THE JACOBIAN CONJECTURE FOR
SYMMETRIC JACOBIAN MATRICES

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Abstract

We show that for all $n \leq 4$ the Jacobian Conjecture holds for all polynomial mappings $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ of the form $F = x + H$, where $H$ is homogeneous of degree $> 1$ and $JF$ is symmetric. It is also shown that the analogous statement for polynomial mappings $\mathbb{R}^n \rightarrow \mathbb{R}^n$ holds for all $n$.

0 Introduction

Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial mapping. Then the Jacobian Conjecture asserts that $F$ is invertible if $\det JF \in \mathbb{C}^*$, where $JF$ denotes the Jacobian matrix of $F$. It is a well-known result, due to Bass-Connell and Wright in [1] and Yagzhev in [7], that it suffices to prove the conjecture for all $n \geq 2$ and all polynomial mappings of the form $F = x + H$ when $H = (H_1, \ldots, H_n)$ and each $H_i$ is either zero or homogeneous of degree 3. It was shown by Wright in [6] that for such maps the conjecture holds in case $n = 3$. The case $n = 4$ was settled affirmatively by Hubbers in [5] (see also [2]). The case $n \geq 5$ remains open.

In this paper we study an apparently overlooked case, namely when $F$ is of the form $x + H$ with $H$ homogeneous of degree $d \geq 2$ and additionally $JF$ is symmetric. Our main result, theorem 3.1, asserts that the Jacobian Conjecture holds for these $F$’s in case $n \leq 4$. The proof is based on a remarkable theorem of Gordan and Noether in [4] (see theorem 1.2 below). It states the following: if $n \leq 4$ and $f \in \mathbb{C}[x] := \mathbb{C}[x_1, \ldots, x_n]$ is a homogeneous polynomial such that $\det h(f) = 0$ ($h(f) := (\frac{\partial^2 f}{\partial x_i \partial x_j})_{1 \leq i, j \leq n}$ is the Hessian of $f$) then after a suitable linear coordinate change $f$ has at most $n - 1$ variables.

The results obtained in this paper also reveal a remarkable difference, not observed earlier, between real polynomial mappings satisfying the Jacobian condition and complex polynomial mappings satisfying the Jacobian condition. In fact, we show at the end of this paper that the analogon of theorem 3.1 for real polynomial mappings holds in all dimensions by showing that the only real $F$ which satisfies the hypothesis of theorem 3.1 is the identity map $F = x!$
1 Preliminaries on symmetric Jacobian matrices

Throughout this paper $\mathbb{C}[x] := \mathbb{C}[x_1, \ldots, x_n]$ denotes the polynomial ring in $n$ variables over $\mathbb{C}$. Furthermore elements of $\mathbb{C}^n$ will be denoted by column vectors. Similarly, if $F : \mathbb{C}^n \to \mathbb{C}^m$ is a polynomial map i.e. its components $F_i$ belong to $\mathbb{C}[x]$, then we also denote the column vector $(F_1, \ldots, F_m)^T$ by $F$. The Jacobian matrix of $F$, denoted $JF$, is the $m \times n$ matrix given by $(JF)_{ij} = \frac{\partial F_i}{\partial x_j}$ for all $i, j$. In particular, viewing an element $f$ in $\mathbb{C}[x]$ as a polynomial map $f : \mathbb{C}^n \to \mathbb{C}$ we get $Jf = \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$.

Instead of $\frac{\partial}{\partial x_j}$ we also write $\partial_j$ and instead of $\partial_j f$ we sometimes write $f_{x_j}$.

**Lemma 1.1** Let $H : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. Then $JH$ is symmetric iff there exists $f$ in $\mathbb{C}[x]$ with $H = (Jf)^T$.

**Proof.** If $JH$ is symmetric, then $\partial_i H_j = \partial_j H_i$ for all $i, j$. So by the well-known Poincaré lemma ([1], 1.3.53) there exists $f \in \mathbb{C}[x]$ such that $H_i = \partial_i f$ for all $i$. The converse is obvious since $\partial_i \partial_j f = \partial_j \partial_i f$ for all $i, j$.

So if $H : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map then $JH$ is symmetric iff $JH$ is a Hessian matrix i.e. a matrix of the form

$$h(f) := \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{1 \leq i, j \leq n}, \text{ for some } f \in \mathbb{C}[x].$$

The proof of the main result of this paper, Theorem 3.1, is based on a remarkable theorem of P. Gordan and M. Noether [4] concerning the determinant of a Hessian matrix of a homogeneous polynomial $f$: a polynomial $f \in \mathbb{C}[x]$ is called degenerate if there exists an invertible linear map $T$ such that $f(Tx) \in \mathbb{C}[x_1, \ldots, x_{n-1}]$. It is easy to see that if $f$ is degenerate then $\det h(f) = 0$. In 1850, in Volume 42 of Crelle’s Journal and again in 1859 in Volume 56 of Crelle’s Journal, Hesse claimed to have proved that conversely if $f$ is homogeneous then $\det h(f) = 0$ implies that $f$ is degenerated. This implication does not hold in general, as Gordan and Noether proved in their paper of 1876 [4], where they obtained the following result.

**Theorem 1.2** Let $f \in \mathbb{C}[x]$ be homogeneous. If $n \leq 4$ and $\det h(f) = 0$ then $f$ is degenerate. However if $n \geq 5$ there exist non-degenerate forms with $\det h(f) = 0$, for example $f = x_1^3 x_3 + x_1 x_2 x_4 + x_2^2 x_5$.

In the proofs given below we use the following notations:

- If both $v$ and $w$ belong to $\mathbb{C}^n$ then we denote $v_1 w_1 + \cdots + v_n w_n (= v^T w)$ by $\langle v, w \rangle$, although this is not a complex inner product!
- Similarly we denote $v_1 x_1 + \cdots + v_n x_n (= v^T x)$ by $\langle v, x \rangle$, where $x = (x_1, \ldots, x_n)^T$ the column vector consisting of the variables of $\mathbb{C}[x]$. One readily verifies that with this terminology we have

$$f \in \mathbb{C}[x] \text{ is degenerate iff there exist } g \in \mathbb{C}[x_1, \ldots, x_{n-1}]
$$

and $v_1, \ldots, v_{n-1}$ in $\mathbb{C}^n$, linearly independent over $\mathbb{C}$

such that $f = g(\langle v_1, x \rangle, \ldots, \langle v_{n-1}, x \rangle).

(1)
Motivated by (1) we consider the following situation: let \( v_1, \ldots, v_{n-1} \in \mathbb{C}^n \) be linearly independent over \( \mathbb{C} \) and \( g \in \mathbb{C}[y_1, \ldots, y_{n-1}] \). Put \( f := g(v_1, x), \ldots, \{v_{n-1}, x\}) \in \mathbb{C}[x] \) and \( V_i := \langle v_i, x \rangle \). Since the \( v_i \) are linearly independent over \( \mathbb{C} \) the \( V_i \) are algebraically independent over \( \mathbb{C} \).

Put \( H := (Jf)^t \). Then one easily verifies that

\[
H = g_{y_1}(V_1, \ldots, V_{n-1})v_1 + \cdots + g_{y_{n-1}}(V_1, \ldots, V_{n-1})v_{n-1}
\]  

and

\[
JH = \sum_{1 \leq i, j \leq n-1} g_{y_i y_j}(V_1, \ldots, V_{n-1})v_iv_j.
\]

Put \( A := (\langle v_i, v_j \rangle)_{1 \leq i, j \leq n-1} \in M_{n-1}(\mathbb{C}) \).

Let \( r := \text{rank } A \). Then there exists \( T \in \text{Gl}_{n-1}(\mathbb{C}) \) such that

\[
T^tAT = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.
\]

More generally, if \( B \in M_{n-1}(\mathbb{C}) \) with rank \( B = r \), then there exists \( T \in \text{Gl}_{n-1}(\mathbb{C}) \) such that

\[
T^tAT = B.
\]

The main object of this paper is to find necessary and sufficient conditions for the matrix \( JH \) of (3) to be nilpotent. Therefore from now on: \textit{we assume that \( JH \) is nilpotent.} Then in particular \( Tr JH = 0 \). Since \( Tr vu^t = \langle v, u \rangle \) for all \( v, u \) in \( \mathbb{C}^n \) we deduce from (3) that

\[
\sum_{1 \leq i, j \leq n-1} \langle v_i, v_j \rangle g_{y_i y_j} = 0
\]

i.e.

\[
\sum_{1 \leq i, j \leq n-1} (v_i, v_j) \partial_i \partial_j g = 0.
\]

So if we put \( y := (y_1, \ldots, y_{n-1})^t \) and \( \partial := (\partial_1, \ldots, \partial_{n-1})^t \) then (6) becomes

\[
(\partial_y^t A \partial_y)g(y) = 0.
\]

Now we are going to show that we can write \( f \) in a particular nice form (Corollary 1.4). Therefore we need

**Lemma 1.3** Let \( T \in \text{Gl}_{n-1}(\mathbb{C}) \). Put \( (\tilde{v}_1, \ldots, \tilde{v}_{n-1}) := (v_1, \ldots, v_{n-1})T, \ z := T^t y \) and \( \tilde{g}(z) := g((T^t)^{-1} z)(= g(y)) \). Then

i) \( T^tAT = ((\tilde{v}_i, \tilde{v}_j))_{1 \leq i, j \leq n-1} \).

ii) \( (\partial_y^t A \partial_y)g(y) = (\partial_z^t (T^t AT) \partial_z)\tilde{g}(z) \).
Proof. i) Observe that $⟨v_i, v_j⟩ = v_i^tv_j$ and hence $A = \begin{pmatrix} v_1^t \\ \vdots \\ v_{n-1}^t \end{pmatrix} \cdot (v_1, \ldots, v_{n-1})$. So

$$T^tAT = T^t \begin{pmatrix} v_1^t \\ \vdots \\ v_{n-1}^t \end{pmatrix} (v_1, \ldots, v_{n-1})T = ((v_1, \ldots, v_{n-1})^t(v_1, \ldots, v_{n-1}))T$$

$$= \begin{pmatrix} \tilde{v}_1^t \\ \vdots \\ \tilde{v}_{n-1}^t \end{pmatrix} (\tilde{v}_1, \ldots, \tilde{v}_{n-1}) = ((\tilde{v}_i, \tilde{v}_j))_{1 \leq i, j \leq n-1}.$$

ii) Observe that from $z = T^ty$ it follows that $\partial_z = T^{-1}\partial_y$. Hence

$$(\partial^2_yA\partial_y)y = (\partial^2_y(T^t)^{-1}(T^tAT)T^{-1}\partial_y)y$$

$$= (T^{-1}\partial_y)^t(T^tAT)T^{-1}\partial_yy = (\partial^2_y(T^tAT)\partial_z)\tilde{y}(z).$$

\[\blacksquare\]

**Corollary 1.4** Notations as in 1.3. Let $T$ be as in (4). Then $f = \tilde{g}(\tilde{v}_1, x), \ldots, (\tilde{v}_{n-1}, x))$ with $⟨\tilde{v}_i, \tilde{v}_i⟩ = 1$ for all $1 \leq i \leq r$ and $⟨\tilde{v}_i, \tilde{v}_j⟩ = 0$ otherwise. Furthermore $(\partial^2_{z_1} + \cdots + \partial^2_{z_r})\tilde{g}(z) = 0$ and $J_{z_1, \ldots, z_r}(J_{z_1, \ldots, z_r} g)$ is nilpotent.

**Proof.** i) Observe that

$$\begin{pmatrix} (v_1, x) \\ \vdots \\ (v_{n-1}, x) \end{pmatrix} = \begin{pmatrix} v_1^t \\ \vdots \\ v_{n-1}^t \end{pmatrix} x = (T^t)^{-1}T^t \begin{pmatrix} v_1^t \\ \vdots \\ v_{n-1}^t \end{pmatrix} x = (T^t)^{-1}((v_1, \ldots, v_{n-1}))^t(T^t)^{-1}(v_1, \ldots, v_{n-1}).$$

Since $g(y) = g(T^t)^{-1}z = \tilde{g}(z)$ it follows that $f = g((v_1, x), \ldots, (v_{n-1}, x)) = \tilde{g}(\tilde{v}_1, x), \ldots, (\tilde{v}_{n-1}, x))$. Furthermore, the statement concerning the “inner products” $(\tilde{v}_i, \tilde{v}_j)$ follows from lemma 1.3 and the hypothesis on $T$.

ii) The statement that $(\partial^2_{z_1} + \cdots + \partial^2_{z_r})\tilde{g}(z) = 0$ follows immediately from (7) and lemma 1.3 ii) since

$$\begin{pmatrix} \partial^2_{z_1} \langle I_r \ 0 \ 0 \ 0 \ 0 \ \partial_{z_r} \end{pmatrix} \tilde{g}(z) = (\partial^2_{z_1} + \cdots + \partial^2_{z_r})\tilde{g}(z).$$

iii) Put $h_r(\tilde{g}) := (\tilde{g}_{z_i, z_j})_{1 \leq i, j \leq r}$. Finally we show that $h_r(\tilde{g})$ is nilpotent. It is well-known that $JH$ is nilpotent iff $Tr(JH)^k = 0$ for all $1 \leq k \leq n$ and similarly to show that $h_r(\tilde{g})$ is nilpotent we need to show that $Tr(h_r(\tilde{g}))^k = 0$ for all $1 \leq k \leq r$. So we are done if we can show

$$Tr(JH)^k = Tr(h_r(\tilde{g}))^k$$

for all $k \geq 1$. (8)
To simplify the notations we write $v_i$ instead of $\tilde{v}_i$, $g$ instead of $\tilde{g}$ and $g_{z,z_j}$ instead of $g_{z,z_j}(V_1,\ldots,V_{n-1})$. So we then have $\langle v_i, v_j \rangle = 1$ for all $1 \leq i \leq r$ and $\langle v_i, v_j \rangle = 0$ otherwise. Furthermore $f = g(\langle v_1, x \rangle, \ldots, \langle v_{n-1}, x \rangle)$. By (3) we get

$$JH = \sum_{1 \leq i,j \leq n-1} g_{z,z_j} v_i v_j^t.$$ Consequently

$$(JH)^k$$ is a finite sum of products of the form

$$g_{z_{i_1}z_{j_1}} g_{z_{i_2}z_{j_2}} \cdots g_{z_{i_k}z_{j_k}} (v_{i_1} v_{j_1} v_{i_2} v_{j_2} \cdots v_{i_k} v_{j_k})$$

Since $v^t w = \langle v, w \rangle$ and $Tr(vv^t) = \langle v, w \rangle$ for all $v, w \in \mathbb{C}^n$ it follows that $Tr(v_1 w_1^t v_2 w_2^t) = (w_1, v_2)(v_1, w_2)$ and more generally

$$Tr(v_1 w_1^t v_2 w_2^t \cdots v_k w_k^t) = (w_1, v_2)(v_1, w_2) \cdots (w_k-1, v_k)(v_1, w_k).$$

So by (9) $Tr(JH)^k$ is a finite sum of products of the form

$$g_{z_{i_1}z_{j_1}} g_{z_{i_2}z_{j_2}} \cdots g_{z_{i_k}z_{j_k}} (v_{i_1} v_{j_1} v_{i_2} v_{j_2} \cdots v_{i_k} v_{j_k}).$$

Now the point is that the expressions $\langle v_{i_1}, v_{i_2} \rangle \cdots \langle v_{i_k}, v_{j_k} \rangle$ do not depend on the form of the vectors $v_i$ but only on the “inner products” $\langle v_i, v_j \rangle$. So the compute $Tr(JH)^k$ we may as well assume that $v_i = e_i$ is the $i$-th standard basis vector of $\mathbb{C}^n$ for all $1 \leq i \leq r$ and $v_1 = 0$ otherwise! Consequently

$$JH = \sum_{1 \leq i,j \leq r} g_{z_{i}z_{j}} (V_1,\ldots,V_{n-1})e_i e_j^t.$$ Since the only non-zero entry of the matrix $e_i e_j^t$ is 1 on the place $(i,j)$ it follows from (10) that $(JH)_{ij} = g_{z_{i}z_{j}}(V_1,\ldots,V_{n-1})$ for all $1 \leq i,j \leq r$ and $(JH)_{ij} = 0$ otherwise i.e.

$$JH = \begin{pmatrix} h_r(\tilde{g}) & 0 \\ 0 & 0 \end{pmatrix}.$$ Consequently $Tr(JH)^k = Tr(h_r(\tilde{g})^k)$ for all $k \geq 1$. □

**Remark 1.5** If $rkA = 2$ then there exists $T \in GL_{n-1}(\mathbb{C})$ and $B \in M_{n-1}(\mathbb{C})$ with $B_{12} = B_{21} = 1$ and $B_{ij} = 0$ otherwise such that $T^a AT = B$. Since $\partial^a_i B \partial^a_j = 2 \partial^a_{z_i} \partial^a_{z_j}$ it follows from lemma 1.3 and the argument given in the proof of i) of Corollary 1.4 that we can write $f$ in the form $f = \tilde{g}((\tilde{v}_1, x), \ldots, (\tilde{v}_{n-1}, x))$ with $\langle \tilde{v}_1, \tilde{v}_2 \rangle = 1$ and $\langle \tilde{v}_1, \tilde{v}_2 \rangle = 0$ otherwise. Furthermore we have $\partial_{z_1} \partial_{z_2} \tilde{g} = 0$ i.e. $\tilde{g}(z) = a(z_1, z_2, \ldots, z_{n-1}) + b(z_2, z_3, \ldots, z_{n-1})$ for some $a \in \mathbb{C}[z_1, z_3, \ldots, z_{n-1}]$ and $b \in \mathbb{C}[z_2, z_3, \ldots, z_{n-1}]$. In other words we can write $f$ in the form

$$f = a((\tilde{v}_1, x), (\tilde{v}_3, x), \ldots, (\tilde{v}_{n-1}, x)) + b((\tilde{v}_2, x), (\tilde{v}_3, x), \ldots, (\tilde{v}_{n-1}, x)).$$

6
Section 2: Special Classes of Symmetric Nilpotent Jacobian Matrices and the Jacobian Conjecture

In this section we show how the results of the previous section can be used to show that the Jacobian Conjecture holds for a large class of polynomial maps whose Jacobian matrix is symmetric.

Throughout this section we have the following situation: \( v_1, \ldots, v_{n-1} \) are linearly independent vectors in \( \mathbb{C}^n \), \( n \geq 2 \) and \( g \in \mathbb{C}[y_1, \ldots, y_{n-1}] \). Furthermore \( f := g((v_1, x), \ldots, (v_{n-1}, x)) \) and \( F := x + H \) where \( H := (Jf)^t \). Finally \( A := ((v_i, v_j))_{1 \leq i, j \leq n-1} \). The main result of this section is

Theorem 2.1 If \( JH \) is nilpotent and \( \text{rk} A \leq 2 \), then \( F \) is invertible.

To prove this result we consider the cases \( \text{rk} A = 1 \) and \( \text{rk} A = 2 \) separately.

Proposition 2.2 If \( JH \) is nilpotent and \( \text{rk} A = 1 \) then there exist \( w_1, \ldots, w_{n-1} \in \mathbb{C}^{n-1} \) linearly independent over \( \mathbb{C} \) and \( g_* \in C[y_1, \ldots, y_{n-1}] \) of the form \( g_* = a(y_2, \ldots, y_{n-1})y_1 + b(y_2, \ldots, y_{n-1}) \) such that \( f = g_*(<w_1, x>, \ldots, <w_{n-1}, x>) \), where \( <w_1, w_1> = 1 \) and \( <w_i, w_j> = 0 \) otherwise. Furthermore \( F = x + H \) is invertible.

Proof. It follows from Corollary 1.4 that \( f = \tilde{g}(<\tilde{v}_1, x>, \ldots, <\tilde{v}_{n-1}, x>) \) with \( \partial^2 \tilde{g} = 0 \). So if we put \( w_i := \tilde{v}_i \) and \( g_* := \tilde{g} \) we get the first part of the proposition. Furthermore it follows from (2) that

\[
F = x + a(W_2, \ldots, W_{n-1})w_1 + \sum_{j=2}^{n-1} (a_j(W_2, \ldots, W_{n-1}) + b_j(W_2, \ldots, W_{n-1})j)w_j
\]

where \( W_i = <w_i, x> \), \( a_j := a_{i,j} \) and \( b_j := b_{i,j} \).

To show that \( F \) is invertible we use the following lemma.

To formulate it we need some notations:
if \( G = (G_1, \ldots, G_n) : \mathbb{C}^n \to \mathbb{C}^n \) is a polynomial map and \( a \in \mathbb{C}[x] \) then \( a(G) := a(G_1, \ldots, G_n) \). If \( D = (D_1, \ldots, D_n) \) is another polynomial map then \( D(G) = (D_1(G), \ldots, D_n(G)) \) denotes the composition \( D \circ G \) of \( D \) and \( G \).

Lemma 2.3 Let \( r, s, n \in \mathbb{N} \) with \( 1 \leq r < s \) and \( w_1, \ldots, w_s \in \mathbb{C}^n \) such that \( <w_i, w_j> = 0 \) for all \( j \geq r + 1 \). Put \( W_i = <w_i, x> \) for all \( i \) and \( E := x + \sum_{j \geq r+1} h_j(W_1, \ldots, W_s)w_j \) with \( h_j \in \mathbb{C}[y_1, \ldots, y_s] \). Then

i) \( W_i(E) = W_i \) for all \( i \).

ii) \( E \) is invertible with inverse \( E' := x - \sum_{j \geq r+1} h_j(W_1, \ldots, W_s)w_j \).

Proof. i) \( W_i(E) = <w_i, E> = <w_i, x> + \sum_{j \geq r+1} h_j(W_1, \ldots, W_s)<w_i, w_j> = <w_i, x> = W_i \) since \( <w_i, w_j> = 0 \) for all \( j \geq r + 1 \).

ii) \( E'(E) = E - \sum_{j \geq r+1} h_j(W_i(E), \ldots, W_s(E))w_j = E - \sum_{j \geq r+1} h_j(W_i, \ldots, W_s)w_j = x \). \( \square \)
Proof of Proposition 2.2 (finished)

Put \( E := x - \sum_{j=2}^{n-1} (a_j(W_2,\ldots,W_{n-1}))W_j + b_j(W_2,\ldots,W_{n-1})w_j \). Then it follows from lemma 2.3 (with \( r = 1 \) and \( s = n-1 \)) that \( E \) is invertible and that \( W_i(E) = W_i \) for all \( i \). Hence

\[
F(E) = E + a(W_2,\ldots,W_{n-1})w_1 + \sum_{j=2}^{n-1} (a_j(W_2,\ldots,W_{n-1})W_1 + b_j(W_2,\ldots,W_{n-1}))w_j
\]

Finally if we put \( E_1 := x + a(W_2,\ldots,W_{n-1})w_1 \) then one readily verifies that \( W_i(E_1) = \langle w_i, E_1 \rangle = 0 \) for all \( i \geq 2 \), which implies that \( E_1 \) is invertible with inverse \( x - a(W_2,\ldots,W_{n-1})w_1 \). So \( F(E)(= E_1) \) is invertible. Since \( E \) is invertible, as observed above, this implies that \( F \) is invertible too. \( \square \)

Proposition 2.4  If \( JH \) is nilpotent and \( rkA = 2 \) then there exist \( w_1,\ldots,w_{n-1} \in \mathbb{C}^n \) linearly independent over \( \mathbb{C} \) and \( g_* \in \mathbb{C}[y_1,\ldots,y_{n-1}] \) of the form \( g_* = a(y_1, y_3,\ldots,y_{n-1}) + b(y_2, y_3,\ldots,y_{n-1}) \) such that \( f = g_* (\langle w_1, x \rangle,\ldots,\langle w_{n-1}, x \rangle) \), where \( \langle w_1, w_2 \rangle = 1 \) and \( \langle w_i, w_j \rangle = 0 \) otherwise. Furthermore \( \tilde{a}_{y_1} \tilde{b}_{y_2} = 0 \) and \( F = x + H \) is invertible.

Proof.  The first part follows from Remark 1.5 by putting \( w_i := \tilde{v}_i \) and \( g_* := \tilde{g} \). Furthermore by (2) we get

\[
F = x + a_1(W_1, W_3,\ldots,W_{n-1}) + b_2(W_2, W_3,\ldots,W_{n-1}) + \sum_{j=3}^{n-1} (a_j(W_1, W_3,\ldots,W_{n-1}) + b_j(W_2,\ldots,W_{n-1}))w_j.
\]

So if we put \( E_1 := x - \sum_{j=3}^{n-1} (a_j(W_1, W_3,\ldots,W_{n-1}) + b_j(W_2,\ldots,W_{n-1}))w_j \) then it follows from lemma 2.3 (with \( r = 2 \) and \( s = n-1 \)) that \( E_1 \) is invertible and \( W_i(E_1) = W_i \) for all \( i \). Consequently

\[
F(E_1) = x + a_1(W_1, W_3,\ldots,W_{n-1})w_1 + b_2(W_2,\ldots,W_{n-1})w_2.
\]

To see that \( F(E_1) \) is invertible (which together with the invertibility of \( E_1 \) implies that \( F \) is invertible) we first observe that \( \tilde{a}_{y_1} \tilde{b}_{y_2} = 0 \); indeed since by (3)

\[
JH = a_{y_1} w_1 w_1 + b_{y_2} w_2 + \cdots
\]

where \( \ldots \) consists of terms of the form \( p(W_1,\ldots,W_{n-1})w_i w_j \) with at least one of \( i \) or \( j \geq 3 \), it follows from the fact that \( \langle w_i, w_j \rangle = 0 \) if at least one of \( i \) or \( j \geq 3 \) that \( \text{Tr}(JH)^2 = 2a_{y_1} b_{y_2} (w_1, w_2)^2 = 2a_{y_1} b_{y_2} \). Since \( JH \) is nilpotent we have that \( \text{Tr}(JH)^2 = 0 \) which gives that \( a_{y_1} b_{y_2} = 0 \). So either \( a_{y_1} = 0 \) or \( b_{y_2} = 0 \).

If for example \( a_{y_1} = 0 \) (the case \( b_{y_2} = 0 \) is treated similarly) then \( (a_{y_1})_{y_1} = 0 \) i.e. \( a_1 \) does not contain \( W_1 \). Consequently

\[
E_2 := F(E_1) = x + a_1(W_3,\ldots,W_{n-1})w_1 + b_2(W_2,\ldots,W_{n-1})w_2.
\]
Put \( E_3 := x - a_1(W_3, \ldots, W_{n-1})w_1 \). Then one easily verifies that \( W_i(E_3) = W_i \) for all \( i \neq 2 \) and that \( E_2 \) is invertible. Consequently

\[
E_2(E_3) = E_3 + a_1(W_3, \ldots, W_{n-1})w_1 + b_2(W_2 - a_2(W_3, \ldots, W_{n-1}), W_3, \ldots, W_{n-1})w_2 = x + b_2(W_2 - a_1(W_3, \ldots, W_{n-1}), W_3, \ldots, W_{n-2})w_2 =: x + b(W_2, \ldots, W_{n-1})w_2.
\]

Hence \( E_2(E_3) \) is invertible by lemma 2.3 with \( r = 1 \). Since \( E_3 \) is invertible this implies that \( E_2 \) is invertible i.e. \( F(E_1)(= E_2) \) is invertible, as desired. \( \square \)

3 Symmetric homogeneous Jacobian matrices in dimension \( \leq 4 \)

In this section we give the main result of this paper:

**Theorem 3.1** Let \( n \leq 4 \) and \( F = x + H : \mathbb{C}^4 \to \mathbb{C}^4 \) a polynomial map such that \( H \) is homogeneous of degree \( d \geq 2 \). If \( \det JF \in \mathbb{C}^* \) and \( JF \) is symmetric, then \( F \) is invertible.

**Proof.** (started) i) Since \( H \) is homogeneous it is well-known that the condition \( \det JF \in \mathbb{C}^* \) implies that \( JH \) is nilpotent ([2], lemma 6.2.11) and hence that \( \det JF = 1 \). If \( n = 2 \) this implies that \( F \) is invertible ([3] or [2], Exercise 4, §2.1).

So from now on we assume that \( 3 \leq n \leq 4 \).

ii) Since \( JH \) is symmetric there exists \( f \in \mathbb{C}[x] \) with \( H = (Jf)^t \) (lemma 1.1) and since \( H \) is homogeneous we can assume \( f \) to be homogeneous. Furthermore, the nilpotency of \( JH \) implies that \( \det JH = 0 \) i.e. \( \det h(f) = 0 \). Then it follows from the Gordan-Noether theorem and (1) that there exist \( v_1, \ldots, v_{n-1} \in \mathbb{C}^n \) linearly independent over \( \mathbb{C} \) and \( g \in \mathbb{C}[y_1, \ldots, y_{n-1}] \) such that \( f = g(v_1, x), \ldots, (v_{n-1}, x) \).

iii) Put \( A := (v_i, v_j)_{1 \leq i, j \leq n-1} \) and let \( r := rkA \). We need

**Lemma 3.2** \( r \geq n - 2 \).

**Proof.** Let \( T \) be as in (4). Then, with the notations of lemma 1.3, we have

\[
(\tilde{v}_i, \tilde{v}_j)_{1 \leq i, j \leq n-1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.
\]

Put \( B := \begin{pmatrix} \tilde{v}_1 \\ \vdots \\ \tilde{v}_{n-1} \end{pmatrix} \). Since \( \tilde{v}_1, \ldots, \tilde{v}_{n-1} \) are linearly independent over \( \mathbb{C} \) we get that \( rkB = n - 1 \). So \( \dim \ker B = 1 \). Now observe that \( \langle \tilde{v}_i, \tilde{v}_j \rangle \) is zero for all \( 1 \leq i \leq n - 1 \) and all \( j \geq r + 1 \). This implies that \( \tilde{v}_{r+1}, \ldots, \tilde{v}_{n-1} \in \ker B \). Since \( \dim \ker B = 1 \) it follows that \( (n - 1) - r \leq 1 \) i.e. \( r \geq n - 2 \) as desired. \( \square \)

**Proof of theorem 3.1.** (continued)

iv) Now assume \( n = 3 \). Then by lemma 3.2 \( rkA = 1 \) or \( rkA = 2 \). Then it follows
In the first case

and $w_1, w_2 \in \mathbb{C}^3$ linearly independent over $\mathbb{C}$ and satisfying $\langle w_1, w_1 \rangle = 1$ and $\langle w_i, w_j \rangle = 0$ otherwise. In case $rkA = 2$ it follows from Proposition 2.4 that we can write $f$ in the form $a(t) + b(t)$ with $a(t) = c_1 t^{d+1}$, $b(t) = c_2 t^{d+1}$, $c_1, c_2 \in \mathbb{C}$ and such that $a''(t)b''(t) = 0$.

Since $d \geq 2$ this implies that either $c_1$ or $c_2$ is zero. So we can write $f$ in the form $c((w, x))^{d+1}$, which is a special case of (11) ($c_1 = 0$). Summarizing we get

If $n = 3$ and $f \in \mathbb{C}[x]$ is homogeneous of degree $d + 1 \geq 3$ such that the Hessian matrix $J((Jf)'')$ is nilpotent then $f$ is of the form (11).

v) Finally we consider the case $n = 4$. Then by lemma 3.2 $rkA = 2$ or $rkA = 3$. In the first case $F$ is invertible by Proposition 2.4. So let’s assume that $rkA = 3$. Then by Corollary 1.4 there exist $\tilde{g} \in \mathbb{C}[z_1, z_2, z_3]$ homogeneous of degree $d + 1$ and $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3 \in \mathbb{C}^4$ linearly independent over $\mathbb{C}$ such that $f = \tilde{g}(\tilde{v}_1, x, \langle \tilde{v}_2, x \rangle, \langle \tilde{v}_3, x \rangle)$. Furthermore $J_2(J_2\tilde{g})$ is nilpotent. But then we can apply (12) to the polynomial $\tilde{g}$. So we obtain that

$$\tilde{g}(z_1, z_2, z_3) = c_1((w_2, z))^{d}(w_1, z) + c_2((w_2, z))^{d+1}$$

for some $c_1, c_2 \in \mathbb{C}$ and $w_1, w_2 \in \mathbb{C}^3$ linearly independent over $\mathbb{C}$ satisfying $\langle w_1, w_1 \rangle = 1$ and $\langle w_i, w_j \rangle = 0$ otherwise. Consequently $f$ is of the form

$$f = c_1((w_2, x))^{d}(u_1, x) + c_2((w_2, x))^{d+1}$$

where $u_i = Sw_i$ for $i = 1, 2$ and $S := (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3) \in M_{4 \times 3}(\mathbb{C})$. Since $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$ are linearly independent over $\mathbb{C}$ it follows that ker$S = 0$. Consequently $u_1$ and $u_2$ are linearly independent over $\mathbb{C}$. Furthermore $SS^t = (\langle \tilde{v}_i, \tilde{v}_j \rangle)_{1 \leq i, j \leq 3} = I_3$. Hence

$$\langle u_1, u_j \rangle = \langle Sw_i, Sw_j \rangle = w_i^tS^tSw_j = w_i^tw_j = \langle w_i, w_j \rangle.$$ In particular $\langle u_1, u_2 \rangle = 0$ and $\langle u_2, u_2 \rangle = 0$.

vi) Put $U_i := \langle u_i, x \rangle$, $i = 1, 2$. Then it follows from (13) and (2) that

$$F = x + c_1 U_1^d + (d+1)c_2 U_2^{d+1}u_2.$$ 

Since $\langle u_1, u_2 \rangle = \langle u_2, u_2 \rangle = 0$ it follows from lemma 2.3 (with $r = 1$) that $E := x - (d+1)c_2 U_2^{d+1}u_2$ is invertible and $U_i(E) = U_i$ for all $i$. Hence $F(E) = x + c_1 U_2^{d+1}u_1$. One easily deduces that $F(E)$ is invertible with inverse $x - c_1 U_2^{d+1}u_1$, since $\langle u_1, u_2 \rangle = 0$. Since, as observed above, $E$ is invertible this implies that $F$ is invertible too. This concludes the proof of theorem 3.1. □

To conclude this section we use some standard techniques (which can be found on page 5-7 of [2]) to generalize theorem 3.1 to the case of polynomial maps over $\mathbb{Q}$-algebras. More precisely, let $R$ be a commutative $\mathbb{Q}$-algebra and $F = x + H$ a polynomial map over $R$ i.e. $H = (H_1, \ldots, H_n)$ where each $H_i$ belongs to $R[x] := R[x_1, \ldots, x_n]$. Denoting the units of $R[x]$ by $R[x]^*$ we have
Theorem 3.3 Let $n \leq 4$ and $F = x + H$ a polynomial map over $R$ such that $H$ is homogeneous of degree $d \geq 2$. If $\det JF \in R[x]^*$ and $JF$ is symmetric, then $F$ is invertible over $R$.

Proof. i) First we assume that $R$ is a domain. Then $d := \det JF \in R^*$. So $d^{-1} \in R$. Let $R_0$ be the $Q$-subalgebra of $R$ generated by the coefficients of $F$ and $d^{-1}$. So $\det JF \in R_0$ and $R_0$ is a finitely generated $Q$-algebra, hence noetherian. Then by the Lefschetz Principle ([2], lemma 1.1.13) we can view $R_0$ as a subring of $C$ and since $\det JF \in R_0$ it follows from [2], lemma 1.1.8 that $F$ is invertible over $R_0$ and hence over $R$. 

ii) Now let $R$ be an arbitrary $Q$-algebra. Replacing $R$ by $R_0$ we may assume that $R$ is noetherian. Furthermore by [2], lemma 1.1.9 we may assume that $R$ is reduced. In particular $(0) = p_1 \cap \ldots \cap p_r$ for some finite set of prime ideals $p_i$ of $R$.

iii) Since for each $i R/p_i$ is a domain it follows from i) that $\overline{F}$ is invertible over $R/p_i$ (where $\overline{F}$ is obtained by reducing its coefficients mod $p_i$). Then a well-known argument (see for example part iii) in the proof of Proposition 1.1.12 in [2]) gives that $F$ is invertible over $R$. $\square$ 

4 A remark on the real case

In this section we show that in contrast with the complex case the real version of theorem 3.1 is almost obvious in all dimensions. More precisely

Theorem 4.1 Let $F = x + H : \mathbb{R}^n \to \mathbb{R}^n$ a polynomial map, where $H \in \mathbb{R}[x_1, \ldots, x_n]^n$ is homogeneous of degree $d \geq 2$. If $\det JF \in \mathbb{R}^*$ and $JF$ is symmetric then $F = x$. In particular $F$ is invertible.

Proof. Since $H$ is homogeneous of degree $d \geq 2$ it follows from $\det JF \in \mathbb{R}^*$ that $\det JF = \det JF(0) = 1$. By [2], lemma 6.2.11 this implies that $JH$ is nilpotent. Now let $a \in \mathbb{R}^n$. Since $JF$ is symmetric it follows that $JH(a)$ is symmetric. So there exists $T \in \text{GL}_n(\mathbb{R})$ with $T^{-1}JH(a)T$ a diagonal matrix, having all its eigenvalues on the diagonal. Since $JH$ and hence $JH(a)$ are nilpotent all eigenvalues of $JH(a)$ are equal to zero. So $T^{-1}JH(a)T$ and hence $JH(a)$ are the zero matrix. Since this holds for all $a \in \mathbb{R}^n$ $JH = 0$. Consequently $H = 0$ and $F = x$. $\square$

References


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