NON-DEGENERACY OF THE HOFER NORM FOR POISSON STRUCTURES

DUŠAN JOKSIMOVIĆ, IOAN MĂRCUȚ

Abstract. We remark that, as in the symplectic case, the Hofer norm on the Hamiltonian group of a Poisson manifold is non-degenerate. The proof is a straightforward application of tools from symplectic topology.

Let (M, π) be a Poisson manifold. The Hamiltonian isotopy associated to a smooth family of compactly supported functions $f \in C_c^\infty([0,1] \times M)$ is denoted by $\{\varphi_f^t\}$. The Hamiltonian group of (M, π) is

$$\mathrm{Ham}(M, \pi) := \left\{ \varphi_f^1 \ : \ f \in C_c^\infty([0,1] \times M) \right\}.$$

The length of a family $f \in C_c^\infty([0,1] \times M)$ is defined as

$$l(f) := \int_0^1 \left(\sup f_t - \inf f_t \right) \, dt.$$

Finally, define the Hofer norm on $\mathrm{Ham}(M, \pi)$ by

$$\nu(\Phi) := \inf \left\{ l(f) \ : \ f \in C_c^\infty([0,1] \times M), \ \varphi_f^1 = \Phi \right\}.$$

The following compatibility properties with the group structure are easily verified

(a) $\nu(\Phi) = \nu(\Phi^{-1})$,
(b) $\nu(\Phi \circ \Psi) \leq \nu(\Phi) + \nu(\Psi)$,
(c) $\nu(\Phi \circ \Psi \circ \Phi^{-1}) = \nu(\Psi)$,

for $\Phi, \Psi \in \mathrm{Ham}(M, \pi)$ (see e.g. [8]), but the non-degeneracy of ν is non-trivial:

(d) $\nu(\Phi) = 0$ if and only if $\Phi = \mathrm{Id}$.

In the symplectic case, Hofer [3] proved non-degeneracy for the standard symplectic structure on $\mathbb{R}^{2n}$, then Polterovich [6] extended it to a larger class of symplectic manifolds, and Lalonde and McDuff [4] completed the proof for all symplectic manifolds. All proofs rely on hard methods from symplectic topology.

Below, we show that the Poisson case can be easily reduced to the sympletic case, by restricting to a symplectic leaf. This was first claimed by Sun and Zhang [8], in the setting of regular Poisson manifolds. Actually, in the proof they do not use regularity, but assume that the restriction of a compactly supported function to a leaf is compactly supported, however, without stating this explicitly. This property is equivalent to the leaves being closed submanifolds (which implies that they are embedded submanifolds, see e.g. [2]). This mistake was noticed by Rybicki [7], who obtained non-degeneracy for Poisson manifolds whose closed leaves form a dense set. Moreover, Rybicki [7] proved non-degeneracy also for integrable Poisson manifolds, by using the displacement energy techniques on the symplectic groupoid. By adapting this proof to a sympelctic leaf, we obtain non-degeneracy in general.

Proof of (d). Let $\Phi \in \mathrm{Ham}(M,\pi)$, $\Phi \neq \mathrm{Id}$, and fix $x \in M$ such that $\Phi(x) \neq x$. Let $i : L \to M$ be the symplectic leaf passing through x, where i denoted the inclusion. Let $B \subset L$ be an open ball with compact closure such that $\Phi(B) \cap B = \emptyset$. Consider $f \in C_c^\infty([0,1] \times M)$ such that $\Phi = \varphi_f^1$. If M is compact, by replacing f_t by $f_t - f_t(x)$, we may assume that $f_t(x) = 0$. Consider a compactly supported smooth function $\lambda : L \to [0,1]$, such that $\lambda = 1$ on the set $\cup_{t\in[0,1]}\varphi_f^t(B) \subset L$. Then $g := \lambda(f \circ i)$ belongs to $C_c^\infty([0,1] \times L)$ and its flow satisfies $\varphi_g^t|_B = \varphi_f^t|_B$; in particular $\varphi_g^1(B) \cap B = \emptyset$.

Notice that since f is compactly supported it follows that $\sup_{y\in M} f_t(y) \geq 0$, for all $t \in [0,1]$. In the compact case this holds by our assumption that $f_t(x) = 0$. Therefore

$$\sup_{y\in M} f_t(y) \geq \sup_{y\in L} \lambda(y) f_t(y) = \sup_{y\in L} g_t(y), \quad t \in [0,1].$$

Applying the same argument to $-f$, we get that

$$\inf_{y\in M} f_t(y) \leq \inf_{y\in L} g_t(y), \quad t \in [0,1].$$

Hence, we obtain that $E(B) \leq l(g) \leq l(f)$, where $E(B)$ is the displacement energy (see for example [5]) of B inside the symplectic manifold L. Thus

$$E(B) \leq \nu(\Phi),$$

and since $0 < E(B)$ (see [4, Theorem 1.1]), we obtain *(d)*. □

References

[1] Y. Eliashberg, L. Polterovich, Bi-invariant metrics on the group of Hamiltonian diffeomorphisms, *Internat. J. Math.* **4** (1993), no. 5, 727–738.

[2] P. Frejlich, I. Mărcuț, The homology class of a Poisson transversal, to appear in *IMRN*.

[3] H. Hofer, On the topological properties of symplectic maps, *Proc. Royal Soc. Edinburgh* **115A** (1990), 25–38.

[4] F. Lalonde, D. McDuff, The geometry of symplectic energy, *Ann. Math.* **141** (1995), 349–371.

[5] D. McDuff, D. Salamon, Introduction to Symplectic Topology, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1995. viii+425 pp. ISBN: 0-19-851177-9.

[6] L. Polterovich, Symplectic displacement energy for Lagrangian submanifolds, *Erg. Th. Dynam. Sys.* **13** (1993), 357–367.

[7] T. Rybicki, On the existence of a Hofer type metric for Poisson manifolds, *Internat. J. Math.* 27 (2016), no. 9, 1650075, 16 pp.

[8] D. Sun, Z. Zhang, A Hofer-type norm of Hamiltonian maps on regular Poisson manifold, *J. Appl. Math.* 2014, Art. ID 879196, 9 pp.

Utrecht University
E-mail address: d.joksimovic@uu.nl

Radboud University Nijmegen
E-mail address: i.marcut@math.ru.nl