling in elasticity often leads to algebraic systems with millions or even hundreds of millions of unknowns due to the complicated shape of engineering constructions and

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because of the need to model singular behavior of the solution in the case of strongly varying material coefficients. Much effort has been devoted to find numerical algorithms, which both are capable of dealing with a wide range of engineering applications and perform efficiently, but still to date this problem remains unsolved. In the present paper we continue the search for a robust solver, which is able to tackle large–scale problems arising in linear elasticity.

To understand better what kind of solver we are looking for, let us briefly outline how the problems of structural mechanics are usually solved by engineers and what limits the use of efficient algorithms, developed by the academic society. A major part of engineering applications is solved either by direct methods or by iterative solvers with incomplete LU, element-by-element, Jacobi (diagonal) or similar preconditioning. From the engineering point of view, the most attractive feature of these methods is that they do not require almost any user–provided adjustments and are relatively easy to implement and use. Most of the above methods allow the use of advanced meshing facilities available today, which makes them widely applicable in the industry. However, the computational cost and/or memory requirements significantly grow when the size of the problem increases, which prohibits their use for problems with millions of unknowns or more. On the other hand, more advanced algorithms, provided by the academic society, are usually more complicated, may involve some user-supplied parameters, require a lot of tuning and are relatively difficult to implement, especially in the case of problems with complicated geometries. Thus, there is a need to move towards a "simple" solver of a black–box type, which is robust with respect to the problem/mesh parameters and scales efficiently with the problem size. The current work is intended to make a step in this direction.

We will use a mixed variables formulation of the elasticity equation to avoid the "locking" phenomenon [11], [14], [21] and make the algorithm applicable even in the case of almost incompressible media, when the contractivity (Poisson) ratio ν tends to 1/2. We will discretize the displacement field $\mathbf{u}$ by piecewise–quadratic functions, while pressure p will be discretized by piecewise linears. Such a choice of the finite element spaces fulfills the familiar Ladyzhenskaya–Babuška–Brezzi inf–sup condition [21], [23], [24] and, therefore, allows us to avoid the locking. Moreover, this pair of finite elements leads to balanced discretization errors for $\mathbf{u}$ and p [8], which is important from the approximation point of view.

The standard finite–element discretization procedure of Stokes equations in linear elasticity leads to the following system of linear equations:

$$\begin{bmatrix} A^h & B^h \\ B^{h^T} & -M^h \end{bmatrix} \begin{bmatrix} \mathbf{u}^h \\ p^h \end{bmatrix} = \begin{bmatrix} \mathbf{g}^h \\ 0 \end{bmatrix}. \tag{1}$$

Further we will reduce this system to the pressure variable

$$S^h p^h = f^h,$$

where

$$S^h = M^h + B^{h^T} A^{h^{-1}} B^h,$$
$$f^h = B^{h^T} A^{h^{-1}} \mathbf{g}^h$$

and solve it iteratively by the preconditioned conjugate gradient method (see [5], [6], [19] and elsewhere). As shown in [17], [22] and by others, the matrix S^h is spectrally equivalent to the mass matrix for the pressure, is well–conditioned with respect to the mesh step size and can be efficiently preconditioned, for example, by the lumped pressure mass matrix. Note, that matrix S^h is never formed explicitly. We will use an inner iterative procedure to compute its action.

To calculate the matrix–vector products $S^h \times v$, which are required during the CG iterations, we need to solve systems with matrix A^h. There is a variety of efficient preconditioning techniques, which can be used to accelerate the convergence of the inner iterative solver for A^h. However, many of them (multigrid, for example) are normally sensitive to the jumps of the problem coefficients and/or quality of the discretization meshes, which makes them less attractive from a practical point of view. Other methods (ILU, for instance) are sufficiently robust, but exhibit bad parallelization properties and are not suitable for the large–scale modeling due to relatively high computational and/or memory requirements.

In the present paper we develop an efficient preconditioner for A^h, which is both scalable and robust. To make the presentation more clear, we describe the two–dimensional version of the method. However, the proposed approach can be also applied in three dimensions. Incidentally, a significant part of problems in structural mechanics concerns the modeling of engineering constructions such as bridges, tunnels, rail- and automobile- roads, pipelines, etc. Such constructions are examples of so-called "long structures", which means that their elastic behavior depends only to a minor extent (or does not depend at all) on one of the spatial dimensions. This allows one to describe the above problems in terms of two–dimensional models, which saves a lot of computational effort compared to the three–dimensional modeling. Therefore, efficient solvers for 2D problems are not only interesting from theoretical point of view, but have a wide range of applications in civil engineering.

The developed preconditioner is a special modification of the multilevel solver proposed in [9] which belongs to the family of block–diagonal AMLI methods suggested in [3]. It can be constructed in a fully automatic way starting from any coarse mesh triangulation and is quasi-optimal with respect to both convergence rate and arithmetic cost per iteration. A performance of the method depends mainly on the single parameter γ_{CBS}, which is the constant in the so-called strengthened Cauchy–Bunyakowski–Schwarz inequality; the parameter γ_{CBS} can be computed at the level of local mesh elements and, therefore, does not depend on the jumps of the coefficients between them. Moreover, for the optimality of the proposed algorithm there is no need to choose a very coarse mesh. The coarsest mesh can be relatively fine and, hence, a significant number of jumps in the coefficient function is allowed. Thus, the method is attractive from the engineering point of view and can be efficiently applied for a wide range of real–life problems.

The remainder of the paper is organized as follows. In Sections 2 and 3 we define our differential problem, construct the finite–element discretization and describe the procedure for solving algebraic problem (1). In Section 4 we present a recently developed extension to the I–AMLI method, used for solving the inner systems with A^h. In Section 5 we test the performance of the algorithm on a number of benchmark problems arising in bridge modeling. Finally, in Section 6 we give some concluding remarks.

2 Variational problem

Consider an equilibrium state of an *isotropic* material body subjected by external volume forces $\mathbf{g}$ and surface forces $\underline{\gamma}$. The equilibrium displacement field $\mathbf{u} = (u_1, u_2, u_3)$ minimizes the following energy functional:

$$\begin{aligned}\Phi(\mathbf{u}) &= \int\limits_{\Omega} \left(\mu \sum_{i,j=1}^{2} (\varepsilon_{ij}(\mathbf{u}))^2 + \frac{\lambda}{2} (\nabla \cdot \mathbf{u})^2 - \mathbf{g} \cdot \mathbf{u} \right) d\Omega - \int\limits_{\Gamma_1} \underline{\gamma} \cdot \mathbf{u} \, d\Gamma \\ &= \int\limits_{\Omega} \left(\mu \sum_{k=1}^{2} (\nabla u_k)^2 - \frac{\mu}{2} (\nabla \times \mathbf{u})^2 + \frac{\lambda}{2} (\nabla \cdot \mathbf{u})^2 - \mathbf{g} \cdot \mathbf{u} \right) d\Omega - \int\limits_{\Gamma_1} \underline{\gamma} \cdot \mathbf{u} \, d\Gamma\end{aligned}$$

in the function space $V = \left\{ \mathbf{u} \in (H^1(\Omega))^2;\ \mathbf{u} = \mathbf{0} \text{ on } \Gamma_0, \int\limits_{\Omega} \nabla \times \mathbf{u} \, d\Omega = 0 \right\}$, where

$$\underline{\underline{\varepsilon}} = [\varepsilon_{ij}],\ \varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

is the strain tensor, Γ_0 denotes a part of the boundary where Dirichlet boundary conditions are imposed and $\Gamma_1 = \partial\Omega / \Gamma_0$ is the rest of the boundary. We assume that $\Gamma_0 \neq \{\emptyset\}$. The parameters λ and μ above are positive material coefficients related to the elasticity modulus $E \geq E_0 > 0$ and Poisson (contractivity) ratio $\nu \in [\nu_0; 1/2)$, $\nu_0 > 0$ by

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \mu \frac{2\nu}{1-2\nu}.$$

According to Korn's inequality (see [18] for example), the energy functional is positive definite in V. This means that there exists a unique minimizer to $\Phi(\mathbf{u})$, $\mathbf{u} \in V$. Clearly, in the equilibrium state the variation of the functional equals zero. Thus, we have the following variational formulation equivalent to the minimization of $\Phi(\mathbf{u})$:

Seek $\mathbf{u} \in V$ such that

$$\delta\Phi(\mathbf{u}) \equiv \Phi(\mathbf{u} + \delta\mathbf{u}) - \Phi(\mathbf{u}) = 0, \quad \forall\ \delta\mathbf{u} \in V. \tag{2}$$

Taking into account the explicit form of $\Phi(\mathbf{u})$ we have the following expression for $\delta\Phi(\mathbf{u})$:

$$\begin{aligned}\delta\Phi(\mathbf{u}) &= \int\limits_{\Omega} \left(2\mu \sum_{k=1}^{2} (\nabla u_k) \cdot (\nabla \delta u_k) - \mu (\nabla \times \mathbf{u}) \cdot (\nabla \times \delta\mathbf{u}) + \lambda (\nabla \cdot \mathbf{u})(\nabla \cdot \delta\mathbf{u}) \right) d\Omega \\ &\qquad - \int\limits_{\Omega} \mathbf{g} \cdot \delta\mathbf{u} \, d\Omega - \int\limits_{\Gamma_1} \underline{\gamma} \cdot \delta\mathbf{u} \, d\Gamma.\end{aligned}$$

As we can see, for $\lambda \to \infty$ ($\nu \to 1/2$) the functional $\Phi(\mathbf{u})$ becomes very sensitive to small perturbations of the displacement field $\mathbf{u}$, which corresponds to the so-called "locking" phenomenon (see for example [10] and [14]). However, the problem can be regularized if we introduce an auxiliary variable p

$$p = \frac{\lambda}{\mu} \nabla \cdot \mathbf{u}$$

and consider the following Lagrangian $\Psi(\mathbf{u}, p)$:

$$\Psi(\mathbf{u}, p) = \int\limits_{\Omega} \left(\mu \sum_{i,j=1}^{2} (\varepsilon_{ij}(\mathbf{u}))^2 + \mu \, (\nabla \cdot \mathbf{u}) \, p - \frac{\mu}{2} \frac{\mu}{\lambda} p^2 - \mathbf{g} \cdot \mathbf{u} \right) d\Omega - \int\limits_{\Gamma_1} \underline{\gamma} \cdot \mathbf{u} \, d\Gamma. \tag{3}$$

Clearly, a saddle point of $\Psi(\mathbf{u}, p)$ corresponds to the minimum of $\Phi(\mathbf{u})$. Moreover, the variation of $\Psi(\mathbf{u}, p)$ is equal zero at the solution and, therefore, the equilibrium displacement field $\mathbf{u}$ can be found by solving the following mixed variable variational problem:

Seek $(\mathbf{u}, p) \in V \times H_0$ such that

$$\delta\Psi(\mathbf{u}, p) \equiv \Psi(\mathbf{u} + \delta\mathbf{u}, p + \delta p) - \Psi(\mathbf{u}, p) = 0 \quad \forall \; (\delta\mathbf{u}, \delta p) \in V \times L^2(\Omega), \tag{4}$$

where

$$H_0 = \left\{ p \in L^2(\Omega) : \int\limits_{\Omega} \mu \, p \, d\Omega = 0 \right\}.$$

The functional $\Psi(\mathbf{u}, p)$ is strongly convex with respect to $\mathbf{u}$ (if p is fixed) and is strongly concave with respect to p (if $\mathbf{u}$ is fixed). This implies, that it has a unique extremum in $V \times H_0$.

For convenience, let us rewrite the functional $\Psi(\mathbf{u}, p)$ in the following form:

$$\Psi(\mathbf{u}, p) = \frac{1}{2} \, a(\mathbf{u}, \mathbf{u}) + b(\mathbf{u}, p) - \frac{1}{2} \, m(p, p) - (\mathbf{u}, \mathbf{g}) - < \mathbf{u}, \underline{\gamma} >, \quad \mathbf{u} \in V, \; p \in H_0,$$

where the bilinear forms $a(\cdot, \cdot)$, $b(\cdot, \cdot)$, $m(\cdot, \cdot)$, $(\cdot, \cdot)$ and $< \cdot, \cdot >$ are defined below:

$$\begin{aligned}
a(\mathbf{u}, \mathbf{v}) &= \int\limits_{\Omega} 2\mu \sum_{i,j=1}^{2} \varepsilon_{ij}(\mathbf{u}) \, \varepsilon_{ij}(\mathbf{v}) \, d\Omega \\
&= \int\limits_{\Omega} \left(2\mu \sum_{k=1}^{2} (\nabla u_k) \cdot (\nabla v_k) - \mu \, (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) \right) d\Omega, \\
b(\mathbf{u}, p) &= \int\limits_{\Omega} \mu \, (\nabla \cdot \mathbf{u}) \, p \, d\Omega \; = \; -(\mathbf{u}, \nabla(\mu p)), \\
m(p, q) &= \int\limits_{\Omega} \mu \, \frac{\mu}{\lambda} \, pq \, d\Omega, \\
(\mathbf{u}, \mathbf{g}) &= \int\limits_{\Omega} \mathbf{u} \cdot \mathbf{g} \, d\Omega, \\
< \mathbf{u}, \underline{\gamma} > &= \int\limits_{\Gamma_1} \mathbf{u} \cdot \underline{\gamma} \, d\Gamma.
\end{aligned} \tag{5}$$

Notice, that Korn's inequality ensures also that there exists a constant $C(\Omega)$, which depends only on the domain and on the boundary conditions, such that

$$C(\Omega) \, \hat{a}(\mathbf{u}, \mathbf{u}) \leq a(\mathbf{u}, \mathbf{u}) \leq 2 \, \hat{a}(\mathbf{u}, \mathbf{u}), \quad \forall \; \mathbf{u} \in V, \tag{6}$$

where the bilinear form $\hat{a}(\cdot,\cdot)$ is defined as follows:

$$\hat{a}(\mathbf{u},\mathbf{v}) = \int_{\Omega} \mu \sum_{k=1}^{2} (\nabla u_k)\cdot(\nabla v_k)\, d\Omega. \tag{7}$$

Moreover, the following inf–sup condition (known in the literature as Ladyzhenskaya–Babuška–Brezzi, or LBB condition) holds (cf. [4],[10],[14]):

$$\sup_{\mathbf{u}\in V-\{0\}} \frac{b(\mathbf{u},p)}{\|\mathbf{u}\|_V} \geq \gamma \|p\|_H\,, \quad \forall\, p\in H_0,\ \mu(\mathbf{x})\geq\mu_0>0,\ \nu_0\leq\nu(\mathbf{x})<\frac{1}{2}, \tag{8}$$

where $\|p\|_H$ is defined as $\|p\|_H = \sqrt{m(p,p)}$. The discrete analog of this condition will be used in the next Section when we will select a pair of finite element spaces for $\mathbf{u}$ and p in order to construct an LBB–stable discretization of (4).

3 Finite element discretization and the choice of an outer iterative procedure

Using definitions (5), the variational problem (4) can be rewritten in the following form:

Seek $\mathbf{u}\in V$ and $p\in H_0$ such that

$$\left\{\begin{array}{rcrcll} a(\mathbf{v},\mathbf{u}) & + & b(\mathbf{v},p) & = & (\mathbf{v},\mathbf{g}) + <\mathbf{v},\underline{\gamma}>, & \forall\ \ \mathbf{v}\in V,\\ b(\mathbf{u},q) & - & m(p,q) & = & 0, & \forall\ \ q\in H. \end{array}\right. \tag{9}$$

The discrete formulation of (4) then reads as follows:

Seek $\mathbf{u}^h\in V^h$ and $p^h\in H_0^h$ such that

$$\left\{\begin{array}{rcrcll} a(\mathbf{v}^h,\mathbf{u}^h) & + & b(\mathbf{v}^h,p^h) & = & (\mathbf{v}^h,\mathbf{g}) + <\mathbf{v}^h,\underline{\gamma}>, & \forall\ \ \mathbf{v}^h\in V^h,\\ b(\mathbf{u}^h,q^h) & - & m(p^h,q^h) & = & 0, & \forall\ \ q^h\in H^h. \end{array}\right. \tag{10}$$

As we see from (10), it is necessary to select a pair of finite element spaces $V^h\subset V$ and $H_0^h\subset H_0$ for the displacements $\mathbf{u}^h$ and pressure p^h. To achieve a stable discrete formulation, it is crucial to choose the spaces V^h and H_0^h such that the discrete form of (8) holds:

$$\sup_{\mathbf{u}^h\in V^h} \frac{b(\mathbf{u}^h,p^h)}{\|\mathbf{u}^h\|_{V^h}} \geq \gamma^h \|p^h\|_{H^h} \geq \hat{\gamma}\|p^h\|_{H^h}, \quad \forall\, p^h\in H_0^h, \tag{11}$$

where $\hat{\gamma}$ is some positive constant, which is not directly related to γ in (8). Moreover, from the practical point of view it is important, that the constant $\hat{\gamma}$ is not very small. This means that the subspace $V^h\subset V$ for $\mathbf{u}$ must be sufficiently "rich" compared to the subspace $H_0^h\subset H_0$ for p. If condition (11) is satisfied, then the finite element solution $\mathbf{u}^h$ of (10) converges to the solution $\mathbf{u}$ of (4) with the rate bounded uniformly with respect to the problem parameters E and ν, see [10], [14], [21].

There is a variety of LBB-stable pairs of finite–element spaces. An extensive survey on them can be found, for example, in [14] or [21]. In this paper we confine ourselves to the case of $u_i^h \in \pi_2^h$, $p^h \in \pi_1^h$, where π_2^h is a space of piecewise–quadratic functions and π_1^h is a space of piecewise–linears. This pair passes the patch test of [24] and is, therefore, LBB–stable. Such choice of spaces V^h and H_0^h is highly recommended from the practical point of view in [23]. Moreover, with this choice of spaces the discretization errors for $\mathbf{u}$ and p are naturally balanced, which is clear from the following error estimate (cf. [8]):

$$\|\mathbf{u}-\mathbf{u}^h\|_V + \|p-p^h\| \le \text{Const}\left[\inf_{\mathbf{u}_I^h \in V^h} \|\mathbf{u}-\mathbf{u}_I^h\|_V + \inf_{p_I^h \in H^h} \|p-p_I^h\|\right].$$

The standard Galerkin procedure, applied to (10), leads to the algebraic system

$$\begin{bmatrix} A^h & B^h \\ B^{h^T} & -M^h \end{bmatrix} \begin{bmatrix} \mathbf{u}^h \\ p^h \end{bmatrix} = \begin{bmatrix} \mathbf{g}^h \\ 0 \end{bmatrix}, \tag{12}$$

where matrix A^h corresponds to the bilinear form $a(\mathbf{u},\mathbf{v})$ and matrices B^h and M^h are generated by the bilinear forms $b(\mathbf{u},p)$ and $m(p,q)$ respectively.

Now we have to choose an iterative procedure for solving system (12). This can be done in several different ways, such as using the exact and inexact Uzawa algorithms [1], [4], [13], preconditioned conjugate residuals method [16], by means of symmetric positive definite reformulation [12] and some others. Aiming at simplicity of the method, we will use a rather straightforward approach of inner/outer iterations for the system, reduced to the pressure variable. This method can be considered as a CG–accelerated version of the exact Uzawa algorithm. As was already mentioned in Section 1, we compute actions of the Schur complement with respect to the pressure,

$$S^h = M^h + B^{h^T} A^{h^{-1}} B^h$$

on vectors occurring during the solution process and solve the system with S^h iteratively. To compute the action of S^h we have to solve the systems with matrix A^h once during each outer iteration. We will use an iterative procedure to perform this. Hence, it is crucial to develop an efficient preconditioner A_0^h for A^h because of its strong impact on the overall performance of the solver. This will be done in the next section. Note also that the proposed approach for constructing A_0^h can be efficiently incorporated in other solvers for (12), mentioned above, where the efficiency of the preconditioner for A^h is also very important.

4 Inner solver for the displacement variables

Consider matrix A^h from (12), which corresponds to the bilinear form $a(\mathbf{u}^h,\mathbf{v}^h)$ from (10). Consider also the auxiliary matrix $\hat{A}^h$, generated by the bilinear form $\hat{a}(\mathbf{u}^h,\mathbf{v}^h)$ defined by (7). It follows from (6), that the matrices $\hat{A}^h$ and A^h are spectrally equivalent with spectral bounds depending only on the domain Ω. Further we construct an efficient preconditioner

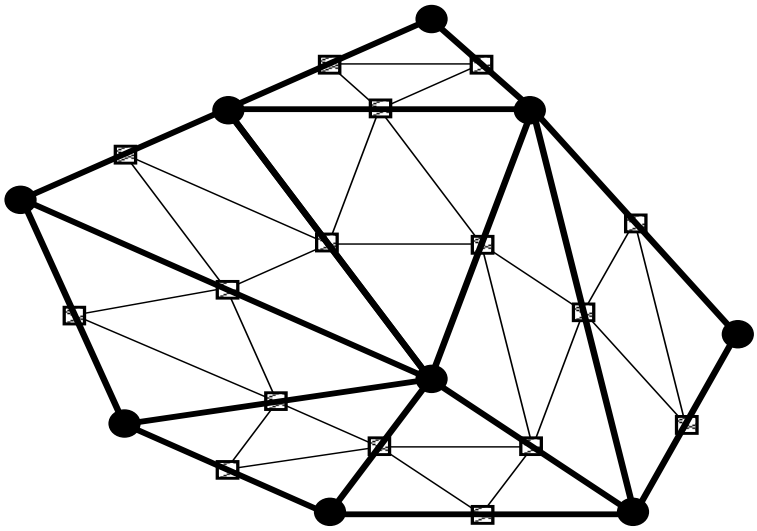

Figure 1: Uniformly refined triangular mesh

for $\hat{A}^h$ which, clearly, will be also a good preconditioner for A^h. To simplify notations, we consider the method in the two–dimensional case.

It follows from (7) that matrix $\hat{A}^h$ can be represented in the following block form

$$\hat{A}^h = \begin{bmatrix} \hat{A}_x & 0 \\ 0 & \hat{A}_y \end{bmatrix},$$

where the diagonal blocks $\hat{A}_x$ and $\hat{A}_y$ correspond to $x-$ and $y-$ components of the displacement field respectively. Clearly, each of the blocks is generated by the bilinear form

$$c(w,z) = \int_\Omega \mu \nabla w \cdot \nabla z \, d\Omega.$$

Although the matrices $\hat{A}_x$ and $\hat{A}_y$ correspond to the same bilinear form, they are not in general equal to each other due to different boundary conditions.

Now we construct spectrally equivalent preconditioners for the matrices $\hat{A}_x$ and $\hat{A}_y$. Since the proposed method for their construction does not depend on the specific choice of boundary conditions, the indices x and y will be omitted further.

Consider the following problem:

Seek w such that

$$c(w,z) = [f,z] \quad \forall\ z \in H_0^1(\Omega) = \{z \in H^1(\Omega)\ :\ z = 0 \text{ on } \Gamma_D\}. \tag{13}$$

Here Γ_D denotes the part of the boundary where the Dirichlet boundary conditions are imposed. The scalar product $[f,z]$ is defined by

$$[f,z] = \int_\Omega f z \, d\Omega.$$

For notational simplicity we assume that the domain Ω is a polygon with boundary $\partial\Omega$ and the coefficient function $\mu(\mathbf{x})$ is uniformly positive for any $\mathbf{x} \in \bar{\Omega}$. We assume also, that the boundary conditions are homogeneous and that the domain Ω is partitioned using triangles $T_i \in \Upsilon_0$ defined on an initially chosen coarse mesh Ω_0. The coefficient function $\mu(\mathbf{x})$ is assumed to be constant in each element $T_i \in \Upsilon_0$ and is allowed to be discontinuous across elements.

Problem (13) leads to the algebraic system

$$A_{\pi_2} x = b, \tag{14}$$

where matrix A_{π_2} corresponds to piecewise–quadratic discretization of (13) since $V^h \subset (\pi_2^h)^2$. Further we construct an efficient preconditioner for A_{π_2}.

If we split the bilinear form $c(u,v)$ from (13) over the mesh elements and introduce a number of local stiffness matrices $A^{(e)}_{\pi_2}$ corresponding to them, then the global stiffness matrix can be represented as the result of assembling the local matrices, i.e.

$$A_{\pi_2} = \sum_e {L^{(e)}}^T A^{(e)}_{\pi_2} L^{(e)}. \tag{15}$$

Here the matrices $L^{(e)}$ correspond to mappings from the global numbering of the unknowns to the local numbering of them within the mesh elements.

Similarly to (15) let the matrix $A_{\pi_1^{ref}}$ be

$$A_{\pi_1^{ref}} = \sum_e {L^{(e)}}^T A^{(e)}_{\pi_1^{ref}} L^{(e)}, \tag{16}$$

where matrices $A^{(e)}_{\pi_1^{ref}}$ arise from the piecewise–linear discretization of (13) on the uniformly refined mesh, see Figure 1. As was shown in [7], matrices $A^{(e)}_{\pi_1^{ref}}$ and $A^{(e)}_{\pi_2}$ can be represented in the following form:

$$A^{(e)}_{\pi_1^{ref}} = \frac{1}{2}\left[\begin{array}{ccc|ccc} c+b & 0 & 0 & 0 & -b & -c \\ 0 & a+c & 0 & -a & 0 & -c \\ 0 & 0 & a+b & -a & -b & 0 \\ \hline 0 & -a & -a & 2\,(a+b+c) & -2\,c & -2\,b \\ -b & 0 & -b & -2\,c & 2\,(a+b+c) & -2\,a \\ -c & -c & 0 & -2\,b & -2\,a & 2\,(a+b+c) \end{array}\right],$$

$$A^{(e)}_{\pi_2} = \frac{8}{3}\left[\begin{array}{ccc|ccc} 3\,(c+b) & c & b & 0 & -4\,b & -4\,c \\ c & 3\,(a+c) & a & -4\,a & 0 & -4\,c \\ b & a & 3\,(a+b) & -4\,a & -4\,b & 0 \\ \hline 0 & -4\,a & -4\,a & 8\,(a+b+c) & -8\,c & -8\,b \\ -4\,b & 0 & -4\,b & -8\,c & 8\,(a+b+c) & -8\,a \\ -4\,c & -4\,c & 0 & -8\,b & -8\,a & 8\,(a+b+c) \end{array}\right],$$

where $a = \cot(\alpha^{(e)})$, $b = \cot(\beta^{(e)})$, $c = \cot(\gamma^{(e)})$, and $\alpha^{(e)}$, $\beta^{(e)}$, $\gamma^{(e)}$ are angles of the patch e, $\gamma^{(e)} = \pi - \alpha^{(e)} - \beta^{(e)}$. Notice, that the matrices $A^{(e)}_{\pi_2}$ and $A^{(e)}_{\pi_1^{ref}}$ have the same kernel

$$\ker A^{(e)}_{\pi_2} = \ker A^{(e)}_{\pi_1^{ref}} = \mathrm{Span}([1,1,1,1,1,1]^T).$$

Consider now the generalized eigenvalue problem

$$A^{(e)}_{\pi_2}\, v = \lambda\, A^{(e)}_{\pi_1^{ref}}\, v \tag{17}$$

in the subspace orthogonal to the kernel. As we can see, there are three equal eigenvalues:

$$\lambda_1 = \lambda_2 = \lambda_3 = \frac{4}{3}.$$

The other two eigenvalues λ_4 and λ_5 are the following:

$$\lambda_4 = \frac{q(a,b,c) + \sqrt{p(a,b,c)}}{z(a,b,c)}, \quad \lambda_5 = \frac{q(a,b,c) - \sqrt{p(a,b,c)}}{z(a,b,c)},$$

where

$$q(a,b,c) = 7\,\left(b\,a^2 + c\,a^2 + a\,b^2 + \frac{18}{7}\,a\,b\,c + a\,c^2 + c\,b^2 + b\,c^2\right),$$

$$p(a,b,c) = (b+c)\,(a+c)\,(a+b)\,(b\,a^2 + c\,a^2 + a\,b^2 - 6\,a\,b\,c + a\,c^2 + b\,c^2 + c\,b^2)$$

and

$$z(a,b,c) = 9\,(b\,a^2 + c\,a^2 + a\,b^2 + \frac{7}{3}\,a\,b\,c + a\,c^2 + c\,b^2 + b\,c^2).$$

Moreover,

$$\lambda_1 = \lambda_2 = \lambda_3 \geq \lambda_4 \geq \lambda_5 \quad \forall\ \alpha^{(e)}, \beta^{(e)}, \gamma^{(e)}$$

and, hence,

$$\lambda_{\max}(A^{(e)}_{\pi_2}, A^{(e)}_{\pi_1^{ref}}) = \frac{4}{3}, \quad \lambda_{\min}(A^{(e)}_{\pi_2}, A^{(e)}_{\pi_1^{ref}}) = \lambda_5.$$

To make the presentation more clear, we plot the minimal eigenvalue of (17) versus the angles $\alpha^{(e)}$ and $\beta^{(e)}$ of the patch e in Figure 2. Since the maximal eigenvalue of (17) is constant, it follows from the definitions of A_{π_2} and $A_{\pi_1^{ref}}$ that the overall condition number $\kappa(A_{\pi_2}, A_{\pi_1^{ref}})$ can be found as

$$\kappa\left(A_{\pi_2}, A_{\pi_1^{ref}}\right) = \frac{\sup\limits_e \left[\lambda_{\max}(A^{(e)}_{\pi_2}, A^{(e)}_{\pi_1^{ref}})\right]}{\sup\limits_e \left[\lambda_{\min}(A^{(e)}_{\pi_2}, A^{(e)}_{\pi_1^{ref}})\right]} = \sup_e \left[\frac{\lambda_{\max}(A^{(e)}_{\pi_2}, A^{(e)}_{\pi_1^{ref}})}{\lambda_{\min}(A^{(e)}_{\pi_2}, A^{(e)}_{\pi_1^{ref}})}\right] = \sup_e \left[\kappa\left(A^{(e)}_{\pi_2}, A^{(e)}_{\pi_1^{ref}}\right)\right].$$

One can also show that the condition number $\kappa(A_{\pi_2}, A_{\pi_1^{ref}})$ depends on the maximal angle in the triangulation as plotted in Figure 3. Clearly, it does not depend on the jumps in the coefficient function $\mu(\mathbf{x})$ from (13).

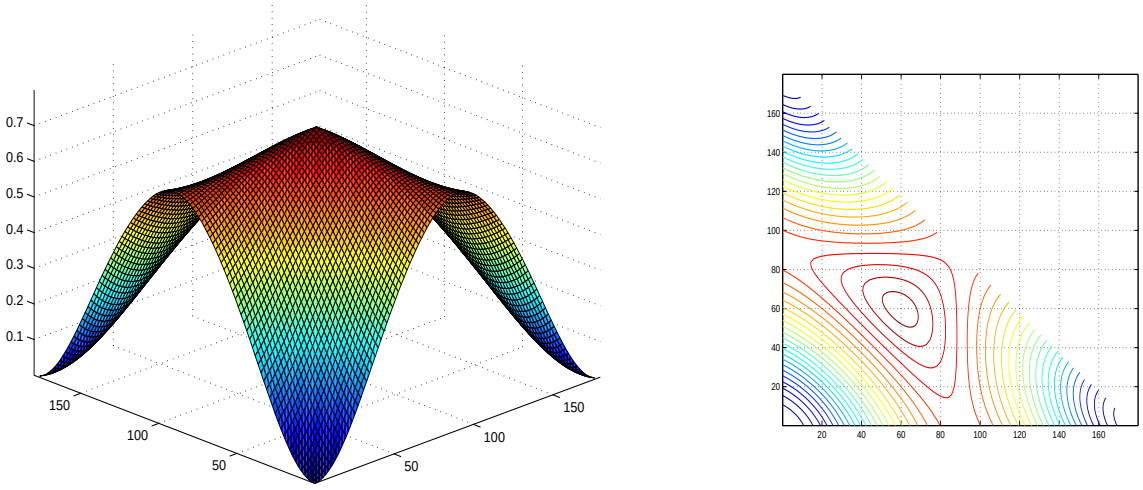

Figure 2: Minimal eigenvalue $\lambda_{\min}(A^{(e)}_{\pi_2}, A^{(e)}_{\pi_1^{ref}})$ vs. $\alpha^{(e)}$ and $\beta^{(e)}$

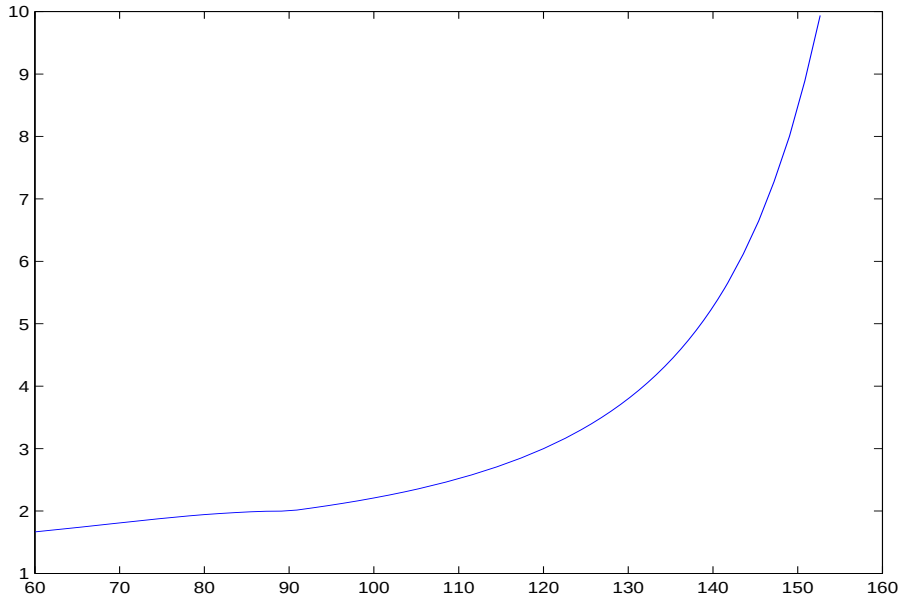

Figure 3: Condition number $\kappa(A_{\pi_2}, A_{\pi_1^{ref}})$ vs. maximal angle in the triangulation

Therefore, if we discretize our problem using a "proper" triangulation without extremely large angles, then the matrix $A_{\pi_1^{ref}}$ can be used as a preconditioner for the matrix A_{π_2}. Notice also that this restriction on the meshing procedure is very weak and can be easily fulfilled in the majority of practical cases. As we see from Figure 3, one can allow the mesh to contain angles up to approximately 150 degrees to get a reasonable value of the condition number less than 10. Thus, if we construct an efficient preconditioner to the matrix $A_{\pi_1^{ref}}$, then with the "proper" choice of triangulation we will have a good preconditioner for the matrix A^h from (12) as well.

As a preconditioner to $A_{\pi_1^{ref}}$ we will use a robust multilevel algorithm of the I–AMLI type suggested in [9]. This method is a black–box solver based on a sequence of nested CG–like iterations, which makes it attractive for industrial use. It works on a hierarchical sequence of finite element meshes to solve the problem with an arithmetic cost, nearly proportional to the dimension of the arising algebraic system. The method imposes no restrictions on the choice of our coarse mesh and, therefore, we can choose any suitable meshing procedure to discretize the problem efficiently. This issue becomes especially important, when the shape of the domain is complicated and the material coefficients E and ν have discontinuities with big jumps. The number of user–supplied adjustments in the method is also small. The only "free" parameters, which may have to be tuned in the solver are the following: a) a set of "regular" levels $k_1, \ldots, k_r$ used to stabilize the overall condition number of the preconditioned system and b) a set of stopping tolerances $\varepsilon_1, \ldots, \varepsilon_r$ on each of the regular levels. Note also, that the I–AMLI method is not sensitive with respect to the above parameters [9]. In all tests we choose a fixed accuracy for ε_i and use the asymptotic formulae from [3] to get a structure of the regular levels, which is sufficiently close to an optimal one.

With a proper choice of the meshing procedure, used for the discretization, the global condition number of the AMLI–preconditioned system (14) can be made arbitrarily close to optimal, i.e. bounded with respect to mesh step size, mesh regularity and problem coefficients. Moreover, by balancing the distance between the regularization points in the multilevel structure, the cost per I–AMLI iteration step can be made nearly proportional to the number of degrees of freedom on the finest mesh (see [3] and [9] for more details).

Hence, we have a robust and scalable preconditioner for the matrix A_{π_2} and, thus, for the matrix A^h from (12) as well. Since the Schur complement matrix S^h is spectrally equivalent to the mass matrix for the pressure variable, we have an efficient solver for the whole system (12). High performance of the proposed algorithm will be shown in the next section on a number of benchmark problems arising in bridge foundation modeling.

5 Numerical experiments

Consider a concrete wall placed in a homogeneous soil layer and subjected by external surface forces $\underline{\gamma}$ and volume forces $\mathbf{g}$, see Figure 4. This type of problems arises, for example, in many typical bridge foundation modeling applications as well as in other applications of civil engineering. Further we assume that the construction technology guarantees the ideal contact between the wall and the soil, i.e. there is no internal friction. We assume also that there is

Problem size	$\nu_s = 0.3$	$\nu_s = 0.4$	$\nu_s = 0.45$	$\nu_s = 0.49$	$\nu_s = 0.499$	$\nu_s = 0.4999$
280	2.1	2.6	2.7	2.9	3.1	3.1
1088	13.4	16.1	17.1	21.0	21.4	20.8
4288	70.1	83.3	96.9	111.3	115.1	113.2
17024	265.6	350.6	396.9	466.3	490.8	486.0
67840	1250.9	1667.9	1838.9	2087.3	2146.8	2112.7

Table 1: Computing time vs. problem size for the different values of Poisson ratio ν_s

Problem size	$\nu_s = 0.3$	$\nu_s = 0.4$	$\nu_s = 0.45$	$\nu_s = 0.49$	$\nu_s = 0.499$	$\nu_s = 0.4999$
280	17	22	23	27	27	27
1088	19	23	26	32	32	32
4288	19	24	28	32	33	33
17024	17	24	28	31	33	33
67840	17	24	27	31	32	32

Table 2: Number of outer Schur complement iterations vs. problem size for the different values of Poisson ratio ν_s

no external force acting in the direction along the wall. Therefore, the displacement field is zero in this direction and, thus, the problem can be described in terms of the two–dimensional model. Clearly, in this case the equilibrium displacement field $\mathbf{u}$ can be found as a minimizer to the energy functional $\Phi(\mathbf{u})$. We will use the mixed variable formulation (4) to avoid the "locking" phenomenon and enable finding of a correct solution even in the case of almost incompressible media, when the Poisson ration approaches 1/2. As was mentioned before, the displacement field will be discretized by means of piecewise–quadratic finite elements while the auxiliary pressure field will be discretized by means of piecewise–linears.

In the computations the volume force $\mathbf{g}$ is taken in the form $(0, -g_v)$. It represents the gravity force acting on the continuous media. A vertical external load $\underline{\gamma} = (0, -\gamma_v)$ is applied to the top of the wall. We assume that the load is uniformly distributed over the loading surface. Since our problem is symmetric with respect to the central plane of the wall, the displacement field is also symmetric and, therefore, the problem can be considered only in one half of the domain, i.e. $\Omega = [0, L_d/2] \times [0, H_d]$. The following boundary conditions are chosen in the experiments: homogeneous Dirichlet b.c. on the bottom side of the domain (displacements are equal zero), homogeneous Neumann b.c. on the rest of the boundary (stresses are equal zero), inhomogeneous Neumann b.c. on the loading surface and "rigid contact" b.c. ($u_x = 0$) on the plane of the symmetry, i.e. when $x = 0$. The Young's modulus

E and the Poisson ratio ν are defined as

$$E = \begin{cases} E_w, & 0 \le x \le L_d/2,\ 0 \le y \le H_d, \\ E_s, & \text{otherwise}, \end{cases}$$

$$\nu = \begin{cases} \nu_w, & 0 \le x \le L_d/2,\ 0 \le y \le H_d, \\ \nu_s, & \text{otherwise}. \end{cases}$$

We fix the values $L_d, H_d, L_w, H_w, E_w, E_s$ and ν_w to be the following:

$$L_d = 37.2\,m, \quad H_d = 31.0\,m, \quad L_w = 1.2\,m, \quad H_w = 15.0\,m,$$

$$E_w = 3.15 \cdot 10^{10}\,Pa, \quad E_s = 1.0 \cdot 10^{7}\,Pa, \quad \nu_w = 0.2,$$

while the value of ν_s is varied to show the robustness of the proposed algorithm with respect to the Poisson ratio. The above parameter setting corresponds to the benchmark problem defined in [15]. We discretized the problem on a sequence of irregular triangular meshes, refined near the bottom end of the wall. An illustration of the used triangulations is given in Figure 5. The inner/outer approach was used to solve a saddle–point system (12) arising in the mixed–variable formulation. A lumped mass matrix for the pressure was used as a preconditioner for the outer Schur complement CG iterations. A fully automatic version of the I–AMLI solver, described in Section 4, was applied to the inner systems with A^h. We used

$$\frac{\|\mathbf{r}\|}{\|\mathbf{r_0}\|} < 10^{-6}$$

as a relative stopping criterion for both the inner and outer iterations. Tables 1 and 2 show that the performance of the algorithm indeed depends very mildly on the Poisson ratio ν_s and computing time is almost proportional to the problem size. One of the computed solutions, corresponding to the case of $\nu_s = 0.4999$, is plotted in Figures 6 and 7. We used a Sun workstation (UltraSparc, 167 Mhz) in all our experiments.

6 Concluding remarks

An efficient inner/outer–type method has been constructed for solving the problems of linear isotropic elasticity. The presented algorithm is essentially a black–box solver, which requires only a few user–supplied adjustments and can be constructed automatically starting from any given initial mesh triangulation. The method shows a good performance in the whole range of material coefficients E and ν with no "locking" and, therefore, is attractive for industrial use. It can be efficiently applied even in the difficult case of nearly incompressible media, which frequently arises in practice for certain rubber–like materials, for instance.

Another important feature of the algorithm is that it imposes only very weak restrictions on the meshing procedure. This allows us to discretize the problem efficiently using a moderate number of refinement steps and resolve the solution with a reasonable arithmetic cost. High computing performance of the algorithm has been demonstrated on a number of benchmark problems, arising in civil engineering.

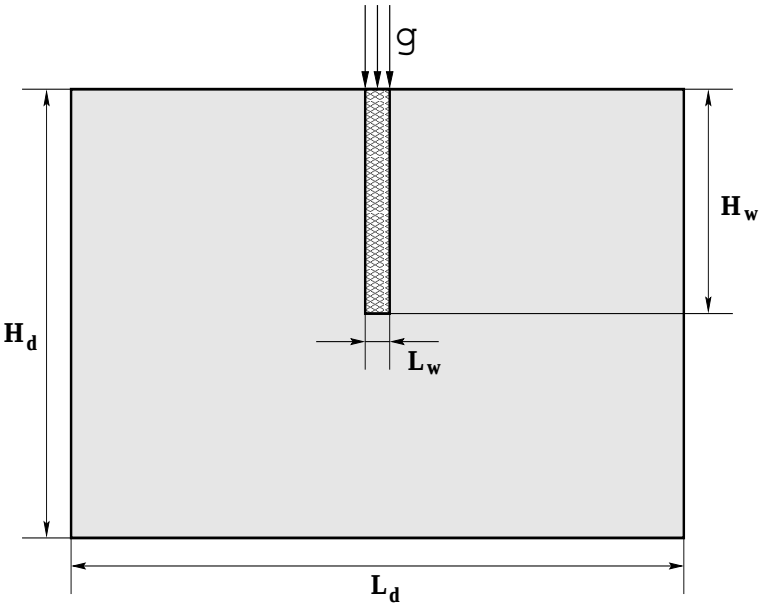

Figure 4: Vertical cross–section of the domain

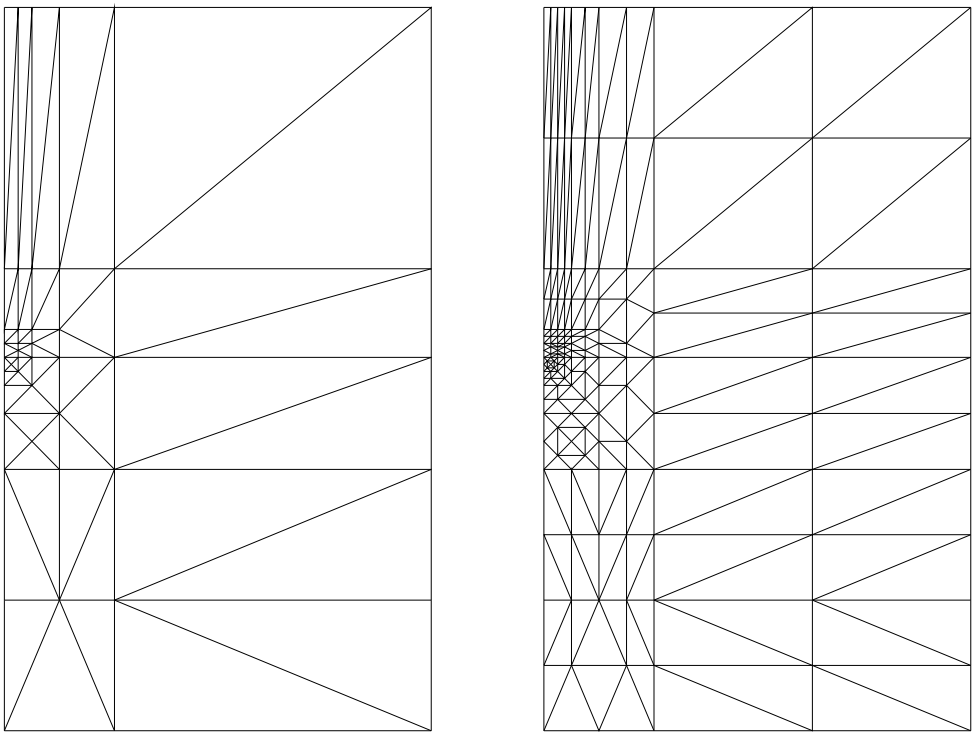

Figure 5: An example of discretization meshes used in our calculations

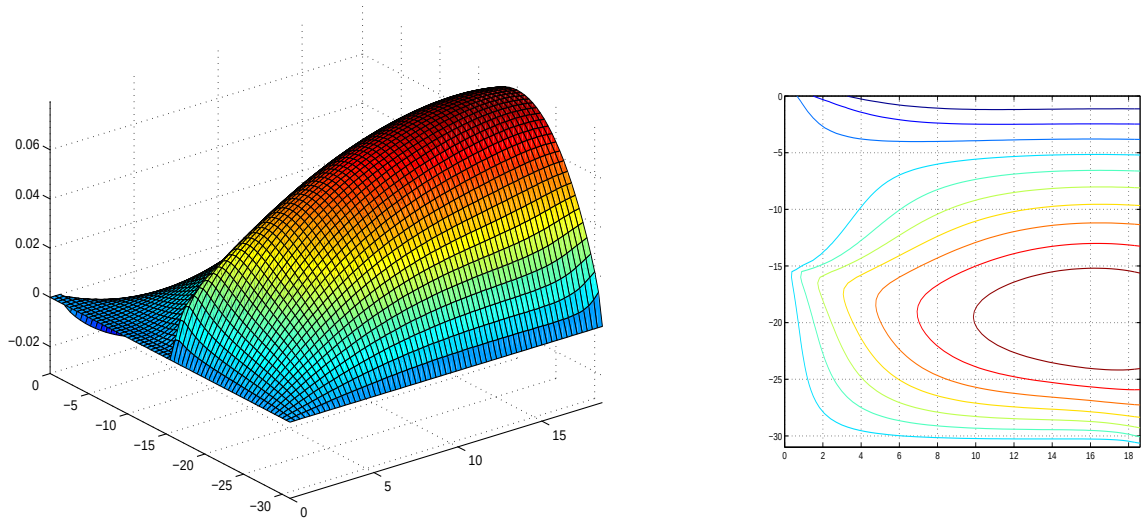

Figure 6: Horizontal component of the displacement field

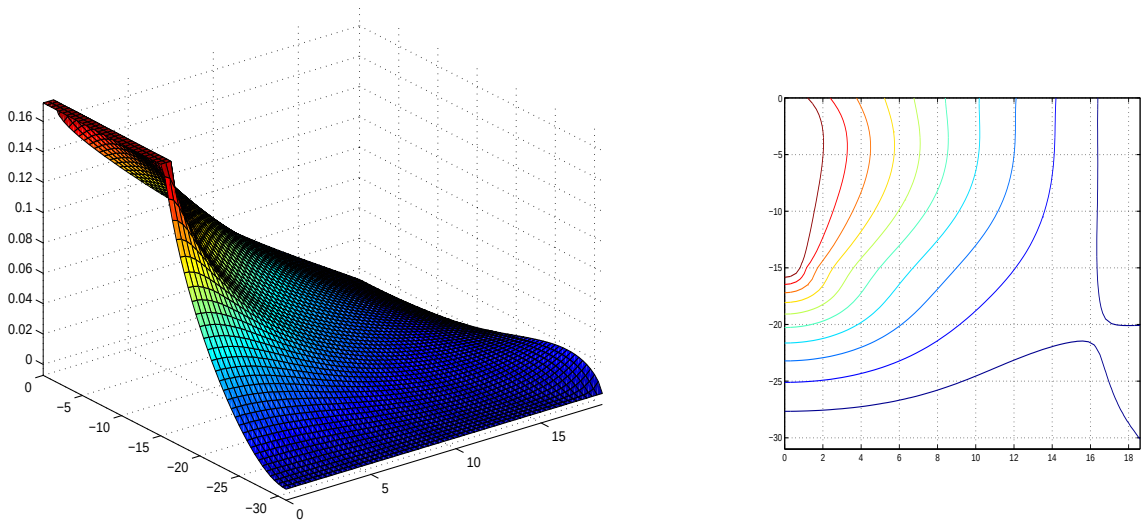

Figure 7: Vertical component of the displacement field

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