Qutrit Metaplectic Gates Are a Subset of Clifford+T

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Abstract

A popular universal gate set for quantum computing with qubits is Clifford+T, as this can be readily implemented on many fault-tolerant architectures. For qutrits, there is an equivalent T gate, that, like its qubit analogue, makes Clifford+T approximately universal, is injectable by a magic state, and supports magic state distillation. However, it was claimed that a better gate set for qutrits might be Clifford+R, where $R = \text{diag}(1, 1, -1)$ is the *metaplectic* gate, as certain protocols and gates could more easily be implemented using the R gate than the T gate. In this paper we show that the qutrit Clifford+R unitaries form a strict subset of the Clifford+T unitaries when we have at least two qutrits. We do this by finding a direct decomposition of $R \otimes \mathbb{I}$ as a Clifford+T circuit and proving that the T gate cannot be exactly synthesized in Clifford+R. This shows that in fact the T gate is more expressive than the R gate. Moreover, we additionally show that it is impossible to find a single-qutrit Clifford+T decomposition of the R gate, making our result tight.

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1 Introduction

Most theoretical work on quantum computing has focussed on qu*bits*, two-dimensional quantum systems. However, many proposed physical types of qubits are actually restricted subspaces of higher-dimensional systems, where the natural dimension can be much higher. The restriction to qubits is made for two reasons: the difficulty of precisely controlling quantum systems and the reliance on analogy to classical computers where two-valued bits reign supreme. However, as quantum control continues to improve, researchers have revisited this design choice. In some cases, higher-dimensional qu*dits* appear to be the superior option.

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For example, interest in qudit algorithms and physical implementations has risen recently, due to the potential advantages in runtime efficiency, resource requirements, computational space, and noise resilience in communication [23].

For qudits to make a good foundation for a quantum computer, we need methods to achieve fault-tolerance. In qubit-based protocols, one popular paradigm is to rely on the Clifford+T gate set. This gate set consists of the efficiently simulable Clifford gates that can be implemented directly on many error correcting codes, and the T gate that can be implemented by distilling and injecting magic states [8]. Analogous constructions have been developed for qudits of all dimensions, each one relying on a specific generalization of the Clifford+T gate set [9].

In this paper we focus on the case of qu*trit*s, three-dimensional quantum systems. While qutrits permit the qutrit Clifford+T gate set, which can be implemented fault-tolerantly on qutrit error correcting codes, analogous to the qubit setting [12], the Clifford+T gate set is not the only proposed universal fault-tolerant gate set for qutrits. In a series of papers [1, 12, 11, 5, 4, 3, 6] and a patent [7], the non-Clifford qutrit gate of choice is the R gate, also referred to as the FLIP gate, *reflection* gate, or *metaplectic* gate. It is defined as $R \coloneqq \mathrm{diag}(1, 1, -1)$. This gate was defined in Ref. [1], where it was shown to admit a magic state distillation and injection protocol. As a non-Clifford gate it achieves approximate universality when added to the Clifford gate set [16], as explicitly proved in Ref. [12, Theorem 2]. It can be implemented in a framework of certain weakly-integral non-abelian anyons via braiding and topological measurement [11, 12].

While the definition of the R gate looks very simple, containing only 1's, 0's and a -1, it is in fact nowhere in the qutrit Clifford hierarchy [10]. This is because for qutrits, Clifford gates are based on the third root of unity $\omega = e^{i2\pi/3}$. Despite this fact, the R gate can still be injected into a qutrit circuit using a repeat-until-success procedure of an R magic state which also allows a distillation protocol [1]. The R gate can hence also be realised fault-tolerantly [1]. Another construction of R is by a measurement-assisted repeat-until-success protocol requiring two ancillary qutrits to probabilistically realise it out of Clifford gates [11]. The R gate has been suggested to be "more powerful in practice" than the T gate [6]. In Ref. [6] they computed the cost of approximating the third level of the Clifford hierarchy in the Clifford+R (which they refer to as the *metaplectic*) gate set, and claimed that constructing the R gate in the Clifford+T gate set requires multiple ancillae and repeat-until-success circuits.

In this paper we find evidence in contradiction to these previous assertions. We show that while no single-qutrit Clifford+T circuit composes to an R gate unitarily[2], rather unexpectedly the R gate is exactly constructible through a unitary two-qutrit Clifford+T circuit with T-count 39, which we construct in Section 3. This demonstrates that $R \in$ Clifford+T. Additionally, we prove that the converse is not true, i.e. that $T \notin$ Clifford+R, and hence Clifford+$R \subsetneq$ Clifford+T. This directly implies any Clifford+T computation can be exactly implemented through Clifford+T gates with constant overhead, whereas there exist Clifford+T circuits whose implementation via Clifford+R must strictly increase with the desired precision.

This result might seem to contradict the fact that R does not belong anywhere in the Clifford hierarchy, while every Clifford gate and the T gate belongs to the third level C_3. But recall that while C_1 and C_2 are closed under composition, this is no longer true for the higher levels of the Clifford hierarchy. In particular, it is not true that any circuit built out of Clifford+T gates is a unitary that belongs to C_3.

[2] Unless stated otherwise, we take "single-qutrit" to mean ancilla-free.

The paper is structured as follows. We cover all the basics on qutrit quantum computation and gate synthesis in Section 2. Then in Section 3 we show how to build the R gate as a two-qutrit unitary using only Clifford+T gates and we prove that it is not possible to do this using just single-qutrit Clifford+T gates. We finish by demonstrating that T is *not* an element of Clifford+R so that Clifford+R is in fact a strict subset of Clifford+T. We end with some concluding remarks in Section 4.

2 Qutrit Clifford+T

A qubit is a two-dimensional Hilbert space. Similarly, a qutrit is a three-dimensional Hilbert space. We will write $|0\rangle$, $|1\rangle$, and $|2\rangle$ for the standard computational basis states of a qutrit. Any normalised qutrit state can then be written as

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle + \gamma\,|2\rangle \tag{1}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$.

For a comprehensive overview of quantum computing based on qudits, we refer to the 2020 review by Wang, Hu, Sanders, and Kais [23]. A qudit quantum processor has been experimentally demonstrated on ion trap systems [20] and superconducting circuits [2, 24, 26].

2.1 Pauli gates and permutation gates

Several concepts for qubits extend to qutrits, or more generally to qu*dits*, which are d-dimensional quantum systems. We are concerned with the qudit generalizations of Paulis and Cliffords.

▶ **Definition 1.** *For a d-dimensional qudit, the Pauli X and Z gates are defined as*

$$X\,|k\rangle = |k+1\rangle \qquad\qquad Z\,|k\rangle = \omega^k\,|k\rangle \tag{2}$$

where $\omega := e^{2\pi i/d}$ such that $\omega^d = 1$, and the addition $|k+1\rangle$ is taken modulo d. We define the Pauli group *as the set of unitaries generated by compositions and tensor products of the X and Z gates. We write $\mathcal{P}_n^d$ for the Pauli group on n qudits [16, 17].*

For qubits this X gate is just the NOT gate while $Z = \mathrm{diag}(1, -1)$. For the duration of this paper we will work solely with qutrits, so we take ω to always be equal to $e^{2\pi i/3}$.

For a qubit there is only one non-trivial permutation of the standard basis states, which is implemented by the X gate. For qutrits there are five non-trivial permutations of the basis states. We call these τ gates and we specify them as τ_L where L is a permutation of the elements $\{0, 1, 2\}$ written in cycle notation. For example, $\tau_{(02)}$ is the permutation which maps $|0\rangle \mapsto |2\rangle$, $|1\rangle \mapsto |1\rangle$, and $|2\rangle \mapsto |0\rangle$. The five non-trivial permutations are then $\tau_{(01)}$, $\tau_{(12)}$, $\tau_{(02)}$, $\tau_{(012)}$, and $\tau_{(021)}$ along with the trivial identity permutation $\mathbb{I} = \tau_{(0)(1)(2)}$. Compositions of these operators are given by $\tau_L \cdot \tau_M = \tau_{L \cdot M}$ with $L \cdot M$ the composition of permutations. Note that $\tau_{(012)} = X$ and $\tau_{(021)} = X^\dagger$.

2.2 Exact synthesis and number rings

One natural question to ask when given a set of gates is to determine which operations can be implemented as a circuit over those gates. This is called the *exact synthesis* problem. One frequently useful notion in addressing exact synthesis is computing the matrix representations of the set of gates in the computational basis and characterizing the *number ring* to which their entries belong. A number ring is a set of numbers which explicitly contains 0 and 1 and is closed under the operations of addition and multiplication. For example, the integers $\mathbb{Z}$ form a number ring.

We can *extend* number rings by considering what happens when we introduce new numbers to the number ring. When we extend the number ring R by α we write $R[\alpha]$ for the ring of formal sums $\sum_j r_j\alpha^j$ where $r_j \in R$. Generally, we extend by an α which is the root of some monic polynomial whose coefficients come from R. If that polynomial has degree p, then all powers of α which are greater than $p-1$ and appear in an element of $R[\alpha]$ can be reduced via that polynomial. For example, the third root of unity ω solves the monic polynomial $1+\omega+\omega^2=0$ over the integers so that we define $\mathbb{Z}[\omega]=\{a+b\omega \mid a,b\in\mathbb{Z}\}$. Any higher-order powers of ω which might appear in an element of $\mathbb{Z}[\omega]$ can be reduced through its polynomial as for example, $\omega^2=-1-\omega$ and $\omega^3=1$.

Another common way to modify number rings is to introduce new denominators by *localizing* a ring. For a number ring R we can take any multiplicatively-closed subset μ of R which contains 1 but not 0 and introduce that set of numbers as denominators:

$$\mu^{-1}R \ := \ \left\{ \frac{r}{m} \;\middle|\; r\in R \text{ and } m\in\mu \right\}.$$

▶ **Definition 2.** *The ring of* triadic fractions *is the number ring defined by localizing $\mathbb{Z}$ at the set $\mu=\{3^k \mid k\in\mathbb{N}\}$, which we denote as $\mathbb{T}:=\mu^{-1}\mathbb{Z}=\{a/3^k \mid a\in\mathbb{Z},k\in\mathbb{N}\}$.*

The use of number rings to help solve the exact synthesis problem stems from the following statement, attributable to many authors in the field but perhaps most notably to Kliuchnikov, Maslov, and Mosca [18]:

▶ **Lemma 3.** *Let $\mathcal{G}=\{G_1,\cdots,G_k\}$ be a quantum gate set. For all $j\in\{1,\ldots,k\}$, let each G_j have the computational basis matrix representation M_j up to a complex global phase such that M_j is a matrix with entries in the number ring R. Then, up to a global phase, the matrix representation of any circuit over $\mathcal{G}$ only has entries in the number ring R.*

It is important to note that Lemma 3 only suffices to *exclude* operations from being representable over a given gate set. To show that a circuit with entries in a particular number ring implies expressibility over a certain gate set is generally equivalent to providing a full solution to the exact synthesis problem.

▶ **Example 4.** Any qutrit Pauli operation in the computational basis has entries in the number ring $\mathbb{Z}[\omega]$. This follows directly from Lemma 3 and the fact that $\mathcal{P}_n^3$ is generated by X and Z.

One interesting aspect of $\mathbb{Z}[\omega]$ (and number rings which contain roots of unity in general) is that it contains elements which square to non-square integers. In particular, $(\omega-\omega^2)^2=(\omega^2-\omega)^2=-3$. Note the minus sign here, which is important as $\pm\sqrt{3}\notin\mathbb{Z}[\omega]$. Due to the ubiquity of the Pauli group and the natural appearance of ω, when working with circuits over qutrits it has become increasingly customary to use $\pm(\omega-\omega^2)=\pm i\sqrt{3}$ in place of $\sqrt{3}$ when possible. We make use of this replacement frequently (see, e.g., the Hadamard gate defined below).

2.3 Clifford gates

Another concept that translates to qutrits (or more general qudits) is that of Clifford unitaries.

▶ **Definition 5.** *Let U be a unitary acting on n qudits. We say that U is* Clifford *when every Pauli is mapped to another Pauli under conjugation by U. I.e., for any $P\in\mathcal{P}_n^d$ we have $UPU^\dagger\in\mathcal{P}_n^d$.*

Note that the set of n-qudit Cliffords forms a group under composition. For qubits, this group is generated by the S, Hadamard, and CX gates. The same is true for qutrits, for the right generalisation of these gates. To define these it will first be helpful to introduce the notion of qutrit phase gates.

▶ **Definition 6.** *We write $Z(a,b)$ for the* phase gate *that acts as $Z(a,b)\,|0\rangle = |0\rangle$, $Z(a,b)\,|1\rangle = \omega^a\,|1\rangle$ and $Z(a,b)\,|2\rangle = \omega^b\,|2\rangle$ where we take $a,b \in \mathbb{R}$.*

We define $Z(a,b)$ in this way, taking a and b to correspond to phases that are multiples of ω, because $Z(a,b)$ will turn out to be Clifford iff a and b are integers. Note that the collection of all $Z(a,b)$ operators constitutes the group of diagonal single-qutrit unitaries modded out by a global phase. Composition of these operations is given by $Z(a,b)\cdot Z(c,d) = Z(a+c, b+d)$.

We will now define the qutrit S gate. For our purposes it will be useful to define it in such a way that it has determinant 1. To do this we will need the ninth-root of unity. Throughout the remainder of the paper, we define $\zeta = e^{2\pi i/9}$.

▶ **Definition 7.** *The qutrit S gate is $S := Z(0,1)$. I.e., it multiplies the $|2\rangle$ state by ω.*

For qubits, the Hadamard interchanges the Z eigenbasis $\{|0\rangle, |1\rangle\}$, and the X basis consisting of the states $|\pm\rangle := \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. The same holds for the qutrit Hadamard. In this case the X basis consists of the following states (where we recall from above that $\frac{1}{\omega^2-\omega} = i/\sqrt{3}$):

$$|+\rangle \;:=\; \frac{1}{\omega^2-\omega}(|0\rangle + |1\rangle + |2\rangle) \tag{3}$$

$$|\omega\rangle \;:=\; \frac{1}{\omega^2-\omega}(|0\rangle + \omega\,|1\rangle + \omega^2\,|2\rangle) \tag{4}$$

$$|\omega^2\rangle \;:=\; \frac{1}{\omega^2-\omega}(|0\rangle + \omega^2\,|1\rangle + \omega\,|2\rangle) \tag{5}$$

▶ **Definition 8.** *The* qutrit Hadamard gate *H is the gate that maps $|0\rangle \mapsto |+\rangle$, $|1\rangle \mapsto |\omega\rangle$ and $|2\rangle \mapsto |\omega^2\rangle$. As a matrix:*

$$H \;:=\; \frac{1}{\omega^2-\omega}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \tag{6}$$

Note that, unlike the qubit Hadamard, the qutrit Hadamard is *not* self-inverse. In fact, we have $H^2 = -\tau_{(12)}$, so that $H^4 = \mathbb{I}$. In particular, $H^\dagger = H^3$. Furthermore, we note that just as the Clifford group in qubits generates certain global phases, the relation $(SH)^3 = -\omega$ implies that global phases of ± 1, $\pm\omega$, and $\pm\omega^2$ naturally appear in the qutrit Clifford group. The Pauli and S gates we defined all have matrix representations with entries over $\mathbb{Z}[\omega]$. We see that H naturally introduces denominators into our matrices, and so we should localize $\mathbb{Z}[\omega]$ to ensure we can characterize circuits which contain H. Since

$$\frac{\omega^k}{\omega^2-\omega} \;=\; \frac{\omega^k(\omega-\omega^2)}{3}$$

we can introduce the appropriate denominators by localizing at $\mu = \{3^k \mid k \in \mathbb{N}\}$ to get the number ring $\mu^{-1}\mathbb{Z}[\omega]$. Note that this is equivalent to the number ring $\mathbb{T}[\omega]$ which consists of elements $a + b\omega$ where $a, b \in \mathbb{T}$ are triadic fractions.

In Definition 6 we defined the Z phase gate. Similarly, we can define the X phase gates, that give a phase to the X basis states.

▶ **Definition 9.** *We define the* X phase gates *to be* $X(a,b) := HZ(a,b)H^\dagger$ *where* $a,b \in \mathbb{R}$.

We already saw examples of such X phase gates: $X = X(2,1)$ and $X^\dagger = X(1,2)$.

Any single-qutrit Clifford can be represented (up to global phase) as a composition of Clifford Z and X phase gates. In particular, we can represent the qutrit Hadamard as follows [15]:

$$H = -Z(2,2)X(2,2)Z(2,2) = -X(2,2)Z(2,2)X(2,2) \tag{7}$$

$$H^\dagger = -Z(1,1)X(1,1)Z(1,1) = -X(1,1)Z(1,1)X(1,1) \tag{8}$$

The final Clifford gate we need is the qutrit CX.

▶ **Definition 10.** *The qutrit CX gate is the two-qutrit gate defined by* $CX|i,j\rangle = |i, i+j\rangle$ *where the addition is taken modulo* 3*.*

▶ **Proposition 11.** *Let* U *be a qutrit Clifford unitary. Then up to global phase* U *can be written as a composition of the* S*,* H *and* CX *gates [16].*

From this it easily follows that the $Z(a,b)$ and $X(a,b)$ gates are Clifford if and only if a and b are integers.

▶ **Corollary 12.** *Let* U *be a qutrit Clifford unitary. Then up to a global phase* U *has a matrix representation in the computational basis with entries in the number ring* $\mathbb{T}[\omega]$*.*

Proof. This follows from Proposition 11, the definitions of S, H, and CX, and Lemma 3. ◀

2.4 T gates and qutrit controlled gates

Clifford unitaries don't suffice for universal computation, so let's introduce the T gate.

▶ **Definition 13.** *The qutrit* T *gate is the* Z *phase gate defined as* $T := Z(1/3, -1/3) = \mathit{diag}(1, \zeta, \zeta^8)$ *[19, 9, 17].*

Like the qubit T gate, the qutrit T gate belongs to the third level of the Clifford hierarchy, can be injected into a circuit using magic states, and its magic states can be distilled by magic state distillation. This means that we can fault-tolerantly implement this qutrit T gate on many types of quantum error correcting codes. Also as for qubits, the qutrit Clifford+T gate set is approximately universal, meaning that we can approximate any qutrit unitary using just Clifford gates and the T gate [12, Theorem 1].

The T gate introduces the phase ζ into matrix representations of circuits and thus we should consider extending the previously-defined $\mathbb{T}[\omega]$ by ζ. Note that ζ is a ninth root of unity which solves the cubic polynomial

$$\zeta^3 - \omega = 0 \tag{9}$$

over $\mathbb{T}[\omega]$. In fact, this polynomial has no solutions over $\mathbb{T}[\omega]$, implying that $\zeta \notin \mathbb{T}[\omega]$ (see Appendix A). We thus define the number ring $\mathbb{T}[\zeta]$:

▶ **Definition 14.** *The extension of* $\mathbb{T}[\omega]$ *by* ζ *is the number ring* $\mathbb{T}[\omega][\zeta] \cong \mathbb{T}[\zeta]$ *defined by*

$$\mathbb{T}[\omega][\zeta] \cong \mathbb{T}[\zeta] := \{a + b\zeta + c\zeta^2 + d\zeta^3 + e\zeta^4 + f\zeta^5 \mid a,b,c,d,e,f \in \mathbb{T}\}.$$

Any higher powers of ζ that might appear in an expression for an element of $\mathbb{T}[\zeta]$ can be reduced using for instance Eq. (9).

▶ **Lemma 15.** *Let U be a qutrit Clifford+T unitary. Then up to a global phase U has a matrix representation in the computational basis with entries in the number ring $\mathbb{T}[\zeta]$.*

Proof. By the definitions of S, H, T, and CX and Lemma 3. ◀

Using T gates, we can construct certain controlled unitaries. When we have an n-qu*bit* unitary U, we can speak of the controlled gate that implements U. This is the $(n+1)$-qubit gate that acts as the identity when the first qubit is in the $|0\rangle$ state, and implements U on the last n qubits if the first qubit is in the $|1\rangle$ state.

For qutrits there are however multiple notions of control.

▶ **Definition 16.** *Let U be a qutrit unitary. Then the* $|2\rangle$-controlled U *is the unitary $|2\rangle$-U that acts as*

$$|0\rangle \otimes |\psi\rangle \mapsto |0\rangle \otimes |\psi\rangle \qquad |1\rangle \otimes |\psi\rangle \mapsto |1\rangle \otimes |\psi\rangle \qquad |2\rangle \otimes |\psi\rangle \mapsto |2\rangle \otimes U\,|\psi\rangle$$

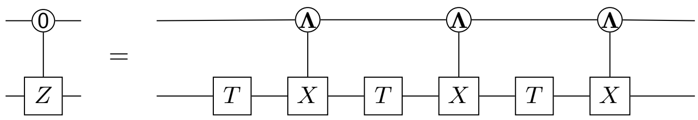
I.e., it implements U on the last qutrits if and only if the first qutrit is in the $|2\rangle$ state.

Note that by conjugating the first qutrit with X or $X^\dagger$ gates we can make the gate also be controlled on the $|1\rangle$ or $|0\rangle$ state.

A different notion of qutrit control was introduced by Bocharov, Roetteler, and Svore [6]: Given a qutrit unitary U they define $\Lambda(U)\,|c\rangle\,|t\rangle = |c\rangle \otimes (U^c\,|t\rangle)$. I.e., apply the unitary U a number of times equal to to the value of the control qutrit, so that if the control qutrit is $|2\rangle$ we apply U^2 to the target qutrits. Note that we can get this notion of control from the former one: just apply a $|1\rangle$-controlled U, followed by a $|2\rangle$-controlled U^2. The Clifford CX gate defined earlier is in this notation equal to $\Lambda(X)$.

Adding controls to a Clifford gate generally makes it non-Clifford. In the case of the CX gate, which is $\Lambda(X)$, it is still Clifford, but the $|2\rangle$-controlled X is *not*.

As shown by Bocharov, Roetteler, and Svore, the $|0\rangle$-controlled Z gate can be constructed by the following 3 T gate circuit [6, Figure 6]:

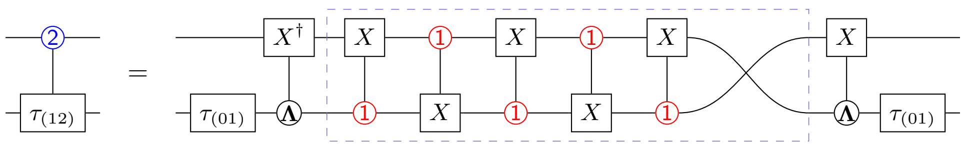


(10)

By conjugating the control qutrit by either $X^\dagger$ or X, the $|1\rangle$- and $|2\rangle$-controlled versions are respectively obtained. Taking the adjoint of Eq. (10) has the effect of changing the target operation from Z to $Z^\dagger$. Finally, note that we can also use this construction for controlled X and $X^\dagger$ gates by conjugating the target qutrit of Eq. (10) by an H or $H^\dagger$ gate. By adapting a different circuit from Ref. [6] we can also construct the other controlled X permutation gates.

▶ **Lemma 17.** *The $|2\rangle$-controlled versions of the $\tau_{(01)}$, $\tau_{(02)}$, and $\tau_{(12)}$ gates can be implemented unitarily using Clifford+T gates without ancillae, with a T-count of 15.*

Proof. The $|2\rangle$-controlled $\tau_{(12)}$ gate can be constructed as follows:



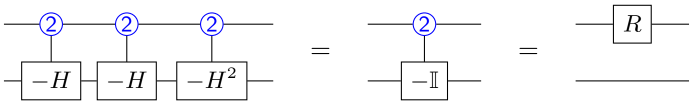
(11)

The dashed box shows the Clifford equivalent gate given by Bocharov, Roetteler, and Svore [6, 4] upon which the construction is based. The $|2\rangle$-controlled $\tau_{(01)}$ and $|2\rangle$-controlled $\tau_{(02)}$ gates follow from Clifford equivalence. ◀

Note that the blue and red color of the controls here is just to visually indicate more clearly which type of control is meant. The colors have no further significance.

3 Results

Previous implementations of the R gate require either distillation [1] or probabilistic creation of the $\mathrm{diag}(1, 1, -1)$ state [11, 6]; both approaches then necessitate injection by a repeat-until-success protocol. Here we present a new approach, which implements R unitarily over the qutrit Clifford+T gate set. As we will discuss later, it is actually impossible to exactly build the R gate from only single-qutrit Clifford+T gates. However, we *can* construct the two-qutrit $R \otimes \mathbb{I}$ unitarily using Clifford+T gates. We will do this[3] by showing how to construct certain $|2\rangle$-controlled gates and then using the following observation:



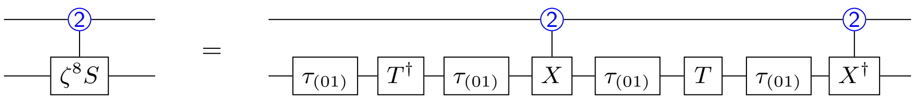
(12)

This works because $H^4 = \mathbb{I}$, and the fact that global phases become *local* phases when adding control wires to them. Here we have a *controlled* global phase because the global phase of -1 is applied to the target if and only if the control is in the $|2\rangle$ state. Therefore, this is an instance of phase kickback: The action of the $|2\rangle$-controlled $-\mathbb{I}$ gate is identical to applying $R \otimes \mathbb{I}$, i.e. the R gate to the control qutrit and identity on the target.

The $|2\rangle$-controlled $-H^2 = \tau_{(12)}$ was constructed as a Clifford+T circuit in Eq. (11). It hence remains to show how we can construct the $|2\rangle$-controlled $-H$ gate in Clifford+T. We can build this using the $|2\rangle$-controlled S gate. Note that for our purposes here our constructions are up to a controlled global phase, arising from our particular choice of global phase convention to define the Clifford gates in relation to a number ring. In forthcoming work [25], we chose a different convention enabling us do away with the controlled global phases here and exactly construct all multiple-controlled Clifford+T gates in Clifford+T.

▶ **Lemma 18.** *The $|2\rangle$-controlled S gate can be constructed unitarily without ancillae, up to a controlled global phase of ζ^8, using only Clifford+T gates, with T-count 8.*

Proof. The correctness can be verified by direct computation of the following circuit.

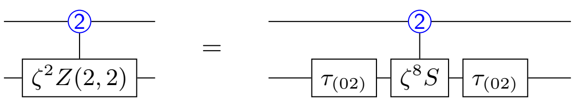


(13)

Alternatively, it is easy to see that this circuit does nothing if the first qutrit is in the $|0\rangle$ or $|1\rangle$ state, as the τ_{01} gates cancel, so that the T can cancel with the $T^\dagger$. Otherwise, if the first qutrit is $|2\rangle$, then the middle permutations combine to $\tau_{(01)} X \tau_{(01)} = \tau_{(012)} = X^\dagger$. When the T is pushed through this, the phases get permuted, and when combined with the $T^\dagger$ gives an S gate up to global phase. ◀

▶ **Corollary 19.** *The $|2\rangle$-controlled $Z(2, 2) = \mathrm{diag}(1, \omega^2, \omega^2)$ gate can be constructed unitarily without ancillae, up to a controlled global phase of ζ^2, using only Clifford+T gates, with T-count 8.*

Proof. Use the following circuit:



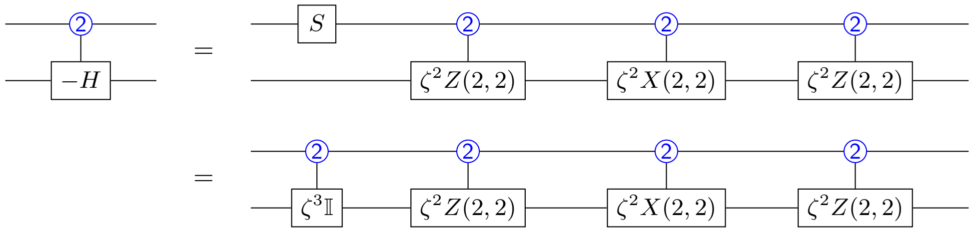
(14)

Its correctness can be verified by direct computation, or by commuting S and $\tau_{(02)}$. ◀

[3] Implemented at `https://github.com/lia-approves/qudit-circuits/tree/main/qutrit_R_from_T`.

▶ **Lemma 20.** *The $|2\rangle$-controlled H gate can be constructed unitarily without ancillae, up to a controlled global phase of -1, using Clifford+T gates with T-count 24.*

Proof. In the construction given below, we use the decomposition of H into alternating Z and X Clifford rotations of Eq. (7).



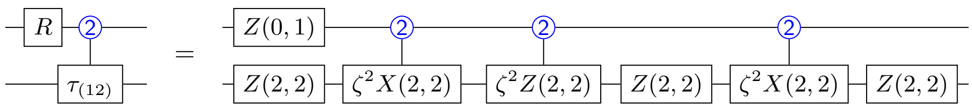
(15)

To construct the controlled H up to a controlled global phase, we apply Eq. (14), conjugating the target by Hadamards for the X rotations per Definition 9. As we require three such gates, their combined $|2\rangle$-controlled global phase becomes $\zeta^{2\cdot 3} = \zeta^6$. As $\zeta^9 = 1$, the necessary correction is to apply the $|2\rangle$-controlled global phase of ζ^3 gate, i.e. the $Z(0,1) \otimes \mathbb{I}$ gate. ◀

We can now construct the R gate in Clifford+T. However, direct substitution of the 24 T-count $|2\rangle$-controlled H gate of Lemma 20 into Eq. (12) yields a T-count 63 construction. We can do better by combining the two iterations of the controlled H in a smarter way.

▶ **Theorem 21.** *The qutrit R gate can be constructed unitarily in Clifford+T with T-count 39, provided there is a borrowed (i.e. returned to its starting state) ancilla available.*

Proof. The equality of the circuits below can be verified by direct computation or by noting that it applies $Z(3,3) = \mathbb{I}$ to the target when the control is $|0\rangle$ or $|1\rangle$, and $H^2 = -\tau_{(12)}$ otherwise.



(16)

To get a circuit for the R gate we simply bring the $|2\rangle$-controlled $\tau_{(12)}$ to the other side (as it is its own inverse). The total T-count of the resulting circuit is then $15 + 3 \cdot 8 = 39$. ◀

As we can construct the R gate as a Clifford+T circuit, any unitary that can be exactly constructed in the Clifford+R gate set can then be exactly (as opposed to approximately) constructed in the Clifford+T gate set. Although do note that our conversion presently seems rather inefficient, as the circuit in Eq. (12) requires 39 T gates.

▶ **Corollary 22.** *The Clifford+R gate set is a subset of the Clifford+T gate set.*

A natural question to ask now is whether we can do better. Do we really need two qutrits to write the R gate as a Clifford+T unitary? The answer is *yes*: it is not possible to construct the R gate using just single-qutrit Clifford+T gates. This follows from the normal form that was found for single-qutrit Clifford+T unitaries in Ref. [14]. Since the proof of this is rather technical we present the details in Appendix B, and just give a sketch here.

The group of 3×3 unitary matrices acts on the 8-dimensional real vector space of traceless Hermitian matrices. This action defines, for each 3×3 unitary matrix U, an 8×8 real matrix $\overline{U}$ known as the *adjoint representation* of U. One can then gather information about U by studying its adjoint representation $\overline{U}$. In particular, it is a consequence of the normal forms for single-qutrit Clifford+T circuits introduced in Ref. [14] that the adjoint representation of a single-qutrit Clifford+T operator has a very specific block matrix form (see Proposition 33 in Appendix B below). It can then be shown by computation that $\overline{R}$ is not of the appropriate form and therefore not Clifford+T .

Another natural question is the converse to Corollary 22: is the T gate included in Clifford+R? I.e., is the inclusion of Clifford+R within Clifford+T strict? We will show that this is indeed the case. We begin by considering matrix representations of circuits over Clifford+R.

▶ **Lemma 23.** *Let U be a qutrit Clifford+R unitary. Then up to a global phase U has a matrix representation in the computational basis with entries in the number ring $\mathbb{T}[\omega]$.*

Proof. By the definitions of S, H, R, and CX and Lemma 3. ◀

▶ **Proposition 24.** *$T \notin$ Clifford+R*

Proof. We have $T \in$ Clifford+R if there exists a unitary circuit over Clifford+R which performs the operation $T \otimes \mathbb{I}_n$ up to a global phase for some $n \in \mathbb{N}$ where $\mathbb{I}_n$ is the n-qutrit identity. In the computational basis, $T \otimes \mathbb{I}$ has a matrix representation with entries from the set $\{0, 1, \zeta, \zeta^8\}$. By Lemma 23, we know that if $T \otimes \mathbb{I}$ permits an exact circuit over Clifford+R we must have $\{0, c, c\zeta, c\zeta^8\} \subset \mathbb{T}[\omega]$ for at least some global phase $c \in \mathbb{C}$ which satisfies $c^*c = 1$. As $\mathbb{T}[\omega]$ is closed under conjugation, we then also have $c^* \in \mathbb{T}[\omega]$, and as it is closed under multiplication we then have $c^*c\zeta = \zeta \in \mathbb{T}[\omega]$. However, it is well-known that $\zeta \notin \mathbb{T}[\omega]$, and so there exists no such global phase c (see Appendix A). Hence, no such suitable c exists. As n was arbitrary, we conclude that no Clifford+R circuit exactly implements T in the computational basis. ◀

▶ **Corollary 25.** *Clifford+$R \subsetneq$ Clifford+T.*

4 Conclusion

In summary, we showed that the universal fault-tolerant qutrit Clifford+R gate set is a subset of Clifford+T, by providing a two-qutrit, T-count 39 unitary Clifford+T construction of the R gate. We prove that our construction is optimal in the number of qutrits by using the single-qutrit Clifford+T normal form of Glaudell, Ross, and Taylor [14] to show there is no single-qutrit construction. Moreover, we prove that Clifford+R is a *strict* subset of Clifford+T by showing that regardless of the number of ancillae qutrits, the T gate is impossible to exactly synthesize unitarily in the Clifford+R gate set.

This result is surprising for several reasons. While a number of papers have studied the Clifford+R gate set, it was not known that it is a subset of Clifford+T, much less a strict subset. Therefore, in contrast to what was previously believed, it looks like the T gate could be more powerful in practice than the R gate. In fact, we find that Clifford+T is strictly more powerful (at least asymptotically and for exact synthesis) than Clifford+R as Clifford+T can exactly synthesise every gate in Clifford+R up to a constant factor of overhead, while the converse is not true. We have reason to believe that the additional gates Clifford+T can exactly represent are important in practice. In Ref. [4] they conjectured that not all ternary classical reversible gates can be exactly represented in Clifford+R, while we have shown in follow-up work that they can all be efficiently constructed in Clifford+T [25]. Further analysis is required to better understand the implications of our results with regards to qutrit algorithms in practice, building upon the comparison between these two gate sets for Shor's algorithm in Ref. [6]. Let us note that using our construction, much of the work done on Clifford+R can now be directly translated to the Clifford+T setting. For example, the universal approximate synthesis algorithms of [5, 3] can now also be used to synthesise Clifford+T circuits.

Our results demonstrate a way in which qutrit Clifford+T is different from that of qubit Clifford+T. While all the one-qubit Clifford+T circuits that can be constructed with and without ancillae coincide [18], our result shows that this is not true for qutrits, as the single-qutrit R gate cannot be constructed in single-qutrit Clifford+T, but can be constructed using one borrowed ancilla.

A natural starting point for future work is to find a lower T-count decomposition of the R gate, as it seems unlikely that the best possible construction would require 39 T gates. It might be possible to find a lower bound on the necessary T-count to prepare the R state by using the resource theory of non-stabiliser states, for instance the mana [21] and thauma [22] measures of magic. Alternatively, there might also be a normal form for multi-qutrit Clifford+T unitaries which is T-optimal, which would then also give us an optimal decomposition of the R gate.

Finally, our results pave the way to deriving a full characterisation of which qutrit unitaries can be exactly implemented over the Clifford+T gate set. We conjecture that, as in the qubit case [13], any qutrit unitary with entries in $\mathbb{T}[\zeta]$ can be exactly synthesised over Clifford+T.

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A $\zeta \notin \mathbb{T}[\omega]$

We provide an elementary proof that primitive ninth roots of unity are not elements of $\mathbb{T}[\omega]$. Note that $\zeta \in \mathbb{T}[\omega]$ only if $\zeta \in \mathbb{Q}[\omega]$, where $\mathbb{Q}[\omega]$ is a field. As $\{1, \omega\}$ forms a basis for $\mathbb{Q}[\omega]$ over $\mathbb{Q}$, if $\zeta \in \mathbb{Q}[\omega]$ we would necessarily require some $a, b \in \mathbb{Q}$ such that

$$\zeta = a + b\omega \implies \zeta^3 = (a + b\omega)^3 = \omega.$$

Expanding, reducing powers using $1 + \omega + \omega^2 = 0$, and collecting terms, we find

$$(a^3 - 3ab^2 + b^3) + (3a^2b - 3ab^2)\omega = \omega.$$

Therefore, by equating coefficients of our basis elements on each side we conclude that we need

$$a^3 - 3ab^2 + b^3 = 0 \tag{17}$$
$$3a^2b - 3ab^2 = 1. \tag{18}$$

Note that clearly $a, b \neq 0$ if Eq. 18 is to be satisfied. Letting $r = a/b \in \mathbb{Q}$, we rearrange Eq. 17 and find

$$r^3 - 3r + 1 = 0. \tag{19}$$

Since $r \neq 0$, let $r = s/t$ for $s, t \in \mathbb{Z}$, $s, t \neq 0$, and $\gcd(s, t) = 1$ without loss of generality. Necessarily, we would have

$$s^3 - 3st^2 + t^3 = 0. \tag{20}$$

For any prime $p \mid s$, we clearly have $p \mid t^3$ implying $p \mid t$. Similarly, for any prime $q \mid t$, we must have $q \mid s^3$ and thus $q \mid s$. As we have assumed $\gcd(s, t) = 1$, we conclude that no prime can divide s nor t and so s, t must be units in $\mathbb{Z}$ as both are necessarily nonzero. No combination of $s, t = \pm 1$ satisfies Eq. 20, and thus we conclude no $r \in \mathbb{Q}$ satisfies Eq. 19. From this, we deduce there are no $a, b \in \mathbb{Q}$ such that

$$\zeta = a + b\omega$$

and thus $\zeta \notin \mathbb{Q}[\omega] \implies \zeta \notin \mathbb{T}[\omega]$.

B The R gate is not a single-qutrit Clifford+T unitary

We start with a set of definitions. These are based on the work done in Ref. [14].

▶ **Definition 26.** *Let $K = \{2^k \mid k \in \mathbb{N}\}$, $\alpha = \sin(2\pi/9)$, and $L = \{\alpha^k \mid k \in \mathbb{N}\}$. We define the following number rings:*

$$\mathbb{D} := K^{-1}\mathbb{Z} = \left\{ \frac{a}{2^k} \;\middle|\; a \in \mathbb{Z} \text{ and } k \in \mathbb{N} \right\}$$
$$\mathbb{D}[\alpha] = \{a + b\alpha + c\alpha^2 + d\alpha^3 + e\alpha^4 + f\alpha^5 \mid a, b, c, d, e, f \in \mathbb{D}\}$$
$$\mathbb{A} := L^{-1}\mathbb{D}[\alpha] = \left\{ \frac{a}{\alpha^k} \;\middle|\; a \in \mathbb{D}[\alpha] \text{ and } k \in \mathbb{N} \right\}$$

Additionally, we will rely on the following quotient ring:

▶ **Definition 27.** *Let $\mathbb{Z}_3 := \mathbb{Z}/(3)$ be the ring of integers modulo 3.*

Using our definitions we can introduce the following ring homomorphism:

▶ **Definition 28.** *Let $\rho : \mathbb{D}[\alpha] \to \mathbb{Z}_3$ be the ring homomorphism defined by $\rho(q) = q \pmod{\alpha}$ for $q \in \mathbb{D}[\alpha]$. In particular, $\rho(1/2) = 2$, $\rho(3) = 0$, and $\rho(\alpha) = 0$.*

To account for powers of α that appear in the denominator of elements of $\mathbb{A}$, we also introduce the following terminology:

▶ **Definition 29.** *Let $q \in \mathbb{A}$. There always exists some $k \in \mathbb{N}$ for which $\alpha^k q \in \mathbb{D}[\alpha]$. We call k a* denominator exponent *of q, and the least such k is called the* least denominator exponent *(LDE). The LDE of a vector or matrix over $\mathbb{A}$ is defined as the largest LDE of their individual elements.*

▶ **Definition 30.** *Let $q \in \mathbb{A}$ and let k be a denominator exponent of q. Then the k-*residue *of q, $\rho_k(q)$ is defined as*

$$\rho_k(q) := \rho(\alpha^k q) \in \mathbb{Z}_3.$$

The k-residue of a vector or matrix is defined component-wise.

The rings we introduced will encompass the entries of Clifford+T matrices in a certain representation called the adjoint representation, which we can describe as follows. Consider the space $\mathbb{H}$ of traceless 3×3 Hermitian matrices. This space forms an 8-dimensional real vector space and can be endowed with an inner product by defining $\langle M, M' \rangle = \mathrm{Tr}\big(M^\dagger M'\big)$, for any $M, M' \in \mathbb{H}$. As the trace is both cyclic and fixed under transposition of arguments, we have $\langle M, M' \rangle^* = \langle M, M' \rangle$ so that inner product of two traceless Hermitian matrices is necessarily real. It is straightforward to verify that if U is a 3×3 unitary matrix, then conjugation by U defines a linear operator on $\mathbb{H}$.

▶ **Definition 31.** *Let U be a 3×3 unitary matrix. We define the linear operator $\overline{U} : \mathbb{H} \to \mathbb{H}$ by $\overline{U}(H) = UMU^\dagger$ for every $M \in \mathbb{H}$. The operator $\overline{U}$ is the* adjoint representation *of U.*

The adjoint representation $U \mapsto \overline{U}$ defines a group homomorphism from $U(3, \mathbb{C})$ to $SO(8, \mathbb{R})$

For U a Clifford+T operator, we will be interested in the matrix representation of $\overline{U}$ in some convenient basis. Following [14], for a single-qutrit Pauli P, we set

$$P_\pm = \frac{P^\dagger \pm P}{\sqrt{\mathrm{Tr}[(P^\dagger \pm P)^2]}}$$

in order to define a basis $\mathcal{B}$ for $\mathbb{H}$.

▶ **Definition 32.** *Let X and Z be the single-qutrit Pauli operators and let $\mathbb{H}$ be the inner product space of 3×3 traceless Hermitian matrices. We define the orthogonal basis $\mathcal{B}$ for $\mathbb{H}$ as follows*

$$\mathcal{B} = \{Z_+, X_+, (XZ)_+, (XZ^2)_+, Z_-, X_-, (XZ)_-, (XZ^2)_-\}.$$

If U is a Clifford+T operator, then the matrix for $\overline{U}$ in the basis $\mathcal{B}$ (ordered as in Definition 32) has several useful properties, as detailed in the following proposition, whose proof can be found in [14, Remark 4.15, Remark 4.18, and Proposition 4.20].

▶ **Proposition 33.** *Let U be a 3×3 unitary matrix and assume that U can be exactly represented by an ancilla-free single-qutrit Clifford+T circuit. Then, in the basis $\mathcal{B}$, the operator $\overline{U}$ has entries in the number ring $\mathbb{A}$. Write*

$$\overline{U} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

where A, B, C, and D are 4×4 matrices. If the minimal T-count of U restricted to single-qutrit circuits is k, then the LDE of submatrix A is $2k$ and the following statements hold:

- *If $k = 0$, then U is a Clifford operator.*
- *If $k > 0$, then up to generalized row and column permutations over $\mathbb{Z}_3$,*

$$\rho_{2k}(A) \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \end{bmatrix} \qquad \textit{and} \qquad \rho_{2k+1}(C) \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

We are now in a position to prove that R cannot be represented over the Clifford+T gate set without using ancillas.

▶ **Proposition 34.** *The R gate cannot be represented by a single-qutrit ancilla-free Clifford+T circuit.*

Proof. Direct computation yields

$$\overline{R} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

where

$$A = D = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -1 & 2 \\ 0 & 2 & 2 & -1 \end{pmatrix} \qquad \text{and} \qquad B = C = 0$$

Thus $\overline{R}$ is a matrix over $\mathbb{A}$. The LDE of A is 6, and thus we compute

$$\rho_6(A) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \end{bmatrix} \qquad \text{and} \qquad \rho_7(C) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In particular, $\rho_7(C)$ is *not* equivalent up to generalized row/column permutations to the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Thus, R cannot be represented by a single-qutrit ancilla-free Clifford+T circuit. ◀