Properties of the Null Distance and Spacetime Convergence

Brian Allen¹ and Annegret Burtscher²,*

¹Department of Mathematics, University of Hartford, 200 Bloomfield Avenue, West Hartford, CT 06117-1599, USA and ²Department of Mathematics, IMAPP, Radboud University, PO Box 9010, Postvak 59, 6500 GL Nijmegen, The Netherlands

*Correspondence to be sent to: e-mail: burtscher@math.ru.nl

The null distance for Lorentzian manifolds was recently introduced by Sormani and Vega. Under mild assumptions on the time function of the spacetime, the null distance gives rise to an intrinsic, conformally invariant metric that induces the manifold topology. We show when warped products of low regularity and globally hyperbolic spacetimes endowed with the null distance are (local) integral current spaces. This metric and integral current structure sets the stage for investigating convergence analogous to Riemannian geometry. Our main theorem is a general convergence result for warped product spacetimes relating uniform, Gromov–Hausdorff, and Sormani–Wenger intrinsic flat convergence of the corresponding null distances. In addition, we show that nonuniform convergence of warping functions in general leads to distinct limiting behavior, such as limits that disagree.

1 Introduction

What is a good notion of convergence for Lorentzian manifolds? Riemannian manifolds naturally carry the structure of metric spaces, and standard notions of metric convergence such as Gromov–Hausdorff (GH) and Sormani–Wenger intrinsic flat (SWIF) convergence [84] naturally interact with the Riemannian structure and (weak) curvature...
bounds. Despite the increasing need to understand Lorentzian manifolds of low regularity, a comprehensive metric theory is still in its infancy.

Shing-Tung Yau first suggested that a notion of spacetime SWIF convergence is needed in light of geometric stability questions arising from spacelike observations in connection with the positive mass theorem by Lee and Sormani [62]. The most natural analog to the Riemannian distance function, the Lorentzian distance, is based on the maximization of curve lengths. But this distance does not even give rise to a metric structure (see [16, Ch. 4] for more details), let alone an integral current space needed for SWIF convergence. With the aim to obtain a metric theory suitable for spacetime convergence Sormani and Vega defined the null distance for Lorentzian manifolds with time functions [83]. In upcoming work, announced in [81], Sormani and Vega analyze big bang spacetimes using the null distance and Sakovich and Sormani consider the null distance with respect to the cosmological time function on maximal developments. Note that Noldus [73] has a different approach toward defining a GH convergence based on the Lorentzian distance.

In this paper, we investigate in detail the metric and integral current structure of Lorentzian manifolds with respect to the null distance and provide spacetime convergence results for warped product spacetimes of low regularity. While as soon as the null distance is shown to be definite it can be readily used for GH convergence, to consider SWIF convergence we also need to prove bi-Lipschitz bounds to show that the spacetimes under consideration are (local) integral current spaces.

A comprehensive spacetime convergence theory will open the door to an exciting new direction of geometric spacetime stability and low-regularity Lorentzian geometry. Causality theory for continuous Lorentzian metrics has already been investigated extensively (see [31, 43, 44, 77, 79], and also [68, 70] for related work). Paired with weak notions of curvature (bounds) new applications in both mathematics and physics are expected. In particular, in connection with notions of distributional curvature (see, for instance, [29, 63]) and curvature bounds (see, for instance, [1, 14, 48, 55, 67, 71, 88]) in general relativity, GH and SWIF spacetime convergence has the potential to transform Lorentzian geometry the same way the metric approach has led to a new era in Riemannian geometry a few decades ago.

1.1 Spacetime metric structure and convergence

In search for a far-reaching metric structure on Lorentzian manifolds Sormani and Vega [83] recently defined the null distance \( \hat{d}_r \) on spacetimes with a (generalized) time
function $\tau$. From a purely geometric point of view, the null distance is thus immediately defined on any noncompact manifold, since any such manifold automatically admits a spacetime structure with a smooth time (or even temporal) function [54, Theorem 2]. The null distance is based on the minimization of the null length of broken causal curves. More precisely, if $\beta : [a, b] \to M$ is a continuous, piecewise smooth causal curve that is either future-directed or past-directed causal on its pieces $a = s_0 < s_1 < \ldots < s_k = b$, then the null length of $\beta$ is given by

$$\hat{L}_\tau(\beta) = \sum_{i=1}^k |\tau(\beta(s_i)) - \tau(\beta(s_{i-1}))|,$$

and the null distance between $p, q \in M$ is

$$\hat{d}_\tau(p, q) = \inf\{\hat{L}_\tau(\beta) : \beta \text{ is a piecewise causal curve from } p \text{ to } q\}.$$

This null distance turns a time-oriented Lorentzian manifold with a time function $\tau$, satisfying mild assumptions, into a metric space in a conformally invariant way. Indeed, in Section 3, we show that spacetimes with suitable time functions possess additional structure.

**Theorem 1.1.** If $(M, g)$ is a spacetime with locally anti-Lipschitz time function $\tau$, then $\hat{d}_\tau$ is an intrinsic metric and hence $(M, \hat{d}_\tau)$ is a length space.

The local influence of different (suitable) time functions on the metric structure obtained by the null distances is minor, in particular, when temporal functions are used (which is no additional restriction [68, Theorem 4.100]). In this paper, we are mostly interested in globally hyperbolic spacetimes and sequences of compact warped product spacetimes. For the latter one can easily verify the global equivalence of null distances $\hat{d}_\tau$ for all time functions $\tau$ as defined in Theorem 1.5 (the more general case is discussed in Section 2).

With the null distance metric space structure at hand some particularly important applications in general relativity can be addressed, such as questions of geometric stability. For instance, a question posed by Sormani in [81] is “In what sense is a black hole spacetime of small mass close to Minkowski space?”

The question of geometric stability has already been studied extensively for spacelike slices of spacetimes by Allen [2, 3, 5], Bray and Finster [19], Bryden, Khuri, and Sormani [20], Cabrera Pacheco [26], Finster [37], Finster and Kath [38], Huang and
Lee [51], Huang, Lee, and Sormani [52], Jauregui and Lee [53], Lee [61], Lee and Sormani [62], LeFloch and Sormani [64], Sakovich and Sormani [76], Sormani and Stavrov Allen [82]. Lee and Sormani [62] conjecture that SWIF convergence is the correct notion of convergence for stability. This conjecture is supported by one of their examples with increasingly many, increasingly small gravity wells of fixed depth which does not have a smooth or GH limit but whose SWIF limit is Euclidean space, which justifies this claim. In the full Lorentzian case the notion of null distance is crucial in order to turn a spacetime into a metric space so that standard metric notions of convergence such as GH and SWIF convergence can be applied to spacetimes.

Since the SWIF notion of distance is defined on integral current spaces (intuitively a metric space with countable many bi-Lipschitz charts to $\mathbb{R}^n$) one of our goals of this paper is to show that globally hyperbolic spacetimes are (local) integral current spaces (recall that GH distance is defined as soon as the null distance is definite). Currents are geometric objects used in the calculus of variations and geometric measure theory that generalize the notion of integration. The theory of currents was developed by Federer and Fleming for Euclidean space [36] and extended to metric spaces by Ambrosio and Kirchheim [8]. The theory of rectifiable currents played a crucial role in establishing the existence of minimal surfaces with given boundary in the higher dimensional Plateau problem. An integral current structure, in particular, guarantees that a metric space is a countably $\mathcal{H}^{n+1}$ ($n + 1$-dimensional Hausdorff measure) rectifiable space, [84] which is an important additional property of a metric space and useful for studying spacetimes of low regularity.

**Theorem 1.2.** Let $I$ be an interval and $(\Sigma, \sigma)$ be a connected complete Riemannian manifold. Suppose $f: I \to (0, \infty)$ is a bounded function that is bounded away from 0. Then there is a natural local integral current space structure on the warped product spacetime $M = I \times_f \Sigma$ with respect to the null distance $\hat{d}_f$. If $I \times \Sigma$ is compact then $(M, \hat{d}_f)$ carries an integral current space structure.

In order to prove Theorem 1.2, we derive explicit expressions for the null distance of Lorentzian products as well as detailed estimates of the null distance on warped products with bounded warping functions. These prototype classes of spacetimes are important to better understand the null distance, and we also use this insight in obtaining a more general result for globally hyperbolic spacetimes.

**Theorem 1.3.** Let $(M, g)$ be a globally hyperbolic spacetime with (smooth) time function $\tau$. Suppose $M$ admits Cauchy hypersurfaces on which the ambient Lorentzian
metric restricts as a complete Riemannian metric and \((M, \hat{d}_\tau)\) is complete as a metric space. Then \((M, \hat{d}_\tau)\) is a local integral current space. If \(M\) is compact then it is an integral current space.

The main ingredient that allows us to extend Theorem 1.2 to the globally hyperbolic setting in Theorem 1.3 is that a (locally) bi-Lipschitz map from a (local) integral current space to a metric space naturally induces a (local) integral current space structure on the target. Since this is an important result in its own right, it is stated separately for integral current spaces in Theorem 2.15 and local integral current spaces in Theorem 2.18.

The metric and integral current structure of spacetimes establishes the basis for proving spacetime convergence results. In this paper, we initiate this investigation of spacetime convergence for the important class of warped product spacetimes. In general relativity, for instance, the cosmological Friedmann–Lemaître–Robertson–Walker (FLRW) metrics describing a homogeneous and isotropic universe are of this class. Our goal is to identify conditions on the sequence of warping functions which guarantee GH and/or SWIF convergence with respect to the null distance with the aim of generalizing these conditions to arbitrary sequences of spacetimes in the future. Due to the physical relevance of Lorentzian geometry it is also crucial to know when limits of spacetimes disagree (or do not exist at all). In the final section of this paper, we therefore present examples of warped product spacetimes with distinct limiting behavior. For instance, we construct an example with degenerate behavior of the limiting warping function which leads to different GH and SWIF limits. Two more examples illustrate that the way in which the warping functions converge and the properties of the limiting warping function itself influence the structure of the limiting integral current space. For instance, \(L^p\)-convergence in the spacetime setting also appears to contain useful information, something that was recently already observed for Riemannian warped products by Allen and Sormani [7], however, with important distinctions coming from the fact that the null lengths are not as directly related to the warping functions as in the Riemannian case. Motivated by these examples, we prove our main theorem on the convergence of warped product spacetimes.

**Theorem 1.4.** Let \((\Sigma, \sigma)\) be a connected compact Riemannian manifold, and \(I\) be a closed interval. Suppose \((f_j)_j\) is a sequence of continuous functions \(f_j : I \to (0, \infty)\) (uniformly bounded away from 0) and \(M_j = I \times f_j \Sigma\) are warped products with Lorentzian metrics

\[
g_j = -dt^2 + f_j(t)^2 \sigma.
\]
Assume that \((f_j)_j\) converges uniformly to a limit function \(f_\infty\). Then the null distances corresponding to the integral current spaces \(M_j\) converge in the uniform, GH, and SWIF sense to the null distance defined with respect to the warped product \(M_\infty = I \times_{f_\infty} \Sigma\).

Note that \(\Sigma\) is explicitly allowed to have a boundary (see Remark 2.22).

Though this theorem involves a stronger notion of convergence than one expects for stability of spacetime questions with gravity wells and black holes, results similar to Theorem 1.4 have already successfully been used in the stability of the Riemannian positive mass theorem [4, 52]. In [4], Allen used a related Riemannian convergence result in order to efficiently handle singular sets, which can be done by showing uniform or smooth convergence on compact sets away from the singular set (see, for instance, the work of Lakzian [56] and Lakzian and Sormani [57]). In [52], Huang, Lee, and Sormani needed a stronger convergence on the boundary of interest in order to show that no points on the outer boundary disappear in the limit. Besides stability questions, uniform convergence can also be employed to analyze structural properties of continuous (semi-)Riemannian metrics (see, for instance, the work of Burtscher [24] and Chruściel and Grant [31]).

1.2 Causality and completeness

Sormani and Vega already proved that the null distance is always a pseudometric and, as long as the time function is anti-Lipschitz, definite. As discussed earlier, in Theorem 1.1 we show that the (local) anti-Lipschitz property of \(\tau\) automatically also yields a length space \((M, \hat{d}_\tau)\).

Since there are, however, many artificial ways to turn a Lorentzian manifold into a metric space it is important that such a metric is connected to the causal structure of the Lorentzian manifold on which it is defined. It is clear that the null distance satisfies

\[ p \leq q \implies \hat{d}_\tau(p, q) = \tau(q) - \tau(p). \]

If the converse also holds \(\hat{d}_\tau\) is said to encode causality, which is thus a property that enables the null distance to capture whether or not points of a spacetime are causally related. Fully understanding when the null distance encodes causality is mentioned as an open problem in [83]. For warped product spacetimes, the answer is already known.

**Theorem 1.5.** [83, Theorem 3.25] Let \(I\) be an interval, \(f: I \to (0, \infty)\) be smooth and \((\Sigma, \sigma)\) be a complete Riemannian manifold. Let \(M = I \times_f \Sigma\) be the warped spacetime with
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metric tensor

\[ g = -dt^2 + f(t)^2 \sigma, \]

and let \( \tau(t, x) = \phi(t) \) be a smooth time function on \( M \) with \( \phi' > 0 \). Then the induced null distance \( \hat{d}_\tau \) is definite and encodes causality of \( M \), i.e., for all \( p, q \in M \),

\[ p \leq q \iff \hat{d}_\tau(p, q) = \tau(q) - \tau(p). \]

We provide two examples in Section 3.4, which show how the null distance can fail to encode causality when it is not complete.

Since metric completeness is generally a useful property, it is desirable to be able to characterize completeness in terms of conditions on the spacetime and time function alone. In this direction, we provide the following sufficient conditions for the null distance to be complete.

Theorem 1.6. Let \((M, g)\) be a spacetime with time function \( \tau \). If \( \tau \) is anti-Lipschitz on \( M \) with respect to a complete metric on \( M \) (that induces the manifold topology), then \((M, \hat{d}_\tau)\) is complete.

Note that the notion of null distance is relatively new and also many interesting and deep questions about its structural properties other than those mentioned here are yet to be fully explored. The conformal invariance and causality-encoding property of the null distance already show that this notion can play a crucial and viable role in general relativity not only in connection with spacetime stability but also in, for instance, Lorentzian causality theory.

1.3 Null distance and weak spacetimes in general relativity

Because the null distance at essence just requires a notion of causality (via the admissible class of piecewise causal curves) and a definition of time function, we note that the null distance can be generalized to very weak notions of spacetimes.

In general relativity, very weak notions of Lorentzian metrics emerge naturally as solutions to the Einstein equations. This is most evident in matter models involving compressible fluids since discontinuities occur both at the matter-vacuum boundary and in the form of shocks within the fluid (see, for instance, [15, 25, 45, 60, 75]). Also for the vacuum Einstein equations and other matter models weak solutions play a prominent role (see, for instance, [23, 30, 34, 78, 85]).
From a geometric as well as causality theoretic perspective, continuous Lorentzian metrics have been studied extensively by Sorkin and Woolgar [79], Chruściel and Grant [31], Galloway, Ling, and Sbierski [43], Graf and Ling [44], and Sämann [77]. It will be particularly insightful to use the null distance in place of the commonly used background Riemannian metric on weak Lorentzian manifolds, for instance, in recent work of Kunzinger and Sämann [55] who investigated a Lorentzian notion of length spaces based on a time-separation function on a given metric space. Using the null distance as background metric for sufficiently regular Lorentzian manifolds in place of an auxiliary Riemannian distance, naturally provides a link to the causal and length space structure.

Riemannian GH and SWIF convergence results have been particularly important in connection with lower curvature bounds (recall the influential work of Gromov [46], Grove and Petersen [47] as well as Burago, Gromov and Perel’ man [22] on lower sectional curvature bounds, and Cheeger and Colding [27, 28] on lower Ricci curvature bounds). Besides, the Cheeger–Gromov collapse theory has already been applied by Anderson [10, 11] to study the long-term behavior of solutions to the vacuum Einstein equations. In the Lorentzian context, distributional notions of curvature in Lorentzian geometry have been considered already by Lichnerowicz [65] and Taub [86], and more systematically by LeFloch and Mardare [63] and Chen and Li [29]. Lorentzian analogs for sectional curvature bounds have been studied by Harris [48], Andersson and Howard [14], Alexander and Bishop [1], and Kunzinger and Sämann [55], and relations to Ricci curvature bounds by Yun [88], McCann [67], and Mondino and Suhr [71]. Paired with (weak) notions of curvature bounds, spacetime convergence results with respect to the null distance can provide new tools to analyze the topology of spacetimes and spacetime singularities.

1.4 Outline

We proceed to give a brief description of each section of this paper.

In Section 2, we recall the definition of the null distance and review important basic properties obtained by Sormani and Vega. Warped product spacetimes are introduced since they will be a rich source of examples throughout this paper. We then review the notions and some results of GH convergence, (local) integral current spaces and prove that this structure can be pushed forward using (locally) bi-Lipschitz maps in Theorem 2.15 and Theorem 2.18, and SWIF convergence of integral current spaces, all of which will be used in Sections 4 and 5.
In Section 3, we explore in detail the length space structure of the metric space defined by the null distance, including Theorem 1.1 mentioned above. In addition, a wealth of examples illustrates what can go wrong when certain properties of the time function or null distance are not assumed. For instance, we give complete examples for which the null distance between certain points cannot be achieved by a piecewise causal curve. We state Theorem 1.6 relating the completeness of the null distance to properties of the time function and a background metric. We conclude the section with a discussion about the relation of the null distance to causality. Two simple examples illustrate that causality generally is not encoded when completeness of the null distance is not satisfied.

In Section 4, we state precisely when the null distance on a spacetime defines a (local) integral current space. The basic idea is to define Lipschitz maps from Minkowski space with the null distance to a particular spacetime of interest with the null distance. Then, since we can show that Minkowski space with the null distance is a local integral current space, we push forward this (local) integral current space structure through the (locally) bi-Lipschitz map to a particular spacetime of interest using our results of Section 2. In particular, this section culminates by showing that warped product spacetimes (of low regularity) and globally hyperbolic spacetimes with a complete Cauchy surface are (local) integral current spaces in Theorem 1.2 and Theorem 1.3, respectively.

In Section 5, we state and prove our main Theorem 1.4 for warped product spacetimes which relates uniform convergence of the warping functions to uniform, GH, and SWIF convergence of the sequence of integral current spaces with the null distance. In addition, we investigate what happens if uniform convergence of the warping functions (and hence Lorentzian manifolds) is not assumed. To this end we provide three examples of sequences of warped product spacetimes with distinct limiting behavior. In particular, we construct a sequence of warped product spacetimes with the corresponding null distances whose GH and SWIF limits disagree (and no pointwise limit exists).

2 Background

We recall various relevant definitions and results related to convergence on metric spaces. We first introduce the null distance, (local) integral current spaces, GH and SWIF distance and then explain what is known about metric convergence results.
2.1 Null distance

Let \((M, g)\) be a spacetime, that is, a time-oriented connected Lorentzian manifold of dimension \(n + 1\). For our purposes \(M\) is always second countable and Hausdorff. Note that these topological assumptions are even superfluous for solutions to the Einstein equations and Lorentzian manifolds with a second countable Cauchy surface due to [87, Lemma 15 & Corollary 22]. In our convention the metric tensor \(g\) has signature \((-,+,+,\ldots,+)\).

By \(\tau\) we denote a time function of \((M, g)\), i.e., a continuous function \(\tau : M \to \mathbb{R}\) that strictly increases along future-directed causal curves [16, p. 64]. Whenever continuity is not assumed, we call \(\tau\) a generalized time function.

We recall the notions of null length and null distance associated to a particular generalized time function \(\tau\) of \((M, g)\) as introduced in [83]. Let \(\beta : [a, b] \to M\) be a piecewise causal curve, i.e., a piecewise smooth curve that is either future-directed or past-directed causal on its pieces \(a = s_0 < s_1 < \ldots < s_k = b\) (moving forward and backward in time is allowed). The null length of \(\beta\) is given by

\[
\hat{L}_\tau(\beta) = \sum_{i=1}^{k} |\tau(\beta(s_i)) - \tau(\beta(s_{i-1}))|.
\]

If \(\tau\) is differentiable along \(\beta\) we can compute the null length of \(\beta\) also via the integral

\[
\hat{L}_\tau(\beta) = \int_{a}^{b} |(\tau \circ \beta)'(s)| \, ds.
\]

For any \(p, q \in M\), the null distance is given by

\[
\hat{d}_\tau(p, q) = \inf \{\hat{L}_\tau(\beta) : \beta \text{ is a piecewise causal curve from } p \text{ to } q\},
\]

where we use that there always exists a piecewise causal curve between two points in \(M\) [83, Lemma 3.5]. Unlike the Lorentzian distance function (see [16]), which is not a true distance because it is neither definite, symmetric nor satisfies the triangle inequality, the null distance \(\hat{d}_\tau\) is a pseudometric that in most cases encodes causality (see [83, Lemmas 3.6 and 3.8] and the discussion in Section 3.4). Note that \(\hat{d}_\tau\) is continuous on \(M \times M\) iff \(\tau\) is continuous [83, Proposition 3.14]. The topology induced by \(\hat{d}_\tau\) coincides with the manifold topology if and only if \(\tau\) is continuous and \(\hat{d}_\tau\) is definite [83, Proposition 3.15]. We discuss the relation of \(\hat{d}_\tau\) to causality in detail in Section 3.4. Note that if temporal functions \(\tau\) are used instead of time functions, the corresponding null distances \(\hat{d}_\tau\) are equivalent on compact sets (for the warped
product case see below, the general case is treated in a forthcoming paper and can be compared to the local bi-Lipschitz property of Riemannian distances [24, Theorem 4.5]; conformal invariance and scaling with $\tau$ was already shown in [83, Proposition 3.9 & Proposition 3.18]).

2.2 Warped product spacetimes

Most importantly for our work, definiteness of the null distance holds for spacetimes with “regular” cosmological time function à la Andersson, Galloway and Howard [13], and warped spacetimes $I \times_f \Sigma$. Such warped spacetimes are $n+1$ dimensional manifolds with metric tensor

$$g = -dt^2 + f^2(t)\sigma,$$

where $I$ is an open interval, $f$ is positive and smooth, and $(\Sigma, \sigma)$ is a complete Riemannian manifold. The result for smooth time functions $\tau(t, x) = \phi(t)$ with $\phi' > 0$ is contained in [83, Theorem 3.25]. For continuous warping functions $f$ with the canonical time function $\tau(t, x) = t$ it is [83, Lemma 3.23]. If $\tau = t$, we write $p = (t_p, p_\Sigma)$ for a point $p \in I \times_f \Sigma$. Lorentzian manifolds of this type are also referred to as \textit{generalized Robertson–Walker (GRW) spacetimes} (see [9, 66, 89] for an overview of some properties). It is easy to see that the null distances $\hat{d}_{\tau_i}$ with respect to different time functions $\tau_i(t, x) = \phi_i(t), \phi_i' > 0$, are equivalent metrics on compact sets.

\textbf{Lemma 2.1.} Let $I$ be a closed interval, $(\Sigma, \sigma)$ be a compact Riemannian manifold and $M = I \times_f \Sigma$ a warped product spacetime. If $\tau$ is the time function defined by $\tau(t, x) = \phi(t)$ with $\phi' > 0$ then there exists a constant $C > 0$ such that

$$\frac{1}{C} \hat{d}_{\tau}(p, q) \leq \hat{d}_{\tau}(p, q) \leq C\hat{d}_{\tau}(p, q), \quad p, q \in M.$$

Moreover, if $\phi'$ is bounded above and below away from zero, the equivalence holds globally on any (also noncompact) $M$.

\textbf{Proof.} Without loss of generality we assume that $p \leq q$. We can write $p = (t_p, p_\Sigma)$ and $q = (t_q, q_\Sigma)$, and assume that $t_p < t_q$ (if equality holds then $p = q$, where $\hat{d}_{\tau}$ and $\hat{d}_{\tau}$ both vanish). By the mean value formula, there exists $t \in (t_p, t_q)$ such that

$$\phi(t_q) - \phi(t_p) = \phi'(t)(t_q - t_p).$$
Since \( \phi' \) is bounded on compact sets, say \( 0 < A \leq \phi'(t) \leq B \), we immediately obtain that
\[
A \hat{d}_t(p, q) \leq \hat{d}_t(p, q) \leq B \hat{d}_t(p, q).
\]

2.3 Gromov–Hausdorff distance

Gromov–Hausdorff convergence \([21, 35, 46]\) is a notion for the convergence of metric spaces based on the Hausdorff distance. For two subsets \( A, B \) of a common metric space \((Z, d_Z)\) the Hausdorff distance between \( A \) and \( B \) is defined by
\[
d_{ZH}(A, B) = \inf \{ r > 0 : A \subseteq U_r(B) \text{ and } B \subseteq U_r(A) \},
\]
where \( U_r(S) \) is the \( r \)-neighborhood of \( S \), i.e., the set of all points \( z \in Z \) such that \( d_Z(z, S) < r \). In general, any pair of (compact) metric spaces \((X_i, d_i), i = 1, 2\), need not be subsets of the same metric space \( Z \). However, for any two separable metric spaces \( A \) and \( B \) such a common metric space \( Z \) can always be constructed, for example, via the Fréchet embedding \( A \) and \( B \) can be isometrically embedded into the Banach space \( l^\infty \) \([49, \text{Section 1.22}]\). Using such isometric embeddings the concept of the Hausdorff distance can be extended to that of the Gromov–Hausdorff distance \( d_{GH} \) between two metric spaces \((X, d_X) \) and \((Y, d_Y)\), given by
\[
d_{GH}((X, d_X), (Y, d_Y)) = \inf \{ r > 0 : \text{there is a metric space } Z \text{ and subspaces } X', Y' \text{ isometric to } X, Y \text{ such that } d_{ZH}^2(X', Y') < r \}.
\]
One can show that \( d_{GH}((X, d_X), (Y, d_Y)) < \infty \) if \( X, Y \) are bounded metric spaces.

A sequence of compact metric spaces \((X_j, d_j)\) is said to Gromov–Hausdorff converge to a compact metric space \((X, d)\) if
\[
d_{GH}((X_j, d_j), (X, d)) \to 0, \quad \text{as } j \to \infty.
\]
In our case, all sets \( X_j \) are the same manifold \( M \), only the null distances \( \hat{d}_j := \hat{d}_{\tau,j} \) differ. Note that the GH distance can also be introduced for pointed (noncompact) metric spaces.

2.4 Integral currents

In Euclidean space integer rectifiable currents arise by integrating differential forms over rectifiable sets, and integral currents are those currents with compatible boundary currents \( \partial T(\omega) = T(d\omega) \) (defined by Stokes’ Theorem). This definition naturally extends
to smooth manifolds. Currents on metric spaces were first introduced by Ambrosio and Kirchheim [8] and integral current spaces were used by Sormani and Wenger to define a notion of distance analogous to the flat distance in $\mathbb{R}^n$ of Federer and Fleming [36]. In what follows we introduce a working definition of integer rectifiable currents on complete metric spaces using a common short-cut via parametrizations which we will then use to define integral current spaces in Section 2.5. See the work of Ambrosio and Kirchheim [8] for a thorough treatment of currents in metric spaces, Sormani and Wenger [84] for a thorough description of the theory of integral current spaces, and [80] for a recent survey. We begin with the definition of a current on a metric space of Ambrosio and Kirchheim.

**Definition 2.2.** Let $n \in \mathbb{N} \cup \{0\}$. An $n$-dimensional current $T$ on a complete metric space $(X, d)$ is a $\mathbb{R}$-valued multilinear functional

$$(f, \pi_1, \ldots, \pi_n) \mapsto T(f, \pi_1, \ldots, \pi_n)$$

on the space of $n + 1$ tuples $(f, \pi_1, \ldots, \pi_n)$, where $f$ is a bounded Lipschitz function from $X \to \mathbb{R}$ and $\pi_i$ are Lipschitz functions from $X \to \mathbb{R}$ satisfying

1. **Locality:** $T(f, \pi_1, \ldots, \pi_m) = 0$ if $\exists i \in \{1, \ldots, m\}$ such that $\pi_i$ is constant on a neighborhood of $\{f \neq 0\}$.
2. **Continuity:** $T$ is continuous with respect to the pointwise convergence of the $\pi_i$ such that $\text{Lip}(\pi_i) \leq 1$.
3. **Finite mass:** there exists a finite Borel measure $\mu$ on $X$ such that

$$|T(f, \pi_1, \ldots, \pi_m)| \leq \prod_{i=1}^m \text{Lip}(\pi_i) \int_X |f| d\mu$$

for all $n + 1$ tuples $(f, \pi_1, \ldots, \pi_n)$.

The mass of a current is defined as follows.

**Definition 2.3.** For a current $T$ on the complete metric space $(X, d)$, the mass measure of $T$, which we will denote $\|T\|_d$, is defined as the smallest Borel measure $\mu$ such that

$$T(f, \pi_1, \ldots, \pi_m) \leq \prod_{i=1}^m \text{Lip}_d(\pi_i) \int_X f d\mu. \quad (2)$$

The mass of $T$ is then defined as $\mathbf{M}(T) = \|T\|_d(X)$. 

We start with defining what it means to restrict a current to a Borel set. Although the first slot of a current is defined to be a bounded Lipschitz function, Ambrosio and Kirchheim explain that one can actually uniquely extend the definition of the current to allow any bounded Borel function \( f \) still satisfying (1) with \( \mu = \| T \| \) [8, p. 12]. This justifies the following definition of the restriction operator.

**Definition 2.4.** Let \( T \) be a current on a complete metric space \((X, d)\), \( A \subseteq X \) a Borel measurable set and \( \chi_A : X \to \mathbb{R} \) the characteristic function of \( A \). Then the current 
\[
(T \downarrow A)(f, \pi_1, ..., \pi_n) = T(f \chi_A, \pi_1, ..., \pi_n)
\]
is called the restriction of \( T \) to \( A \).

**Example 2.5** (Euclidean space). Let \( h : U \subseteq \mathbb{R}^n \to \mathbb{Z} \) be an \( L^1 \) function then one can define the \( n \)-dimensional current \([h]\) as 
\[
[h](f, \pi_1, ..., \pi_n) = \int_U hf \, d\pi_1 \wedge ... \wedge \pi_n.
\]

A common way to define a current structure on a metric space \((X, d)\) is to cover \( X \) by images of bi-Lipschitz maps \( \varphi_i : U_i \to X \), \( U_i \subseteq \mathbb{R}^n \), and induce the current structure on \( X \) through the pushforward of the usual current structure on \( \mathbb{R}^n \) (see [80, Section 9.3.2]). We now review the necessary definitions required to understand this approach.

**Definition 2.6.** Let \( X \) and \( Y \) be complete metric spaces and \( \varphi : X \to Y \) a Lipschitz map. If \( T \) is a current \( T \) on \( X \) then the pushforward current \( \varphi_\# T \) on \( X \) is given by 
\[
\varphi_\# T(f, \pi_1, ..., \pi_n) = T(f \circ \varphi, \pi_1 \circ \varphi, ..., \pi_n \circ \varphi),
\]
where \( \pi_i : Y \to \mathbb{R} \) are Lipschitz maps and \( f : Y \to \mathbb{R} \) is a bounded Lipschitz function.

A useful example of an integral current on a metric space which makes use of Definition 2.6 is the following.

**Example 2.7** (Metric spaces). If \((X, d)\) is a complete metric space, \( \varphi : \mathbb{R}^n \to X \) a bi-Lipschitz map, and a Lebesgue function \( \theta \in L^1(U, \mathbb{Z}) \) where \( U \subseteq \mathbb{R}^n \), then \( \varphi_\# [\theta] \) is an \( n \)-dimensional current on \( X \) such that 
\[
\varphi_\# [\theta](f, \pi_1, ..., \pi_n) = \int_U \theta(f \circ \varphi) \, d(\pi_1 \circ \varphi) \wedge ... \wedge d(\pi_n \circ \varphi),
\]
where by Rademacher’s Theorem \(d(\pi_i \circ \varphi)\) is well defined since \(\pi_i \circ \varphi\) is differentiable almost everywhere.

**Remark 2.8.** In general, for pushforward currents defined via Definition 2.6 Ambrosio and Kirchheim, in equation (2.4) of [8], showed that

\[
\|\varphi_\# T\|_d \leq (\text{Lip}(\varphi))^n \|\varphi_\# T\|_d,
\]

\[
M(\varphi_\# T) \leq (\text{Lip}(\varphi))^m M(T).
\]

For a current given as in Example 2.7 we can deduce

\[
\|\varphi_\# [\theta]\|_d \leq (\text{Lip}(\varphi))^n |\theta| d\mathcal{L}^n,
\]

which implies the mass of \(\varphi_\# T\) is bounded by

\[
M(\varphi_\# [\theta]) \leq (\text{Lip}(\varphi))^m \int_U \theta \, d\mathcal{L}^n,
\]

where \(d\mathcal{L}^n\) is the Lebesgue measure on \(\mathbb{R}^n\). See the discussion around [84, Remark 2.21 and Lemma 2.42] for more details.

By combining Definition 2.6 with Example 2.7 one can define a parametrization of a current structure on a metric space by an atlas of bi-Lipschitz charts. We will use this as the definition of an integer rectifiable current on a metric space, as is commonly done.

**Definition 2.9.** Let \((X, d)\) be a complete metric space. By choosing a countable collection of bi-Lipschitz maps \(\varphi_i: U_i \to X, U_i \subset \mathbb{R}^n\) precompact Borel measurable such that

\[
\varphi_i(U_i) \cap \varphi_j(U_j) = \emptyset, \quad \text{for } i \neq j,
\]

and weight functions \(\theta_i \in L^1(U_i, \mathbb{Z})\), we define an integer rectifiable current \(T\) on \((X, d)\) by

\[
T = \sum_{i=1}^{\infty} \varphi_i^\# [\theta_i],
\]

with mass

\[
M(T) = \sum_{i=1}^{\infty} M(\varphi_i^\# [\theta_i]).
\]
In particular, the current in Example 2.7 is an integer rectifiable current.

The boundary of a current can also be defined by mimicking and extending the smooth case (based on differential forms and Stokes’ Theorem).

**Definition 2.10.** The boundary $\partial T$ of an $n$-dimensional current $T$ is defined as

$$
\partial T(f, \pi_1, \ldots, \pi_{n-1}) = T(1, f, \pi_1, \ldots, \pi_{n-1}).
$$

By construction, $\varphi_{\#}(\partial T) = \partial(\varphi_{\#} T)$ and $\partial \partial T = 0$.

**Definition 2.11.** An integral current $T$ on a complete metric space $(X, d)$ is an integer rectifiable current $T$ such that the boundary $\partial T$ is a current itself.

We now also review the definition of the set of a current which will be important to defining integral current spaces in Section 2.5 below.

**Definition 2.12.** Given an $n$-dimensional current on a complete metric space $(X, d)$ the canonical set of $T$ is the collection of points

$$
\text{set}_d(T) = \left\{ p \in X : \liminf_{r \to 0} \frac{\|T\|_d(B^d_r(p))}{\omega_n r^n} > 0 \right\},
$$

where $\omega_n$ is the volume of the unit ball in $n$-dimensional Euclidean space.

### 2.5 Integral current spaces

Based on the work by Ambrosio and Kirchheim these spaces were introduced by Sormani and Wenger [84] and are important for defining the SWIF distance in Section 2.7. We also briefly discuss locally integer rectifiable currents introduced by Lang and Wenger [59] and local integral current spaces by Jauregui and Lee [53] in Section 2.6, which are relevant in the noncompact setting. Our main results show that integral current structures can be pushed forward from one metric space to another via (locally) bi-Lipschitz maps.

**Definition 2.13.** We say that $(X, d, T)$ is an integral current space if $(X, d)$ is a metric space and $T$ is an $n$-dimensional integral current on the completion, $(\overline{X}, \overline{d})$, of $X$ so that $\text{set}_d(T) = X$. 
Example 2.14 (Riemannian manifolds). The standard current structure for an oriented compact smooth manifold $M$ (with boundary) is the standard integration of $n$-forms over $M$ and thus $T$ is given by

$$T(f, \pi_1, \ldots, \pi_n) = \int_M f \, d\pi_1 \wedge \ldots \wedge d\pi_n.$$ 

Together with a Riemannian metric $g$, and induced distance function $d_g$, this yields an integral current space with boundary current $\partial T$ agreeing with the boundary $\partial M$ of $M$ (due to Stokes’ Theorem). Note that for precompact manifolds one can work with metric completions instead.

The following theorem allows one to induce an integral current space structure on one metric space from another metric space via a bi-Lipschitz map.

**Theorem 2.15.** Let $(X, d_X)$ be a metric space and $(X, d_X, T)$ be an integral current space. If $(Y, d_Y)$ is a metric space and there is a bi-Lipschitz map $\psi : X \to Y$, then $(Y, d_Y, \psi_* T)$ is also an integral current space.

**Proof.** Let $(X, d_X, T)$ be an integral current space and $\psi : X \to Y$ a bi-Lipschitz map.

By Definition 2.13 and Definition 2.9 there exist bi-Lipschitz maps $\varphi_i : U_i \to X$, $U_i \subset \mathbb{R}^n$ precompact Borel measurable with pairwise disjoint images and weight functions $\theta_i \in L^1(U_i, Z)$. We can induce an integer rectifiable current on $(Y, d_Y)$ by considering $\tilde{\varphi}_i = \psi \circ \varphi_i$ in Definition 2.9. Since $\psi$ is bi-Lipschitz we know it is a homeomorphism and hence $\tilde{U}_i = \psi(U_i)$ are precompact Borel measurable with pairwise disjoint images. Hence, we define the integral current structure on $(Y, d_Y)$ by

$$\tilde{T} = \sum_{i=1}^{\infty} \tilde{\varphi}_i #[\tilde{\theta}_i].$$

We also notice that, since $(\psi \circ \varphi_i)_# = \psi_* \circ \varphi_i#$, 

$$\psi_* (T)(f, \pi_1, \ldots, \pi_n) = T(f \circ \psi, \pi_1 \circ \psi, \ldots, \pi_n \circ \psi)$$

$$= \sum_{i=1}^{\infty} \varphi_i #[\theta_i](f \circ \psi, \pi_1 \circ \psi, \ldots, \pi_n \circ \psi)$$

$$= \sum_{i=1}^{\infty} \tilde{\varphi}_i #[\tilde{\theta}_i](f, \pi_1, \ldots, \pi_n) = \tilde{T}(f, \pi_1, \ldots, \pi_n).$$
Since Cauchy sequences in $X$ are mapped to Cauchy sequences in $Y$, and vice versa, $\psi$ extends as a bi-Lipschitz map $\psi : \overline{X} \to \overline{Y}$ between the completions of $X$ and $Y$, and hence $\widetilde{T}$ is defined on $\overline{Y}$ since $T$ is defined on $\overline{X}$ by definition.

From this, Definition 2.6, Example 2.7, Remark 2.8, and the fact that measures are additive over a countable disjoint collection of sets we can deduce that

$$\|\widetilde{T}\|_{d_Y} = \sum_{i=1}^{\infty} \|\widetilde{\varphi}_{i#}\theta_i\|_{d_Y} \leq (\text{Lip}(\psi))^n \sum_{i=1}^{\infty} \|\varphi_{i#}\theta_i\|_{d_X}.$$  

The mass of $\widetilde{T}$ is finite since

$$M(\widetilde{T}) = \sum_{i=1}^{\infty} M(\widetilde{\varphi}_{i#}\theta_i)$$

$$\leq (\text{Lip}(\psi))^n \sum_{i=1}^{\infty} M(\varphi_{i#}\theta_i) = (\text{Lip}(\psi))^n M(T) < \infty,$$

where we take advantage of the fact that $M(T)$ is finite by definition.

Using the fact that boundary and pushforwards commute for currents we note that

$$\partial(\widetilde{T}) = \psi_#(\partial T),$$

and hence $\partial(\widetilde{T})$ is a current and has finite mass since

$$M(\partial(\widetilde{T})) \leq (\text{Lip}(\psi))^{n-1} M(\partial T).$$

Hence, $\widetilde{T}$ is an integral current on $\overline{Y}$.

It remains to be shown that set $d_Y(\widetilde{T}) = Y$. Since $\psi$ is bi-Lipschitz, for any $p, q \in X$

$$(\text{Lip}(\psi^{-1}))^{-1} d_X(p, q) \leq d_Y(\psi(p), \psi(q)) \leq \text{Lip}(\psi) d_X(p, q),$$

which implies that for any $r > 0$

$$\psi^{-1} \left( B_{\text{Lip}(\psi^{-1})^{-1}}^{d_Y}(\psi(p)) \right) \subseteq B_r^{d_X}(p) \subseteq \psi^{-1} \left( B_{\text{Lip}(\psi)}^{d_Y}(\psi(p)) \right).$$

By combining these inclusions with Remark 2.8, we obtain

$$\psi^{-1}_{#} \|\widetilde{T}\|_{d_Y} \geq (\text{Lip}(\psi^{-1}))^{-n} \|\psi^{-1}_{#}\widetilde{T}\|_{d_X} = (\text{Lip}(\psi^{-1}))^{-n} \|T\|_{d_X}. \quad (3)$$
where the second equality in (3) follows from the earlier observation \( \psi^{-1}_Y T = T \). Putting all of this together we find

\[
\text{set}_{d_Y}(\tilde{T}) = \left\{ \psi(p) \in Y : \liminf_{r \to 0} \frac{\|\tilde{T}\|_{d_Y} \left( B^d_{rY}(\psi(p)) \right)}{\omega_n r^n} > 0 \right\}
\]

\[
= \psi \left( \left\{ p \in X : \liminf_{r \to 0} \frac{\psi^{-1}_Y \|\tilde{T}\|_{d_Y} \left( \psi^{-1}(B^d_{rY}(\psi(p))) \right)}{\omega_n r^n} > 0 \right\} \right)
\]

\[
\supseteq \psi \left( \left\{ p \in X : \liminf_{r \to 0} \frac{\|T\|_{dX} \left( B^d_{\text{Lip}(\psi)^{-1}r}(p) \right)}{\omega_n r^n} > 0 \right\} \right)
\]

\[
= \psi \left( \left\{ p \in X : \text{Lip}(\psi)^{-n} \liminf_{r \to 0} \frac{\|T\|_{dX} \left( B^d_{\text{Lip}(\psi)^{-1}r}(p) \right)}{\omega_n (\text{Lip}(\psi)^{-1}r)^n} > 0 \right\} \right)
\]

\[
= \psi(\text{set}_{d_X}(T)) = \psi(X) = Y,
\]

and hence \( \text{set}_{d_Y}(\tilde{T}) = Y \). Thus \( (Y, d_Y, \tilde{T}) \) is an integral current space. \( \blacksquare \)

### 2.6 Local integral current spaces

When considering spacetimes one is mostly interested in the noncompact case and hence we also introduce the notion of locally integral currents defined by Lang and Wenger [59] and local integral current spaces defined by Jauregui and Lee [53]. Lang and Wenger define a notion of a \textit{n-dimensional metric functional} \( T \) which satisfies properties analogous to those in Definition 2.9 but for \( f: X \to \mathbb{R} \) bounded Lipschitz functions with bounded support and \( \pi_i: X \to \mathbb{R} \) functions that are Lipschitz on every bounded subset of \( X \) (see [59, Definition 2.1]). They also define a notion of mass, \( M_V(T) \in [0, \infty] \), for \( T \) on any open set \( V \). In [59, Definition 2.2] such an \textit{n-dimensional metric functional} \( T \) on a metric space \((X, d)\) is called an \textit{n-dimensional metric current with locally finite mass} if for every bounded, open set \( V \subseteq X \)

\[
M_V(T) < \infty,
\]

and for any \( \epsilon > 0 \), there exists a compact set \( C \subseteq V \) so that

\[
M_{V \setminus C}(T) < \epsilon.
\]
Lang and Wenger provide analogous definitions of locally integer rectifiable currents in [59, Definition 2.4] and locally integral currents in [59, Definition 2.5].

Using the results of [59], Jauregui and Lee [53, Definition 13] recently gave the following definition of local integral current spaces.

**Definition 2.16.** We say that \((X, d, T)\) is a local integral current space if \((X, d)\) is a metric space and \(T\) is an \(n\)-dimensional locally integral current on the completion, \((\bar{X}, \bar{d})\), of \(X\) so that \(\text{set}_d(T) = X\).

**Example 2.17.** A complete, connected, oriented manifold with continuous Riemannian metric with the canonical choice of \(T\) defined as in Example 2.14 is a local integral current space [59, Section 2.8].

Locally integral currents can be pushed forward by maps that are Lipschitz on bounded sets and which also satisfy that the preimages of bounded sets are bounded [59, p. 160]. Hence, we can conclude this section by showing that a local integral current space structure can be induced via maps that are bi-Lipschitz on bounded sets.

**Theorem 2.18.** Let \((X, d_X), (Y, d_Y)\) be metric spaces, \((X, d_X, T)\) be a local integral current space, and \(\psi : X \to Y\) a map which is bi-Lipschitz on bounded subsets. Then \((Y, d_Y, \psi_#T)\) is also a local integral current space.

**Proof.** Let \((X, d_X, T)\) be a local integral current space and \(\psi\) be bi-Lipschitz on bounded sets, in particular, \(\psi\) and \(\psi^{-1}\) are Lipschitz on metric balls. By the discussion in [59, p. 171f] we know that \(\psi_#T\) is a metric current with locally finite mass. By [53, Lemma 14], for any \(p \in X\) and a.e. \(r > 0\) the restriction \(T_\cap B_r(p)\) to the open ball \(B_r(p)\) is an integral current on \((\bar{X}, \bar{d}_X)\). By Theorem 2.15,

\[
\psi_#T \cap \psi(B_r(p)) = \psi_#(T \cap B_r(p))
\]

is then an integral current on \(\bar{Y}\) for a.e. \(r > 0\) since \(\psi\) is bi-Lipschitz on all balls \(B_r(p)\). In particular, those integral currents \(\psi_#T \cap \psi(B_r(p))\) can be viewed as locally integral currents on \(\bar{Y}\) by [59, top of p. 175]. Hence, [59, Lemma 2.3] implies that \(\psi_#T\) is a locally integral current on \((\bar{Y}, \bar{d}_Y)\).

Finally, checking whether a point in \(Y\) is in \(\text{set}_{d_Y}(\psi_#T)\) is a local calculation for metric balls in \(Y\) of arbitrarily small size. Then, since \(\psi^{-1}\) is continuous, we know that \(\psi(B_r(p))\) is open in \(Y\) for any \(r > 0\) and hence for all \(\varepsilon > 0\) sufficiently small
$B_e^{d_Y}(\psi(p)) \subseteq \psi(B_r(p))$. Since $\psi$ is bi-Lipschitz on bounded sets of $X$ we can repeat the same argument as in Theorem 2.15 to find that set $d_Y(\psi \# T) = Y$. ■

**Remark 2.19.** On proper (and hence locally compact) metric spaces locally bi-Lipschitz maps $\psi$ can be used in place of maps that are bi-Lipschitz on bounded sets in Theorem 2.18. This is because on locally compact metric spaces locally Lipschitz maps are Lipschitz on compact sets, and due to the properness the closure of every bounded set is compact. Recall, in particular, that any locally compact complete length space is proper (see, e.g., [21, Proposition 2.5.22]).

**Remark 2.20.** For general locally compact metric spaces related by locally bi-Lipschitz maps it should be straightforward to prove a result analogous to Theorem 2.18 by using an earlier notion of locally integral currents proposed by Lang [58, Definition 8.6]. This is since the push forward for such currents on locally compact spaces only requires locally Lipschitz maps that are proper, i.e., the preimages of compact sets must be compact (see [58, Definition 2.6]). As pointed out in [59], however, local compactness alone may not persist in the limit and the local currents of Lang thus seem less suitable for general pointed convergence results.

It is by using Theorem 2.15 and Theorem 2.18 (together with Remark 2.19) that we will induce a (local) integral current space structure on warped products and globally hyperbolic spacetimes endowed with the null distance in Section 4.

### 2.7 Sormani–Wenger intrinsic flat distance

Intrinsic flat convergence is a notion of convergence for sequences of integral current spaces which was coined by Sormani and Wenger [84] to handle the convergence of “hairy” Riemannian manifolds.

The construction of Ambrosio and Kirchheim [8] allows one to define the flat distance between currents $T_1, T_2$ of a metric space $Z$ by

$$d_F^Z(T_1, T_2) = \inf \{M^n(A) + M^{n+1}(B) : A + \partial B = T_1 - T_2\},$$

where $A$ and $B$ are $n$- and $(n + 1)$-dimensional integral currents, respectively. The Sormani–Wenger intrinsic flat distance (SWIF distance) is then defined between pairs of integral current spaces $M_1 = (X_1, d_1, T_1)$ and $M_2 = (X_2, d_2, T_2)$ as

$$d_F(M_1, M_2) = \inf \{d_F^Z(\varphi_1 \# T_1, \varphi_2 \# T_2) : \varphi_j : X_j \to Z\}$$
where the infimum is taken over all complete metric spaces $Z$ and all metric isometric embeddings $\varphi_j$, that is, $\varphi_j: X_j \to Z$ that satisfy
\[
d_Z(\varphi_j(x), \varphi_j(y)) = d_{X_j}(x, y), \quad x, y \in X_j.
\]

In [84] many important properties about SWIF (or even shorter $\mathcal{F}$) convergence are shown. Since this work in 2011 a deeper understanding of this notion of convergence has been obtained. See [57, 80, 84] for many interesting examples of SWIF convergence and its relationship to GH convergence.

2.8 Metric convergence results

When considering sequences of integral current spaces our goal will be to show bi-Lipschitz bounds on the sequence in terms of a background metric space. This then allows us to use the following compactness result of Huang, Lee, and Sormani [52, Appendix], which guarantees that a subsequence will converge in the uniform, GH, and SWIF sense to the same metric space. The GH part of this theorem was already shown by Gromov in [46].

**Theorem 2.21.** [52, Theorem A.1] Let $(X, d_0, T)$ be a precompact $n$-dimensional integral current space without boundary (e.g., $\partial T = 0$) and fix $\lambda > 0$. Suppose that $d_j, j \in \mathbb{N}$, are metrics on $X$ such that
\[
\frac{1}{\lambda} \leq \frac{d_j(p, q)}{d_0(p, q)} \leq \lambda.
\]
Then there exists a subsequence of $(d_j)_j$, also denoted $(d_j)_j$, and a length metric $d_\infty$ satisfying (6) such that $d_j$ converges uniformly to $d_\infty$, that is,
\[
\varepsilon_j = \sup \left\{ |d_j(p, q) - d_\infty(p, q)| : p, q \in X \right\} \to 0.
\]
Furthermore
\[
\lim_{j \to \infty} d_{GH} \left( (X, d_j), (X, d_\infty) \right) = 0,
\]
and
\[
\lim_{j \to \infty} d_{\mathcal{F}} \left( (X, d_j, T), (X, d_\infty, T) \right) = 0.
\]
In particular, \((X,d_{\infty},T)\) is an integral current space and \(\text{set}(T) = X\) so there are no disappearing sequences of points \(x_j \in (X,d_j)\).

In fact, we have

\[
d_{\text{GH}}\left( (X,d_j), (X,d_{\infty}) \right) \leq 2\varepsilon_j,
\]

and

\[
d_{\mathcal{F}}\left( (X,d_j,T), (X,d_{\infty},T) \right) \leq 2^{(n+1)/2}\lambda^{n+1} 2\varepsilon_j M(X,d_0)(T).
\]

**Remark 2.22.** In Theorem 2.21, it was assumed that the integral current spaces have no boundary. We note that this assumption is not necessary and the proof given in the appendix of [52] goes through in the boundary case as well. In the proof, a metric space \((Z_j,d'_j) = (I_{\varepsilon} \times X,d'_j), I_{\varepsilon} = [-\varepsilon_j,\varepsilon_j]\), is constructed into which one can embed \((X,d_j)\) and \((X,d_{\infty})\) isometrically. This space is used to estimate the SWIF distance

\[
d_{\mathcal{F}}\left( (X,d_j,T), (X,d_{\infty},T) \right) \leq d_{\mathcal{F}}^Z\left( \phi_j#T, \phi'_j#T \right)
\]

\[
\leq M_{(Z_j,d'_j)}(B) + M_{(Z_j,d'_j)}(A),
\]

where \(\phi_j, \phi'_j\) are the embedding maps. In the proof of Theorem A.1 in [52] \(A = 0\) since there was no boundary. In the case where one allows a boundary, the term \(A = I_{\varepsilon} \times \partial T\) is nontrivial, however, \(M_{(Z_j,d'_j)}(A)\) can be estimated exactly in the same way as \(M_{(Z_j,d'_j)}(B)\) was estimated because the bi-Lipschitz bounds are global. By doing so one obtains the SWIF convergence in the case where \(\partial T \neq 0\). In Section 5, we can therefore apply Theorem 2.21 also in the case where the integral current spaces of the sequence have a boundary.

Theorem 2.21 guarantees that the limit space will be an integral current space since the assumed bi-Lipschitz bounds persist in the limit. In fact, due to these bi-Lipschitz bounds one can use the same current structure on the entire sequence and limit space as the background integral current space. The mass of each integral current space will be different though since this depends explicitly on the distance function.

Our goal in the analysis of warped product spacetimes in Section 5 of this paper is to prove pointwise convergence of the corresponding null distances to the desired limit space, which when combined with Theorem 2.21 will imply uniform, GH, and SWIF convergence to this limit space.
3 \((M, \hat{d}_\tau)\) as a Length Space

In this section, we assume that the spacetime \((M, g)\) admits a time function \(\tau\) such that the induced null distance \(\hat{d}_\tau\) is a metric. We first discuss sufficient assumptions for this to hold.

3.1 Metric structure

Certain causality assumptions imply the existence of time functions. By [16, Proposition 3.24] and [69, Theorem 3.51] (see also discussion in [83, Sec. 2.3]), a sufficient condition for the existence of a generalized time function is a past-distinguishing spacetime (e.g., a strongly causal spacetime), while a stably causal spacetime furthermore admits a (continuous) time function. Since only continuous time functions lead to a null distance that induces the manifold topology [83, Proposition 3.15], we restrict our attention to those.

Furthermore, the null distance \(\hat{d}_\tau\) is definite, and hence a metric if and only if the (generalized) time function \(\tau\) is locally anti-Lipschitz [83, Proposition 4.5, Theorem 4.6]. Alternatively, definiteness follows if \(\hat{d}_\tau\) encodes causality (see [83, Lemma 3.12] and Section 3.4).

**Definition 3.1.** Let \((M, g)\) be a spacetime and \(f: M \to \mathbb{R}\). For any \(U \subseteq M\) we say that \(f\) is anti-Lipschitz on \(U\) if there exists a (definite) distance function \(d_U\) on \(U\) so that for any \(p, q \in U\) we have

\[
p \leq q \implies f(q) - f(p) \geq d_U(p, q).
\]  

We say that \(f: M \to \mathbb{R}\) is locally anti-Lipschitz if \(f\) is anti-Lipschitz on a neighborhood \(U\) of each point \(p \in M\).

**Remark 3.2.** In order to obtain definiteness of \(\hat{d}_\tau\) it is sufficient to simply demand the anti-Lipschitz property with respect to a *any* metric \(d_U\) on \(M\) (see [83, Sec. 4.1]), however, for other global properties of \((M, \hat{d}_\tau)\) a background Riemannian structure may be more useful. If one (continuous) Riemannian metric exists with respect to which \(\tau\) is locally anti-Lipschitz function, then it is automatically locally anti-Lipschitz with respect to *any* Riemannian metric due to the fact that the associated distance functions are equivalent on compact sets [24, Theorem 4.5].

We recall an important result by Sormani and Vega.
**Theorem 3.3.** [83, Theorem 4.6] Let $M$ be a spacetime and $\tau$ a time function on $M$. If $\tau$ is locally anti-Lipschitz, then the induced null distance $\hat{d}_\tau$ is a definite and conformally invariant metric on $M$ that induces the manifold topology.

**Remark 3.4.** Time functions $\tau$ whose gradient vectors $\nabla \tau$ are defined almost everywhere and which are locally bounded away from the light cone [83, Def. 4.13] are anti-Lipschitz, and so is the regular cosmological time function of Andersson, Galloway and Howard [13].

**Remark 3.5.** Due to the conformal invariance of the null distance we note that many of the theorems of this paper are automatically true in the conformal to warped product case as well.

### 3.2 Length structure

We show that the null length $\hat{L}_\tau$ on the set $\hat{A}$ of piecewise (smooth) causal paths defines a length structure on $M$, and thus $M$ together with the null distance $\hat{d}_\tau$ is a length space. We follow the conventions of Burago, Burago, and Ivanov [21].

**Theorem 1.1.** Let $(M, g)$ be a spacetime with locally anti-Lipschitz time function $\tau$. Then $(M, \hat{A}, \hat{L}_\tau)$ defines a length structure on $M$, i.e., $\hat{A}$ is an admissible class of paths according to [21, Sec. 2.1.1] and

1. The length of paths is additive, i.e., $\hat{L}_\tau(\beta|_{[a,c]}) = \hat{L}_\tau(\beta|_{[a,b]}) + \hat{L}_\tau(\beta|_{[c,b]})$ for any $c \in [a, b]$.
2. The length of a piece of a path continuously depends on the piece, i.e., for a fixed path $\beta : [a, b] \to M$ the function $t \mapsto \hat{L}_\tau(\beta|_{[a,t]})$ is continuous.
3. The length is invariant under reparametrizations, i.e., $\hat{L}_\tau(\beta \circ \varphi) = \hat{L}_\tau(\beta)$ for any diffeomorphism $\varphi$ such that $\beta, \beta \circ \varphi \in \hat{A}$.
4. The length structure agrees with the topology, i.e., for $U_p$ a neighborhood of $p$ there is $\inf(\hat{L}_\tau(\beta) : \beta(a) = p, \beta(b) \in M \setminus U_p) > 0$.

Thus $\hat{d}_\tau$ is the intrinsic metric of $M$ with respect to this length structure, and hence $(M, \hat{d}_\tau)$ is a length space.

**Proof.** It is clear that the class $\hat{A}$ of piecewise causal paths is closed under restrictions, concatenations and (smooth) reparametrizations. Thus $\hat{A}$ is admissible.
(i) Suppose $\beta : [a, b] \to M$ is a piecewise causal curve, i.e., on the pieces $a = s_0 < s_1 < \ldots < s_k = b$ the path is smooth and either future- or past-directed causal and $\hat{L}_\tau(\beta) = \sum_{i=1}^k |\tau(\beta(s_i)) - \tau(\beta(s_{i-1}))|$. If $c = s_j$, then the result is trivial, so assume that $c \in (s_{j-1}, s_j)$. Since $\beta$ is either future- or past-directed on $[s_{j-1}, s_j]$ it is either future- or past-directed on both pieces $s_{j-1} < c < s_j$ and the result follows by artificially introducing a breaking point $c$ and subsequent cancellation of $\tau(\beta(c))$ in the null length.

(ii) This follows immediately from the continuity of $\tau$ (and $\beta$).

(iii) This is clear because a diffeomorphism maps an interval to another interval in a bijective smooth way. A change of orientation does not matter because the length is computed using the absolute value.

(iv) Suppose $p \in M$, $U_p$ is a neighborhood of $M$ and $q \in M \setminus U_p$. By [83, Theorem 4.6], $\hat{d}_\tau$ then encodes the manifold topology. Hence, there exists an $r > 0$ such that $\hat{B}_\tau(p) = \{x \in M : \hat{d}_\tau(p, x) < r\} \subseteq U_p$, and therefore $\inf[\hat{L}_\tau(\beta) : \beta(a) = p, \beta(b) \in M \setminus U_p] \geq r > 0$.

Remark 3.6. Note that property (ii) in Theorem 1.1 does not hold for discontinuous generalized time functions (see Example 3.9 and Example 3.18 below). Hence, only time functions $\tau$ yield a length structure. On the other hand, property (iv) requires an interaction with the manifold topology, so $\hat{d}_\tau$ needs to be furthermore definite, that is, a metric (see Example 3.10). To express the metric property of $\hat{d}_\tau$ in terms of $\tau$ we have assumed that $\tau$ is furthermore locally anti-Lipschitz.

Remark 3.7. Note that for every globally hyperbolic manifold with $C^{2,1}$ metric a continuously differentiable, locally anti-Lipschitz time functions exists due to Chruściel, Grant and Minguzzi [32, Theorem 1.1, Cor. 4.4]. Furthermore, for every spacetime metric with a time function there exists an $\varepsilon$-close locally anti-Lipschitz time function [32, Proposition 5.3]. For a more general setup using cone fields see the work of Bernard and Suhr [18].

By construction $\hat{d}_\tau$ is an intrinsic metric. Thus it coincides with the metric induced by the length structure $L_{\hat{d}_\tau}$ induced by $\hat{d}_\tau$.

**Proposition 3.8.** Let $M$, $g$ and $\tau$ be as in Theorem 1.1. Then the null distance $\hat{d}_\tau$ is an **intrinsic metric**, that is, it coincides with the metric

$$d(p, q) = \inf \{ L_{\hat{d}_\tau}(\gamma) : \gamma \text{ is a rectifiable curve between } p \text{ and } q \} ,$$
induced by the length structure

\[ L_{\hat{d}_\tau} (\gamma) = \sup \left\{ \sum_{i=1}^{k} \hat{d}_\tau (\gamma(s_i), \gamma(s_{i-1})) : a = s_0 < s_1 < \ldots < s_k = b \right\} . \]

Moreover, \( L_{\hat{d}_\tau} = \hat{L}_\tau \) holds on the set \( \hat{A} \) of piecewise causal curves.

**Proof.** We follow the proof of [21, Proposition 2.4.1]. Let \( \beta : [a, b] \to M \) be a path in \( \hat{A} \) and \( a = s_0 < s_1 < \ldots < s_k = b \) be an arbitrary partition of \([a, b]\). By definition of the null distance, for each \( i = 1, \ldots, k \),

\[ \hat{d}_\tau (\beta(s_i), \beta(s_{i-1})) \leq L_\tau (\beta|_{[s_{i-1}, s_i]}), \]

and hence we obtain

\[ \sum_{i=1}^{k} \hat{d}_\tau (\beta(s_i), \beta(s_{i-1})) \leq \sum_{i=1}^{k} \hat{L}_\tau (\beta|_{[s_{i-1}, s_i]}) = \hat{L}_\tau (\beta) \]

due to the additivity of null lengths obtained in Theorem 1.1 (i). Since the partition was arbitrary, the inequality holds also for the supremum, i.e.,

\[ L_{\hat{d}_\tau} (\beta) \leq \hat{L}_\tau (\beta). \]

A smaller length function immediately implies a smaller metric, i.e., \( d \leq \hat{d}_\tau \).

The converse inequality \( \hat{d}_\tau \leq d \) holds automatically since for any \( p, q \in M \) and any rectifiable curve \( \gamma \) between these endpoints

\[ \hat{d}_\tau (p, q) \leq L_{\hat{d}_\tau} (\gamma). \]

Thus the metrics \( \hat{d}_\tau \) and \( d \) coincide.

It remains to show equality of lengths for curves \( \beta \in \hat{A} \). If \( a = s_0 < s_1 < \ldots < s_k = b \) are the future- and past-directed causal pieces of \( \beta \), we have that

\[ \hat{L}_\tau (\beta) = \sum_{i=1}^{k} |\tau (\beta(s_i)) - \tau (\beta(s_{i-1}))| \]

\[ = \sum_{i=1}^{k} \hat{d}_\tau (\beta(s_i), \beta(s_{i-1})) \leq L_{\hat{d}_\tau} (\beta). \]

This proves the final statement on the equality of lengths for piecewise causal curves \( \beta \). \( \blacksquare \)
To conclude our discussion of Theorem 1.1 we provide simple (counter) examples mentioned in Remark 3.6 for more general time functions in two-dimensional Minkowski space $\mathbb{M}^{1+1} = \mathbb{R}^{1,1}$.

**Example 3.9** (Discontinuous time function not satisfying 1.1(iii)). Consider $\mathbb{M}^{1+1}$ with discontinuous time function

$$\tau(t) = \begin{cases} 
t + 1 & t > 0 \\
0 & t = 0 \\
t - 1 & t < 0
\end{cases}$$

then any causal curve, $\beta : [a, b] \to \mathbb{M}^{1+1}$, connecting a point $p \in \{(x, t) \in \mathbb{M}^{1+1} : t < 0\}$ to a point $q \in \{(x, t) \in \mathbb{M}^{1+1} : t > 0\}$ will be such that $t \mapsto \hat{L}_\tau(\beta|_{[a,t]})$ is not continuous. We also observe that $\hat{d}_\tau$ is not complete since the sequence $p_i = (0, -\frac{1}{i})$ is a Cauchy sequence since

$$\hat{d}_\tau(p_i, p_j) = \left|\frac{i - j}{ij}\right|,$$

which does not converge to $p = (0, 0)$ since $\hat{d}_\tau(p_i, p) = 1 + \frac{1}{i}$.

**Example 3.10** (Non-anti-Lipschitz time function). Consider $\mathbb{M}^{1+1}$ with time function $\tau(t) = t^3$

which is continuous but not anti-Lipschitz. Sormani–Vega [83, Proposition 3.4] have shown that the resulting null distance $\hat{d}_\tau$ is not definite and does not encode causality in this case. We just also note that this metric is not a length space either since the resulting length structure would not agree with the manifold topology.

Since $\hat{L}_\tau = L_{\hat{d}_\tau}$ on $\hat{A}$ by Proposition 3.8, and $L_{\hat{d}_\tau}$ is lower semicontinuous on the larger set of rectifiable paths, by [21, Proposition 2.3.4] it follows immediately that $\hat{L}_\tau$ is lower semicontinuous on $\hat{A}$.

**Lemma 3.11.** Let $\tau$ be a time function on the Lorentzian manifold $(M, g)$. The null length functional $\hat{L}_\tau$ is lower semicontinuous, that is, for any sequence $(\beta_i)_i$ of piecewise causal path that converge pointwise to a piecewise causal path $\beta$ we have that

$$\hat{L}_\tau(\beta) \leq \liminf_{i \to \infty} \hat{L}_\tau(\beta_i).$$
Fig. 1. A sequence of piecewise timelike curves $\beta_i$ in $\mathbb{M}^{1+1}$ converges uniformly to a null curve $\beta$ such that $\liminf_{i \to \infty} \hat{L}_\tau(\beta_i) > \hat{L}_\tau(\beta)$ in Example 3.12.

One can construct selfsimilar “fractal” counterexamples in order to show that upper semicontinuity does not hold in general.

Example 3.12 (Timelike counterexample). We define piecewise causal curves $\beta_i : [0, 1] \rightarrow \mathbb{M}^{1+1}$ recursively by the graphs of

$$\tilde{\beta}_1(s) = \begin{cases} 3s & s \in [0, \frac{1}{2}], \\ 2 - s & s \in [\frac{1}{2}, 1], \end{cases}$$

and

$$\tilde{\beta}_{i+1}(s) = \begin{cases} \frac{1}{2} \tilde{\beta}_i(2s) & s \in [0, \frac{1}{2}], \\ \frac{1}{2} \tilde{\beta}_i(2s - 1) + \frac{1}{2} & s \in [\frac{1}{2}, 1]. \end{cases}$$

See also Figure 1. It follows that for all $i \in \mathbb{N}$,

$$\hat{L}_\tau(\beta_{i+1}) = \hat{L}_\tau(\beta_i) = \ldots = \hat{L}_\tau(\beta_1) = 2.$$

On the other hand, the limit curve $\beta(s) = s$ satisfies $\hat{L}_\tau(\beta) = 1 < 2$. 
Fig. 2. A sequence of piecewise null curves $\beta_i$ in $\mathbb{M}^{1+1}$ converges uniformly to a null curve $\beta$ such that $\liminf_{i \to \infty} \hat{L}_\tau(\beta_i) > \hat{L}_\tau(\beta)$ in Example 3.13.

Example 3.13 (Null counterexample). In a similar fashion one can define a sequence of piecewise null curves $\beta_i$ on $\mathbb{M}^{1+1}$ that uniformly converge to the curve $\beta(s) = s$ so that

$$\hat{L}_\tau(\beta_i) = 5 > 1 = \hat{L}_\tau(\beta).$$

See Figure 2.

3.3 Completeness

Length spaces with complete metrics have additional features. We provide conditions on the Lorentzian manifold and the continuous time function $\tau$ which imply that $(M, \hat{d}_\tau)$ is a complete metric space.

Proposition 3.14. Let $(M, g)$ be a spacetime with time function $\tau$. If there exists a complete background metric $d$ on $M$ (that induces the manifold topology) and a constant
$C > 0$ such that

$$\hat{d}_\tau(p, q) \geq Cd(p, q), \quad p, q \in M,$$

then $(M, \hat{d}_\tau)$ is complete.

**Proof.** Let $(p_i)_i$ be a Cauchy sequence of points in $M$ with respect to $\hat{d}_\tau$. By assumption,

$$\hat{d}_\tau(p_i, p_j) \geq Cd(p_i, p_j),$$

where $d$ is the distance function associated with a complete metric $d$. In particular, since $(p_i)_i$ is also a Cauchy sequence with respect to $d$ there exists $p \in M$ so that

$$\lim_{i \to \infty} d(p_i, p) = 0.$$

It remains to be shown that $\lim_{i \to \infty} \hat{d}_\tau(p_i, p) = 0$ to prove completeness of $M$ with respect to $\hat{d}_\tau$.

Let $\epsilon > 0$. Since both $d$ and $\hat{d}_\tau$ are metrics that induce the manifold topology (in the latter case it is due to [83, Proposition 3.15 and Proposition 4.5]) there exists a $\epsilon' \in (0, \epsilon)$ such that $B^{\hat{d}_\tau}_{\epsilon'}(p) \subseteq B^d_{\epsilon''}(p)$, and a $\epsilon'' > 0$ such that

$$B^d_{\epsilon''}(p) \subseteq B^{\hat{d}_\tau}_{\epsilon'}(p) \subseteq B^d_{\epsilon''}(p).$$

Since $p_i \to p$ with respect to $d$, for all $i$ sufficiently large we have that $p_i \in B^d_{\epsilon''}(p) \subseteq B^{\hat{d}_\tau}_{\epsilon'}(p)$. In particular, for $i$ sufficiently large we have $p_i \in B^{\hat{d}_\tau}_{\epsilon'}(p)$ since $\epsilon' < \epsilon$, and therefore

$$\lim_{i \to \infty} \hat{d}_\tau(p_i, p) = 0. \quad \Box$$

We now relate the anti-Lipschitz time function and background metric assumptions of Proposition 3.14 more directly.

**Theorem 1.6.** Let $(M, g)$ be a spacetime with time function $\tau$. If $\tau$ is anti-Lipschitz on $M$ with respect to a complete metric $d$ (that induces the manifold topology), then $(M, \hat{d}_\tau)$ is complete.
Proof. We show that the assumptions of Proposition 3.14 are satisfied. If \( p \) and \( q \) are causally related, then

\[
|\tau(p) - \tau(q)| = \hat{d}_\tau(p, q).
\]

Otherwise, for every \( \epsilon > 0 \), we can choose a piecewise null curve \( \beta \) so that

\[
\hat{L}(\beta) \leq \hat{d}_\tau(p, q) + \epsilon,
\]

with breaking points \( \beta(s_i) = p_i \) so that

\[
\hat{d}_\tau(p, q) \geq \hat{L}(\beta) - \epsilon = \sum_i \hat{d}_\tau(p_i, p_{i+1}) - \epsilon = \sum_i |\tau(p_i) - \tau(p_{i-1})| - \epsilon.
\]

Since \( \tau \) is globally anti-Lipschitz with respect to \( d \) we find

\[
\hat{d}_\tau(p, q) > C \sum_i d(p_i, p_{i-1}) - \epsilon \geq Cd(p, q) - \epsilon.
\]

Since this inequality holds for all \( \epsilon \), we have

\[
\hat{d}_\tau(p, q) \geq Cd(p, q)
\]

and can apply Proposition 3.14 to finish the proof.

The Hopf–Rinow Theorem implies the following result.

Corollary 3.15. Let \((M, g)\) be a spacetime with time function \( \tau \). If \( \tau \) is anti-Lipschitz on \( M \) with respect to a complete Riemannian metric \( h \), then \((M, \hat{d}_\tau)\) is complete.

Remark 3.16. Recall that the standard Hopf–Rinow Theorem does not hold for Lorentzian manifolds (see the example of the Clifton–Pohl torus in [74, p. 193]). The Hopf–Rinow–Cohn-Vossen Theorem for length spaces (see, for instance, [21, Theorem 2.5.28] or [46, Section 1.9]) implies that if the length space \((M, \hat{d}_\tau)\) is complete then the Heine–Borel property holds (every closed and bounded set is compact) and that between any two points a shortest rectifiable curves exists. Examples 3.17 and 3.18 show that completeness does not guarantee that the distance is achieved by a piecewise causal curve.
Example 3.17 (Warped product with distance not achieved in \(\hat{A}\)). Consider the Lorentzian warped product \(\mathbb{R} \times f \mathbb{R}\) with \(f(t) = t^2 + 1\), i.e.,

\[
g = -dt^2 + (t^2 + 1)^2 dx^2.
\]

We argue that the null distance between \(p = (x, t) = (-1, 0)\) and \(q = (1, 0)\) is \(\hat{d}_t(p, q) = 2\), and that it is not achieved by any piecewise causal curve. Indeed, this is true for any two points at the \(\{t = 0\}\) level.

In order to see that \(\hat{d}_t(p, q) = 2\) we refer to Proposition 4.10, which implies that on each time interval \([-\frac{1}{t}, \frac{1}{t}] \times \mathbb{R}\) (see Figure 3 for a visualization of null curves emanating from \(t = 0\), we obtain for \(p = (-1, 0)\) and \(q = (1, 0)\) that

\[
\hat{d}_t(p, q) = \hat{d}_M(p, q) = 2.
\]

This distance cannot be achieved by any piecewise causal curve, because any such curve \(\beta\) leaves the \(\{t = 0\}\) line and beyond the \(\{t = 0\}\) line the light cones of Minkowski space are strictly larger than that of \(g\).
Fig. 4. A sequence of piecewise causal curves $\beta_i$ between $p$ and $q$ showing that the null distance is not achieved in Example 3.18.

Note that, however, the metric $g$ still encodes causality by Theorem 1.5 of Sormani and Vega [83] stated in Section 3.4. (We would like to thank Christina Sormani for sharing this example with us.)

**Example 3.18** (Minkowski space and distance not achieved in $\mathcal{A}$). Consider $\mathbb{M}^{1+1}$ with discontinuous time function

$$
\tau(t) = \begin{cases} 
\sqrt{t} + 1 & t > 1 \\
\sqrt{t} & 0 \leq t \leq 1 \\
t & t \leq 0
\end{cases}
$$

then the distance between $p = (-1, 1)$ and $q = (1, 1)$, $\hat{d}_\tau(p, q) = 1$, is not achieved by any piecewise causal curve. One can define a sequence of piecewise causal curves $\beta_i: [-1, 1] \to \mathbb{M}^{1+1}$ connecting $p$ and $q$ as graph of $\tilde{\beta}_i$ (see Figure 4), given inductively by

$$
\tilde{\beta}_1(s) = |s|,
$$

$$
\tilde{\beta}_i(s) = \begin{cases} 
\frac{1}{2} + \frac{1}{2} \tilde{\beta}_{i-1}(2s + 1) & s \in [-1, 0], \\
\frac{1}{2} + \frac{1}{2} \tilde{\beta}_{i-1}(2s - 1) & s \in [0, 1].
\end{cases}
$$

Then $\hat{L}_\tau(\beta_1) = 2$ and

$$
\hat{L}_\tau(\beta_i) = 2^i \left(1 - \frac{(2^{i-1} - 1)\frac{1}{2}}{2^{i+1}}\right) = 2^\frac{i}{2} \left(2^{i-1} - (2^{i-1} - 1)\frac{1}{2}\right) \to 1
$$
Note that if, on the other hand, $\bar{\tau}(t) = \sqrt{t}$ for all $t \geq 0$ then $\hat{d}_\tau(p, q) = 2(\sqrt{2} - 1) < 1$ which is achieved by a piecewise causal curve connecting $p$ to $(0, 2)$ and subsequently to $q$. In the initial example, the discontinuity of $\tau$ forces the sequence of distance-approximating curves to eventually lie below $t = 1$. The distance cannot be achieved because the limit curve $\beta(s) = (s, 1)$ is not piecewise causal. Note that the time function $\tau$ is locally anti-Lipschitz and hence the issue is truly caused by the discontinuity.

**Remark 3.19.** We conjecture that having a (possibly locally anti-Lipschitz but) generalized time function $\tau$ that is discontinuous along any causal curve already implies that distances are not achieved by piecewise causal curves in $(M, \hat{d}_\tau)$.

### 3.4 Causality-encoding property

One of the major advantages of the null distance over auxiliary metrics (for instance, induced via a background Riemannian metric) is its relationship to the causal structure of a spacetime, a feature already investigated in [83]. In this section, we recall the concept and discuss known results and counterexamples.

**Definition 3.20.** Let $(M, g)$ be a spacetime with generalized time function $\tau$. We say that the null distance $\hat{d}_\tau$ encodes causality if

$$p \leq q \iff \hat{d}_\tau(p, q) = \tau(p) - \tau(q). \quad (9)$$

In some cases, for instance, Minkowski space $\mathbb{M}^{n+1} = \mathbb{R}^{n,1}$ with time function $\tau = t$ and smooth warped products, it was already shown by Sormani and Vega that causality is encoded. Since we are mostly interested in warped product spacetimes, we recall this result here.

**Theorem 1.5** [83, Theorem 3.25] Let $I$ be an interval, $f : I \to (0, \infty)$ smooth and $(\Sigma, \sigma)$ a complete Riemannian manifold. Let $M = I \times_f \Sigma$ be the warped spacetime with metric tensor

$$g = -dt^2 + f(t)^2 \sigma,$$

and let $\tau(t, x) = \phi(t)$ be a smooth time function on $M$ with $\phi' > 0$. Then the induced null distance $\hat{d}_\tau$ is definite and encodes causality of $M$. 

Remark 3.21. Note that, if one drops completeness of \((\Sigma, \sigma)\) in Theorem 1.5, one can still prove that
\[
q \in \overline{I^+(p)} \iff \hat{d}_\tau(p, q) = \tau(q) - \tau(p),
\]
see [83, Theorem 3.28].

On more general spacetimes, it is clear that
\[
p \leq q \implies \hat{d}_\tau(p, q) = \tau(q) - \tau(p)
\]
is always satisfied, while the converse
\[
p \leq q \iff \hat{d}_\tau(p, q) = \tau(q) - \tau(p) \quad (10)
\]
guarantees that \(\hat{d}_\tau\) is definite [83, Lemma 3.12], and hence \(\hat{d}_\tau\) is a metric that induces the manifold topology for continuous \(\tau\) [83, Proposition 3.15]. In [83, Proposition 4.5] it has been shown that locally anti-Lipschitz time functions \(\tau\) are exactly those for which the corresponding null distances \(\hat{d}_\tau\) are definite.

Remark 3.22. Suppose that \((M, \hat{d}_\tau)\) is such that for any two points \(p, q \in M\), the distance is achieved by a piecewise causal curve \(\beta\). Then \(\hat{L}(\beta) = \hat{d}_\tau(p, q) = \tau(q) - \tau(p)\) implies by [83, Lemma 3.6] that \(p \leq q\), i.e., the converse (10) holds and \(\hat{d}_\tau\) encodes causality.

Summing up the above discussion, the following relations for (generalized) time functions \(\tau\) are known:

\[
\begin{array}{ccc}
\hat{d}_\tau \text{ encodes causality} & \rightarrow & \hat{d}_\tau \text{ is definite} \\
\text{if distances are achieved in } \hat{A} & \uparrow & \\
\tau \text{ is locally anti-Lipschitz}
\end{array}
\]

In particular, for spacetimes where all null distances between points are realized by piecewise causal curves, all three important notions are equivalent. In Riemannian geometry, such a condition would be guaranteed by completeness of the length space via the Hopf–Rinow Theorem. In Section 3.3, we have observed that completeness of \((M, \hat{d}_\tau)\) does in general not guarantee that a shortest path is piecewise
null distance and spacetime convergence

Fig. 5. A sequence of piecewise null curves $\beta_i$ in $\mathbb{M}^{1+1}$ whose length converges to $\hat{d}_\tau(p, q) = 2$ in an incomplete spacetime, described in Example 3.23.

causal, however, the null distances of the counterexamples still encode causality. As such, completeness still plays an important role in connection with the causality encoding property. In Examples 3.23 and 3.24, we discuss two simple incomplete (and not globally hyperbolic) spacetimes with globally anti-Lipschitz function that do not encode causality.

Example 3.23. Consider two-dimensional Minkowski space $\mathbb{M}^{1+1}$ with the point $(1, 1)$ removed and time function $\tau = t$ (see [74, Ex. 4, p. 403]). For $p = (0, 0)$ and $q = (2, 2)$ we obtain that $\hat{d}_\tau(p, q) = \tau(q) - \tau(p) = 2$, which can be seen by taking a sequence of piecewise causal curves in the future light cone of $p$ (see Figure 3.23). On the other hand, it is clear that $p \not\leq q$.

Example 3.24. Consider again $\mathbb{M}^{1+1}$ with time function $\tau = t$, but now we remove the whole line $\{(x, 1) : -1 \leq x \leq 1\}$. Let $p = (0, 0)$ and $q = (0, 2)$ which satisfy $\hat{d}_\tau(p, q) = \tau(q) - \tau(p) = 2$ as can be seen by taking (approximating) paths which connect $p$ via null geodesics to $p_\epsilon = (\epsilon, 0)$, through $(1 + \epsilon, 1)$ and then proceed to $q_\epsilon = (\epsilon, 2)$ (see Figure 3.24). It is also clear in this case that $p \not\leq q$. 
Fig. 6. A sequence of piecewise null curves $\beta_i$ in $\mathbb{M}^{1+1}$ whose null length converges to $\hat{d}_\tau(p, q) = 2$ in an incomplete spacetime, described in Example 3.24.

4 (M, $\hat{d}_\tau$) as an Integral Current Space

Our goal in this section is to rigorously introduce the current structure on spacetimes (including those of low regularity), which carry a metric structure obtained via the null distance. In particular, we focus here on warped products and globally hyperbolic spacetimes, since they constitute the most important classes of Lorentzian manifolds in general relativity. The strategy is to induce a (local) integral current space structure by using (locally) bi-Lipschitz maps to Lorentzian products and then defining bi-Lipschitz maps from regions of Lorentzian products to regions of a suitable Riemannian space which allows us to apply Theorem 2.15 or Theorem 2.18. If the Lorentzian metrics are continuous a null distance analog of [24, Theorem 4.5] suffices, however, we also allow metrics of less regularity if they have a warped product structure. Thus we prove a Lorentzian analog of the Riemannian result in [7, Lemma 2.3] instead. We start by investigating properties of Lorentzian products, and then move on to warped products and globally hyperbolic spacetimes.

4.1 Lorentzian product manifolds

We analyze Lorentzian products $\mathbb{M}_\sigma = \mathbb{R} \times \Sigma$, where $(\Sigma, \sigma)$ is a suitable Riemannian manifold, and show that its canonical null distance $\hat{d}_\sigma = \hat{d}_t$ has the same features as
the null distance of Minkowski space $\mathbb{M}^{n+1} = \mathbb{R}^{n,1}$. Moreover, we prove that $\mathbb{M}_σ$ is a (local) integral current space.

Lorentzian products naturally include all Lorentzian manifolds for which a Lorentzian splitting theorem applies (see, for instance, [35, 41, 42, 72], and [40] for an overview of Lorentzian splitting theorems). In particular, they include globally hyperbolic (or timelike geodesically complete) spacetimes with nonnegative Ricci curvature bounds and a timelike line.

We recall the metric and locally integral current structure of the simplest Lorentzian product.

**Example 4.1 (Minkowski space).** By $\mathbb{M}^{n+1}$ we denote the $(n+1)$-dimensional Minkowski space. The Minkowski metric can be written as

$$\eta = -dt^2 + g_{\mathbb{E}^n},$$

where $g_{\mathbb{E}^n}$ is the standard metric on the Euclidean $n$-space. For points $p \in \mathbb{M}^{n+1}$ we write $p = (t_p, p_E)$, where $t_p = t(p) \in \mathbb{R}$ and $p_E \in \mathbb{R}^n$. The null distance of $\mathbb{M}^{n+1}$ (with respect to the canonical time function $\tau = t$) is denoted by $\hat{d}_\mathbb{M}$. Several properties of this null distance have already been established by Sormani and Vega in [83]. Among others, it was shown in [83, Proposition 3.3] that $\hat{d}_\mathbb{M}$ is definite, translation invariant, and satisfies

$$\hat{d}_\mathbb{M}(p, q) = \begin{cases} |t(q) - t(p)| & q \in J^\pm(p), \\ \|q_E - p_E\| & q \notin J^\pm(p), \end{cases} \quad (11)$$

where $\|\cdot\|$ denotes the Euclidean norm and $J^\pm(p)$ the causal future/past of $p$. Note that many of these properties are lost if one uses another time function [83, Proposition 3.4].

The standard locally integral current $T$ on $\mathbb{M}^{n+1}$ is that of $\mathbb{R}^{n+1}$, that is,

$$T(f, π_1, \ldots, π_{n+1}) = \int_{\mathbb{R}^{n+1}} f \, dπ_1 \wedge \ldots \wedge dπ_{n+1}.$$

**Definition 4.2.** Let $(\Sigma, σ)$ be a Riemannian manifold. The Lorentzian product manifold $\mathbb{M}_σ$ is the Lorentzian manifold $\mathbb{R} \times \Sigma$ with metric tensor

$$\eta_σ = -dt^2 + σ. \quad (12)$$
Remark 4.3 (Regularity). In Definition 4.2, we do not need to assume that $\sigma$ is smooth, however, the induced Riemannian distance function $d_\sigma$ should be well-defined to obtain a meaningful null distance $\hat{d}_\sigma$. We can, for instance, assume that $\sigma$ is continuous [24].

The form (11) of the null distance in Minkowski space extends to (weak) Lorentzian products in the obvious way.

Lemma 4.4. Let $(\Sigma, \sigma)$ be a connected Riemannian manifold and $(\mathbb{M}_\sigma, \eta_\sigma)$ be the corresponding Lorentzian product manifold of Definition 4.2. Then the null distance $\hat{d}_\sigma$ induced by the canonical time function $t$ satisfies

$$\hat{d}_\sigma(p, q) = \begin{cases} |t(p) - t(q)| & q \in J^\pm(p), \\ d_\sigma(p, q) & q \notin J^\pm(p), \end{cases}$$

where $d_\sigma$ is the Riemannian distance function induced by $\sigma$ on $\Sigma$ and $J^\pm(p)$ is the causal future/past of $p$ with respect to $\eta_\sigma$.

Proof. Let $p \in \mathbb{M}_\sigma$. If $q \in J^\pm(p)$, then any causal curve $\beta$ between $p$ and $q$ is minimal and hence $\hat{d}_\sigma(p, q) = \hat{L}_t(\beta) = |t(q) - t(p)|$.

Suppose $q \notin J^\pm(p)$. By [83, Lemma 3.5] there is a piecewise causal curve $\beta$ joining $p$ and $q$. The curve $\beta$ thus consists of a finite number of (unbroken) causal curves which we call $\beta_1, \ldots, \beta_k$ and assume to be parametrized by time and without loss of generality of the form $\beta_i: [t_{i-1}, t_i] \to M$ (if $t_{i-1} \geq t_i$ we may change orientation). That is, they are of the form

$$\beta_i(t) = (t, \alpha_i(t)), \quad t \in [t_{i-1}, t_i],$$

where $\alpha_i([t_{i-1}, t_i]) \subseteq \Sigma$. Causality of $\beta_i$ implies that

$$\eta_\sigma(\beta_i', \beta_i') = -1 + \sigma(\alpha_i', \alpha_i') \leq 0,$$

hence $\sigma(\alpha_i', \alpha_i') \leq 1$. Thus

$$L_\sigma(\alpha) = \sum_{i=1}^k L_\sigma(\alpha_i) \leq \sum_{i=1}^k |t_i - t_{i-1}| = \hat{L}_t(\beta).$$
Since \( d_\sigma(p_\Sigma, q_\Sigma) \leq L_\sigma(\tilde{\alpha}) \) holds for any spatial curve \( \tilde{\alpha} \), this implies that

\[
d_\sigma(p_\Sigma, q_\Sigma) \leq \hat{d}_\sigma(p, q). \tag{14}
\]

It remains to be shown that equality holds. Let \( \varepsilon > 0 \) and \( \alpha : [0, L] \to \Sigma \) be a (non-selfintersecting) curve from \( p_\Sigma \) to \( q_\Sigma \), parametrized by arc length, such that

\[
L_\sigma(\alpha) < d_\sigma(p_\Sigma, q_\Sigma) + \varepsilon. \tag{15}
\]

We construct a broken null curve \( \beta \) from \( p \) to \( q \) by suitably lifting \( \alpha \) to \( M_\sigma \) such that \( \hat{L}_t(\beta) = L_\sigma(\alpha) \). Consider the connected manifold \( M_\tilde{\sigma} = \mathbb{R} \times \text{Im}(\alpha) \) with induced Lorentzian metric \( \tilde{\sigma} = \sigma|_{\text{Im}(\alpha)} \) and the same canonical time function \( t \). By [83, Lemma 3.5 & Remark 3.7] (making use of [39]) we know there exists a piecewise null curve \( \beta \) in \( M_\tilde{\sigma} \) connecting \( p \) and \( q \). Since \( \text{Im}(\beta) \subseteq \mathbb{R} \times \text{Im}(\alpha) \) we can assume without loss of generality that \( \beta \) is parametrized such that

\[
\beta(s) = (t(s), \alpha(s))
\]

and has breaking points \( s_i, i = 0, \ldots, k \) (since we can always restrict to \( \mathbb{R} \times \text{Im}(\alpha|_{[s_i, b_i]} \) in the \( i \)-th step, we can assume that \( \beta \) does not go “back and forth” along \( \alpha \)). Since \( \beta \) is piecewise null with respect to \( \tilde{\sigma} \), and hence with respect to \( \sigma \), we have that

\[
\hat{L}_t(\beta) = \sum_{i=1}^{k} |t(\beta(s_i)) - t(\beta(s_{i-1}))| = \sum_{i=1}^{k} L_{\tilde{\sigma}}(\alpha|_{[s_{i-1}, s_i]}) = L_\sigma(\alpha).
\]

Together with (15), we obtain

\[
\hat{d}_\sigma(p, q) \leq \hat{L}_t(\beta) = L_\sigma(\alpha) < d_\sigma(p_\Sigma, q_\Sigma) + \varepsilon,
\]

and therefore, together with (14),

\[
\hat{d}_\sigma(p, q) = \hat{d}_\sigma(p_\Sigma, q_\Sigma) \tag{16}
\]

whenever \( p \) and \( q \) are not causally related, i.e., if \( q \not\in J^\pm(p) \) and vice versa. This proves the second case for (13).
Remark 4.5. In the above proof of Lemma 4.4 we considered $\mathbb{M}_\sigma = \mathbb{R} \times \Sigma$. The same result holds for $\mathbb{M}_\sigma = I \times \Sigma$, where $I$ is an interval. Note that $\Sigma$ does not necessarily need to be complete.

Remark 4.6 (Causality encoding property). Lorentzian products $\mathbb{M}_\sigma = I \times (\Sigma, \sigma)$ always satisfy the anti-Lipschitz property for the canonical time function due to the explicit structure (13) of the null distance obtained in Lemma 4.4.

In order to prove that we have a (locally) integral current structure on a warped product $M_f = I \times_f \Sigma$ it therefore remains to be shown that the Lorentzian product manifold $\mathbb{M}_\sigma$ with null distance $\hat{d}_\sigma$ is a (local) integral current space, and to establish a connection between $M_f$ and $\mathbb{M}_\sigma$. This is shown in Lemma 4.7, Proposition 4.10, and Theorem 1.2.

Lemma 4.7. Let $I \subseteq \mathbb{R}$ be an interval and $(\Sigma, \sigma)$ be a connected complete Riemannian manifold. Let $\mathbb{M}_\sigma$ be the corresponding Lorentzian product manifold, endowed with the null distance $\hat{d}_\sigma$ with respect to the canonical time function $\tau = t$. Let $\tilde{d}_\sigma$ be the distance function with respect to the Riemannian warped product

$$\tilde{\eta}_{\sigma} = dt^2 + \sigma.$$

Then the identity map

$$\text{id}: (I \times \Sigma, \hat{d}_\sigma) \rightarrow (I \times \Sigma, \tilde{d}_\sigma),$$

is bi-Lipschitz and a natural local integral current space structure is induced on $\mathbb{M}_\sigma$. If, in addition, $I \times \Sigma$ is compact then $\mathbb{M}_\sigma$ is an integral current space.

Remark 4.8. The proof requires knowledge of the metric structure of products of Riemannian manifolds. Let $M = \Sigma_1 \times \Sigma_2$ be the product of the Riemannian manifolds $(\Sigma_1, h_1)$ and $(\Sigma_2, h_2)$ with induced distance functions $d_{h_1 \times h_2}, d_{h_1}, d_{h_2}$. Suppose $\Sigma_1$ is a uniquely geodesic metric space, that is, any two points are the endpoints of a unique minimizing geodesic. Then the distance function on $M$ is given by the Pythagorean formula

$$d_{h_1 \times h_2}(p, q)^2 = d_{h_1}(p_1, q_1)^2 + d_{h_2}(p_2, q_2)^2, \quad p, q \in M.$$

See [33, Section 3.1] for a proof when $\Sigma_1$ and $\Sigma_2$ are complete Riemannian manifolds.
Proof. We first assume that \( q \in J^\pm(p) \). Then by (the proof of) Lemma 4.4 we have that

\[
\hat{d}_\sigma(p, q) = |t(q) - t(p)| \geq d_\sigma(p_\Sigma, q_\Sigma),
\]

which together with Remark 4.8 implies that

\[
\hat{d}_\sigma(p, q) = |t(q) - t(p)| \leq \bar{d}_\sigma(p, q).
\]

Next assume that \( q \notin J^\pm(p) \). Then by (the proof of) Lemma 4.4 we know that

\[
\hat{d}_\sigma(p, q) = d_\sigma(p_\Sigma/\Sigma_1, q_\Sigma/\Sigma_1) \leq \bar{d}_\sigma(p, q).
\]

Note that (17) and (18) imply that

\[
\hat{d}_\sigma(p, q) \leq \bar{d}_\sigma(p, q) \leq \sqrt{2} \hat{d}_\sigma(p, q).
\]

for any \( p, q \in I \times \Sigma \). Hence, the identity map on \( I \times \Sigma \) is bi-Lipschitz from \( \hat{d}_\sigma \) to \( \bar{d}_\sigma \).

Recall that connected complete Riemannian manifolds with continuous metric are local integral current spaces. Let \( T \) be this locally integral current on \((I \times \Sigma, \bar{d}_\sigma)\). Then by Theorem 2.18 we find that \((M_\sigma, \hat{d}_\sigma, \text{id}_\# T)\) is a local integral current space. By Theorem 2.15 it is an integral current space when \( I \times \Sigma \) is compact.

4.2 Spacetime warped products

The formula (13) can be generalized to warped products with merely bounded warping function \( f \) if we estimate the Riemannian distance function using the bounds of \( f \). Since we always use the canonical time function \( \tau(t, x) = t \), we simply write \( \hat{d}_f \) for the null
distance \( \hat{d}_t \) corresponding to the causal structure of the warped product \( I \times_f \Sigma \). We also write \( \hat{L} \) for the corresponding null length \( \hat{L}_t \).

**Lemma 4.9.** Let \( I \) be an interval, \( (\Sigma, \sigma) \) be a connected Riemannian manifold, and \( f \) be a bounded function such that

\[
0 < f_{\text{min}} \leq f(t) \leq f_{\text{max}}, \quad t \in I.
\]

Then the warped product \( M_f = I \times_f \Sigma \) with Lorentzian metric

\[
g_f = -dt^2 + f(t)^2 \sigma
\]

is such that the null distance \( \hat{d}_f \) of \( g_f \) with canonical time function \( t \) satisfies

\[
\hat{d}_f(p, q) = |t(p) - t(q)|, \quad q \in J^\pm(p),
\]

\[
f_{\text{min}} d_\sigma(p_\Sigma, q_\Sigma) \leq \hat{d}_f(p, q) \leq f_{\text{max}} d_\sigma(p_\Sigma, q_\Sigma), \quad q \notin J^\pm(p).
\]  

**Proof.** Essentially we can apply the same proof as in Section 4.1 above. Let \( p, q \in M \) and \( \beta(s) = (t(s), \alpha(s)) \) be a null causal curve joining \( p \) and \( q \). The fact that

\[
g_f(\beta'(t), \beta'(t)) = -1 + f(t)^2 \sigma(\alpha'(t), \alpha'(t)) = 0,
\]

however, implies that \( \sigma(\alpha', \alpha') \leq \frac{1}{f_{\text{min}}} \) and thus

\[
L_\sigma(\alpha) \leq \frac{1}{f_{\text{min}}} \hat{L}(\beta).
\]

Therefore,

\[
f_{\text{min}} d_\sigma(p_\Sigma, q_\Sigma) \leq \hat{d}_f(p, q).
\]

In the second part of the proof, where a null curve \( \beta \) is constructed from an (almost) \( \Sigma \)-minimizing \( \alpha \), (20) implies that \( \sigma(\alpha', \alpha') \geq \frac{1}{f_{\text{max}}} \). Therefore,

\[
\frac{1}{f_{\text{max}}} \hat{d}_f(p, q) \leq \frac{1}{f_{\text{max}}} \hat{L}(\beta) \leq L_\sigma(\alpha) \leq d_\sigma(p, q) + \varepsilon,
\]
which shows that
\[
\hat{d}_f(p, q) \leq f_{\text{max}} \hat{d}_\sigma(p, q)
\]
if \(q \notin J^\pm(p)\). 
\[
\tag{21}
\]
Case 2: \( q \in J^+_f(p) \setminus J^+_I(p) \). Suppose \( \beta \) is a future-directed causal curve in \( J^+_f(p) \) connecting \( p \) and \( q \). Then

\[
\hat{d}_f(p, q) = \hat{L}_f(\beta) = t(q) - t(p) \leq \hat{d}_\sigma(p, q).
\] (22)

Since \( \beta \) is causal with respect to \( g_f \), we may assume that it is parametrized by time, i.e., \( \beta(t) = (t, \alpha(t)) \) for some curve \( \alpha \) in \( \Sigma \). Thus

\[
g_f(\beta'(t), \beta'(t)) = -1 + f(t)^2 \sigma(\alpha'(t), \alpha'(t)) \leq 0,
\]

and hence \( \|\alpha'\|_\sigma \leq \frac{1}{f_{\min}} \). This implies

\[
d_\sigma(p, q) \leq L_\sigma(\alpha) \leq \frac{1}{f_{\min}} |t(q) - t(p)|.
\]

By Lemma 4.4 and Lemma 4.9 together with (22) we obtain that

\[
\hat{d}_f(p, q) \leq \hat{d}_\sigma(p, q) = d_\sigma(p, q) \leq \frac{1}{f_{\min}} \hat{d}_f(p, q).
\]

Case 3: \( q \in J^+_I(p) \setminus J^+_f(p) \). Then for a future-directed causal curve \( \beta(t) = (t, \alpha(t)) \) in \( J^+_I(p) \) we have

\[
\hat{d}_\sigma(p, q) = \hat{L}_I(\beta) = t(q) - t(p) \leq \hat{d}_f(p, q).
\] (23)

Causality of \( \beta \) implies that

\[
\eta_\sigma(\beta'(t), \beta'(t)) = -1 + \sigma(\alpha'(t), \alpha'(t)) \leq 0,
\]

and therefore that \( \|\alpha'\|_\sigma \leq 1 \). Thus

\[
d_\sigma(p, q) \leq L_\sigma(\alpha) \leq |t(q) - t(p)|.
\]

By Lemma 4.4 and Lemma 4.9 together with (23) we hence obtain

\[
\frac{1}{f_{\max}} \hat{d}_f(p, q) \leq d_\sigma(p, q) \leq \hat{d}_\sigma(p, q) \leq \hat{d}_f(p, q).
\]
Case 4: \( q \not\in J^+_1(p) \cup J^+_2(p) \). If \( q \) is in neither of the light cones, then by Lemma 4.4

\[
\hat{d}_\sigma(p,q) = d_\sigma(p_{\Sigma}, q_{\Sigma}),
\]

and by Lemma 4.9 we know that

\[
f_{\min} d_\sigma(p_{\Sigma}, q_{\Sigma}) \leq \hat{d}_f(p,q) \leq f_{\max} d_\sigma(p_{\Sigma}, q_{\Sigma}).
\]

Hence, the statement follows immediately in this case. \( \blacksquare \)

Remark 4.12. We expect that Proposition 4.10 can be extended to multi-warped products and to spacetimes \( N_h = I_h \times (\Sigma, \sigma) \) of the form

\[
g_h = -h^2 dt^2 + \sigma,
\]

where \( h \) is a positive bounded function on \( \Sigma \). Singular static metrics appear, for instance, when the Einstein equations are coupled to matter fields [12, 23]. Using spacetime convergence results, it may be possible to study their spacetime stability using techniques analogous to [20, 52, 62, 64].

Combining the above results, Proposition 4.10, Theorem 2.15, and Theorem 2.18 yields that we can induce a natural (local) integral current space structure on warped product spacetimes \( I \times_f (\Sigma, \sigma) \) with warping functions \( f \) of low regularity and continuous Riemannian metrics \( \sigma \).

Theorem 1.2. Let \( I \) be an interval and \( (\Sigma, \sigma) \) be a connected complete Riemannian manifold. Suppose \( f: I \to (0, \infty) \) is a bounded function that is bounded away from 0. Then there is a natural local integral current space structure on the warped product spacetime \( M = I \times_f \Sigma \) with respect to the null distance \( \hat{d}_f \). If \( I \times \Sigma \) is compact then \( (M, \hat{d}_f) \) carries an integral current space structure.

Proof. By Lemma 4.7 the Lorentzian product \( I \times \Sigma \) is a (local) integral current space with respect to \( \hat{d}_\sigma \). Since the identity \( \text{id}: (I \times \Sigma, \hat{d}_\sigma) \to (I \times \Sigma, \hat{d}_f) \) is bi-Lipschitz by Proposition 4.10, the pushforward Theorems 2.15 and 2.18 imply that the warped product spacetime \( M \) with null distance \( \hat{d}_f \) can be induced with a (local) integral current structure as well. \( \blacksquare \)
4.3 Globally hyperbolic spacetimes

We use the results of the previous subsections to show that globally hyperbolic spacetimes are (local) integral current spaces with respect to their corresponding null distances. We begin with a result which allows us to compare null distance functions between general Lorentzian metrics and warped product metrics. We adapt the following (slightly modified) ≼ relation from [31, Section 1.2].

**Definition 4.13.** Let $M$ be a connected manifold with Lorentzian metrics $g_1, g_2$. We say that $g_1$ is *smaller* than $g_2$, denoted by $g_1 \preceq g_2$, if for all tangent vectors $v \neq 0$ the implication

$$g_1(v, v) \leq 0 \implies g_2(v, v) \leq 0$$

is satisfied.

In other words, if $g_1 \preceq g_2$ then the light cones with respect to $g_2$ are wider than those of $g_1$.

**Example 4.14.** If $(\Sigma, \sigma)$ is a connected Riemannian manifold and $f_1, f_2$ are two warping functions that satisfy $0 < f_1 \leq f_2$ everywhere, then the warped product metrics on $I \times \Sigma$ satisfy $g_{f_2} \preceq g_{f_1}$ and the corresponding null distances satisfy $\hat{d}_{f_1}(p, q) \leq \hat{d}_{f_2}(p, q)$.

The observation regarding the null distance in Example 4.14 can be generalized.

**Lemma 4.15.** Let $M$ be a connected manifold and $g_1, g, g_2$ be Lorentzian metrics on $M$ such that

$$g_1 \preceq g \preceq g_2. \quad (24)$$

Then for a time function $\tau : M \to \mathbb{R}$ the corresponding null distances satisfy

$$\hat{d}_{g_2}(p, q) \leq \hat{d}_{g}(p, q) \leq \hat{d}_{g_1}(p, q), \quad p, q \in M.$$

**Proof.** Property (24) makes sure that the light cones of the different Lorentzian manifolds are contained in one another. In particular, if $\alpha$ is a piecewise causal curve for $g_1$ then $\alpha$ is a piecewise causal curve for $g$. Similarly, if $\beta$ is a piecewise causal curve for $g$, then $\beta$ is a piecewise causal curve for $g_2$. Since the null distance between $p$ and $q$ is
defined as an infimum over piecewise causal curves these inclusions imply the desired conclusion,

\[ \hat{d}_{g_2}(p, q) \leq \hat{d}_g(p, q) \leq \hat{d}_{g_1}(p, q). \]

We generalize Lemma 4.15 to a local version which is utilized below and also interesting in its own right. The proof is based on a similar argument as the proof of [24, Theorem 4.5] in the Riemannian case, however, the assumptions are necessarily stronger.

**Proposition 4.16.** Let \( M \) be a connected manifold, equipped with Lorentzian metrics \( g \) and \( h \). Let \( U \) be an open set such that \( g \preceq h \) holds on \( U \). Suppose \( \tau \) is a time function with respect to both \( g \) and \( h \) such that the null distances \( \hat{d}_g \) and \( \hat{d}_h \) with respect to \( \tau \) are metrics. Then for every compact set \( K \subset U \) there exists a constant \( C > 0 \) (possibly depending on \( K \)) such that

\[ \hat{d}_h(p, q) \leq C\hat{d}_g(p, q), \quad p, q \in K. \]  

**(25)**

**Proof.** We proceed by contradiction. Suppose (25) does not hold. Then for all \( n \in \mathbb{N} \) there exist points \( p_n, q_n \in K \) such that

\[ \hat{d}_h(p_n, q_n) > n\hat{d}_g(p_n, q_n). \]  

**(26)**

Since \( K \) is compact, and both \( \hat{d}_g, \hat{d}_h \) are metrics that induce the manifold topology, we can assume that the sequences \( (p_n)_n \) and \( (q_n)_n \) converge to the same limit \( p \in K \) by passing to subsequences.

Since \( M \) is locally compact there exists an \( r_0 > 0 \) such that the ball \( B^g_{r_0}(p) = \{ q \in M : \hat{d}_g(p, q) \leq r_0 \} \) is compact and contained in \( U \). Consider \( r := \frac{r_0}{4} \) and arbitrary points \( x, y \in B^g_r(p) \). Then, for every \( \varepsilon \in (0, r) \), there exists a (piecewise) causal curve \( \beta_\varepsilon \) such that

\[ \hat{L}^g_\beta \beta_\varepsilon < \hat{d}_g(x, y) + \varepsilon. \]

This curve \( \beta_\varepsilon \) does not leave \( B^g_{r_0}(p) \) since for all \( s \)

\[ \hat{d}_g(p, \beta_\varepsilon(s)) \leq \hat{d}_g(p, x) + \hat{d}_g(x, \beta_\varepsilon(s)) \leq \hat{d}_g(p, x) + \hat{d}_g(x, y) + \varepsilon < r + 2r + r = 4r = r_0. \]
By assumption, \( g \preceq h \) on \( B^g_{r_0}(p) \subseteq U \), hence \( \beta_\varepsilon \) is also piecewise causal with respect to \( h \) and therefore \( \hat{L}_g(\beta_\varepsilon) = \hat{L}_h(\beta_\varepsilon) \) and

\[
\hat{d}_h(x, y) \leq \hat{d}_g(x, y) = \hat{d}_g(x, y) + \varepsilon.
\]

Since this estimate holds for all \( \varepsilon \leq r \) this implies that

\[
\hat{d}_h(x, y) \leq \hat{d}_g(x, y), \quad x, y \in B^g_r(p).
\]

In particular, since the points \( p_n \) and \( q_n \) converge to \( p \) we have that for sufficiently large \( n \)

\[
\hat{d}_h(p_n, q_n) \leq \hat{d}_g(p_n, q_n),
\]

which contradicts (26).

\[\Box\]

We apply an orthogonal splitting and use the above relation on chart neighborhoods and for particular Lorentzian product metrics \( g_1 \) and \( g_2 \). This induces a current on \((M, \hat{d}_\tau)\) via locally bi-Lipschitz maps.

**Proposition 4.17.** Let \((M, g)\) be a globally hyperbolic spacetime. Then there exists a (smooth) time function \( \tau \), \((M, g)\) is isometric to \((\mathbb{R} \times \Sigma, \tilde{g})\) where \( \tilde{g} \) is a Lorentzian metric with Cauchy surfaces \( (\Sigma_\tau, \sigma_\tau) = \sigma|_{\Sigma_\tau} \), and \( \text{id} : (\mathbb{R} \times \Sigma, \hat{d}_{\sigma_0}) \to (\mathbb{R} \times \Sigma, \hat{d}_{\tilde{g}}) \) is locally bi-Lipschitz.

**Proof.** Bernal and Sánchez prove in [17] that any globally hyperbolic spacetime admits a smooth time function \( \tau \) such that \( \nabla \tau \) is everywhere past-pointing timelike. More precisely, \((M, g)\) is isometric to \((\mathbb{R} \times \Sigma, \tilde{g})\) with

\[
\tilde{g} = -h^2 d\tau^2 + \sigma,
\]

where \( \Sigma \) is a smooth spacelike Cauchy hypersurface, \( \tau : \mathbb{R} \times \Sigma \to \mathbb{R} \) is the natural projection (and time function), \( h^2 : \mathbb{R} \times \Sigma \to (0, \infty) \) is a smooth function, and \( \sigma \) is a symmetric 2-tensor field on \( \mathbb{R} \times \Sigma \). Moreover, on each constant-\( \tau \) hypersurface \( \Sigma_\tau \) the restriction \( \sigma_\tau := \sigma|_{\Sigma_\tau} \) is a Riemannian metric.

Let \((U_i, u_i)_i\) be a countable atlas of precompact open sets on \( \mathbb{R} \times \Sigma \) and let \( V_i \) be precompact open sets so that \( \overline{U_i} \subseteq V_i \). Since \( h^2 > 0 \) and \( V_i \) is precompact, there exists a
constant $c_i$ such that $h^2|_{V_i} \in [\frac{1}{c^2_i}, c^2_i]$. Moreover, we can compare each Riemannian metric $\sigma_\tau$ on $V_i \cap \Sigma_\tau$ to $\sigma_0$ since there exist continuous functions $\lambda_0, \mu_0 > 0$ such that

$$\lambda_0(\tau)^2 \sigma_0(v,v) \leq \sigma_\tau(v,v) \leq \mu_0(\tau)^2 \sigma_0(v,v),$$

for all $v \in T_p \Sigma, (\tau, p') \in U_i \cap \Sigma_\tau$, which follows immediately from the proofs in [24, Proposition 4.1 & (4.2)]. For $a, b \in \mathbb{R} \setminus \{0\}$ we define the reference product metric

$$\tilde{g}_{a,b} := -a^2 d\tau^2 + b^2 \sigma_0,$$

Due to the pre-compactness of $V_i$ we have positive constants

$$\lambda_i := \inf \lambda_0(\tau(V_i)) > 0,$$
$$\mu_i := \sup \mu_0(\tau(V_i)) \geq \lambda_i > 0,$$

which implies that for $w \in T_p(\mathbb{R} \times \Sigma), p \in V_i$,

$$\tilde{g}_{c_i, \lambda_i}(w,w) \leq \tilde{g}(w,w) \leq \tilde{g}_{\frac{1}{c_i}, \mu_i}(w,w),$$

which in the notation of Definition 4.13 says that locally on $V_i$ we have

$$\tilde{g}_{\frac{1}{c_i}, \mu_i} \preceq \tilde{g} \preceq \tilde{g}_{c_i, \lambda_i}.$$ 

Since $U_i$ is precompact, by Proposition 4.16 there exists a constant $C_i > 0$ such that

$$\frac{1}{C_i} \hat{d}_{\tilde{g}_{c_i, \lambda_i}}(p,q) \leq \hat{d}_{\tilde{g}}(p,q) \leq C_i \hat{d}_{\tilde{g}_{\frac{1}{c_i}, \mu_i}}(p,q), \quad p, q \in U_i.$$ 

By [83, Proposition 3.9] the null distance is invariant under conformal changes, hence

$$\frac{1}{C_i} \hat{d}_{\tilde{g}_{\frac{1}{c_i}, \lambda_i}}(p,q) \leq \hat{d}_{\tilde{g}}(p,q) \leq C_i \hat{d}_{\tilde{g}_{1, c_i \mu_i}}(p,q), \quad p, q \in U_i.$$ 

By Proposition 4.10, we can estimate the left and right null distances further to obtain for the constant $A_i = C_i \max\{c_i \mu_i, \frac{c_i}{\lambda_i}, 1\} > 0$ that

$$\frac{1}{A_i} \hat{d}_{\sigma_0}(p,q) \leq \hat{d}_{\tilde{g}}(p,q) \leq A_i \hat{d}_{\sigma_0}(p,q), \quad p, q \in U_i.$$ 

which means that the identity map $\text{id}: (\mathbb{R} \times \Sigma, \hat{d}_{\sigma_0}) \to (\mathbb{R} \times \Sigma, \hat{d}_{\tilde{g}})$ is bi-Lipschitz on $U_i$. 

$\blacksquare$
Theorem 1.3. Let \((M, g)\) be a globally hyperbolic spacetime with (smooth) time function \(\tau\) of Proposition 4.17. Suppose \(M\) admits Cauchy hypersurfaces on which the ambient Lorentzian metric restricts as a complete Riemannian metric and \((M, \hat{d})\) is complete as a metric space. Then \((M, \hat{d})\) is a local integral current space. If \(M\) is compact then it is an integral current space.

Proof. By Proposition 4.17 \(M = \mathbb{R} \times \Sigma\) and the identity \(\text{id}: (M, \hat{d}_{\sigma_0}) \to (M, \hat{d}_g)\) is locally bi-Lipschitz. Recall that \(\hat{d}_{\sigma_0}\) is the null distance of the product spacetime \(M_{\sigma_0}\) and thus carries a natural (local) integral current space structure by Lemma 4.7. By Theorem 1.1 it is a locally compact length space, because it carries the manifold topology. Moreover, since \((\Sigma, \sigma_0)\) is complete, so is the Riemannian product \((\mathbb{R} \times \Sigma, d\tau^2 + \sigma_0)\) and hence the Lorentzian product \((M, \hat{d}_{\sigma_0})\) is complete by Corollary 3.15. Hence, \((M, \hat{d}_{\sigma_0})\) is a proper metric space (see Remark 2.19). Hence, the locally Lipschitz identity \(\text{id}: (M, \hat{d}_{\sigma_0}) \to (M, \hat{d}_g)\) is Lipschitz on bounded sets by Remark 2.19. Similarly, the assumption that \((M, \hat{d})\) is complete implies that it is proper, and the inverse map \(\text{id}: (M, \hat{d}_g) \to (M, \hat{d}_{\sigma_0})\) is also Lipschitz on bounded sets. Hence, by Theorem 2.18 we can push forward the (local) integral current structure via the locally bi-Lipschitz identity to \((M, \hat{d}_g)\).

The compact case follows immediately from Proposition 4.17 and Theorem 2.15.

Remark 4.18. Recall that a time-oriented Lorentzian manifold \((M, g)\) is called globally hyperbolic if and only if it is causal and if for every pair of points \(p, q \in M\) the set \(J^+(p) \cap J^-(q)\) is compact (in fact, Hounnonkpe and Minguzzi recently showed the surprising result that the causal condition is not even needed for noncompact manifolds with spacetime dimensions larger than three [50, Theorem 2.8]). Lorentzian products \(I \times \Sigma\) and warped products \(I \times_f \Sigma\) are globally hyperbolic if and only if the Riemannian manifold \((\Sigma, \sigma)\) is complete (see [16, Theorem 3.66]). This is the setting in which we studied both cases earlier, hence Theorem 1.3 naturally generalizes Lemma 4.7 and Theorem 1.2 on the integral current structure of \(I \times \Sigma\) and \(I \times_f \Sigma\), respectively.

Remark 4.19. We believe Theorem 1.3 can be extended to weaker notions of causality or related local assumptions. In particular, the regularity assumptions may be weakened.

Moreover, it should be straightforward to drop the completeness assumptions in Theorem 1.3 (as well as in the earlier (warped) product results Theorem 1.2 and Lemma 4.7) by using a definition of local integral current spaces that is based on the locally integral currents of Lang [58] (see also our earlier Remark 2.20).
5 Convergence of Warped Product Spacetimes

By Section 4 we now know that warped product spacetimes (of low regularity) and globally hyperbolic spacetimes with complete Cauchy surfaces are (local) integral current spaces with respect to the null distance. We are thus in a position to study the limits of such sequences of spacetimes under GH and SWIF convergence. In this section, we initiate the study of spacetime convergence for the class of warped products. In particular, we provide a general statement for uniformly converging warping functions (see Theorem 1.4) and address the distinct limiting behavior for nonuniformly converging sequences in Section 5.2.

5.1 Uniform convergence for warped product spacetimes

In this subsection, we state sufficient conditions on a sequence of warping functions \( f_j, f_j \to f_\infty \), which guarantee that the sequence of corresponding warped product spacetimes with associated null distance structure converges in the GH and SWIF topology to the warped product spacetime with warping function \( f_\infty \). The strategy of the proof is to show a general pointwise convergence result and then use the bi-Lipschitz bounds of Section 4 to obtain the uniform, GH, and SWIF convergence result via a compactness result of Huang, Lee and Sormani (discussed in Section 2.8).

**Proposition 5.1.** Let \( I \) be an interval and \( (\Sigma, \sigma) \) be a connected Riemannian manifold. Suppose \((f_j)_j\) is a sequence of bounded continuous functions \( f_j : I \to (0, \infty) \) (uniformly bounded away from 0) and \( M_j = I \times f_j \Sigma \) are warped products with Lorentzian metric tensors

\[
g_j = -dt^2 + f_j(t)^2 \sigma.
\]

Assume that \((f_j)_j\) converges uniformly to a limit function

\[
f_\infty(t) = \lim_{j \to \infty} f_j(t).
\]

Then the corresponding null distances \( \hat{d}_j \) of \( g_j, j \in \mathbb{N} \cup \{\infty\} \), with respect to the canonical time function \( \tau(t, x) = t \) converge pointwise, i.e., for any \( p, q \in M = I \times \Sigma \) it follows that

\[
\lim_{j \to \infty} \hat{d}_j(p, q) = \hat{d}_\infty(p, q).
\]
Proof. Due to the uniform convergence $f_j \to f_\infty$, it follows that $f_\infty$ is continuous and bounded as well. In particular, $f_\infty$ is bounded away from 0 by some constant $f_{\infty, \text{min}} > 0$.

Let $p, q \in M$ and $\varepsilon \in (0, \frac{f_{\infty, \text{min}}}{4})$. Then, uniform convergence of $(f_j)_j$ implies for all $j$ sufficiently large, say, $j \geq j_0$, 

$$\|f_j - f_\infty\|_\infty < \varepsilon. \quad (28)$$

We start with a general observation on suitably broken causal curves which holds for all $g_j$ and the limit $g_\infty$. For each $j \geq j_0$ we construct in Step 1 a particular set of subintervals related to a (broken) causal curve, which will be used in Step 2 and 3 to estimate the difference of $\hat{d}_j(p, q)$ to $\hat{d}_\infty(p, q)$.

**Step 1: Construction of subintervals.** Fix $j \geq j_0$. Let $\beta_j$ be a (possibly broken) $g_j$-causal curve such that 

$$\hat{L}(\beta_j) < \hat{d}_j(p, q) + \varepsilon. \quad (29)$$

Consider the time projection of $\beta_j$, more precisely, the compact interval $I_j := t(\text{im}(\beta_j))$. Since $f_\infty$ is uniformly continuous on $I_j$ there exists a $\delta > 0$ such that for all $s, t \in I_j$

$$|t - s| < 2\delta \implies |f_\infty(t) - f_\infty(s)| < \varepsilon. \quad (30)$$

We can cover $I_j$ by finitely many $\delta$-intervals $(t_{2i} - \delta, t_{2i} + \delta)$ (numbered with increasing time). Choose furthermore $t_{2i+1} \in (t_{2i} - \delta, t_{2i} + \delta) \cap (t_{2i+2} - \delta, t_{2i+2} + \delta) \neq \emptyset$. Add the breaking points of $\beta_j$, i.e., the times $t_j = t(\beta_j(s_j))$ of the parameter values $s_j$ where $\beta_j$ changes from past- to future-directed and vice versa. Now, we consider all preimages $s_j$ of points on $\beta_j$ with times $t_j$ on $\beta_j$, and number the (possibly repeating) times according to their appearance on $\beta_j$, i.e., such that

$$t(\beta_j(s_i)) = t_i.$$ 

Without loss of generality we can assume that all breaking points have even parameter indices. Recall that due to the construction of the $s_i$’s and $t_i$’s and (30) we know, in particular, that

$$f_\infty([t_{2i}, t_{2i+2}]) \subseteq [f_\infty(t_{2i+1}) - \varepsilon, f_\infty(t_{2i+1}) + \varepsilon].$$

Since $j \geq j_0$, (28) implies

$$f_j([t_{2i}, t_{2i+2}]) \subseteq [f_\infty(t_{2i+1}) - 2\varepsilon, f_\infty(t_{2i+1}) + 2\varepsilon].$$
Note that for every \( j \in \mathbb{N} \cup \{ \infty \} \) we have

\[
\mathcal{L}(\beta_j) = \sum_i \hat{d}_j(\beta_j(s_{2i}), \beta_j(s_{2i-2})) = \sum_i |t_{2i} - t_{2i-2}|.
\]

In order to obtain upper and lower bounds of \( \hat{d}_j(p, q) \) involving \( \hat{d}_\infty(p, q) \) and \( \varepsilon \), we estimate all summands \( \hat{d}_j \) by employing Proposition 4.10 in a clever way.

**Step 2: Upper bound.** Fix \( j \geq j_0 \) and consider the associated path \( \beta_j \) and subintervals (indexed by \( i \)) from Step 1. Consider \( \beta_\infty \) as constructed above with respect to \( g_\infty \) and \( f_\infty \). We aim at estimating all parts \( \hat{d}_\infty(\beta_\infty(s_{2i}), \beta_\infty(s_{2i-2})) \) by \( \hat{d}_j(\beta_\infty(s_{2i}), \beta_\infty(s_{2i-2})) \), for all \( j \) sufficiently large. To this end we note that

\[
g_j = -dt^2 + f_j(t)^2 \sigma = -dt^2 + \frac{f_j(t)^2}{(f_\infty(t_{2i-1}) - \varepsilon)^2} (f_\infty(t_{2i-1}) - 2\varepsilon)^2 \sigma.
\]

Here,

\[
\tilde{\sigma} = (f_\infty(t_{2i-1}) - 2\varepsilon)^2 \sigma
\]

is a conformal Riemannian metric, and by construction \( g_j \preceq \eta_{\tilde{\sigma}} \) and \( g_\infty \preceq \eta_{\tilde{\sigma}} \). Thus we can apply the right inequality of (21) in Proposition 4.10 to obtain

\[
\hat{d}_j(\beta_\infty(s_{2i}), \beta_\infty(s_{2i-2})) \leq \max \left\{ 1, \max_{t \in [t_{2i-2}, t_{2i}]} \frac{f_j(t)}{f_\infty(t_{2i-1}) - 2\varepsilon} \right\} \hat{d}_\tilde{\sigma}(\beta_\infty(s_{2i}), \beta_\infty(s_{2i-2}))
\]

\[
\leq \frac{f_\infty(t_{2i-1}) + 2\varepsilon}{f_\infty(t_{2i-1}) - 2\varepsilon} \hat{d}_\tilde{\sigma}(\beta_\infty(s_{2i}), \beta_\infty(s_{2i-2})).
\]

Note that in the second line it is enough to consider the maximum on \([t_{2i-2}, t_{2i}]\) because the only relevant cases in the proof of Proposition 4.10 are Cases 1 and 3 where \( \beta_\infty \) is causal with respect to \( \eta_{\tilde{\sigma}} \) by construction. Hence, also

\[
\hat{d}_\tilde{\sigma}(\beta_\infty(s_{2i}), \beta_\infty(s_{2i-2})) = \hat{d}_\infty(\beta_\infty(s_{2i}), \beta_\infty(s_{2i-2})),
\]

and thus together

\[
\hat{d}_j(\beta_\infty(s_{2i}), \beta_\infty(s_{2i-2})) \leq \frac{f_\infty(t_{2i-1}) + 2\varepsilon}{f_\infty(t_{2i-1}) - 2\varepsilon} \hat{d}_\infty(\beta_\infty(s_{2i}), \beta_\infty(s_{2i-2})).
\]
The constant can be estimated globally by

\[ f_\infty(t_{2i-1}) + 2\varepsilon \over f_\infty(t_{2i-1}) - 2\varepsilon = 1 + \frac{4\varepsilon}{f_\infty(t_{2i-1}) - 2\varepsilon} \leq 1 + \frac{4\varepsilon}{f_\infty,\text{min} - 2\varepsilon} \leq 1 + \frac{8\varepsilon}{f_\infty,\text{min}}. \]

Summing up we have

\[ \hat{d}_j(p, q) \leq \sum_i \hat{d}_j(\beta_\infty(s_{2i}), \beta_\infty(s_{2i-2})) \]

\[ \leq \left( 1 + \frac{8\varepsilon}{f_\infty,\text{min}} \right) \sum_i \hat{d}_\infty(\beta_\infty(s_{2i}), \beta_\infty(s_{2i-2})) \]

\[ = \left( 1 + \frac{8\varepsilon}{f_\infty,\text{min}} \right) \hat{L}(\beta_\infty) \]

Thus by the choice of \( \beta_\infty \), more precisely (29), this yields an upper bound for \( \hat{d}_j \),

\[ \hat{d}_j(p, q) \leq \left( 1 + \frac{8\varepsilon}{f_\infty,\text{min}} \right) \left( \hat{d}_\infty(p, q) + \varepsilon \right) \]

\[ \leq \hat{d}_\infty(p, q) + \varepsilon \left( 1 + \frac{8\varepsilon}{f_\infty,\text{min}} + \frac{8}{f_\infty,\text{min}} \hat{d}_\infty(p, q) \right). \] (32)

Since \( p, q \) is assumed to be fixed, the second summand is small for all sufficiently large \( j \).

**Step 3: Lower bound.** Fix \( j \geq j_0 \) and consider the associated path \( \beta_j \) and subintervals (indexed by \( i \)) from Step 1. As in the case of the upper bound we can write \( g_j \) and \( g_\infty \) in terms of \( \bar{\sigma} \) via (31). Hence, for all parts of \( \beta_j \) we know that

\[ \hat{d}_j(\beta_j(s_{2i}), \beta_j(s_{2i-2})) = |t_{2i} - t_{2i-2}| = \hat{d}_\bar{\sigma}(\beta_j(s_{2i}), \beta_j(s_{2i-2})). \]

Moreover, since only the Cases 1 and 3 are relevant on \([t_{2i-2}, t_{2i}]\) in the proof of Proposition 4.10 we obtain that

\[ \hat{d}_\bar{\sigma}(\beta_j(s_{2i}), \beta_j(s_{2i-2})) \]

\[ \geq \frac{1}{\max \left\{ 1, \max_{t \in [t_{2i-2}, t_{2i}]} \frac{f_\infty(t)}{f_\infty(t_{2i-1}) - 2\varepsilon} \right\}} \hat{d}_\infty(\beta_j(s_{2i}), \beta_j(s_{2i-2})) \]

\[ \geq \frac{f_\infty(t_{2i-1}) - 2\varepsilon}{f_\infty(t_{2i-1}) + \varepsilon} \hat{d}_\infty(\beta_j(s_{2i}), \beta_j(s_{2i-2})). \]
Similar to the previous case we know that
\[
\frac{f_\infty(t_{2i-1}) - 2\varepsilon}{f_\infty(t_{2i-1}) + \varepsilon} = 1 - \frac{3\varepsilon}{f_\infty(t_{2i-1}) + \varepsilon} \geq 1 - \frac{3\varepsilon}{f_\infty,\min}
\]
and thus
\[
\hat{d}_j(\beta_j(s_{2i}), \beta_j(s_{2i-2})) \geq \left(1 - \frac{3\varepsilon}{f_\infty,\min}\right) \hat{d}_\infty(\beta_j(s_{2i}), \beta_j(s_{2i-2})).
\]
Due to the choice of \(\beta_j\) in (29) and the triangle inequality we obtain the pointwise lower bound
\[
\hat{d}_j(p, q) > \hat{L}(\beta_j) - \varepsilon
\]
\[
= \sum_i \hat{d}_\sigma(\beta_j(s_{2i}), \beta_j(s_{2i-2})) - \varepsilon
\]
\[
\geq \left(1 - \frac{3\varepsilon}{f_\infty,\min}\right) \sum_i \hat{d}_\infty(\beta_j(s_{2i}), \beta_j(s_{2i-2})) - \varepsilon
\]
\[
\geq \left(1 - \frac{3\varepsilon}{f_\infty,\min}\right) \hat{d}_\infty(p, q) - \varepsilon,
\]
and hence
\[
\hat{d}_j(p, q) \geq \hat{d}_\infty(p, q) - \varepsilon \left(1 + \frac{3\varepsilon}{f_\infty,\min} \hat{d}_\infty(p, q) \right). \tag{33}
\]
Combining the bounds (32) and (33) we thus obtain the desired pointwise convergence
\[
\lim_{j \to \infty} \hat{d}_j(p, q) = \hat{d}_\infty(p, q). \quad \blacksquare
\]
In what follows we establish the bi-Lipschitz bounds needed to for Theorem 2.21. While continuity of the warping functions \(f_j\) and completeness of \(\Sigma\) is crucial for the convergence result, they are not necessary to obtain these bounds.

**Proposition 5.2.** Let \((\Sigma, \sigma)\) be a connected Riemannian manifold, and \(I\) be an interval. Suppose \((f_j)\) is a sequence of uniformly bounded functions \(f_j : I \to \mathbb{R}\) (bounded away from 0) and \(M_j = I \times f_j \Sigma\) are warped products with Lorentzian metric tensors
\[
g_j = -dt^2 + f_j(t)^2 \sigma.
\]
Then there exists a constant $\lambda > 1$ such that for any $j \in \mathbb{N}$

$$
\frac{1}{\lambda} \leq \hat{d}_j(p, q) \leq \lambda \tag{34}
$$

on $M$, where $\hat{d}_j$ denotes the null distance corresponding to $f_j$ with respect to the canonical time function $\tau(t) = t$ and $\hat{d}_\sigma$ denotes the null distance with respect to the reference Lorentzian product metric $\eta_\sigma = -dt^2 + \sigma$.

**Proof.** By assumption of uniform boundedness of $(f_j)_j$ there exists $\lambda > 1$ such that

$$
\frac{1}{\lambda} \leq f_j(t) \leq \lambda, \quad t \in I, j \in \mathbb{N}.
$$

Proposition 4.10 then implies the bi-Lipschitz bounds

$$
\frac{1}{\lambda} \hat{d}_\sigma(p, q) \leq \hat{d}_j(p, q) \leq \lambda \hat{d}_\sigma(p, q), \quad p, q \in M, j \in \mathbb{N},
$$

which implies (34) for $p \neq q$. \hfill \blacksquare

**Remark 5.3.** Instead of $\hat{d}_\sigma$ we could have used any other reference null distance of which we know it yields a (local) integral current space, for instance, the null distance corresponding to Minkowski space or any of the $g_j$ themselves.

Using the bi-Lipschitz bounds (34), a theorem of Huang, Lee and Sormani [52, Theorem A.1] implies convergence of a subsequence. Even the rate of convergence can be estimated (see the restated Theorem 2.21 and Remark 2.22 in the Background Section 2 of this paper for more details).

**Corollary 5.4.** Let $I$ be a closed interval, $(\Sigma, \sigma)$ a connected compact Riemannian manifold and $(f_j)_j$ be given as in Proposition 5.2. Then there exists a subsequence $(\hat{d}_{jk})_k$ of $(\hat{d}_j)_j$ and a length metric $d_\infty$ satisfying (34) such that this subsequence converges uniformly to $d_\infty$, i.e.,

$$
\sup_{p, q \in M} | \hat{d}_{jk}(p, q) - d_\infty(p, q) | \to 0, \quad \text{as } k \to \infty.
$$

Moreover, the metric spaces $(M, \hat{d}_{jk})_k$ converge to $(M, d_\infty)$ with respect to the GH distance, i.e.,

$$
d_{GH}((M, \hat{d}_{jk}), (M, d_\infty)) \to 0, \quad \text{as } k \to \infty,
$$
and the integral current spaces \((M, \hat{d}_k, T)\) (with respect to the integral current structure of \(\hat{d}_g\) obtained in Theorem 1.2) converge to the integral current space \((M, d_\infty, T)\) with respect to the SWIF distance, i.e.,

\[
d_F((M, \hat{d}_k, T), (M, d_\infty, T)) \to 0, \quad \text{as } k \to \infty.
\]

At this point we know nothing about the limiting distance function \(d_\infty\), not even whether it arises as a null distance related to some kind of causal structure on \(M\). We thus assume, in addition, uniform convergence and continuity of \((f_j)\) to be able to employ Proposition 5.1.

**Theorem 1.4.** Let \(I\) be a closed interval and \((\Sigma, \sigma)\) be a connected compact Riemannian manifold. Suppose \((f_j)\) is a sequence of continuous functions \(f_j: I \to (0, \infty)\) (uniformly bounded away from 0) and \(M_j = I \times f_j \Sigma\) are warped products with Lorentzian metric tensors

\[
g_j = -dt^2 + f_j(t)^2 \sigma.
\]

Assume that \((f_j)\) converges uniformly to a limit function

\[
f_\infty(t) = \lim_{j \to \infty} f_j(t).
\]

Then the corresponding null distances \(\hat{d}_j\) of \(g_j, j \in \mathbb{N}\), with respect to the canonical time function \(\tau(t, x) = t\), converge uniformly to \(\hat{d}_\infty\) on \(M\). Moreover, \((M, \hat{d}_j)\) converge to \((M, \hat{d}_\infty)\) in the GH and SWIF topology, with \(T\) the canonical integral current obtained in Theorem 1.2.

**Proof.** We show that \(\hat{d}_\infty\) is the metric \(d_\infty\) obtained in Corollary 5.4. Note that uniform boundedness is implied by the assumption of uniform convergence \(f_j \to f_\infty\), and that continuity and boundedness of \(f_\infty > 0\) follows as well.

More precisely, Corollary 5.4 implies that a subsequence \((\hat{d}_j)_k\) converges to some metric \(d_\infty\) uniformly. Since we have shown pointwise convergence \(\hat{d}_j \to \hat{d}_\infty\) in Proposition 5.1, we know that \(d_\infty = \hat{d}_\infty\) is the uniform limit of \((\hat{d}_j)_k\). Assume, for contradiction, that the full sequence does not converge uniformly. Hence, there must exist a subsequence \((\hat{d}_{j_l})\) so that

\[
d_{\text{unif}}(\hat{d}_{j_l}, \hat{d}_\infty) \geq C > 0
\]
for all $l \in \mathbb{N}$ (we omit $M$ since the space does not change). This subsequence still satisfies all the properties we have shown for the original pointwise convergent sequence and hence by the argument above we know that we can take a further subsequence so that

$$d_{\text{unif}}(\hat{d}_{j_{i_{n}}}, \hat{d}_{\infty}) \to 0,$$

which contradicts (35) for infinitely many $l$. Hence, we find the desired uniform convergence of $\hat{d}_{j}$ to $\hat{d}_{\infty}$. The GH and SWIF convergence follows from Theorem 2.21 since uniform convergence implies GH and SWIF convergence. \hfill \blacksquare

Remark 5.5. Note that the proof of Proposition 5.1 and Theorem 1.4 crucially relies on the continuity of the warping functions $f_{j}$ and that of the limit warping function $f_{\infty}$. In Section 5.2, we will see that uniform convergence and continuity are not always necessary to obtain uniform, GH or SWIF convergence of $\hat{d}_{j}$ to $\hat{d}_{\infty}$, but none of those assumptions can be dropped for a general result of the type of Theorem 1.4.

Remark 5.6. Note that in Theorem 1.4 we assume $\Sigma$ is compact (possibly with boundary). In the case of a noncompact sequence this result can be used to show convergence on compact sets in the standard way.

5.2 Examples with nonuniformly converging warping functions

In this subsection, we explore a few examples of sequences of warped product spacetimes in order to exhibit the variety of behavior we can observe in the limit under uniform, GH, and SWIF convergence. In particular, the sequences of warping functions $f_{j}$ we are investigating do not satisfy all assumptions in Theorem 1.4 and only converge pointwise and in the $L^{p}$ sense to semicontinuous limit functions $f_{\infty}$. In the first example we construct a sequence of warped product spacetimes with lower semicontinuous limit warping function which does not converge to the null distance of the limiting warped product spacetime. On the contrary, in the second example the (pointwise) limit warping function is upper semicontinuous and we do still obtain a uniform (and GH and SWIF) convergence result of the corresponding metric spaces. Finally, in the third example a sequence of warping functions converges to a degenerate limit warping function and while we show that the GH and SWIF limits still exist, they disagree.

Example 5.7 (Pointwise convergence of $(f_{j})_{j}$ is not enough). Fix a constant $h_{0} \in (0,1)$ and a smooth increasing function $h: [0,1] \to (0,1]$ satisfying $h(t) = h_{0}$ for $t \in [0, \frac{1}{2}]$ and
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Fig. 7. Sequence of pointwise converging warping functions \( f_j \) in Example 5.7.

\[ h(1) = 1 \text{ with } h'(1) = 0. \]

Consider the sequence of smooth warping functions \( f_j : [0, 2] \rightarrow [h_0, 1] \) (see also Figure 7), given by

\[
f_j(t) = \begin{cases} 
  h(jt) & t \in [0, \frac{1}{j}], \\
  1 & t \in (\frac{1}{j}, 2].
\end{cases}
\]

For any fixed connected compact Riemannian manifold \((\Sigma, \sigma)\), the sequence \((f_j)_j\) defines a sequence of smooth warped product Lorentzian metrics \( g_j \) with induced null distances \( \hat{d}_j \) on the manifold \( M = [0, 2] \times \Sigma \).

The pointwise limit of \((f_j)_j\) is the bounded function (see Figure 7)

\[
f_\infty(t) = \begin{cases} 
  h_0 & t = 0, \\
  1 & t \in (0, 2],
\end{cases}
\]

with the corresponding null distance \( d_\infty = \hat{d}_\sigma \).

A uniform limit (even of a subsequence) of \((f_j)_j\) clearly does not exist, and hence Corollary 5.4 and Theorem 1.4 are not applicable. We will show that \( d_\infty = \hat{d}_\sigma \) is not a limit of any subsequence of null distances \((\hat{d}_j)_j\).

In fact, we will show that \((\hat{d}_j)_j\) converges to a another metric space with metric \( d_0 \).

**Claim.** The sequence of null distances \( \hat{d}_j \), induced by the warping functions \( f_j \) converges uniformly to the metric \( d_0 \neq \hat{d}_\infty = \hat{d}_\sigma \), for points \( p = (t(p), p_\Sigma), q = (t(q), q_\Sigma) \in [0, 2] \times \Sigma \) given by

\[
d_0(p, q) := \min \left\{ \hat{d}_\sigma(p, q), t(p) + t(q) + h_0 \hat{d}_\sigma(J_{\eta_\sigma}(p) \cap \Sigma_0, J_{\eta_\sigma}(q) \cap \Sigma_0) \right\}
\]

where \( \Sigma_t = \{t\} \times \Sigma \). In addition \((M, \hat{d}_j)\) converges in the GH and SWIF sense to \((M, d_0)\).
Proof. First we notice that since \( f_j \) is uniformly bounded from above and below we have Lipschitz bounds on \( \hat{d}_j \) by Proposition 4.10, and hence by Theorem 2.21 we know that a subsequence must converge in the uniform, GH, and SWIF sense to a metric \( d_\infty \) satisfying the same bi-Lipschitz bounds (see Corollary 5.4). In what follows we show that \( d_\infty = d_0 \) is the pointwise limit of \( (\hat{d}_j)_j \), and subsequently prove uniform convergence. Notice that it is clear that \( d_0 \neq \hat{d}_\infty \) since they only agree when \( t(p) \) and \( t(q) \) are positive.

Let \( p = (t(p), p_\Sigma) \) and \( q = (t(q), q_\Sigma) \) for \( p_\Sigma, q_\Sigma \in \Sigma \). First notice that since \( h_0 \leq f_j \leq 1 \) we have by Proposition 4.10 that

\[
h_0 \hat{d}_\sigma(p, q) \leq \hat{d}_j(p, q) \leq \hat{d}_\sigma(p, q).
\]

(Case 1: \( t(p) = t(q) = 0 \). In this case, \( d_0(p, q) = h_0 d_\sigma(p_\Sigma, q_\Sigma) \) and we show that \( \hat{d}_j \rightarrow d_0 \) pointwise. We can construct a piecewise null curve \( \beta_j \) with respect to \( g_j \) which remains within the strip \( S_j := [0, \frac{1}{2^j}] \times \Sigma \) in order to calculate

\[
\hat{d}_j(p, q) \leq \hat{L}_j(\beta_j) = h_0 d_\sigma(p_\Sigma, q_\Sigma),
\]

so together with (36) we immediately have that

\[
\lim_{j \rightarrow \infty} \hat{d}_j(p, q) = h_0 d_\sigma(p_\Sigma, q_\Sigma) = d_0(p, q)
\]

is satisfied.

(Case 2: \( t(p) > 0 \) and \( t(q) = 0 \). In this case,

\[
d_0(p, q) = t(p) + h_0 \hat{d}_\sigma(J^-_\eta(p) \cap \Sigma_0, q).
\]

The proof of pointwise convergence is obtained by curves minimizing their contributions in both the \( \Sigma_0 \) and the \( (0, 2] \times \Sigma \) region, i.e.,

\[
d_0(p, q) = \inf_{q' \in [0] \times \Sigma} \left( \hat{d}_\sigma(p, q') + h_0 d_\sigma(q'_\Sigma, q_\Sigma) \right).
\]

Note that if \( q \in J^-_\eta(p) \) then the minimum is obtained by choosing \( q' = q \) because the second term vanishes and the first one is minimal due to (36). If \( q \notin J^-_\eta(p) \) then both terms are minimal for \( q' \) at the intersection of the light cone with \( \Sigma \). It remains to show that (37) is indeed the limit of \( (\hat{d}_j)_j \).
We consider two cases: (a) If \( q \in J_{\eta}^-(p) \), then for any \( j \in \mathbb{N} \) also \( q \in J_{\eta}^- \) (since \( f_j \leq f_{j+1} \leq f_\infty \) and thus \( g_\infty \preceq g_{j+1} \preceq g_j \) by Example 4.14) and therefore

\[
\hat{d}_j(p, q) = t(p) - t(q) = t(p) = d_0(p, q).
\]

In particular, \( d_\infty(p, q) := \lim_{j \to \infty} \hat{d}_j(p, q) = d_0(p, q) \).

(b) Suppose now that \( q \not\in J_{\eta}^-(p) \). For any \( q' \in (J_{\eta}^-(p) \setminus I_{\eta}^-(p)) \cap (\{0\} \times \Sigma) \) there exists a \( g_\infty \)-null curve \( \beta^1 \) such that

\[
\hat{L}(\beta^1) = t(p) - t(q') = t(p) = \hat{d}_\sigma(p, q') = d_0(p, q'),
\]

Since \( q' \in J_{M\sigma}^-(p) \subseteq J_j^-(p) \), also

\[
\hat{d}_j(p, q') = t(p).
\] (38)

Moreover, for every such \( q' \) there exists a broken causal curve \( \beta_j^2 \) in \([0, \frac{1}{j}] \times \Sigma\) such that by Lemma 4.9

\[
\hat{L}(\beta_j^2) \leq \hat{d}_j(q', q) + \frac{1}{j} = h_0 d_\sigma(q'_\Sigma, q_\Sigma) + \frac{1}{j}.
\]

Thus by the triangle inequality

\[
\hat{d}_j(p, q) \leq \inf_{q' \in \Sigma_0} (\hat{d}_j(p, q') + \hat{d}_j(q', q))
\]

\[
\leq t(p) + h_0 \inf_{q' \in \Sigma_0} \hat{d}_\sigma(q'_\Sigma, q_\Sigma) + \frac{1}{j}
\]

\[
= d_0(p, q) + \frac{1}{j},
\]

and therefore

\[
\limsup_{j \to \infty} \hat{d}_j(p, q) \leq d_0(p, q). \tag{39}
\]

On the other hand, there is a broken causal curve \( \beta_j \) from \( p \) to \( q \) with respect to \( g_j \) that satisfies

\[
\hat{L}(\beta_j) \leq \hat{d}_j(p, q) + \frac{1}{j}
\]
Due to the continuity of $\beta_j$ and $t$ (and since $t(p) > 0 = t(q)$), for $j$ sufficiently large, there exists a last point $q_j$ along the image of $\beta_j$ in $M$ satisfying $t(q_j) = \frac{1}{j}$. Then

$$\hat{d}_j(q_j, q) \geq t(q_j) - t(q) = \frac{1}{j}.$$ 

We add $q_j$ to the break points $\{x_i\}_{i=0}^N$ of $\beta_j$. Thus

$$\hat{d}_j(p, q) + \frac{1}{j} \geq \hat{L}(\beta_j) = \sum_i |t(x_i) - t(x_{i+1})| \geq t(p) - t(q_j) + \hat{d}_j(q_j, q) \geq t(p) - \frac{1}{j} + h_0\hat{d}_\sigma(q_j, q) \geq d_0(p, q) - \frac{1}{j},$$

and therefore

$$\liminf_{j \to \infty} \hat{d}_j(p, q) \geq d_0(p, q). \quad (40)$$

Together, (39) and (40) imply the desired result

$$\lim_{j \to \infty} \hat{d}_j(p, q) = d_0(p, q)$$

for all $p, q$ in Case 2 pointwise.

**Case 3:** $t(p), t(q) > 0$. We distinguish two cases. If $q \in J_{\eta_0}^+(p) \subseteq J_j^+(p)$ (or vice versa), then it follows immediately that

$$\hat{d}_j(p, q) = |t(p) - t(q)| = d_0(p, q).$$

Next we assume that $q \not\in J_{\eta_0}^+(p)$. Since $\eta_\sigma \leq g_j$ we know by Lemma 4.15 that $\hat{d}_j(p, q) \leq \hat{d}_\sigma(p, q)$. For any $p' \in J_{\eta_0}^-(p) \cap \Sigma_0$ and $q' \in J_{\eta_0}^-(q) \cap \Sigma_0$ the triangle inequality implies

$$\hat{d}_j(p, q) \leq \hat{d}_j(p, p') + \hat{d}_j(p', q') + \hat{d}_j(q', q).$$
and hence by Case 1 and Case 2 we find

\[ \limsup_{j \to \infty} \hat{d}_j(p, q) \leq t(p) + h_0 \hat{d}_\sigma(p', q') + t(q), \]

which implies

\[ \limsup_{j \to \infty} \hat{d}_j(p, q) \leq d_0(p, q). \]  \hspace{1cm} (41)

It remains to show that the value is attained. If \( J_{\eta_a}(p) \cap J_{\eta_a}(q) \cap [\frac{1}{2}, 2] \neq \emptyset \) then for \( j \) (and all larger indices)

\[ \hat{d}_j(p, q) = \hat{d}_\sigma(p, q), \]

and thus furthermore

\[ \lim_{j \to \infty} \hat{d}_j(p, q) = \hat{d}_\sigma(p, q) \geq d_0(p, q). \]  \hspace{1cm} (42)

and we are done. If, on the other hand, \( J_{\eta_a}(p) \cap J_{\eta_a}(q) \cap (0, 2] = \emptyset \), then for \( j \) sufficiently large such that \( t(p), t(q) > \frac{1}{j} \) we consider two points \( p_j \in J_{\eta_a}(p) \cap \Sigma_j \), \( q_j \in J_{\eta_a}(q) \cap \Sigma_j \) and the auxiliary function \( \tilde{f}_j : [0, 2] \to \mathbb{R} \), defined by

\[ \tilde{f}_j(t) = \begin{cases} h_0 & \text{if } t \leq \frac{1}{j}, \\ 1 & \text{if } t < \frac{1}{j}. \end{cases} \]

This function \( \tilde{f}_j \) defines a bounded warping function that satisfies \( \tilde{f}_j \leq f_j \), and hence by Example 4.14 we know that

\[ \hat{d}_j(p, q) \leq \hat{d}_j(p, q). \]

Let \( \beta_j \) be a curve that connects \( p \) to \( p_j \) to \( q_j \) (the latter within \( [0, \frac{1}{j}] \times \Sigma \)) to \( q \). Then

\[
\begin{aligned}
\inf_{p_j, q_j} \hat{L}(\beta_j) &= t(p) - \frac{1}{j} + h_0 \inf_{p_j, q_j} \hat{d}_\sigma(p_j, q_j) + t(q) - \frac{1}{j} \\
&= t(p) + t(q) - \frac{2}{j} + h_0 \hat{d}_\sigma(J_{\eta_a}(p) \cap \Sigma_j, J_{\eta_a}(q) \cap \Sigma_j)
\end{aligned}
\]
and

\[ \hat{d}_j = \min(\hat{d}_\sigma(p, q), \inf_{\beta_j} \hat{L}(\beta_j)) \]

thus

\[
\liminf_{j \to \infty} \hat{d}_j(p, q)
\geq \min\{\hat{d}_\sigma(p, q), t(p) + t(q) + h_0 \hat{d}_\sigma(J_{\eta}(p) \cap \Sigma_0, J_{\eta}(q) \cap \Sigma_0)\}
= d_0(p, q).
\]

Together, (41)–(43) establish that also if \( q \not\in J_{\eta}(p) \) (or vice versa) we have

\[
\lim_{j \to \infty} \hat{d}_j(p, q) = d_0(p, q).
\]

Since a subsequence of \( \hat{d}_j \) must converge in the uniform, GH, and SWIF sense and we have now shown pointwise convergence of \( \hat{d}_j(p, q) \to d_0(p, q) \) in all cases, we can conclude that \( (\hat{d}_j)_j \) itself must converge in the uniform, GH, and SWIF sense to \( d_0 \). For the sake of contradiction assume that \( \hat{d}_j \) does not converge in the uniform, GH, or SWIF sense to \( d_0 \) and hence there must exist a subsequence \( (\hat{d}_{j_l})_l \) that is bounded away from \( (M, d_0) \) in the uniform, GH, and SWIF sense. However, the subsequence \( (\hat{d}_{j_l})_l \) still satisfies all the properties shown for the original Example 5.7, and thus we can take a further subsequence \( (\hat{d}_{j_{l_n}})_n \) of \( (\hat{d}_{j_l})_l \) must converge in the uniform, GH, and SWIF topology—and yet again to the only possible limit \( d_0 \), a contradiction. \( \blacksquare \)

**Remark 5.8.** The above Example 5.7 shows that pointwise convergence of \( (f_j)_j \) is generally not enough to obtain the desired convergence of the corresponding null distances \( (\hat{d}_j)_j \) to the null distance of the limiting function \( f_\infty \). This illustrates our assumption of uniform convergence in Theorem 1.4. The setting can easily be extended to an interval \([-2, 2]\) by extending \( h \) as an even function on \([-2, 0]\). Similarly, it can be extended to \( \mathbb{R} \times \Sigma \) by extending it with constant value 1.

Besides, note that \( L^p \) convergence, for \( p \geq 1 \) is also not enough, since the \( L^p \) limit of \( (f_j)_j \) is

\[ f_\infty^p(t) = 1 \quad \text{a.e.} \]

with the same corresponding null distance \( \hat{d}_\infty^p = \hat{d}_\sigma \) as the pointwise limit.
Fig. 8. Sequence of pointwise converging warping functions $f_j$ of Example 5.9.

While in Example 5.7 we have seen that the convergence does not hold if the null cone of $\hat{d}_\infty$ is narrower than those of the sequence $(\hat{d}_j)_j$, this is not the case when the null cone of $\hat{d}_\infty$ is wider than the null cone of the sequence $(\hat{d}_j)_j$. Indeed, we will be able to deduce that the sequence does converge to the null distance of a warped product spacetime in such a setting. As such, uniform convergence of $(f_j)_j$ is not necessary to obtain uniform convergence of the corresponding null distances.

Example 5.9 (Uniform convergence of $(f_j)_j$ is not necessary). Let $h_0 > 1$ and consider a smooth function $h: [0, 1] \to [1, h_0]$ defined as in Example 5.7 except that it is now decreasing instead of increasing between $h(t) = h_0$ for $t \in [0, \frac{1}{2}]$ and $h(1) = 1$, with $h'(1) = 0$. Consider the sequence of smooth functions $f_j(t): [0, 2] \to [1, h_0]$ (see also Figure 8), given by

$$f_j(t) = \begin{cases} h(jt) & t \in [0, \frac{1}{j}], \\ 1 & t \in (\frac{1}{j}, 2]. \end{cases}$$

For any fixed connected compact Riemannian manifold $(\Sigma, \sigma)$, the sequence $(f_j)_j$ defines a sequence of smooth Lorentzian metrics $g_j$ with induced null distances $\hat{d}_j$ on the manifolds $M = [0, 2] \times \Sigma$.

The pointwise limit of $(f_j)_j$ is the bounded function (see Figure 8)

$$f_\infty(t) = \begin{cases} h_0 & t = 0, \\ 1 & t \in (0, 2], \end{cases}$$

with induced null Lorentzian metric $g_\infty$ and corresponding null distance $\hat{d}_\infty = \hat{d}_\sigma$. 
Claim. The null distances $\hat{d}_j$ converge uniformly to $\hat{d}_\infty$, and $(M, \hat{d}_j)$ converges in the GH and SWIF sense to $(M, \hat{d}_\infty)$.

Proof. First we notice that since $f_j$ is uniformly bounded from above and below we have Lipschitz bounds on $\hat{d}_j$ by Proposition 4.10 and hence by Theorem 2.21 we know that a subsequence must converge in the uniform, GH, and SWIF sense to a metric $d$ which satisfies the same bi-Lipschitz bounds. Our goal now is to show that the full sequence $(\hat{d}_j)_j$ converges pointwise to $d = \hat{d}_\infty$, which implies uniform convergence using the same the proof as in Example 5.7.

Let $p = (t(p), p_\Sigma)$ and $q = (t(q), q_\Sigma)$ in $M$, where $p_\Sigma, q_\Sigma$ is the projection onto $\Sigma$. First notice that since $1 \leq f_j \leq h_0$ we have by Proposition 4.10 that

$$\hat{d}_\infty(p, q) = \hat{d}_\sigma(p, q) \leq \hat{d}_j(p, q) \leq h_0 \hat{d}_\sigma(p, q).$$

(44)

and thus

$$\hat{d}_\infty(p, q) \leq \liminf_{j \to \infty} \hat{d}_j(p, q).$$

It remains to be shown that

$$\limsup_{j \to \infty} \hat{d}_j(p, q) \leq \hat{d}_\infty(p, q).$$

(45)

Case 1: $t(p), t(q) > 0$. For $j$ sufficiently large, $t(p)$ and $t(q)$ are larger than $\frac{1}{j}$. For any such $j$ there exists a broken causal curve $\beta_j$ such that

$$\hat{L}_\infty(\beta_j) \leq \hat{d}_\infty(p, q) + \frac{1}{j}.$$ 

Without loss of generality we can assume that $\text{im}(\beta_j) \subseteq [\frac{1}{j}, 2] \times \Sigma$, which implies that

$$\hat{L}_j(\beta_j) = \hat{L}_\infty(\beta_j)$$

and therefore, for all $j$ sufficiently large,

$$\hat{d}_j(p, q) \leq \hat{L}_\infty(\beta_j) \leq \frac{1}{j} + \hat{d}_\infty(p, q),$$

which implies (45).
Case 2: \( t(p) = t(q) = 0 \). Let \( p_j = \left( \frac{1}{j}, p_\Sigma \right) \) and \( q_j = \left( \frac{1}{j}, q_\Sigma \right) \). By the triangle inequality, and the fact that \( p_j \in J^+(p) \), \( q_j \in J^+(q) \) (with respect all \( g_i \) and also \( g_\infty \))

\[
\hat{d}_j(p, q) \leq \hat{d}_j(p, p_j) + \hat{d}_j(p_j, q_j) + \hat{d}_j(q_j, q) = \frac{2}{j} + \hat{d}_j(p_j, q_j).
\]

(46)

Moreover, we can connect \( p_j \) and \( q_j \) by broken \( g_\infty \)-causal curves \( \beta_j \) in \([\frac{1}{j}, 2] \times \Sigma \) (which have the same null length as those in \([0, 2] \times \Sigma \)), hence by Lemma 4.4

\[
\hat{d}_j(p_j, q_j) \leq \hat{d}_\sigma(p_j, q_j) = d_\sigma(p_\Sigma, q_\Sigma) = d_\infty(p, q).
\]

Thus together with (46) we have

\[
\hat{d}_j(p, q) \leq \frac{2}{j} + d_\infty(p, q),
\]

which implies (45).

Case 3: \( t(p) > 0 \), \( t(q) = 0 \). For any \( j \) sufficiently large such that \( t(p) > \frac{1}{j} \) we define \( q_j = \left( \frac{1}{j}, q_\Sigma \right) \). Then due to the triangle inequality,

\[
\hat{d}_j(p, q) \leq \hat{d}_j(p, q_j) + \hat{d}_j(q_j, q) \\
\leq \hat{d}_\sigma(p, q_j) + \frac{1}{j},
\]

since we can connect \( p \) and \( q_j \) by broken \( g_\infty \)-causal curves in \([\frac{1}{j}, 2] \times \Sigma \) which have the same null length as those in \([0, 2] \times \Sigma \). Applying again the triangle inequality for \( \hat{d}_\sigma \) implies

\[
\hat{d}_j(p, q) \leq \hat{d}_\sigma(p, q) + \hat{d}_\sigma(q_j, q) + \frac{1}{j} \\
= \hat{d}_\infty(p, q) + \frac{2}{j},
\]

and thus the desired estimate (45).

As in the previous Example 5.7 we can show that the pointwise limit \( \hat{d}_\infty \) is the uniform limit of \( \{\hat{d}_j\} \).
Remark 5.10. As Example 5.7, also Example 5.9 can be extended to $[-2, 2] \times \Sigma$ or even $\mathbb{R} \times \Sigma$. Note that considering the $L^p$ limit $f^p_\infty(t) = 1$ a.e. also yields uniform convergence of null distances, since $\hat{d}_\infty$ is the same as the corresponding pointwise null distance $\hat{d}_\infty = \hat{d}_\sigma$.

In Example 5.7 and Example 5.9, we have seen that uniform, GH, and SWIF convergence hold and that all limiting distance functions are the same (although, as in Example 5.7, not necessarily related to the null distance of the limiting warped product). In the last example we construct a sequence of warped product spacetimes for which the GH and SWIF limits disagree. For general metric spaces we know that the GH and SWIF limits can disagree (see examples in [57, 80, 84]), so it is important to have examples where this happens for spacetimes equipped with the null distance. Note that for such a counterexample the conditions of Corollary 5.4 must be violated.

Example 5.11 (GH and SWIF limits need not agree). Let $h: [0, 1] \to [0, 1]$ be a smooth increasing function that satisfies $h(0) = 0$, $h(1) = 1$ and $h'(0) = h'(1) = 0$. Consider the sequence $(f_j)_j$ of smooth increasing functions $f_j: [0, 2] \to [\frac{1}{j}, 1]$ (see Figure 9), given by

$$
\begin{align*}
    f_j(t) & = \begin{cases} 
    1 & t \in [0, 1 - \frac{1}{j}], \\
    \frac{1}{j} + \frac{t-1}{j} h(jt - (j - 1)) & t \in (1 - \frac{1}{j}, 1), \\
    1 & t \in [1, 2].
    \end{cases}
\end{align*}
$$

For any fixed connected compact Riemannian manifold $(\Sigma, \sigma)$, the sequence $(f_j)_j$ defines a sequence of smooth Lorentzian warped product metrics $g_j$ with induced null distances $\hat{d}_j$ on the manifold $M = [0, 2] \times \Sigma$.

Since $f_{j+1}(t) \leq f_j(t)$ for all $j$, by Example 4.14 we have $g_j \preceq g_{j+1}$ and

$$
\hat{d}_{j+1}(p, q) \leq \hat{d}_j(p, q), \quad p, q \in M. \quad (47)
$$
The pointwise limit of \( (f_j)_j \) is the bounded discontinuous function (see also Figure 9)

\[
f_\infty(t) = \begin{cases} 
0 & t \in [0, 1), \\
1 & t \in [1, 2],
\end{cases}
\]

which clearly does not induce a nondegenerate warped product metric on \( M \), and hence no null distance. We investigate whether the sequence \( (\hat{d}_j)_j \) converges to a limiting distance function with respect to different notions of convergence.

Fix \( p_0 \in \Sigma \) and let

\[
M_\infty = ([0, 1] \times \{p_0\}) \cup (1, 2) \times \Sigma,
M' = \{1\} \times \{p_0\} \cup ((1, 2) \times \Sigma),
\]

with distance function

\[
d_\infty(p, q) = \begin{cases} 
\min\{\hat{d}_\sigma(p, q), t(p) - 1 + t(q) - 1\} & t(p) > 1, t(q) > 1, \\
|t(p) - t(q)| & \text{else}.
\end{cases}
\]

Claim. We have that

\[
(M, \hat{d}_j) \xrightarrow{\text{GH}} (M_\infty, d_\infty),
(M, \hat{d}_j) \xrightarrow{\mathcal{F}} (X, d_\infty),
\]

where \( X \subseteq M' \), possibly the zero space.

Proof. By Example 4.14 we know that \( (\hat{d}_j)_j \) is a decreasing sequence of null distances on \( M = [0, 2] \times \Sigma \), which is bounded below by 0. Thus the Monotone Convergence Theorem and (47) implies that \( \hat{d}_j \) converges pointwise to a function \( d_0 \) on \( M \times M \) such that for any \( p, q \in M \)

\[
d_0(p, q) \leq \hat{d}_j(p, q) \leq \hat{d}_1(p, q) = \hat{d}_\sigma(p, q).
\] (48)

If we can show that \( d_0 \) is continuous, then Dini’s Theorem implies uniform convergence \( \hat{d}_j \to d_0 \) on \( M \times M \). We distinguish several cases to prove this. Let \( p = (t(p), p_\Sigma), q = (t(q), q_\Sigma) \in [0, 2] \times \Sigma \).
Case 1: $t(p), t(q) < 1$. By choosing $j$ large enough we can ensure that $t(p), t(q) < 1 - \frac{1}{j}$ and hence we know that by Proposition 4.10 and Remark 4.11 (note that case 3 cannot occur because $\eta_\sigma = g_1 \preceq g_j$)

$$\hat{d}_j(p, q) = \max \left\{ |t(p) - t(q)|, \frac{1}{j} \hat{d}_\sigma(p, q) \right\},$$

since we can always restrict our attention to broken causal curves with image in $[0, 1 - \frac{1}{j}] \times \Sigma$. Since $\frac{1}{j} \hat{d}_\sigma(p, q) \to 0$, we obtain a pointwise limit

$$\lim_{j \to \infty} \hat{d}_j(p, q) = |t(p) - t(q)|.$$

Case 2: $t(p), t(q) \geq 1$. Let $p'_j = (1 - \frac{1}{j}, p_\Sigma)$ and $q'_j = (1 - \frac{1}{j}, q_\Sigma)$. Then due to the triangle inequality

$$|t(p) - t(q)| \leq \hat{d}_j(p, q) \leq \hat{d}_j(p, p'_j) + \hat{d}_j(p'_j, q_j) + \hat{d}_j(q'_j, q)$$

$$= |t(p) - 1| + \frac{1}{j} d_\sigma(p_\Sigma, q_\Sigma) + |t(q) - 1|,$$

thus

$$\limsup_{j \to \infty} \hat{d}_j(p, q) \leq |t(p) - 1| + |t(q) - 1|.$$

in this case. Since by (48),

$$\limsup_{j \to \infty} \hat{d}_j(p, q) \leq \hat{d}_\sigma(p, q),$$

we obtain

$$\limsup_{j \to \infty} \hat{d}_j(p, q) \leq \min \left\{ \hat{d}_\sigma(p, q), |t(p) - 1| + |t(q) - 1| \right\}. \quad (49)$$

It remains to be shown that these values are indeed attained. If $\hat{d}_\sigma(p, q) \leq |t(p) - 1| + |1 - t(q)|$ then we note that a $\hat{d}_\sigma$ (almost) length-minimizing curve is contained in $[1, 2] \times \Sigma$ with the same null length with respect to $g_j$. Furthermore, for any piecewise causal curve $\beta$ which enters the region $[0, 1] \times \Sigma$ the null length with respect to $g_j$ is at least as long as

$$\hat{L}_j(\beta) \geq |t(p) - 1| + |t(q) - 1| \geq \hat{d}_\sigma(p, q).$$
Since these two classes exhaust all possible curves we find

\[ \hat{d}_j(p, q) \geq \hat{d}_\sigma(p, q). \]

If, on the other hand, \( \hat{d}_\sigma(p, q) \geq |t(p) - 1| + |1 - t(q)| \) then any curve which remains in \([1, 2] \times \Sigma\) will have \( g_j \) null length equal to \( \hat{d}_\sigma(p, q) \) and any curve which enters the region \([0, 1] \times \Sigma\) will have \( g_j \) null length at least as long as \( |t(p) - 1| + |t(q) - 1| \) so that

\[ \hat{d}_j(p, q) \geq |t(p) - 1| + |1 - t(q)|. \] (50)

Hence we obtain a limiting statement involving the right hand side of 49, and hence together

\[ \lim_{j \to \infty} \hat{d}_j(p, q) = \min \left\{ \hat{d}_\sigma(p, q), |t(p) - 1| + |1 - t(q)| \right\}. \]

**Case 3:** \( t(p) \geq 1, \ t(q) < 1 \). For sufficiently large \( j \) we know that \( t(q) < 1 - \frac{1}{j} \). We include the auxiliary point \( p' = (t(q), p_{\Sigma}) \). Then

\[ |t(p) - t(q)| \leq \hat{d}_j(p, q) \leq \hat{d}_j(p, p') + \hat{d}_j(p', q) \]
\[ = |t(p) - t(q)| + \frac{1}{j} \hat{d}_\sigma(p', q), \]

which yields

\[ \lim_{j \to \infty} \hat{d}_j(p, q) = |t(p) - t(q)|. \]

Combining Cases 1 to 3 we see that \( \hat{d}_j \to d_0 \) uniformly and, therefore, \( d_0 \) is also continuous. Note that symmetry and the triangle inequality follow for \( d_0 \) by taking limits but \( d_0 \) is clearly not definite on \([0, 1] \times \Sigma\). Hence, we identify points whose \( d_0 \) distance is 0 in order to define the metric space \( M_\infty \) with the distance function \( d_\infty \).

Now by the observation in [21, Example 7.4.4] we find that

\[ (M, \hat{d}_j) \overset{GH}{\to} (M_\infty, d_\infty). \] (51)
Finally, we show that the SWIF limit exists and is a proper subset of the GH limit. To this end we note that by [84, Lemma 2.43]

\[
M \left( (M, \hat{d}_j) \right) \leq \frac{2^m}{\omega_m} \mathcal{H}^m(M, \hat{d}_j), \\
M \left( (\partial M, \hat{d}_j) \right) \leq \frac{2^{m-1}}{\omega_{m-1}} \mathcal{H}^{m-1}(\partial M, \hat{d}_j),
\]

where \( \omega_m, m \in \mathbb{N} \), is the volume of the unit ball in \( \mathbb{R}^m \) and \( \mathcal{H}^m(X,d) \) is the \( m \)-dimensional Hausdorff measure of \((X,d)\). Due to (48) we see that for any \( U \subseteq M \)

\[
\text{Diam}_{\hat{d}_j}(U) \leq \text{Diam}_{\hat{d}_\sigma}(U),
\]

which implies

\[
\mathcal{H}^m(M, \hat{d}_j) \leq \mathcal{H}^m(M, \hat{d}_\sigma) \leq V_0, \\
\mathcal{H}^{m-1}(\partial M, \hat{d}_j) \leq \mathcal{H}^{m-1}(\partial M, \hat{d}_\sigma) \leq A_0.
\]

By [84, Theorem 3.20] we know that a SWIF limit \((X,d_\infty)\) of the sequence \((M, \hat{d}_j)_j\) exists where \( X \) is possibly the zero space. Since the SWIF limit must be \( n+1 \)-dimensional we know that \( X \subseteq M' \) and hence a proper subset of the GH limit.

Remark 5.12. In Example 5.11, we note that the expected SWIF limit is \( X = (M', d_\infty) \). We do not go through the work of explicitly showing this result here since new machinery which can be used to estimate the SWIF distance between spacetimes with the null distance should be developed first. Since a lot of the literature on estimating the SWIF distance has involved Riemannian manifolds (see [6, 7, 56, 57, 62]) it is necessary to first investigate extensions of these results for Lorentzian manifolds equipped with the null distance. Nonetheless, our results on warped product spacetimes are a first major step toward understanding spacetime convergence with respect to the null distance, and highlight the potential of this notion in geometric analysis and general relativity.

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