Bounded variation and tensor products of Banach lattices

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Abstract We introduce bilinear maps of order bounded variation, bounded semivariation and norm bounded variation. We use these notions to extend the knowledge of the projective tensor product of Banach lattices.

1. Introduction

Tensor products have been an important staple in the general theory of Banach spaces ever since Schatten's paper [9] and, of course, Grothendieck's memoir [6]. A study of tensor products arose much later in the theory of Banach lattices. In fact, the inception of a general tensor product for Archimedean Riesz spaces by Fremlin dates back only to 1972, followed by a theory of tensor products for Banach lattices in 1974. The amount of information about tensor products of Riesz spaces still appears rather limited compared to the explosive growth of that subject in the theory of Banach spaces. One reason for this lack of information lies in the difficulty of the subject matter. Seemingly innocent and natural questions turn out to be surprisingly hard. To give just one example, it is known that the tensor product of two Riesz subspaces is a Riesz subspace of the tensor product; but it is unknown, whether the tensor product of two ideals is an ideal in the tensor product. Also, the very construction of tensor products of Riesz spaces raises questions about what should be their defining universal property. It is, in part, the latter question that we address here.

In [4] Fremlin defined the Archimedean Riesz space tensor product $E\bar{\otimes}F$ of the Archimedean Riesz spaces E and F. His definition required that for every Archimedean Riesz space G and for every Riesz bimorphism $\phi : E \times F \to G$ there exists a unique Riesz homomorphism $\phi^{\otimes} : E\bar{\otimes}F \to G$ such that $\phi^{\otimes}(x \otimes y) = \phi(x, y)$ for all $x \in E$, $y \in F$. As Fremlin himself remarked, the existence of this Archimedean tensor product is less remarkable than the additional universal property that it possesses: For every uniformly complete Riesz space G and every bipositive bilinear map $T : E \times F \to G$ there exists a unique positive linear map $T^{\otimes} : E\bar{\otimes}F \to G$ such that $T^{\otimes}(x \otimes y) = T(x, y)$ for all $x \in E$, $y \in F$. Is there a similar property for bilinear maps that are not necessarily bipositive? A theory of tensor products should link the bilinear maps that appear in the universal property with a natural space of linear mappings on the tensor product. For instance, in case of the Archimedean Riesz space tensor product

$$L_b(E\bar{\otimes}F, G),$$

the space of all order bounded linear maps $E\bar{\otimes}F \to G$ is the natural space of linear mappings. We use this as a guide to answer the above question. We discuss separately the case of the Archimedean Riesz space tensor product (Section 3) and the case of the Banach lattice tensor product (Section 5). The Archimedean Riesz space tensor product leads naturally to the projective norm on the normed Riesz space tensor product. As it turns out, the operator norm is no longer necessarily preserved when one considers nonpositive mappings (see Section 7 for an example). As a consequence one is forced to introduce another norm. Thus we introduce the notions of order bounded variation, bounded semivariation and norm bounded variation. In a follow-up paper we will show that these cases are special instances of a much more general situation involving bornological Riesz spaces. However, that generalized and unified setting requires considerable preparatory work and complicates the proofs of theorems that already are rather delicate. We have made a preliminary report on our findings in [2]. Our results generalize the results of Fremlin in [4] and [5], enable us to unify these results in a forthcoming paper and allow us to consider Banach spaces rather than Banach lattices (in Section 4) for the spaces G above. We remark that there are obvious connections to the theory of vector measures of bounded variation and of bounded semivariation (see [3]). Our terminology is standard and closely follows the monographs [1], [7] and [8].

2. A convenient notation

Let E be an Archimedean Riesz space. The order bounded linear functions $E \to \mathbf{R}$ form a Riesz space $E^\sim$. If $f \in E^\sim$, then $|f|$ is determined by the formula

$$|f|(a) = \sup\{\Sigma|f(x_n)| : x_1, x_2, ..., x_N \in E^+, \Sigma x_n = a\} \qquad (a \in E^+).$$

Constructions such as in the right hand member will occur so frequently that it pays to introduce a less heavy-handed notation.

Take $a \in E^+$. A *partition* of a is a finite sequence of elements of E^+ whose sum equals a. The partitions of a form a set Πa. We often denote a partition $(x_1, x_2, ..., x_N)$ of a by just the letter "x" and write, e.g.,

$$|f|(a) = \sup_{x \in \Pi a} \Sigma|f(x_n)| \qquad (a \in E^+).$$

If $x = (x_1, x_2, ..., x_N)$ and $y = (y_1, y_2, ..., y_M)$ are partitions of a we call x a *refinement* of y if the set $\{1, ..., N\}$ can be written as a disjoint union of sets $I_1, I_2, ..., I_M$ in such a way that

$$y_m = \sum_{n \in I_m} x_n \qquad (m = 1, ..., M).$$

Any two partitions of a have a common refinement. Thus, in a natural way Πa is a directed set. If S is a linear map of E into a Riesz space F, if $x, y \in \Pi a$ and x is a refinement of y, then

$$\Sigma|S(x_n)| \ \geq \ \Sigma|S(y_m)|.$$

Hence, for every linear $S : E \to F$,

$$(\Sigma|S(x_n)|)_{x\in\Pi a}$$

is an increasing net in F. In particular, for $f \in E^\sim$ we can rewrite the formula earlier in this section as

$$|f|(a) = \lim_{x\in\Pi a} \Sigma|f(x_n)| \qquad (a \in E^+).$$

Similarly, for $x \in E$:

$$f(|x|) = \lim_{g\in\Pi f} \Sigma|g_n(x)| \qquad (f \in E^{\sim +}).$$

3. The Riesz space case

Let E, F, G be Archimedean Riesz spaces. We say that a bilinear map $T : E\times F \to G$ is of *order bounded variation* if for all $a \in E^+$ and all $b \in F^+$ the set

$$\{\sum_{n,m} |T(x_n, y_m)| \ : \ x \in \Pi a, \ y \in \Pi b\}$$

is order bounded.

The bilinear maps $E \times F \to G$ that are of order bounded variation form a vector space $\mathrm{Bil}_{bv}(E, F; G)$ containing all positive bilinear maps. $\mathrm{Bil}_{bv}(E, F; G)$ is an ordered vector space if we define $T_1 \le T_2$ to mean that $T_2 - T_1$ is increasing.

In what follows $E\bar{\otimes}F$ is the Archimedean Riesz space tensor product as introduced by Fremlin in [4]. As mentioned in the introduction, the theory about tensor products such as $E\bar{\otimes}F$ ought to connect the space $L_b(E\bar{\otimes}F, G)$ with a space of bilinear maps $E \times F \to G$. The space $\mathrm{Bil}_{bv}(E, F; G)$ does the job:

Theorem 1. *Let E, F, G be Archimedean Riesz spaces; let G be Dedekind complete.*

(i) Let T be a bilinear map $E \times F \to G$ that is of order bounded variation. Then there exists a unique order bounded linear $T^\otimes : E\bar{\otimes}F \to G$ for which

$$T(x, y) = T^\otimes(x \otimes y) \qquad (x \in E, \ y \in F).$$

(ii) $\mathrm{Bil}_{bv}(E, F; G)$ is a Dedekind complete Riesz space. The correspondence

$$T \mapsto T^\otimes$$

is a Riesz isomorphism from $\mathrm{Bil}_{bv}(E, F; G)$ onto $L_b(E\bar{\otimes}F, G)$.
(iii) For $T \in \mathrm{Bil}_{bv}(E, F; G)$, $|T|$ is determined by

$$|T|(a, b) = \sup_{x\in\Pi a, \ y\in\Pi b} \sum_{n,m} |T(x_n, y_m)| \qquad (a \in E^+, b \in F^+).$$

Proof. (i) Suppose S_1 and S_2 are order bounded linear maps $E\bar{\otimes}F \to G$ such that $T = S_1 \circ \otimes$ and $T = S_2 \circ \otimes$. Then $S_1 - S_2$ vanishes on the algebraic tensor product $E \otimes F$. But this $E \otimes F$ is relatively uniformly dense in $E\bar{\otimes}F$ (Theorem 4.2(iii) in [4]) whereas $S_1 - S_2$ is order bounded. It follows that $S_1 - S_2 = 0$.

This takes care of the uniqueness part of (i); now we turn to the existence. Define

$$\bar{T}_+(a,b) := \sup_{x\in\Pi a,\ y\in\Pi b} \sum_{n,m} |T(x_n, y_m)| \qquad (a \in E^+, b \in F^+).$$

From the observation that for all a and b the net (see Section 2)

$$\left(\sum_{n,m} |T(x_n, y_m)|\right)_{(x,y)\in\Pi a\times\Pi b}$$

is increasing one easily shows that $\bar{T}_+(a_1+a_2, b) = \bar{T}_+(a_1, b) + \bar{T}_+(a_2, b)$ and $\bar{T}_+(a, b_1 + b_2) = \bar{T}_+(a, b_1) + \bar{T}_+(a, b_2)$ for all $a, a_1, a_2 \in E^+$ and all $b, b_1, b_2 \in F^+$. By a routine reasoning, $\bar{T}_+$ extends to a positive bilinear map $\bar{T} : E \times F \to G$. Now $\bar{T} - T$ is also bilinear and positive. By Fremlin's result (Th. 5.3 in [4]) there exist positive linear maps $\bar{T}^{\otimes}$ and $(\bar{T} - T)^{\otimes}$ of $E\bar{\otimes}F$ into G with $\bar{T} = \bar{T}^{\otimes} \circ \otimes$ and $\bar{T} - T = (\bar{T} - T)^{\otimes} \circ \otimes$. Set $T^{\otimes} := \bar{T}^{\otimes} - (\bar{T} - T)^{\otimes}$.

(ii) We now have a (linear and order preserving) map $T \mapsto T^{\otimes}$ from $\mathrm{Bil}_{bv}(E, F; G)$ into $L_b(E\bar{\otimes}F, G)$. It is clear that for every positive linear $S : E\bar{\otimes}F \to G$ there is a unique (positive) bilinear $T : E \times F \to G$ with $T^{\otimes} = S$. Consequently,

$$T \mapsto T^{\otimes}$$

is an isomorphism of ordered vector spaces. In particular, $\mathrm{Bil}_{bv}(E, F; G)$ is a Dedekind complete Riesz space.

(iii) Let $T \in \mathrm{Bil}_{bv}(E, F; G)$ and let $\bar{T}$ be as above: we wish to show that $\bar{T} = |T|$. That is a simple matter. $\bar{T}$ is bilinear and positive, so $\bar{T} \in \mathrm{Bil}_{bv}(E, F; G)$. Clearly, $\bar{T} \geq T$ and $\bar{T} \geq -T$, whence $\bar{T} \geq |T|$. On the other hand, if $a \in E^+$, $b \in F^+$, $x \in \Pi a$, $y \in \Pi b$, then $\sum_{n,m} |T(x_n, y_m)| \leq \sum_{n,m} |T|(x_n, y_m) = |T|(a, b)$. Therefore, $\bar{T} \leq |T|$.

The above leads to the following corollary.

Corollary 2. *If E, F and G are Archimedean Riesz spaces and G is Dedekind complete, then the Riesz spaces $Bil_{bv}(E, F; G)$, $L_b(E\bar{\otimes}F, G)$ and $L_b(E, L_b(F, G))$ are naturally isomorphic.*

Proof. The natural bijection between the positive cones of $\mathrm{Bil}_{bv}(E, F; G)$ and $L_b(E, L_b(F, G))$ extends to a Riesz isomorphism.

We will have use for the special case $G = \mathbf{R}$ of Theorem 1, formulated in terms of convergence of an increasing net.

Corollary 3. *Let E and F be Archimedean Riesz spaces. Then $Bil_{bv}(E,F;\mathbf{R})$ is a Dedekind complete Riesz space, isomorphic to $(E\bar{\otimes}F)^\sim$. For $f\in Bil_{bv}(E,F;\mathbf{R})$, $|f|$ is given by*

$$|f|(a,b)=\lim_{(x,y)\in\Pi a\times\Pi b}\sum_{n,m}|f(x_n,y_m)| \qquad (a\in E^+, b\in F^+).$$

Theorem 1 generalizes Fremlin's Theorem [4], 5.3 in that T is not necessarily bipositive. On the other hand, Fremlin's theorem doesn't require Dedekind completeness of the end space G, but only uniform completeness. In this vein we have:

Theorem 4. *Let E,F,G be Archimedean Riesz spaces; let G be uniformly complete. Then for every bilinear $T:E\times F\to G$ that is of order bounded variation there is a unique order bounded linear map $T^\otimes : E\bar{\otimes}F\to G$ for which*

$$T(x,y)=T^\otimes(x\otimes y) \qquad (x\in E,\ y\in F).$$

$T\mapsto T^\otimes$ is a linear order isomorphism between $Bil_{bv}(E,F;G)$ and $L_b(E\bar{\otimes}F,G)$.

Proof. Let G^δ be a Dedekind completion of G viewed as a space containing G. Any T in $\mathrm{Bil}_{bv}(E,F;G)$ lies in $\mathrm{Bil}_{bv}(E,F;G^\delta)$ and so induces a $T^\otimes$ in $L_b(E\bar{\otimes}F,G^\delta)$. All we really have to show is that $T^\otimes$ maps $E\bar{\otimes}F$ into G. But that is easy: $T^\otimes$ maps the algebraic tensor product $E\otimes F$ into G; $E\otimes F$ is relatively uniformly dense in $E\bar{\otimes}F$ (Theorem 4.2 (iii) in [4]); $T^\otimes$ preserves relative uniform convergence and G is uniformly closed in G^δ.

4. The Banach space case

Though the previous sections were relatively smooth sailing, the case in which E and F are normed Riesz spaces is much harder.

For Banach lattices E and F, in [5] Fremlin defines a norm $||\ ||_{|\pi|}$ on $E\bar{\otimes}F$, the "projective-product norm" and shows that for all $u\in E\bar{\otimes}F$:

$$||u||_{|\pi|}=\inf\{\Sigma||a_k||\,||b_k|| : a_1,...,a_k\in E^+,\ b_1,...,b_k\in F^+,\ |u|\le\Sigma a_k\otimes b_k\}.$$

Taking a slightly different approach, for any two normed Riesz spaces E and F (not necessarily Banach) we use the above formula to *define* a function $||\ ||_{|\pi|}$ on $E\bar{\otimes}F$. This $||\ ||_{|\pi|}$ clearly is a Riesz seminorm. The reasoning that Fremlin gives in Parts (a) and (b) of the proof of Theorem 1E in [5] shows that $||\ ||_{|\pi|}$ actually is a norm.

Relative to this norm we have:

(i) *$E\otimes F$ is dense in $E\bar{\otimes}F$. (It is even relatively uniformly dense; see* [4], *Theorem* 4.2 (iii)).

(ii) *The cone generated by* $\{x \otimes y : x \in E^+,\ y \in F^+\}$ *is dense in* $(E\bar{\otimes}F)^+$; *see* [5], 1B(b).

Now we can get to work. Indeed, let E and F be normed Riesz spaces; G a normed vector space. We endow $E\bar{\otimes}F$ with its projective product norm, $||\ ||_{|\pi|}$. For a bilinear map $T : E \times F \to G$ we define its *semivariation* to be

$$|||T|||\ :=\ \sup\{||\Sigma_{n,m}\varepsilon_{nm}T(x_n, y_m)||:\ x_1, x_2, ..., x_N \in E^+,\ ||\Sigma x_n|| \leq 1,$$

$$y_1, y_2, ..., y_M \in F^+,\ ||\Sigma y_n|| \leq 1,\ \varepsilon_{nm} \in \{-1, 1\} \text{ for all } n, m\}.$$

The bilinear maps $T : E \times F \to G$ for which $|||T|||$ is finite form a vector space $\mathrm{Bil}_{|||\ |||}(E, F; G)$ on which $|||\ |||$ is a norm. All elements of this space are norm continuous. Indeed, $||T|| \ \leq\ |||T|||$ for every T. The elements of $\mathrm{Bil}_{|||\ |||}(E, F; G)$ are said to have *finite semivariation.*

Theorem 5. *Let E and F be normed Riesz spaces and let G be a Banach space. The map*

$$S \mapsto S \circ \otimes \qquad (S \in L(E\bar{\otimes}F, G))$$

is a Banach space isomorphism of $L(E\bar{\otimes}F, G)$ onto Bil$_{|||\ |||}(E, F; G)$. In particular, for every bilinear map $T : E \times F \to G$ that is of finite semivariation there exists a unique continuous linear $T^{\otimes} : E\bar{\otimes}F \to G$ with

$$T(x, y) = T^{\otimes}(x \otimes y) \qquad (x \in E,\ y \in F).$$

We have $||T^{\otimes}|| = |||T|||$.

Proof. (Step I). Take $S \in L(E\bar{\otimes}F, G)$. Obviously, $S\circ\otimes$ is bilinear. If $x_1, x_2, ..., x_N \in E^+$ and $y_1, y_2, ..., y_M \in F^+$ and $\varepsilon_{nm} \in \{-1, 1\}$ $(n \leq N,\ m \leq M)$, then

$$||\Sigma\varepsilon_{nm}S \circ \otimes(x_n, y_m)|| \leq ||\Sigma\varepsilon_{nm}S(x_n \otimes y_m)|| =$$

$$= ||S(\Sigma\varepsilon_{nm}(x_n \otimes y_m))|| \leq ||S||\ ||\Sigma x_n \otimes y_m||_{|\pi|} =$$

$$= ||S||\ ||(\Sigma_n x_n) \otimes (\Sigma_m y_m)||_{|\pi|} \leq ||S||\ ||\Sigma_n x_n||\ ||\textstyle\sum_m y_m||.$$

It follows that $|||S \circ \otimes||| \leq ||S||$.

Thus, $S \mapsto S\circ\otimes$ $(S \in L(E\bar{\otimes}F, G))$ is a (linear) map $L(E\bar{\otimes}F, G) \to \mathrm{Bil}_{|||\ |||}(E, F; G)$. If $S \in L(E\bar{\otimes}F, G)$ and $S \circ \otimes = 0$, then S vanishes identically on the vector space tensor product $E \otimes F$, so that $S = 0$, since $E \otimes F$ is dense in $E\bar{\otimes}F$. Thus the map $S \mapsto S \circ \otimes$ is injective. It remains to show that for every $T \in \mathrm{Bil}_{|||\ |||}(E, F; G)$ there is a linear $S : E\bar{\otimes}F \to G$ with $T = S \circ \otimes$ and $||S|| \leq |||T|||$. Let $T \in \mathrm{Bil}_{|||\ |||}(E, F; G)$.

(Step II). Take $f \in G^{'}$. Then $f \circ T$ is a bilinear map $E \times F \to \mathbf{R}$. If $a \in E^+$ and $b \in F^+$, then for all $x = (x_1, ..., x_N) \in \Pi a$ and all $y = (y_1, ..., y_M) \in \Pi b$, setting $\varepsilon_{n,m} :=$ sign $f(T(x_n, y_m))$, we obtain that

$$\begin{aligned}\Sigma_{n,m}|(f \circ T)(x_n, y_m)| &= f(\Sigma_{n,m}\varepsilon_{n,m}T(x_n, y_m)) \leq \\ &\leq ||f|| \, ||\Sigma_{n,m}\varepsilon_{n,m}T(x_n, y_m)|| \\ &\leq ||f|| \, ||\Sigma x_n|| \, ||\Sigma y_m|| \, |||T||| = ||f|| \, ||a|| \, ||b|| \, |||T|||\end{aligned}$$

It follows that $f \circ T \in \mathrm{Bil}_{bv}(E, F; \mathbf{R})$ and (using Cor.3) that

$$|f \circ T|(a, b) \leq ||f|| \, ||a|| \, ||b|| \, |||T||| \qquad (a \in E^+, b \in F^+).$$

$f \circ T$ and $|f \circ T|$ induce elements $(f \circ T)^{\otimes}$ and $|f \circ T|^{\otimes}$ of $(E\bar{\otimes}F)^{\sim}$; we have $|(f \circ T)^{\otimes}| = |f \circ T|^{\otimes}$ (see Corollary 3). We proceed to estimate the norm of $(f \circ T)^{\otimes}$ as a continuous linear function on on the normed Riesz space $E\bar{\otimes}F$. Take $u \in E\bar{\otimes}F$. If $x_1, ..., x_k \in E^+$, $y_1, ..., y_k \in F^+$ and $|u| \leq \sum x_k \otimes y_k$, then

$$\begin{aligned}|(f \circ T)^{\otimes}(u)| \leq |f \circ T|^{\otimes}(|u|) \leq |f \circ T|^{\otimes}(\Sigma x_k \otimes y_k) = \\ = \Sigma|f \circ T|(x_k, y_k) \leq \Sigma||f|| \, ||x_k|| \, ||y_k|| \, |||T|||.\end{aligned}$$

Thus, $|(f \circ T)^{\otimes}(u)| \leq ||f|| \, ||u||_{|\pi|} \, |||T|||$. Consequently,

$$||(f \circ T)^{\otimes}|| \leq ||f|| \, |||T|||.$$

(Step III). From the above we obtain a continuous linear map $f \mapsto (f \circ T)^{\otimes}$ of $G^{'}$ into $(E\bar{\otimes}F)^{'}$. This map induces a continuous linear map $T^{\circ} : E\bar{\otimes}F \to G^{''}$ by

$$(T^{\circ}u)(f) := (f \circ T)^{\otimes}(u) \qquad (u \in E\bar{\otimes}F, \ f \in G^{'}).$$

Note that, by the result of Step II,

$$||T^{\circ}|| \leq |||T|||.$$

If $x \in E$, $y \in F$, then for all $f \in G^{'}$

$$T^{\circ}(x \otimes y)(f) = (f \circ T)^{\otimes}(x \otimes y) = f(T(x, y)).$$

Hence, T° maps the vector space tensor product $E \otimes F$ into the canonical image $\hat{G}$ of G in $G^{''}$. As T° is continuous and $E \otimes F$ is norm dense in $E\bar{\otimes}F$ (even relatively uniformly dense) we see that T° maps all of $E\bar{\otimes}F$ into $\hat{G}$. Thus we can define $S : E\bar{\otimes}F \to G$ by

$$f(Su) = (T^{\circ}u)(f) \qquad (u \in E\bar{\otimes}F, \ f \in G^{'}).$$

Then S is linear and continuous and $T = S \circ \otimes$ and $||S|| \leq ||T^{\circ}|| \leq |||T|||$.

5. The Banach lattice case

5.1. Ordered Banach spaces. Our main interest lies in the case where G is a Banach lattice. An intermediate stage between Banach spaces and Banach lattices is formed by the Banach spaces G that are simultaneously ordered vector spaces, the norm and ordering being connected by

$$(*) \qquad \text{if } x, a \in G \text{ and } -a \leq x \leq a \text{ then } ||x|| \leq ||a||.$$

Spaces of that type appear in two places in the paper [5] by Fremlin. The condition $(*)$ above is equivalent to

$$\text{if } a, b \in G^{+}, \text{ then } ||a-b|| \leq ||a+b||$$

(where $G^{+} = \{x \in G : x \geq 0\}$) and also to

$$\text{if } x_1, ..., x_N \in G^{+} \text{ and } \varepsilon_1, ..., \varepsilon_N \in \{-1, 1\}, \text{ then } ||\Sigma \varepsilon_n x_n|| \leq ||\Sigma x_n||.$$

Examples are of course all closed linear subspaces of Banach lattices, but also $L(E, F)$ for Banach lattices E and F. The signed measures on a given $\sigma-$algebra form a Riesz space and (under the sup-norm) a Banach space that satisfies $(*)$ but is not a Banach lattice. From our Theorem 5 we obtain the following result (basically generalizing Fremlin's Corollary 2L of [5] to our level of discussion).

Theorem 6. *Let E and F be normed Riesz spaces, G an ordered Banach space satisfying $(*)$, above. Let $T : E \times F \to G$ be bilinear, continuous, positive (i.e., $T(E^{+} \times F^{+}) \subset G^{+}$). Then the semivariation of T is finite and, indeed, $||T|| = |||T|||$. With $T^{\otimes}$ as in Theorem 5, $T^{\otimes}$ is a positive linear map and $||T|| = ||T^{\otimes}||$. (It follows that $L(E\bar{\otimes}F, G)$ and $Bil_{|||\ |||}(E, F; G)$ are isomorphic as ordered Banach spaces.)*

Proof. If $x_1, ..., x_N \in E^{+}$, $y_1, ..., y_M \in F^{+}$ and $\varepsilon_{n,m} \in \{-1, 1\}$ $(n \leq N,\ m \leq M)$, then

$$||\Sigma \varepsilon_{n,m} T(x_n, y_m)|| \leq ||\Sigma T(x_n, y_m)|| = ||T(\Sigma x_n, \Sigma y_m)|| \leq ||T||\ ||\Sigma x_n||\ ||\Sigma y_m||.$$

It follows that $|||T||| \leq ||T||$ and therefore $||T|| = |||T|||$.

Hence, $T^{\otimes}$ exists. Its positivity follows from the fact that the cone generated by $\{x \otimes y : x \in E^{+}, y \in F^{+}\}$ is norm dense in $(E\bar{\otimes}F)^{+}$ (see 1A (h) in [5]).

5.2. Linear maps. Let E and F be normed Riesz spaces. If S is a linear map $E \to F$, we define the *norm variation* of S to be

$$\text{Var } S := \sup\{||\ \Sigma |Sx_n|\ || : x_1, x_2, ..., x_N \in E^{+},\ ||\Sigma x_n|| \leq 1\}$$

and we say that S is *of norm bounded variation* if Var S is finite.

If $S : E \to F$ is linear, then $||S|| \leq \text{Var } S$, equality holding as soon S is positive. Moreover, for all $a \in E^+$,

$$||\Sigma\, |Sx_n|\, || \leq ||a|| \text{Var } S \qquad (x \in \Pi a).$$

The linear maps that are of order bounded variation form a vector space containing all positive continuous linear maps. For further examples the reader is invited to read [3] where examples of vector measures of bounded variation on certain classical spaces easily translate into operators of norm bounded variation.

5.3. Bilinear maps. The previous subsection has a straightforward analogue for bilinear maps. Indeed, let E, F, G be normed Riesz spaces. The *norm variation* of a bilinear map $T : E \times F \to G$ is

$$\text{Var } T := \sup\{||\textstyle\sum_{n,m} |T(x_n, y_m)|\, || : x_1, x_2, ..., x_N \in E^+,\ ||\sum x_n|| \leq 1,$$

$$y_1, y_2, ..., y_M \in F^+,\ ||\textstyle\sum y_m|| \leq 1\};$$

T is of *norm bounded variation* if Var $T < \infty$.

For every bilinear map $T : E \times F \to G$ we have $||T|| \leq |||T||| \leq \text{Var } T$, whereas $||T|| = \text{Var } T$ if T is positive (in the sense of Fremlin's paper [5]). Also, if $a \in E^+$ and $b \in F^+$, then

$$||\, \Sigma\, |T(x_n, y_m)|\, || \leq ||a||\ ||b|| \text{ Var } T \qquad (x \in \Pi a,\ y \in \Pi b).$$

The bilinear maps $E \times F \to G$ of norm bounded variation form a vector space containing all bilinear maps that are bipositive and continuous.

6. The main Theorem

We can now state and prove the main theorem of this paper.

Theorem 7. *Let E and F be normed Riesz spaces; G a Banach lattice; let T be a bilinear map $E \times F \to G$ that is of norm bounded variation. Then there exists a unique continuous linear $T^{\otimes} : E\bar{\otimes}F \to G$ with*

$$T(x, y) = T^{\otimes}(x \otimes y) \qquad (x \in E,\ y \in F).$$

We have that $T^{\otimes} \geq 0$ if and only if T is positive. Furthermore, $T^{\otimes}$ is of norm bounded variation and

$$\text{Var } T^{\otimes} = \text{Var } T.$$

Proof. (Step I). First, if $a \in E^+$ and $b \in F^+$, then for all $x \in \Pi a$ and all $y \in \Pi b$ we see, using Theorem 5 that

$$\begin{aligned} || \textstyle\sum_{n,m} |T(x_n, y_m)| \, || &= || \textstyle\sum_{n,m} |T^{\otimes}(x_n \otimes y_m)| \, || \leq \\ &\leq ||a \otimes b||_{|\pi|} \text{Var } T^{\otimes} \leq \\ &\leq ||a|| \, ||b|| \text{Var } T^{\otimes}. \end{aligned}$$

It follows that Var $T \leq$ Var $T^{\otimes}$. For the reverse inequality, take $u_1, ..., u_k \in (E \bar{\otimes} F)^+$, $|| \sum u_k ||_{|\pi|} \leq 1$; we prove that $|| \sum |T^{\otimes} u_k| \, || \leq$ Var T. Take $f \in (G')^+$. In view of [8] Formula (2) on page 87, it suffices to prove that

$$f(\Sigma |T^{\otimes} u_k|) \leq ||f|| \text{Var } T.$$

Now using the Banach lattice isomorphism $G \to \hat{G}$ with notation as in the proof of Theorem 5, Step III, we have all the ingredients to prove this. Indeed,

$$\begin{aligned} f(\Sigma_k |T^{\otimes} u_k|) = \Sigma_k f(|T^{\otimes} u_k|) &= \Sigma_k |T^{\circ} u_k|(f) \\ &= \Sigma_k \lim_{g \in \Pi f} \Sigma_j |(T^{\circ} u_k)(g_j)| = \\ &= \Sigma_k \lim_{g \in \Pi f} \Sigma_j |(g_j \circ T^{\otimes})(u_k)| = \\ &= \lim_{g \in \Pi f} \Sigma_{k,j} |(g_j \circ T^{\otimes})(u_k)| \leq \\ &\leq \sup_{g \in \Pi f} \textstyle\sum_j |g_j \circ T|^{\otimes} (\sum_k u_k) = \\ &= \sup_{g \in \Pi f} (\textstyle\sum_j |g_j \circ T|)^{\otimes} (\sum_k u_k). \end{aligned}$$

Thus, we are done if we can prove that

$$||(\Sigma_j |g_j \circ T|)^{\otimes}|| \leq ||f|| \text{Var } T \qquad (g \in \Pi f),$$

i.e. (because of Theorem 6)

$$(*) \qquad ||\Sigma_j |g_j \circ T| \, || \leq ||f|| \text{Var } T \qquad (g \in \Pi f).$$

Take $a \in E^+, b \in F^+$. Applying Corollary 3 we obtain for every $g \in \Pi f$:

$$\begin{aligned}
\Sigma_j |g_j \circ T|(a,b) &= \Sigma_j \lim_{(x,y)\in\Pi a\times\Pi b} \Sigma_{n,m} |(g_j \circ T)(x_n, y_m)| = \\
&= \lim_{(x,y)\in\Pi a\times\Pi b} \Sigma_{n,m} \Sigma_j |(g_j(T(x_n, y_m))| \leq \\
&\leq \sup_{(x,y)\in\Pi a\times\Pi b} \Sigma_{n,m} \Sigma_j g_j(|T(x_n, y_m)|) = \\
&= \sup_{(x,y)\in\Pi a\times\Pi b} \Sigma_{n,m} f(|T(x_n, y_m)|) \leq \\
&\leq \sup_{(x,y)\in\Pi a\times\Pi b} ||f|| \; || \Sigma_{n,m} |T(x_n, y_m)| \; || \leq \\
&\leq ||f|| \; ||a|| \; ||b|| \text{Var } T
\end{aligned}$$

and $(*)$ follows.

So far we have proved that Var $T = \text{Var} T^{\otimes}$. As for the final part of the theorem, it is clear that positivity of $T^{\otimes}$ implies positivity of T. The converse is true by an argument similar to 1E (iii) in [5].

7. The operator norm versus the variation norm: A counterexample.

In our main theorem we have seen that Var T = Var $T^{\otimes}$, i.e. the variation norm is preserved in tensoring a bilinear map of bounded variation. Interestingly, the operator norm is not necessarily preserved. The easiest example of that behavior is the following.

Example 8. Take $\mathbf{R}^2$ with the supremum norm. The formula

$$\begin{pmatrix} x \\ y \end{pmatrix} \otimes \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} xu \\ yu \\ xv \\ yv \end{pmatrix}$$

defines a Riesz isomorphism between $\mathbf{R}^2 \bar{\otimes} \mathbf{R}^2$ and $\mathbf{R}^4$. Moreover, it easily follows that $|| \quad ||_{|\pi|} = || \quad ||_\infty$. Define $T : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$ by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right) = xu - xv + yu + yv.$$

Then for $\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| \leq 1, \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| \leq 1$ we have

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right) \leq |u - v| + |u + v| \leq 2,$$

hence $\|T\| \leq 2$. But $T^{\otimes}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = p - q + r + s$ and $\|T^{\otimes}\| \geq 4$.

It would be interesting to characterize situations in which $\|T^{\otimes}\| = \|T\|$ for all regular T.

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