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Variations of Weyl's Tube Formula

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Abstract
In 1939 Weyl showed that the volume of spherical tubes around compact submanifolds M of Euclidean space depends solely on the induced Riemannian metric on M. Can this intrinsic nature of the tube volume be preserved for tubes with more general cross sections $\mathbb{D}$ than the round ball? Under sufficiently strong symmetry conditions on $\mathbb{D}$ the answer turns out to be yes.

Keywords Tubes · Volumes · Curvature · Invariant theory · Finite reflection groups

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1 Introduction

Let us be given a compact connected manifold M (possibly with a boundary) of dimension n embedded in $\mathbb{R}^{n+m}$ as submanifold of codimension m. For each $r \in M$ we have an orthogonal decomposition $T_r M \oplus N_r M$ of $\mathbb{R}^{m+n}$ into tangent space and normal space at r of M. It was shown by Weyl [15] that the Euclidean volume of the spherical tube

$$\{r + n\,;\, r \in M,\ n \in N_r M,\ |n| \le a\}$$

around M with radius $a > 0$ sufficiently small is equal to

$$V_M(a) = \Omega_m \sum_{d=0}^{n} \frac{k_d(M)\, a^{m+d}}{(m+2)\cdots(m+d)} \qquad (d \text{ even})$$

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with Ω_m the volume of the unit ball $\mathbb{B}^m = \{t\,;\,|t| \le 1\}$ in $\mathbb{R}^m$.

The remarkable insight of Weyl is that the coefficients $k_d(M)$ are integral invariants of M only determined by the *intrinsic* metric nature of M. For example, the initial coefficient $k_0(M) = \int_M \mathrm{d}s$ is the Riemannian volume of M and the next coefficient is $k_2(M) = \frac{1}{2}\int_M S\,\mathrm{d}s$ with S the scalar curvature of M. If M has empty boundary and is of even dimension it was proved by Allendoerfer and Weil [1] in their approach towards the Gauss–Bonnet theorem that the top coefficient $k_n(M) = (2\pi)^{n/2}\chi(M)$ with $\chi(M)$ the Euler characteristic of M is even of topological nature. See also the text books of Gray on tubes [6] and of Morvan on generalized curvatures [13] for further details.

Due to the *local* nature of the tube formula we can assume that the submanifold M of $\mathbb{R}^{n+m}$ comes with a chosen orthonormal frame in the normal bundle NM of M in $\mathbb{R}^{n+m}$. In turn this gives for all $r \in M$ an identification of the normal space N_rM with $\mathbb{R}^m$, and so for $\mathbb{D}^m$ a compact domain around 0 in $\mathbb{R}^m$ we can consider the *generalized tube*

$$\{r + n\,;\, r \in M,\ n \in a\mathbb{D}^m\}$$

around M of type $a\mathbb{D}^m$ for $a > 0$ sufficiently small. The main result of this paper is that under sufficiently strong (relative to the dimension n of M) symmetry requirements on the domain $\mathbb{D}^m$ a similar intrinsic formula for the volume $V(a)$ of the above generalized tube remains valid as in Weyl's case where $\mathbb{D}^m$ equals the unit ball $\mathbb{B}^m$.

Our generalized tubes share the feature that the domains $\mathbb{D}^m$ are invariant under the following subgroups of the orthogonal group.

Definition 1.1 A subgroup G_m of the orthogonal group $\mathrm{O}_m(\mathbb{R})$ on $\mathbb{R}^m$ is called *orthogonal of degree n* if any polynomial $p(t) \in \mathbb{R}[t]$ on $\mathbb{R}^m$ of degree $\le n$ that is invariant under G_m is in fact invariant under the full orthogonal group $\mathrm{O}_m(\mathbb{R})$.

Our principal result is the following generalized tube formula.

Theorem 1.2 *Let M be a compact connected manifold of dimension n embedded in Euclidean space $\mathbb{R}^{n+m}$. If the compact domain $\mathbb{D}^m$ around 0 in $\mathbb{R}^m$ has a symmetry group G_m inside $\mathrm{O}_m(\mathbb{R})$ that is orthogonal of degree n then the volume of the generalized tube of type $a\mathbb{D}^m$ for $a > 0$ sufficiently small is given by*

$$V_M(a) = \sum_{d=0}^{n} \frac{\{\int_{\mathbb{D}^m} |t|^d\,\mathrm{d}t\}\, k_d(M)\, a^{m+d}}{m(m+2)\cdots(m+d-2)} \qquad (d \text{ even})$$

with intrinsic coefficients $k_d(M) = \int_M H_d\,\mathrm{d}s$ as specified in Theorem 4.1.

In Sects. 2–4 we shall review the proof of the tube formula following Weyl's original approach and along the way obtain variations of the tube volume formula for polyhedral (and related) tubes rather than spherical tubes. We discuss several examples in Sect. 5 and counterexamples in Sect. 6, as well as causal tubes in Minkowski space in Sect. 7.

After this paper was finished we learned that tube volume formulas with more general cross sections $\mathbb{D}^m$ had already been studied before by Domingo-Juan and

Miquel [4]. We decided to leave our paper as it was, but add Sect. 8 in order to briefly survey their approach and compare their results with ours.

2 The Volume of Tubes

Locally a submanifold M of dimension n in $\mathbb{R}^{n+m}$ is given in the Gaussian approach by a parametrization

$$r\colon U^n \to M \subset \mathbb{R}^{n+m}, \quad u \mapsto r(u)$$

with $u = (u^1, \dots, u^n) \in U^n \subset \mathbb{R}^n$ and $\partial_i = \partial/\partial u^i$ in the usual notation of differential geometry. A summation sign $\sum$ without explicit mention of indices always means a summation over all indices, which occur both as upper and as lower index. The first fundamental form (or Riemannian metric) is given by

$$ds^2 = \sum g_{ij} \mathrm{d}u^i \mathrm{d}u^j, \quad g_{ij} = \partial_i r \cdot \partial_j r,$$

with $\cdot$ the scalar product on the ambient Euclidean space $\mathbb{R}^{n+m}$ and $g_{ij} = g_{ij}(u)$ a positive definite symmetric matrix for all $u \in U^n$.

Let us choose an orthonormal frame field $u \mapsto n_1(u), \dots, n_m(u)$ in the normal bundle of M, and so $\partial_i r \cdot n_p = 0$ and $n_p \cdot n_q = \delta_{pq}$ along M for all $i = 1, \dots, n$ and $p, q = 1, \dots, m$. Let $t = (t^1, \dots, t^m)$ be Cartesian coordinates on $\mathbb{R}^m$. Let us be given a compact domain $\mathbb{D}^m$ around 0 in $\mathbb{R}^m$ such that the map

$$x\colon U^n \times \mathbb{D}^m \to \mathbb{R}^{n+m}, \quad (u,t) \mapsto x(u,t) = r(u) + \sum t^p n_p(u)$$

is a diffeomorphism of $U^n \times \mathbb{D}^m$ onto its image in $\mathbb{R}^{n+m}$. This image is called a tube of type $\mathbb{D}^m$ around $r(U^n)$.

We are interested in the Euclidean volume $V_{U^n}(a)$ of the local tube

$$\{x(u,t) = r(u) + \sum t^p n_p(u)\,;\, u \in U^n,\ t \in a\mathbb{D}^m\} \subset \mathbb{R}^{n+m}$$

of type $a\mathbb{D}^m$ as a function of a small positive parameter $a > 0$. By the Jacobi substitution theorem we have

$$V_{U^n}(a) = \int_{U^n} \left\{ \int_{a\mathbb{D}^m} J(u,t)\,\mathrm{d}t \right\} \mathrm{d}u$$

with J the absolute value of the determinant

$$\det(\partial_1 x \ \cdots \ \partial_n x \ n_1 \ \cdots \ n_m)$$

and $\partial_i x = \partial_i r + \sum t^p \partial_i n_p$ for $i = 1, \dots, n$.

Recall that

$$\partial_i \partial_j r = \sum \Gamma_{ij}^k \partial_k r + \sum h_{ij}^p n_p$$

with $\Gamma_{ij}^k = \Gamma_{ij}^k(u)$ the Christoffel symbols and $h_{ij}^p = h_{ij}^p(u)$ the coefficients of the second fundamental form h_{ij} relative to the orthonormal normal frame n_p. Here indices $p, q = 1, \ldots, m$ are coordinate indices in the normal direction, while the other indices $i, j, k = 1, \ldots, n$ are coordinate indices on the submanifold M. Since $\partial_j r \cdot n_p = 0$ we get

$$\partial_i n_p \cdot \partial_j r = -n_p \cdot \partial_i \partial_j r = -\sum \delta_{pq} h_{ij}^q$$

with δ_{pq} the Kronecker symbol. Writing $t_p = \sum \delta_{pq} t^q$ we find

$$\partial_i x = \partial_i r - \sum \delta_{pq} t^p h_{ik}^q g^{kj} \partial_j r + \ldots = \sum (\delta_i^j - \sum t_p h_{ik}^p g^{kj}) \partial_j r + \ldots$$

with $g_{ij} = \partial_i r \cdot \partial_j r$, g^{ij} its inverse matrix, $\det g = \det(g_{ij})$ its determinant and $\ldots$ stands for a linear combination of the normal fields n_p. If in the usual notation we write $h_i^{jp} = \sum g^{jk} h_{ik}^p$ for $i, j = 1, \ldots, n$ and $p = 1, \ldots, m$ then we get

$$\begin{aligned} &\det(\partial_1 x \ \cdots \ \partial_n x \ n_1 \ \cdots \ n_m) \\ &= \det(\delta_i^j - \sum t_p h_i^{jp}) \det(\partial_1 r \ \cdots \ \partial_n r \ n_1 \ \cdots \ n_m) \end{aligned}$$

which in turn implies

$$V_{U^n}(a) = \int_{U^n} \left\{ \int_{a\mathbb{D}^m} \det(\delta_i^j - \sum t_p h_i^{jp}) \, \mathrm{d}t \right\} \sqrt{\det g_{ij}} \, \mathrm{d}u$$

for all $a > 0$ sufficiently small.

For fixed $u \in U^n$ the integrand $\det(\delta_i^j - \sum t_p h_i^{jp})$ is a polynomial in t of degree n and so, after integration and patching together the locally defined tubes, we conclude that

$$V_M(a) = \sum_{d=0}^{n} v_d \, a^{m+d}$$

is a polynomial in a of degree $m + n$ and with coefficient $v_0 = \mathrm{vol}(\mathbb{D}^m)\mathrm{vol}(M)$. If the domain $\mathbb{D}^m$ is centrally symmetric with respect the origin, that is if $-\mathbb{D}^m = \mathbb{D}^m$, then the integrals of odd degree monomials in t over $\mathbb{D}^m$ vanish and so $v_d = 0$ for d odd.

In order to show that the volume $V_M(a)$ of a generalized tube of type $\mathbb{D}^m$ depends only on intrinsic quantities of M, two steps are necessary. Firstly, by assuming that M is embedded in flat $\mathbb{R}^{n+m}$, one observes that certain combinations of the second fundamental forms are intrinsic curvature quantities (this was already done by Weyl). Secondly, by imposing certain symmetry conditions on $\mathbb{D}^m$, we show that only those intrinsic combinations remain in the volume formula $V_M(a)$ for the generalized tube (done by Weyl for the ball $\mathbb{B}^m$). These steps are carried out in Sects. 3 and 4, respectively.

3 The Gauss Equations

As before, we write

$$\partial_i \partial_j r = \sum \Gamma_{ij}^k \partial_k r + h_{ij}$$

with Γ_{ij}^k the Christoffel symbols given by

$$\tfrac{1}{2} \sum g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$$

and $h_{ij} = \sum h_{ij}^p n_p$ the second fundamental form relative to the orthonormal frame n_p in the normal bundle along M. Given scalar functions g_{ij} and h_{ij}^p, the integrability conditions for the existence of an embedding of M into flat Euclidean space with these functions as coefficients of the first and second fundamental forms are given by

$$\partial_i \Big(\sum \Gamma_{jk}^l \partial_l r + \sum h_{jk}^p n_p \Big) - \partial_j \Big(\sum \Gamma_{ik}^l \partial_l r + \sum h_{ik}^p n_p \Big) = 0$$

for all i, j, k (by working out $\partial_i (\partial_j \partial_k r) - \partial_j (\partial_i \partial_k r) = 0$).

In the normal directions this leads to the Codazzi–Mainardi equations

$$\partial_i h_{jk}^p - \partial_j h_{ik}^p + \sum (\Gamma_{jk}^l h_{il}^p - \Gamma_{ik}^l h_{jl}^p) = 0$$

for all i, j, k and all p. In the tangential directions this amounts to the Gauss equations

$$R_{kij}^l = \sum \delta_{pq} g^{ln} (h_{in}^p h_{jk}^q - h_{jn}^p h_{ik}^q)$$

for all i, j, k, l with

$$R_{kij}^l = \partial_i \Gamma_{kj}^l - \partial_j \Gamma_{ki}^l + \sum (\Gamma_{kj}^m \Gamma_{mi}^l - \Gamma_{ki}^m \Gamma_{mj}^l)$$

the coefficients of the Riemann curvature tensor. As mentioned earlier, by raising indices $h_i^{jp} = \sum g^{jk} h_{ki}^p$ and $R_{ij}^{kl} = \sum g^{ln} R_{nij}^k$ the Gauss equations take the form

$$R_{ij}^{kl} = \sum \delta_{pq} (h_i^{kp} h_j^{lq} - h_j^{kp} h_i^{lq}) = h_i^k \cdot h_j^l - h_j^k \cdot h_i^l$$

for all i, j, k, l and $h_i^j = \sum h_i^{jp} n_p = \sum g^{jk} h_{ki}$ normal vectors along M.

4 Averaging the Integrand

For $p(t) \in \mathbb{R}[t_1, \cdots, t_m]$ a polynomial on Euclidean space $\mathbb{R}^m$ and G_m a closed subgroup of the orthogonal group $\mathrm{O}_m(\mathbb{R})$ let us write

$$\langle p(t) \rangle_{G_m} = \int_{G_m} p(gt)\,\mathrm{d}\mu(g)$$

for the average of p over G_m, with μ the normalized Haar measure on G_m. Clearly

$$\langle p(t) \rangle_{\mathrm{O}_m(\mathbb{R})} \in \mathbb{R}[t \cdot t]$$

with $t \cdot t = |t|^2$ the norm squared of $t \in \mathbb{R}^m$. The crucial step for the intrinsic nature of the coefficients of the tube volume formula is the following result (see the Lemma on page 470 of Weyl's paper [15]).

Theorem 4.1 *We have (with $1 \le i, j \le n$ and $1 \le p \le m$)*

$$\left\langle \det\left(\delta_i^j - \sum t_p h_i^{jp}\right)\right\rangle_{\mathrm{O}_m(\mathbb{R})} = \sum_{d=0}^{n} \frac{H_d\,|t|^d}{m(m+2)\cdots(m+d-2)}$$

with H_d intrinsic functions on M given by

$$H_d = \begin{cases} 0 & \text{if } d \text{ odd}, \\ 1 & \text{if } d = 0, \\ \sum \varepsilon_{i_1 \ldots i_d}^{j_1 \ldots j_d}\, R_{i_1 i_2}^{j_1 j_2} \cdots R_{i_{d-1} i_d}^{j_{d-1} j_d} & \text{if } d > 0 \text{ even}, \end{cases}$$

and

$$R_{ij}^{kl} = H_{ij}^{kl} - H_{ji}^{kl} \quad , \quad H_{ij}^{kl} = h_i^k \cdot h_j^l = \sum \delta_{pq} h_i^{kp} h_j^{lq}$$

for $i, j, k, l = 1, \ldots, n$. In the expression for H_d with $d > 0$ even the sum runs over all cardinality d subsets $\mathcal{D}$ of $\{1, \ldots, n\}$ and over all possible couplings of pairs

$$\begin{matrix} j_1 j_2 \\ i_1 i_2 \end{matrix} \left| \begin{matrix} j_3 j_4 \\ i_3 i_4 \end{matrix} \right| \cdots \left| \begin{matrix} j_{d-1} j_d \\ i_{d-1} i_d \end{matrix} \right|$$

taken from $\mathcal{D} = \{i_1, \ldots, i_d\} = \{j_1, \ldots, j_d\}$. Here a pair $i_1 i_2$ means two distinct numbers i_1, i_2 irrespective of their order.

Proof Averaging the characteristic polynomial $\det(\lambda\delta_i^j - \sum t_p h_i^{jp})$ over the orthogonal group $\mathrm{O}_m(\mathbb{R})$ acting on $t \in \mathbb{R}^m$ yields

$$\left\langle \det\left(\lambda\delta_i^j - \sum t_p h_i^{jp}\right)\right\rangle_{\mathrm{O}_m(\mathbb{R})} = \sum_{d=0}^{n} \lambda^{n-d}\,|t|^d \sum_{\mathcal{D}} A_{\mathcal{D}}(h_i^j)$$

with the sum over all even d and all cardinality d subsets $\mathcal{D}$ of $\{1, \ldots, n\}$ and with $A_{\mathcal{D}}(h_i^j)$ given by

$$\left\langle \det(t \cdot h_i^j)_{i,j\in\mathcal{D}} \right\rangle_{\mathrm{O}_m(\mathbb{R})} = |t|^d A_{\mathcal{D}}(h_i^j)$$

as degree d polynomial on $\mathbb{R}^{m\times d^2}$, which is invariant under the diagonal action of $\mathrm{O}_m(\mathbb{R})$. By the first fundamental theorem of invariant theory for $\mathrm{O}_m(\mathbb{R})$ (see Corollary 4.2.3 of [5], which is a modern reincarnation of Weyl's classic [16]) we have

$$A_{\mathcal{D}}(h_i^j) = B_{\mathcal{D}}(H_{ij}^{kl})$$

with $H_{ij}^{kl} = h_i^k \cdot h_j^l$ and $i, j, k, l \in \mathcal{D}$ with $i \neq j, k \neq l$. From the explicit determinantal form it follows that $B_{\mathcal{D}}(H_{ij}^{kl})$ is in fact a linear combination of monomials of the form

$$H_{i_1 i_2}^{j_1 j_2} \cdots H_{i_{d-1} i_d}^{j_{d-1} j_d}$$

with $\mathcal{D} = \{i_1, \ldots, i_d\} = \{j_1, \ldots, j_d\}$. Moreover under the action of the symmetric group $\mathfrak{S}_d$ acting on both the lower and the upper indices $B_{\mathcal{D}}(H_{ij}^{kl})$ transforms under the sign character. Therefore $B_{\mathcal{D}}(H_{ij}^{kl}) = C_{\mathcal{D}}(R_{ij}^{kl})$ with $R_{ij}^{kl} = H_{ij}^{kl} - H_{ji}^{kl}$ and by symmetry for $\mathfrak{S}_d$ we arrive at

$$C_{\mathcal{D}}(R_{ij}^{kl}) = c(m, d) \sum \varepsilon_{i_1 \ldots i_d}^{j_1 \ldots j_d} \; R_{i_1 i_2}^{j_1 j_2} \cdots R_{i_{d-1} i_d}^{j_{d-1} j_d}$$

with the sum over all possible couplings of pairs from $\mathcal{D}$ and $c(m, d)$ a constant depending solely on m and d. The conclusion is that

$$\left\langle \det \left(\delta_i^j - \sum t_p h_i^{jp} \right) \right\rangle_{\mathrm{O}_m(\mathbb{R})} = \sum_{d=0}^{n} c(m, d) H_d \, |t|^d$$

and all that is left is the computation of the constant $c(m, d)$.

For this computation we take the special choice $h_i^{jp} = \delta_i^j$ for $p = 1$ and $h_i^{jp} = 0$ for $p \geq 2$. In that case

$$A_{\mathcal{D}}(h_i^j) = \frac{\int_{S^{m-1}} t_1^d \, \mathrm{d}\mu(t)}{\int_{S^{m-1}} \mathrm{d}\mu(t)}$$

with μ the Euclidean measure on the unit sphere S^{m-1} in $\mathbb{R}^m$. The integral in the numerator becomes

$$\int_{-1}^{1} r^d (1 - r^2)^{(m-3)/2} \mathrm{d}r = \int_0^1 s^{(d-1)/2} (1 - s)^{(m-3)/2} \mathrm{d}s$$

(apart from a factor volume ω_{m-1} of S^{m-2}) and so equals

$$\mathrm{B}((d+1)/2,(m-1)/2)=\frac{\Gamma((d+1)/2)\Gamma((m-1)/2)}{\Gamma((d+m)/2)}\,.$$

In turn this implies

$$A_{\mathcal{D}}(h_i^j)=\frac{\Gamma((d+1)/2)\Gamma(m/2)}{\Gamma(1/2)\Gamma((d+m)/2)}=\frac{1\cdot 3\cdots(d-1)}{m(m+2)\cdots(m+d-2)}\,.$$

On the other hand $R_{ij}^{kl}=\delta_i^k\delta_j^l-\delta_j^k\delta_i^l$ and so equal to ε_{ij}^{kl} if the pairs ij and kl coincide and 0 otherwise, and hence

$$C_{\mathcal{D}}(R_{ij}^{kl})=c(m,d)\,\frac{d!}{2^{d/2}(d/2)!}=c(m,d)\cdot 1\cdot 3\cdots(d-1)\,.$$

Hence $c(m,d)=1/m(m+2)\cdots(m+d-2)$ as desired. □

Recall that the volume ω_m of the unit sphere S^{m-1} and the volume Ω_m of the unit ball $\mathbb{B}^m$ are related by $\Omega_m=\omega_m/m$. The tube formula of Weyl can now be easily derived.

Corollary 4.2 *If the domain $\mathbb{D}^m$ in $\mathbb{R}^m$ is equal to the unit ball $\mathbb{B}^m$ then the tube volume is given by*

$$V_M(a)=\Omega_m\sum_{d=0}^{n}\frac{k_d(M)\,a^{m+d}}{(m+2)\cdots(m+d)}\qquad(d\text{ even})$$

for $a>0$ small and $k_d(M)=\int_M H_d\,\mathrm{d}s$ with H_d the intrinsic expression on M in the previous theorem and $\mathrm{d}s$ the Riemannian measure on M.

Proof By Sect. 2 and the symmetry of $\mathbb{B}^m$ we have

$$\begin{aligned}V_{U^n}(a)&=\int_{U^n}\left\{\int_{a\mathbb{B}^m}\det(\delta_i^j-\sum t_p h_i^{jp})\,\mathrm{d}t\right\}\sqrt{\det g_{ij}}\,\mathrm{d}u\\&=\int_{U^n}\left\{\int_{a\mathbb{B}^m}\sum_{d=0}^{n}\frac{H_d\,|t|^d}{m(m+2)\cdots(m+d-2)}\,\mathrm{d}t\right\}\sqrt{\det g_{ij}}\,\mathrm{d}u\\&=\int_{U^n}\left\{\omega_m\int_0^a\sum_{d=0}^{n}\frac{H_d\,r^{m+d-1}}{m(m+2)\cdots(m+d-2)}\,\mathrm{d}r\right\}\sqrt{\det g_{ij}}\,\mathrm{d}u\\&=\Omega_m\sum_{d=0}^{n}\frac{\{\int_{U^n}H_d\,\mathrm{d}s\}\,a^{m+d}}{(m+2)\cdots(m+d-2)(m+d)}\end{aligned}$$

and the result follows. □

If we consider domains $\mathbb{D}^m$ with symmetry groups $G_m < \mathrm{O}_m(\mathbb{R})$ such that the invariant polynomials of degree $\leq n = \dim M$ for both groups agree, then we can prove Theorem 1.2.

Proof By the Fubini theorem we have for $H < G$ compact groups and f a continuous function on G that

$$\int_G f(g)\,\mathrm{d}\mu_G(g) = \int_{G/H} \left\{ \int_H f(gh)\,\mathrm{d}\mu_H(h) \right\} \mathrm{d}\mu_{G/H}(gH)$$

with μ_G, μ_H and $\mu_{G/H}$ the normalized invariant measures on G, H and G/H respectively. Hence by the assumption on G_m we have

$$\left\langle \det\left(\delta_i^j - \sum t_p h_i^{jp}\right)\right\rangle_{G_m} = \left\langle \det\left(\delta_i^j - \sum t_p h_i^{jp}\right)\right\rangle_{\mathrm{O}_m(\mathbb{R})}$$

and so we can just argue as in the previous proof. □

Using the discussion in Sect. 2, for $n = 1$ the tube formula is intrinsic as long as $\mathbb{D}^m$ is centrally symmetric, the case already covered by Hotelling [10] if $\mathbb{D}^m = \mathbb{B}^m$.

Corollary 4.3 *If M is a curve of finite length in $\mathbb{R}^{m+1}$ and $\mathbb{D}^m$ is centrally symmetric, then for $a > 0$ sufficiently small*

$$V_M(a) = \mathrm{length}(M)\,\mathrm{vol}(\mathbb{D}^m)\,a^m$$

and hence is intrinsic. □

The following example shows that central symmetry is not a necessary condition.

Example 4.4 Let $\mathbb{D}^m$ be the union of half the unit ball $\{|t| \leq 1, t_1 \leq 0\}$ and the cone $\{t_2^2 + \cdots + t_m^2 \leq (1 - t_1/b)^2, 0 \leq t_1 \leq b\}$ with top $(b, 0, \ldots, 0)$ for some $b > 0$. By symmetry the average of any linear function of $t_2, \ldots, t_m$ over $\mathbb{D}^m$ equals zero. By direct computation the average of t_1 over $\mathbb{D}^m$ is equal to zero if $b = \sqrt{m}$, and so the previous corollary remains valid for this domain as well.

Indeed for curves it is sufficient that the center of mass of $\mathbb{D}^m$ is at the origin by the generalized Pappus centroid theorem. See Sect. 8 for the higher dimensional case.

5 Examples of Polyhedral Domains $\mathbb{D}^m$

If we are looking for domains $\mathbb{D}^m$ in $\mathbb{R}^m$ with a sufficiently large symmetry group $G_m < \mathrm{O}_m(\mathbb{R})$ it is natural to consider regular polytopes $\mathbb{D}^m$ in $\mathbb{R}^m$. It is well known that the symmetry group G_m in that case is an irreducible finite reflection group. Such groups are classified by their Coxeter diagrams or by letters X_m with $\mathrm{X} = \mathrm{A}, \mathrm{B}, \mathrm{D}, \mathrm{E}, \mathrm{F}, \mathrm{H}, \mathrm{I}(k)$ for $k \geq 5$. The corresponding reflection groups are denoted by $G_m = W(\mathrm{X}_m)$.

It is a well known theorem due to Shephard and Todd [14] (with a case by case proof) and Chevalley [3] (with a proof from the Book) that the algebra of polynomial invariants for a finite reflection group $W < \mathrm{O}(\mathbb{R}^m)$ is itself a polynomial algebra.

Table 1 Degrees of the generating polynomial invariants for the irreducible types

type	m	$d_1, d_2, \ldots, d_m$
A_m	≥ 1	$2, 3, \ldots, m+1$
B_m	≥ 2	$2, 4, \ldots, 2m$
D_m	≥ 4	$2, 4, \ldots, 2m-2, m$
E_6	6	2, 5, 6, 8, 9, 12
E_7	7	2, 6, 8, 10, 12, 14, 18
E_8	8	2, 8, 12, 14, 18, 20, 24, 30
F_4	4	2, 6, 8, 12
H_3	3	2, 6, 10
H_4	4	2, 12, 20, 30
$\mathrm{I}_2(k)$	2	$2, k \geq 5$

Theorem 5.1 *The algebra $\mathbb{R}[\mathbb{R}^m]^W$ of polynomial invariants for W is of the form $\mathbb{R}[p_1, \ldots, p_m]$ with $p_1, \ldots, p_m$ algebraically independent homogeneous invariants of degrees $d_1, d_2, \ldots d_m$ respectively.*

For each of the irreducible types these degrees can be calculated and are given in Table 1. The proof of these results can be found in the standard text books by Bourbaki [2] or by Humphreys [11].

So for $m \geq 2$ the irreducible finite reflection group $W(\mathrm{X}_m) < \mathrm{O}(\mathbb{R}^m)$ is orthogonal of degree $d_2 - 1$ in the sense of Definition 1.1.

Corollary 5.2 *If $\mathbb{D}^m$ is a domain in $\mathbb{R}^m$ invariant under a finite reflection group $W(\mathrm{X}_m)$ then the tube formula of Theorem 1.2 does hold with intrinsic coefficients if $n = \dim M < d_2$, that is, the dimension n of M is strictly smaller than the second fundamental degree d_2.* □

For example, if $\mathbb{D}^3$ is an icosahedron with symmetry group $W(\mathrm{H}_3)$ then the tube formula is intrinsic for submanifolds M of dimension $n \leq 5$ in $\mathbb{R}^{n+3}$, and if $\mathbb{D}^4$ is a 600-cell with symmetry group $W(\mathrm{H}_4)$ then the tube formula is intrinsic for $n \leq 11$. For any dimension n of $M \hookrightarrow \mathbb{R}^{n+2}$ with $\mathbb{D}^2$ a regular k-gon with $k > n$ the tube formula is intrinsic, since its symmetry group is $W(\mathrm{I}_2(k))$. For dimension $n = 2$ or 3 we find in this way examples of intrinsic tube formulas for arbitrary codimension m via domains $\mathbb{D}^m$ with symmetry groups $W(\mathrm{A}_m)$ $(n = 2)$ and $W(\mathrm{B}_m)$ $(n = 2, 3)$, respectively. However, for dimension $n \geq 4$ we obtain in this way only examples of intrinsic tube formulas with relatively small codimension $m \leq 8$.

Examples with larger codimension m can be obtained by the following construction.

Corollary 5.3 *Let G be a noncompact simple Lie group acting on its Lie algebra $\mathfrak{g}$, and let θ be a Cartan involution of G and $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the decomposition in $+1$ and -1 eigenspaces of θ on $\mathfrak{g}$. If the domain $\mathbb{D} \subset \mathfrak{p}$ is the convex hull of a nonzero orbit of $K = G^\theta$ on $\mathfrak{p}$ then the tube formula of Theorem 1.2 does hold with intrinsic coefficients under the assumption that the dimension n of $M \hookrightarrow \mathbb{R}^{n+m}$ (with $m = \dim \mathfrak{p}$) is strictly smaller than the second fundamental degree d_2 of the Weyl group W of the pair $(\mathfrak{g}, \theta)$.*

Proof The Killing form $(\cdot, \cdot)$ on $\mathfrak{p}$ is positive definite and the fixed point group $K = G^\theta$ of θ on G acts on $\mathfrak{p}$ as a subgroup of $\mathrm{SO}(\mathfrak{p})$. If $\mathfrak{a} \subset \mathfrak{p}$ is a maximal Abelian subspace then each orbit of K on $\mathfrak{p}$ intersects $\mathfrak{a}$ in an orbit of the Weyl group $W = \mathrm{N}_K(\mathfrak{a})/\mathrm{Z}_K(\mathfrak{a})$ of the pair $(\mathfrak{g}, \theta)$. Hence each invariant polynomial $p \in \mathbb{R}[\mathfrak{p}]^K$ for K on $\mathfrak{p}$ restricts to a Weyl group invariant polynomial on $\mathfrak{a}$. It is a theorem of Chevalley (see Lemma 7 in [7]) that the restriction map

$$\mathbb{R}[\mathfrak{p}]^K \to \mathbb{R}[\mathfrak{a}]^W$$

is an isomorphism of algebras. Since W acts on $\mathfrak{a}$ as a finite reflection group the latter algebra is described by Theorem 5.1. The possible finite reflection groups that can occur as such a Weyl group W are those reflection groups, which can be defined over $\mathbb{Z}$. This means that H_3 and H_4 are excluded and only the dihedral types $\mathrm{I}_2(k) = \mathrm{A}_2, \mathrm{B}_2, \mathrm{G}_2$ for $k = 3, 4, 6$ respectively are allowed. The text books [8,9] by Helgason give a thorough exposition of the theory. Using the convexity theorem of Kostant [12] it is easy to see that the convex hull of an orbit of K on $\mathfrak{p}$ intersects $\mathfrak{a}$ in the convex hull of an orbit of W on $\mathfrak{a}$. □

For example, if G is the complex Lie group of type E_8 (and so K is the compact Lie group of type E_8 acting on $\mathfrak{p} = i\mathfrak{k}$) then we do find in this way examples of local submanifolds M of Euclidean space of dimension $n \leq 7$ and of codimension $m = 248$ for which the tube formula of Theorem 1.2 has intrinsic coefficients. Presumably this large codimension relative to the small dimension of M allows for an abundance of room for isometric deformations for the embedding of M in such a Euclidean space.

6 No-Go Results for Diamond Domains $\widehat{\mathbb{D}}^m$

In this section we shall denote by $\widehat{\mathbb{D}}^m$ the convex hull of the subset

$$\{t_1^2 + \ldots + t_{m-1}^2 \leq 1, t_m = 0\} \sqcup \{(0, \ldots, 0, \pm 1)\}$$

in $\mathbb{R}^m$, $m \geq 2$. Any multiple $a\widehat{\mathbb{D}}^m$ for $a > 0$ will be called a diamond domain. The symmetry group G_m of $\widehat{\mathbb{D}}^m$ is equal to $\mathrm{O}_{m-1}(\mathbb{R}) \times \mathrm{O}_1(\mathbb{R})$ for $m \geq 3$ while for $m = 2$ the symmetry group G_2 is equal to the dihedral group $W(\mathrm{B}_2)$ of order 8. The essential point of Weyl's argument for the intrinsic nature of the volume formula for tubes is the computation of the integral

$$\int_{a\widehat{\mathbb{D}}^m} \det\left(\delta_i^j - \sum t_p h_i^{jp}\right) \mathrm{d}t$$

as a polynomial in a, by first averaging over the symmetry group G_m of the domain $\widehat{\mathbb{D}}^m$ in $\mathbb{R}^m$. The outcome should hopefully be a polynomial expression in Riemann curvature components R_{ij}^{kl} as in Theorem 4.1. We will work out two examples, one with $n = 2$ and $m \geq 3$ the other with $n = 4$ and $m = 2$, where this does not work.

Example 6.1 Let us first consider the case of surfaces of codimension m at least 3. Since the symmetry group of $\widehat{\mathbb{D}}^m$ is then $G_m = \mathrm{O}_{m-1}(\mathbb{R}) \times \mathrm{O}_1(\mathbb{R})$ the invariants of degree 2 in $\mathbb{R}[t_1, \dots, t_m]$ are linear combinations of $R = (t_1^2 + \dots + t_{m-1}^2)/(m-1)$ and $S = t_m^2$. The above determinant for $n = 2$ becomes

$$\det\left(\delta_i^j - \sum t_p h_i^{jp}\right) = 1 - \sum t_p\left(h_1^{1p} + h_2^{2p}\right) + \sum t_p t_q\left(h_1^{1p} h_2^{2q} - h_1^{2p} h_2^{1q}\right)$$

and averaging over the symmetry group G_m yields

$$1 + A(h_i^j)R(t) + B(h_i^j)S(t)$$

with

$$A = \sum_{p=1}^{m-1}\left(h_1^{1p} h_2^{2p} - h_1^{2p} h_2^{1p}\right) \qquad B = \left(h_1^{1m} h_2^{2m} - h_1^{2m} h_2^{1m}\right)$$

and $A + B = R_{12}^{12}$ intrinsic. Thus by the above, if the integrals of R and S over $\widehat{\mathbb{D}}^m$ agree, then the generalized tube volume is intrinsic as well. The integrals of $R(t)$ and $S(t)$ over $\widehat{\mathbb{D}}^m$ amount respectively to (put $r = \sqrt{R}$ and $s = \sqrt{S}$)

$$\frac{2\,\omega_{m-1}}{m-1}\int r^2 r^{m-2}\,\mathrm{d}r\,\mathrm{d}s \quad \text{and} \quad 2\,\omega_{m-1}\int s^2 r^{m-2}\,\mathrm{d}r\,\mathrm{d}s,$$

integrated over the triangle $\{(r, s)\,;\, r, s \geq 0, r + s \leq 1\}$, and we will show that for $m \geq 3$ these are distinct. Apart from the factor $2\,\omega_{m-1}$ the left integral becomes

$$\frac{1}{m-1}\int_0^1 (1-r)r^m\,\mathrm{d}r = \frac{1}{(m-1)(m+1)(m+2)}$$

while the right integral equals

$$\int_0^1 \tfrac{1}{3}(1-r)^3 r^{m-2}\,\mathrm{d}r = \frac{2}{(m-1)m(m+1)(m+2)}$$

and for the difference we find

$$\frac{m-2}{(m-1)m(m+1)(m+2)}$$

which is nonzero for $m \geq 3$, as claimed. Hence the tube volume formula for a general surface M in $\mathbb{R}^{2+m}$ with diamond domain $\widehat{\mathbb{D}}^m$ is no longer intrinsic for $m \geq 3$. For $m = 2$ it still is intrinsic as should, because $G_2 = W(\mathrm{B}_2)$ is orthogonal of degree 3 (in fact, it is not only intrinsic for $n = 2$ but also for $n = 3$ since odd exponents vanish for the centrally symmetric diamond $\widehat{\mathbb{D}}^m$).

Example 6.2 Let us next consider the case that $n = 4$ and $m = 2$. The symmetry group of the diamond domain $\widehat{\mathbb{D}}^2 = \{(t_1, t_2)\,;\, |t_1| + |t_2| \le 1\}$ is the dihedral group $W(\mathrm{B}_2)$ of order 8 generated by the two reflections $s_1(t_1, t_2) = (-t_1, t_2)$ and $s_2(t_1, t_2) = (t_2, t_1)$. The invariant polynomials for this group $W(\mathrm{B}_2)$ are generated as an algebra by the quadratic invariant $P(t) = t_1^2 + t_2^2$ and the quartic invariant $Q(t) = t_1^2 t_2^2$. Hence any quartic invariant is a unique linear combination of Q and $R(t) = t_1^4 + t_2^4 = P^2 - 2Q$.

We would like to know if Weyl's averaging trick (over the dihedral group $W(\mathrm{B}_2)$ this time) remains valid for any pencil of second fundamental forms. In order to keep the calculation as simple as possible we look at the special case that $h_i^{jp} = 0$ for $i \neq j$ and $p = 1, 2$. If we write $h_i^{i1} = a_i$ and $h_i^{i2} = b_i$ we get

$$\det\left(\delta_i^j - \sum t_p h_i^{jp}\right) = \prod_{i=1}^{4} (1 - t_1 a_i - t_2 b_i)$$

and averaging over the dihedral group $W(\mathrm{B}_2)$ yields

$$1 + A(a, b) P(t) + B(a, b) Q(t) + C(a, b) R(t)$$

with A, B, C homogeneous polynomials of degree 2, 4, 4 respectively. A direct calculation gives

$$\begin{aligned} A &= \tfrac{1}{2} \sum_{i<j} (a_i a_j + b_i b_j) = \tfrac{1}{2} \sum_{i<j} R_{ij}^{ij} \\ B &= \sum_{i<j,k<l} a_i a_j b_k b_l = \sum_{i<j,k<l} R_{ij}^{ij} R_{kl}^{kl} - 3(a_1 a_2 a_3 a_4 + b_1 b_2 b_3 b_4) \\ C &= \tfrac{1}{2}(a_1 a_2 a_3 a_4 + b_1 b_2 b_3 b_4) \end{aligned}$$

with $\{i, j, k, l\} = \{1, 2, 3, 4\}$ in the sum for B and $R_{ij}^{ij} = h_i^i \cdot h_j^j$ as before. Note that A as well as $B + 6C$ are intrinsic quantities. For the integrals of Q and R over $\widehat{\mathbb{D}}^2$ we find (put $r = |t_1|$ and $s = |t_2|$)

$$\int Q(t)\, \mathrm{d}t_1\, \mathrm{d}t_2 = 4 \int_0^1 \left\{ \int_0^{1-r} r^2 s^2\, \mathrm{d}s \right\} \mathrm{d}r = \tfrac{1}{45}$$

and

$$\int R(t)\, \mathrm{d}t_1\, \mathrm{d}t_2 = 4 \int_0^1 \left\{ \int_0^{1-r} (r^4 + s^4)\, \mathrm{d}s \right\} \mathrm{d}r = \tfrac{4}{15} \neq \tfrac{6}{45}.$$

Hence for fourfolds in $\mathbb{R}^6$ with diamond domain $\widehat{\mathbb{D}}^2$ we see that the tube volume formula need no longer be intrinsic.

Table 2 Admissible dimensions and codimensions for intrinsic tube formulas

$m = \text{codim } M$	Symmetry group of the diamond in $\mathbb{R}^m$	$n = \dim M$
1	$\mathrm{O}_1(\mathbb{R})$	any
2	$W(\mathrm{B}_2)$	≤ 3
any	$\mathrm{O}_{m-1}(\mathbb{R}) \times \mathrm{O}_1(\mathbb{R})$	1

The conclusion therefore is that the tube formula for submanifolds M in $\mathbb{R}^{n+m}$ of dimension n with cross section the diamond $\widehat{\mathbb{D}}^m$ will in general no longer be intrinsic, unless we are in one of the cases of Table 2.

Our motivation for looking at diamond tubes in a Euclidean vector space came from the analogous causal tubes in a Lorentzian vector space, which are discussed in the next section.

7 Riemannian Submanifolds of a Lorentzian Vector Space

Let us suppose that M is a compact connected n-dimensional Riemannian submanifold of an ambient Cartesian space $\mathbb{R}^{n+m}$, equipped with a nondegenerate but possibly indefinite scalar product denoted by a dot. Let $\mathbb{D}^m$ be a compact domain around 0 in $\mathbb{R}^m$. Say we have a local parametrization around M given by

$$x : U^n \times \mathbb{D}^m \to \mathbb{R}^{n+m}, \quad (u, t) \mapsto x(u, t) = r(u) + \sum t^p n_p(u)$$

with $u = (u^1, \ldots, u^n) \in U^n$, $t = (t^1, \ldots, t^m) \in \mathbb{D}^m$ while $n_1(u), \ldots, n_m(u)$ are vectors in $\mathbb{R}^{n+m}$ depending smoothly on $u \in U^n$ and

$$\partial_i r(u) \cdot n_p(u) = 0, \quad n_p(u) \cdot n_q(u) = \eta_{pq}$$

for all $u \in U^n$, all $i = 1, \ldots, n$, all $p, q = 1, \ldots, m$ and η_{pq} a $m \times m$ diagonal matrix with entries ± 1 (so in particular constant, that is independent of $u \in U^n$). Observe that the choice of such an orthonormal frame for the normal bundle of M in $\mathbb{R}^{n+m}$ is in principle only possible locally. Indeed if $0 \in U^n$ then by linear algebra we can choose a basis $n_1(0), \ldots, n_m(0)$ for the orthogonal complement of the tangent vectors $\partial_1 r(0), \ldots, \partial_n r(0)$ with $n_p(0) \cdot n_q(0) = \eta_{pq}$ and subsequently apply Gram–Schmidt to the vectors $\partial_1 r(u), \ldots, \partial_n r(u), n_1(0), \ldots, n_m(0)$ for u small.

As in Sect. 2 we can write

$$\partial_i x = \sum_j (\delta_i^j - \sum t^p n_p \cdot h_i^j) \partial_j r + \ldots$$

with second fundamental form normal vectors $h_i^j = \sum g^{jk} h_{ik}$ and the dots $\ldots$ stand for a linear combination of the normal fields n_p. Likewise writing $t_p = \sum \eta_{pq} t^q$ we arrive at the generalized tube volume formula

$$V_{U^n}(a) = \int_{U^n} \left\{ \int_{a\mathbb{D}^m} \det\left(\delta_i^j - \sum t_p h_i^{jp}\right) \mathrm{d}t \right\} \sqrt{\det g_{ij}}\, \mathrm{d}u$$

with $\mathbb{D}^m$ a compact domain around 0 and $a > 0$ sufficiently small. Hence $V_M(a)$ is a polynomial in a of degree $m + n$ with $\mathrm{vol}(M)\mathrm{vol}(\mathbb{D}^m)a^m$ as lowest order term. The Gauss equations

$$R_{ij}^{kl} = h_i^k \cdot h_j^l - h_j^k \cdot h_i^l$$

as derived in Sect. 3 remain valid for an indefinite scalar product.

For Riemannian curves M of dimension $n = 1$ and a centrally symmetric domain $\mathbb{D}^m$ around 0 in $\mathbb{R}^m$ we get

$$V_M(a) = \mathrm{length}(M)\mathrm{vol}(\mathbb{D}^m)a^m$$

just like the original case of Hotelling [10]. Also, if $\eta_{pq} = \delta_{pq}$ then we are essentially in the original setting of Weyl and his spherical tube formula and our variations hold without change.

Let us suppose for the rest of this section that M is a compact Riemannian submanifold of a Lorentzian vector space $\mathbb{R}^{n+m-1,1}$ with scalar product $\cdot$ of signature $(n + m - 1, 1)$ and thus $\eta_{pq} = \mathrm{diag}(1, \ldots, 1, -1)$ in $\mathbb{R}^m$. If we denote by $\mathbb{J} = \{x \in \mathbb{R}^{n+m-1,1}\,;\, x \cdot x \leq 0\}$ the causal future and past of the origin then for e a unit timelike vector the domain $\widehat{\mathbb{D}}^{n+m}(e) = \{e + \mathbb{J}\} \cap \{-e + \mathbb{J}\}$ is called the *causal diamond* around 0 with unit timelike normal e. It is the locus traced out by all causal curves between e and $-e$. Any two causal diamonds around 0 can be transformed into each other by an element of the Lorentz group $\mathrm{O}_{n+m-1,1}(\mathbb{R})$, while the symmetry group of a causal diamond is isomorphic to $\mathrm{O}_{n+m-1}(\mathbb{R}) \times \mathrm{O}_1(\mathbb{R})$. The set

$$\{r + n\,;\, r \in M,\ n \in N_r M \cap a\widehat{\mathbb{D}}^{n+m}(n_m(r))\}$$

will be called the *causal tube* with radius $a > 0$ (sufficiently small) around M relative to the unit timelike normal field n_m. Its volume is given by

$$V_M(a) = \int_M \left\{ \int_{a\widehat{\mathbb{D}}^m} \det(\delta_i^j - \sum t_p h_i^{jp})\, \mathrm{d}t \right\} \mathrm{d}s$$

with $\widehat{\mathbb{D}}^m$ the diamond domain in $\mathbb{R}^m$ in the notation of the previous section.

In accordance with Weyl's tube formula, apart from the $\pm$ sign, we obtain the following version of the tube formula for Riemannian hypersurfaces.

Corollary 7.1 *For a spacelike hypersurface M of codimension $m = 1$ in a Lorentzian vector space $\mathbb{R}^{n,1}$ the causal tube volume formula takes the form*

$$V_M(a) = 2 \sum_{d=0}^{n} \frac{(-1)^{d/2} k_d(M) a^{1+d}}{3 \cdot 5 \cdots (1 + d)} \qquad (d \text{ even}).$$

Indeed if h_{ij} is the scalar valued second fundamental form then $H_{ij}^{kl} = -h_i^k h_j^l$ and so R_{ij}^{kl} in Theorem 4.1 also picks up a minus sign, that is H_d and $k_d(M) = \int_M H_d \, \mathrm{d}s$ pick up a factor $(-1)^{d/2}$.

There is yet another case, where the causal tube formula has an intrinsic form, namely in case $M \hookrightarrow \mathbb{R}^{n+m-1} \hookrightarrow \mathbb{R}^{n+m-1,1}$. This can be checked easily using Weyl's tube formula in a straightforward way.

The next example shows, however, that the positive result for diamond tubes for $\dim M = \operatorname{codim} M = 2$ of Sect. 6 cannot be extended to the Lorentzian setting.

Example 7.2 If we specialize to the case $n = m = 2$ and $\eta_{pq} = \operatorname{diag}(1, -1)$ of a compact spacelike surface M in Minkowski spacetime $\mathbb{R}^{3,1}$ then the integrand

$$\det\left(\delta_i^j - \sum t_p h_i^{jp}\right) = 1 - \sum t_p\left(h_1^{1p} + h_2^{2p}\right) + \sum t_p t_q\left(h_1^{1p} h_2^{2q} - h_1^{2p} h_2^{1q}\right)$$

averages over the symmetry group $W(\mathrm{B}_2)$ of the square $\widehat{\mathbb{D}}^2$ as in Example 6.1 to the expression

$$1 + (A(h_i^j) + B(h_i^j))(t_1^2 + t_2^2)/2$$

with $A = h_1^{11} h_2^{21} - h_1^{21} h_2^{11}$ and $B = h_1^{12} h_2^{22} - h_1^{22} h_2^{12}$. Since

$$\int_{a\widehat{\mathbb{D}}^2} \mathrm{d}t_1 \, \mathrm{d}t_2 = 2a^2, \quad \int_{a\widehat{\mathbb{D}}^2} t_1^2 \, \mathrm{d}t_1 \, \mathrm{d}t_2 = \int_{a\widehat{\mathbb{D}}^2} t_2^2 \, \mathrm{d}t_1 \, \mathrm{d}t_2 = a^4/3$$

we find

$$V_M(a) = \int_M \left\{2a^2 + (A+B)a^4/3\right\} \mathrm{d}s = \operatorname{area}(M)\, 2a^2 + \int_M (A+B)\, \mathrm{d}s\, a^4/3$$

for the volume of the causal tube along M.

On the other hand, the Gauss equation (for n=2 there is just a single one) in this particular case of spacelike surfaces in Minkowski spacetime becomes

$$R_{12}^{12} = h_1^1 \cdot h_2^2 - h_1^2 \cdot h_2^1 = A - B.$$

Since $\int_M (A+B)\, \mathrm{d}s$ enters in the tube volume formula while $\int_M (A-B)\, \mathrm{d}s$ is the total Gauss curvature for M, the volume formula for causal tubes around surfaces need not be intrinsic.

The conclusion is that for spacelike submanifolds of Minkowski spacetime $\mathbb{R}^{3,1}$ the causal tube volume formula will in general no longer be intrinsic, except for the obvious cases of spacelike curves ($n = 1$) or hypersurfaces ($m = 1$). This question about the intrinsic nature of causal tube volume formulas was the starting point for our work.

8 Pappus Type Theorems

Let us denote the graded commutative algebra $\mathbb{R}[t_1, \ldots, t_m]$ by $P = \oplus P^d$. The subalgebra of invariants for $\mathrm{O}_m(\mathbb{R})$ is equal to $\mathbb{R}[t_1^2+\ldots+t_m^2]$ and is denoted $I = \oplus I^d$. The graded subspace

$$C = \left\{ p \in P\,;\, \int_{\mathrm{O}_m(\mathbb{R})} g(p)\,\mathrm{d}\mu(g) = 0 \right\} = \oplus C^d$$

is the unique invariant complement of I in P. Here μ is the normalized Haar measure on $\mathrm{O}_m(\mathbb{R})$. Hence $P = I \oplus C$ and clearly $C^d = P^d$ for d odd while C^d has codimension one in P^d for d even.

Definition 8.1 A compact domain $\mathbb{D}^m$ in $\mathbb{R}^m$ is called *symmetric of degree n* if

$$\int_{\mathbb{D}^m} p(t)\,\mathrm{d}t = 0$$

for all polynomials $p \in C^1 \oplus \ldots \oplus C^n$.

If the compact domain $\mathbb{D}^m$ has a symmetry group G_m that is orthogonal of degree n (in the sense of our Definition 1.1) then

$$\int_{\mathbb{D}^m} p(t)\,\mathrm{d}t = \int_{\mathbb{D}^m} \langle p(t)\rangle_{G_m}\,\mathrm{d}t = \int_{\mathbb{D}^m} \langle p(t)\rangle_{\mathrm{O}_m(\mathbb{R})}\,\mathrm{d}t$$

for all polynomials $p(t)$ of degree $\leq n$. In particular, if the symmetry group G_m of $\mathbb{D}^m$ is orthogonal of degree n then the domain $\mathbb{D}^m$ is necessarily symmetric of degree n. From the discussions in Sects. 2 and 4 it follows that our Theorem 1.2 holds with the condition on the symmetry group G_m of $\mathbb{D}^m$ being orthogonal of degree n replaced by the condition on $\mathbb{D}^m$ being symmetric of degree n. This more general form of Theorem 1.2 was obtained as Theorem 4.4 in [4].

A compact domain $\mathbb{D}^m$ in $\mathbb{R}^m$ is symmetric of degree 1 if and only if the center of mass of $\mathbb{D}^m$ lies at the origin. Hence the condition for $\mathbb{D}^m$ to be symmetric of degree 1 is a good deal more general than the condition for the symmetry group G_m to be orthogonal of degree 1. If the manifold M is a circle in $\mathbb{R}^3$ then the tube volume formula boils down to the ancient Pappus's centroid theorem. For this reason the higher dimensional tube volume formulas are sometimes also called Pappus type theorems.

The next example shows that for a planar domain $\mathbb{D}^2$ and for all $n \geq 1$ the notion for $\mathbb{D}^2$ to be symmetric of degree n is strictly weaker than the notion for the symmetry group G_2 of $\mathbb{D}^2$ being orthogonal of degree n.

Example 8.2 Consider in polar coordinates $t_1 = r\cos\phi, t_2 = r\sin\phi$ the planar domain $\mathbb{D}^2 = \{(r, \phi)\,;\, 0 \leq r \leq a(\phi), \phi \in \mathbb{R}/2\pi\mathbb{Z}\}$ for some continuous function $a\colon \mathbb{R}/2\pi\mathbb{Z} \to (0, \infty)$. The space C^d is spanned by the functions $r^d\cos(e\phi)$ and $r^d\sin(e\phi)$ with $1 \leq e \leq d$ and $e \equiv d \pmod 2$. The condition that $\mathbb{D}^2$ is symmetric of degree n amounts to

$$\int_0^{2\pi}\int_0^{a(\phi)} r^{d+1}\,\mathrm{d}r\cos(e\phi)\,\mathrm{d}\phi = \int_0^{2\pi}\int_0^{a(\phi)} r^{d+1}\,\mathrm{d}r\sin(e\phi)\,\mathrm{d}\phi = 0$$

or equivalently

$$\int_0^{2\pi}(a(\phi))^{d+2}\cos(e\phi)\,\mathrm{d}\phi = \int_0^{2\pi}(a(\phi))^{d+2}\sin(e\phi)\,\mathrm{d}\phi = 0$$

for all $1 \leq d \leq n$, $1 \leq e \leq d$ and $e \equiv d \pmod 2$. Clearly these conditions are satisfied if for some $k > n$ the function $a(\phi)$ is invariant under the cyclic group C_k of order k acting on the circle $\mathbb{R}/2\pi\mathbb{Z}$ by rotations. Indeed, in that case the Fourier coefficients of all functions $a(\phi)^{d+2}$ vanish for modes not contained in $k\mathbb{Z}$. This is in accordance with our Theorem 1.2 since the symmetry group C_k of this domain $\mathbb{D}^2$ is orthogonal of degree $k > n$.

However, if for a fixed $n \geq 1$ one chooses integers $p > n$ and $q > (n+3)p$ then the function $a(\phi) = b(\phi)(2 + \cos(p\phi))$ with $b > 0$ invariant under C_q has the property that the Fourier coefficients of all functions $a(\phi)^{d+2}$ for $1 \leq d \leq n$ vanish for modes $\pm 1, \ldots, \pm n$. Hence this domain $\mathbb{D}^2$ is certainly symmetric of degree n. On the other hand, if we pick p and q relatively prime then the symmetry group G_2 of $\mathbb{D}^2$ will be trivial in case $b(\phi)$ is chosen sufficiently general (so that the symmetry group for $b(\phi)$ is not larger than C_q), and $G_2 = \{1\}$ is not orthogonal of any degree $n \geq 1$.

The examples obtained in Proposition 4.3 of [4] of compact domains $\mathbb{D}^m$ in $\mathbb{R}^m$ that are symmetric of degree n are for $n \geq 2$ domains $\mathbb{D}^2$ with dihedral symmetry and for $n = 2, 3$ domains $\mathbb{D}^m$ with hyperoctahedral symmetry, besides of course the unit ball $\mathbb{B}^m$ for all n. Hence apart from giving a pedestrian exposition of Weyl's tube volume formula and also a discussion of tube volume formulas for Riemannian submanifolds of a Lorentzian vector space our paper gives a more complete and transparent discussion in Sect. 5 of examples based on symmetry of cross sections $\mathbb{D}^m$ for which the intrinsic tube volume formula holds.

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