Inexact Newton solvers in plasticity. Theory and experiments.

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Abstract

Applications of inexact Newton and inexact Newton-like solvers are described and analyzed for the solution of nonlinear systems appearing in the solution of problems of elasto-plasticity. For the numerical solution, both explicit and return mapping incremental finite element algorithms are considered.

Keywords: Elasto-plasticity, inexact Newton and Newton-like solvers, explicit incremental finite element method, return mapping algorithms.

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1 Introduction

Numerical solution of problems of elasto-plasticity appears in many engineering applications. In geomechanics the elastoplastic behaviour is traditionally assumed for soils and the elastoplastic analysis is frequently used namely for estimation of limit states for footings, embankments, slopes etc., see e.g.[11]. There is also enhanced interest in using elasto-plastic models for describing the behaviour of rocks including processes of rock failure, cf. again [11]. In this respect, we can also mention relatively new applications of the model of continuum damage mechanics [20], [21] which is mathematically very close to elasto-plasticity [18].

The applications in geomechanics demand research and development mainly in two fields (a) development of (complicated) plasticity models which fit well the behaviour of geomaterials, (b) development of efficient numerical techniques, i.e. discretization methods preserving the plasticity conditions and solution methods for large-scale nonlinear systems.

Our paper deals with the numerical solution of problems of elasto-plasticity which are described as the initial value problem of so called incremental or flow theory of plasticity, see e.g. [13]. Discretization of these problems leads to incremental finite elements schemes appearing in many variants in the literature, see e.g. [24] and the references given there.

In our paper, we shall consider incremental finite element schemes with both explicit and implicit (return mapping) computation of stresses. The nonlinear systems appearing in the course of application of the incremental finite element method can be solved by Newton-like methods such as initial stiffness and tangential stiffness methods. Also the Newton method can be applied for the schemes with implicit stress computation. For solving large-scale (especially 3D) problems, it is efficient to use inexact variants of these methods with some lower accuracy solution of the arising linear problems by a suitable inner iterative method. This gives rise to inexact Newton methods, see e.g. [1].

We shall concentrate mainly on convergence and efficiency of the inexact iterative solvers in elasto-plasticity. The analysis will be performed mainly for simple plasticity models. Some further research will be necessary for analysis of more complicated plasticity models but we believe that most of our conclusions remain valid also in that case.

The present paper can be viewed as a survey of the results described in more detail in the papers [8],[7],[10] and [9]. It also attempts to explain the differences between incremental finite element algorithms with explicit and return mapping stress computations and shows some further possibilities for the construction of efficient solvers.

2 Elasto-plasticity

Let us consider a deformable body $\Omega \subset R^3$ whose state after loading is described by the *displacement* (u) , the *small strain tensor* and the *Cauchy stress tensor.* In the sequel, we shall work with *strain vector* (ε) and *stress vector* (σ) . These vectors contain components of the corresponding tensors, e.g.

$$\varepsilon = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{21}, \varepsilon_{23}, \varepsilon_{32}, \varepsilon_{13}, \varepsilon_{31})^T$$

where

$$\varepsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \quad \text{for} \quad \varepsilon = \varepsilon(u).$$

Thus, $\varepsilon, \sigma \in S$,

$$S = \{ v = (v_1, \cdots, v_9)^T \in R^9, \; v_4 = v_5, \, v_6 = v_7, \, v_8 = v_9 \}.$$

Further, we divide the strain vector into an elastic and a plastic part and assume that there is a linear relation between the stress and the elastic part of the strain vector, i.e.

$$\begin{aligned} \varepsilon &= \varepsilon^e + \varepsilon^p , \\ \sigma &= D \varepsilon^e \end{aligned} \tag{1}$$

where D is the 9×9 Hooke's matrix for isotropic elastic material. The relation (1) can then also be expressed in the classical form

$$\sigma_{ij} = \lambda(\varepsilon^e_{11} + \varepsilon^e_{22} + \varepsilon^e_{33})\delta_{ij} + 2\mu \varepsilon^e_{ij} \tag{2}$$

where λ, μ are Lamè moduli and δ_{ij} is the Kronecker's symbol.

The onset of the inelastic behaviour is given by the *yield condition* which can be written in the form

$$P(\sigma, \kappa) = 0$$

where the function P depends on the stress and some *hardening parameters,* denoted by κ.

Example 1 (von Mises plasticity with isotropic strain hardening) :

$$P(\sigma\,,\,\kappa) = \sqrt{\frac{3}{2}}\parallel s \parallel - H(\kappa)\,, \tag{3}$$

where s is the deviator of σ, $s\;=\;\mathrm{dev}(\sigma)$,

$$s = (s_{ij}) : s_{ij} = \sigma_{ij} - \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})\delta_{ij}\,,$$

$\parallel s \parallel = \sqrt{s^T s}$, $\kappa = \sqrt{\frac{2}{3}}\parallel \varepsilon^p \parallel$, H is a given differentiable function.

Example 2 (von Mises condition with kinematic hardening):

$$P(\sigma\,,\,\kappa) = \sqrt{\frac{3}{2}}\parallel s - \kappa \parallel - H_0 \tag{4}$$

where κ is now defined by

$$\kappa = c_o\,\varepsilon^p\,.$$

Note that c_o and H_0 are two material constants.

For describing the development of plastic strains, we must follow the history of loading. For this reason, we introduce a continuation parameter $t\epsilon[0,T]$. In the sequel, we shall denote by dot the derivation with respect to this continuation parameter.

Now, the development of plastic strains can be described by the *flow rule*

$$\dot{\varepsilon}^p = \gamma\, p \,,\; \gamma \geq 0 \tag{5}$$

where γ is a plastic multiplier and p is the flow direction. In our paper, we shall consider only *associative plasticity* for which

$$p = \frac{\partial P}{\partial \sigma}\,.$$

The hardening parameters are also related to the plastic strains so that

$$\dot{\kappa} = \gamma\, z \tag{6}$$

where z will be scalar or vector depending on the hardening rule. We shall also denote

$$q = \frac{\partial P}{\partial \kappa}\,.$$

Example 1 (continuation):

$$p = \frac{\partial P}{\partial \sigma} = \sqrt{\frac{3}{2}}\,\frac{s}{\parallel s \parallel}\,, \quad q = \frac{\partial P}{\partial \kappa} = -H^{'}\,, \quad z = 1\,.$$

Example 2 (continuation):

$$p = \frac{\partial P}{\partial \sigma} = \sqrt{\frac{3}{2}}\,\frac{s-\kappa}{\parallel s-\kappa \parallel}\,, \quad q = \frac{\partial P}{\partial \kappa} = -p\,, \quad z = c_o p\,.$$

The problem of elasto-plasticity can now be formulated as the following *initial value problem*: find

$$u = u(x,t), \quad \sigma = \sigma(x,t), \quad \kappa = \kappa(x,t)$$

such that

$$\int_\Omega \dot{\sigma}^T \varepsilon(v)\, dx = (\dot{F}, v) \quad \text{for all} \quad v \epsilon V, \quad t \epsilon (0,T]$$

$$\dot{\sigma} = D_{ep}(\sigma, \kappa, \dot{\varepsilon})\, \dot{\varepsilon}$$

$$\dot{\kappa} = G(\sigma, \kappa, \dot{\varepsilon})\, \dot{\varepsilon}$$

$$\dot{\varepsilon} = \varepsilon(\dot{u}), \quad \dot{u} \epsilon V$$

with initial conditions

$$u(x,0) = 0, \qquad \sigma(x,0) = 0, \qquad \kappa(x,0) = 0 \quad \text{for all } x \epsilon \Omega.$$

Note that V is the space of admissible displacements, $V \subset [H^1(\Omega)]^3$ and V contains functions vanishing on the part Γ_0 of the boundary $\partial\Omega$ which is fixed. F is the functional of external forces.

Moreover, D_{ep} and G are incremental constitutive relations derived from the assumption that the differential $dP = 0$ during plastic yielding, cf. [13], [24]. They satisfy

$$D_{ep}(\sigma, \kappa, \dot{\varepsilon}) = D - \rho(\sigma, \kappa, \dot{\varepsilon})\, D_p(\sigma, \kappa), \tag{7}$$

$$G(\sigma, \kappa, \dot{\varepsilon}) = \rho(\sigma, \kappa, \dot{\varepsilon})\, G_p(\sigma, \kappa), \tag{8}$$

where

$$D_p = \frac{Dpp^T D}{p^T Dp - q^T z}, \qquad G_p = \frac{zp^T D}{p^T Dp - q^T z},$$

$$\rho(\sigma, \kappa, \dot{\varepsilon}) = \begin{cases} 0 & \text{if} \quad p^T \dot{\varepsilon} < 0 \\ 1 & \text{if} \quad p^T \dot{\varepsilon} \geq 0 \end{cases}$$

For the numerical solution, we shall approximate

$$u(x, t_k),\ \sigma(x, t_k),\ \kappa(x, t_k) \quad \text{where} \quad t_k = k \triangle t$$

by

$$u_h^k(x) \quad , \sigma_h^k(x), \quad , \kappa_h^k(x)$$

where $u_h^k \, \epsilon V_h$ which is some finite element subspace of V. The unknown functions can be computed by the following *incremental finite element algorithm*:

Initial step: $\quad u_h^0 = 0, \quad \sigma_h^0 = 0, \quad \kappa_h^0 = 0$

Load steps: $\quad$ for $k = 0, \cdots, T/\triangle t - 1$

Given u_h^k, σ_h^k, κ_h^k , compute $\triangle u_h$, $\triangle \sigma_h$, $\triangle \kappa_h$:

$$\int_\Omega \triangle \sigma_h^T \, \varepsilon(v_h) \, dx = (\triangle F^k, v_h) \qquad \text{for all } v_h \epsilon V_h \, , \tag{9}$$

$$\triangle \sigma_h = T(\sigma_h^k, \kappa_h^k, \triangle \varepsilon) \tag{10}$$

$$\triangle \kappa_h = G(\sigma_h^k, \kappa_h^k, \triangle \varepsilon) \tag{11}$$

$$\triangle \varepsilon = \varepsilon(\triangle u_h), \quad \triangle u_h \epsilon V_h.$$

Put $\quad u_h^{k+1} = u_h^k + \triangle u_h, \quad \sigma_h^{k+1} = \sigma_h^k + \triangle \sigma_h, \quad \kappa_h^{k+1} = \kappa_h^k + \triangle \kappa_h$.

End

3 Explicit stress computation

For computation of the stress increment $\triangle \sigma$ from the given strain increment $\triangle \varepsilon$ in the $k-$th load step, we can exploit the incremental constitutive relation, i.e.

$$\triangle \sigma_h = \int_{t_k}^{t_{k+1}} \dot{\sigma}_h \, dt \quad \text{with} \quad \dot{\sigma}_h = D_{ep}(\sigma, \kappa, \dot{\varepsilon})\dot{\varepsilon} \qquad \text{for} \quad t \epsilon [t_k, t_{k+1}].$$

The simplest stress computation procedure derived from this relation is the following,

$$\triangle\sigma_h = T(\sigma_h^k, \kappa_h^k, \triangle\varepsilon_h) = D\triangle\varepsilon_h - \tilde{\rho}\, D_p(\sigma_h^k, \kappa_h^k)\triangle\varepsilon_h, \tag{12}$$

where $\tilde{\rho} = \tilde{\rho}(\sigma_h^k, \kappa_h^k, \triangle\varepsilon_h)$ is the state multiplicator defined by

$$\tilde{\rho}(\sigma_h^k, \kappa_h^k, \triangle\varepsilon_h) = \begin{cases} 0 & \text{if} \quad P(\sigma_h^k + D\triangle\varepsilon_h, \kappa_h^k) \leq 0 \\ 1 & \text{if} \quad P(\sigma_h^k + D\triangle\varepsilon_h, \kappa_h^k) > 0 \end{cases}$$

This formula is of course not very accurate especially for truly finite load increments. This low accuracy can be understood from violation of the consistency condition

$$P_{k+1} = 0\,, \quad P_{k+1} = P(\sigma_h^{k+1}\,, \kappa_h^{k+1}).$$

Thus, there is a need for using a more accurate stress computation procedure

$$\triangle\sigma_h = T^{(m)}(\sigma_h^k, \kappa_h^k, \triangle\varepsilon_h) = \triangle\sigma_h^m \tag{13}$$

where

$$\triangle\sigma_h^0 = 0, \quad \triangle\sigma_h^j = \sum_{i=1}^{j} T(\sigma_h^k + \triangle\sigma_h^{i-1}, \kappa_h^k + \triangle\kappa_h^{i-1}, \triangle\varepsilon/m) \quad \text{for}\, j = 1, \ldots, m.$$

Here, the increments of hardening parameters $\triangle\kappa_h^j$ are computed from (8) in the same manner as the stress increments. Note that this more accurate stress computation procedure is known as subincrementation or substepping, see [19], [24]. This procedure resembles the Euler forward method and accordingly, we shall refer to them as *explicit stress computation procedure.* Naturally, we let $T^1 = T$.

A drawback of the explicit stress computation procedure, which is not so apparent, lies in the discontinuity of T or $T^{(m)}$ which may cause difficulties in numerical solution of the nonlinear system given by (9),(10). This discontinuity arises in transition from elastic to plastic state by a sudden change of the state multiplicator $\tilde{\rho}$. However, note that the size of this discontinuity (jump) can be reduced by taking smaller load increments or subincrements.

This discontinuity can be fully removed, see [13], [8], if we use a *regularized state multiplicator* $\bar{\rho} = \bar{\rho}_\delta$ defined by

$$\bar{\rho}_\delta\,(\sigma_h^k, \kappa_h^k, \triangle\varepsilon_h) = \begin{cases} 0 & \text{if} \quad P = P(\sigma_h^k + D\triangle\varepsilon_h, \kappa_h^k) \le \delta \\ 1 + P/\delta & \text{if} \quad -\delta < P \le 0\,, \\ 1 & \text{if} \quad P > 0 \end{cases} \tag{14}$$

where δ is a positive constant. Using the regularized state multiplicator, we obtain continuous stress operators $\bar{T}$ and $\bar{T}^{(m)}$.

Two questions arise. The first one is the convergence of the discrete solution to the solution of the plasticity problem. In this respect, we can refer to the book [13]. The second question is how to solve efficiently the nonlinear system given by (9), (10). This question will be discussed in the next Section.

4 Iterative solution of the nonlinear systems

Using an isomorphism $R^n \longrightarrow V_h$, the equations (9) and (10) can be written as a system of nonlinear equations

$$\mathcal{F}_k(\triangle u) = \triangle f^k\,, \qquad \triangle u,\, \triangle f^k \epsilon\, R^n\,. \tag{15}$$

This system can be solved by Newton-like iterative methods in the form

$$\triangle u^{i+1} \;=\; \triangle u^i + \delta^i, \quad \triangle u^0 \quad \text{given} \tag{16}$$

$$A_{k\,i}\,\delta^i \;=\; r^{k,i} \;=\; \triangle f^k - \mathcal{F}_k(\triangle u^i) \tag{17}$$

where $A_{k\,i}$ are $n \times n$ matrices defined by the identity

$$< A_{k\,i}\,u, v >= \int_\Omega < D_{k\,i}\,\varepsilon(u_h),\, \varepsilon(v_h) \;>\; dx \tag{18}$$

valid for all $u, v \epsilon R^n$.

Note that here and in the sequel u_h, v_h, etc. denote the finite element functions corresponding respectively to the vectors u, v etc. from R^n. Moreover, $< u, v >= u^T v$.

Natural choices for $D_{k\,i}$ are

$$D_{k\,i} = D \tag{19}$$

$$D_{k\,i} = D - \tilde{\rho}(\sigma_h^k, \kappa_h^k, \varepsilon(\triangle u_h^i))\, D_p(\sigma_h^k, \kappa_h^k) \tag{20}$$

$$D_{k\,i} = D - \bar{\rho}(\sigma_h^k, \kappa_h^k, \varepsilon(\triangle u_h^i))\, D_p(\sigma_h^k, \kappa_h^k) \tag{21}$$

The first choice gives the *initial stiffness (IS) method* the other choices give the *tangential stiffness (TS) method* , see [8], [13]. We shall not consider other possible techniques for solving the nonlinear system (15), such as e.g. quasi-Newton methods, see [22], [24].

For the IS method the matrix $A_{k\,i}$ remains constant and equal to the elasticity stiffness matrix A_e. This fact is advantageous if (15) is solved by some direct solution method or if we work with a complicated plasticity model for which the TS method gives $A_{k\,i}$ with some properties which are not suitable (e.g. because the non-associated plasticity, $A_{k\,i}$ will be not symmetric). A great advantage of the TS method is the in general faster convergence; some comparisons can be found in [9] , [10], for instance.

For large scale problems , it is advantageous to use an iterative method for solving the auxiliary linear systems (15). In this case, a great deal of the computational effort can be saved by inexact solution of the system (15), i.e. by solving this system with some lower accuracy. This idea gives the *inexact initial stiffness (IIS)* and the *inexact tangential stiffness (ITS) methods.*

The convergence analysis of the inexact IS and inexact TS methods, under assumption that the regularized state multiplicator $\bar{\rho}$ is used, can be found in [8]. We now briefly summarize the convergence results, which have been obtained.

Let us introduce two norms in R^n :

$$\| u \|_E = \sqrt{< A_e u, u >}, \quad \| u \|_{E^{-1}} = \sqrt{< A_e^{-1} u, u >} \tag{22}$$

where A_e is the elasticity stiffness matrix given by the identity

$$< A_e u, v >= \int\limits_{\Omega} < D\varepsilon(u_h), \varepsilon(v_h) > dx \tag{23}$$

for all $u, v \epsilon R^n$.

Theorem 1. *Let the load increments $\triangle f^k$ be sufficiently small. Then for $\triangle u^0 = 0$ or for $\triangle u^0$ sufficiently close to the zero vector, both the inexact initial stiffness and the inexact tangential stiffness methods converge to the unique solution of the nonlinear system (15) .*

Moreover, there is a constant $\zeta < 1$ such that

$$\| r^{k,i+1} \|_{E^{-1}} \leq \zeta \ \| r^{k,i} \|_{E^{-1}} \tag{24}$$

for the residuals $r^{k,i}$ from (17).

Theorem 2. *Let the assumptions of Theorem 1 be fulfilled and let us consider the IIS and ITS methods for which the correction $\tilde{\delta}^{\,i}$ fulfils the inequality*

$$\| A_{k\,i}\,\tilde{\delta} - r^{k\,i} \|_{E^{-1}} \leq \eta \ \| r^{k,i} \|_{E^{-1}} \ . \tag{25}$$

Then both the inexact initial stiffness and the inexact tangential stiffness methods converge if $\zeta + \eta + \zeta\eta \ < \ 1$ where ζ is the reduction factor from (24).

Note: For the rate of convergence it is important that we have not required any damping of the corrections $\delta^{\,i}$ or $\tilde{\delta}^{\,i}$ in the above Theorems.

The proofs of both Theorems can be found in [8]. We shall only mention the following important facts exploited in the proofs.

1. Assuming that

$$\frac{p^T\,Dp}{p^T Dp - q^T z} \ \leq \ \nu_0 \ < \ 1\,, \tag{26}$$

we can prove that

$$\int_\Omega < D_p \varepsilon(u_h),\, \varepsilon(v_h) > dx \;\le\; \nu_0 \parallel u \parallel_E \parallel v \parallel_E$$

for all $u, v \epsilon R^n$.

The assumption (26) can be easily verified, see the following examples.

Example 1 (continuation):

$$\frac{p^T D p}{p^T D p - q^T z} \;=\; \frac{3\mu}{3\mu + H'} \;\le\; \frac{3\mu}{3\mu + H_0} \;=\; \nu_0 \;<\; 1 \quad \text{if} \quad H^{'}(x) \;\ge\; H_0 \;>\; 0.$$

Example 2 (continuation):

$$\frac{p^T\, D p}{p^T\, D p - q^T z} \;=\; \frac{3\mu}{3\mu + \frac{3}{2}\, c_0} = \nu_0 \;<\; 1$$

2. For the regularized state multiplicator $\bar{\rho}(x, u) \;=\; \bar{\rho}_\delta(\sigma(x), \kappa(x), \varepsilon(u_h))$ with the smoothing parameter δ (see (14)), we obtain

$$\max_{x \epsilon \Omega} \;\mid\; \bar{\rho}(x, u) - \bar{\rho}(x, v) \mid \le\; K \parallel\; u - v \;\parallel_E$$

for all $u, v \epsilon R^n$. Note that the constant $K \;=\; K(\delta, h)$ is increasing with $h \to 0_+$ which implies that the convergence of the IIS and ITS does not deteriorate with mesh refinement if some kind of stability condition holds. For the linear finite elements, we obtain $K = O(h^{-3/2})$ and the stability condition is then $\parallel \triangle f^k \parallel = O(h^{3/2})$.

3. We can prove the following *approximation condition*

$$\parallel \mathcal{F}_k(u) - \mathcal{F}_k(v) - A_e(u - v) \parallel_{E^{-1}} \le\; \zeta_v \parallel A_e(u - v) \parallel_{E^{-1}} , \qquad (27)$$

$$\zeta_v \;=\; \nu_0 + \nu_0 K \parallel v \parallel_E \le \zeta \;< 1 \quad \text{for a sufficiently small} \quad v\,.$$

4. For the inexact IS method, we then have (cf. [4], [8])

$$\| r^{k,i+1} \|_{E^{-1}} = \| r^{k,i} - A_e\tilde{\delta}^i + r^{k,i+1} - r^{k,i} + A_e\tilde{\delta}^i \|_{E^{-1}}$$

$$\leq \quad \| r^{k,i} - A_e\tilde{\delta}^i \|_{E^{-1}} + \| -\mathcal{F}_k(\triangle u^i + \tilde{\delta}) + \mathcal{F}_k(\triangle u^i) + A_e\tilde{\delta}^i \|_{E^{-1}}$$

$$\leq \quad \eta \| r^{k,i} \|_{E^{-1}} + \zeta \| A_e\tilde{\delta}^i \|_{E^{-1}} \leq [\eta + \zeta(1+\eta)] \; \| r^{k,i} \|_{E^{-1}}$$

Note that the use of the approximate condition in the above estimate is possible if we guarantee that $\| \triangle u^i \|_E$ is sufficiently small. This can be done by the assumption of sufficiently small load increments.

5. For the inexact TS method, we can use a somewhat different approximation condition, cf. [8].

5 Return mapping stress computation

In this Section, we describe a new procedure for stress computation within the incremental finite element scheme. This procedure is based on trial elastic stress (elastic predictor) and return of this trial stress onto the yield surface (plastic corrector), cf. e.g. [16],[24]. This new return mapping (RM) stress computation procedure guarantees that the consistency condition holds and enhances accuracy and stability [17].

The RM stress computation procedure proceeds in the following two steps:

1. *elastic predictor*: from a given strain increment $\triangle\varepsilon_h$ compute the *elastic stress increment* $\triangle\sigma_h^e$ and *the trial stress* σ_h^t,

$$\triangle\sigma_h^e = D\,\triangle\varepsilon_h\,, \quad \sigma_h^t \;=\; \sigma_h^k + \triangle\sigma_h^e \tag{28}$$

2. *plastic corrector*: the increments of stress and hardening parameters are computed by mapping the trial stress back to the yield surface

$$\triangle\sigma_h = \triangle\sigma_h^e - \gamma D p \tag{29}$$

$$\triangle\kappa_h = \gamma\, z \tag{30}$$

with γ given by the consistency condition

$$P(\sigma_h^k + \triangle\sigma_h,\, \kappa_h^k + \triangle\kappa_h) = 0. \tag{31}$$

The RM stress computation procedure defines a new *stress operator* T_k,

$$T_k(\triangle\varepsilon) = T_k^{RM}(\sigma_h^k, \kappa_h^k, \triangle\varepsilon),$$

$$T_k(\triangle\varepsilon) \;=\; \begin{cases} D\triangle\varepsilon & \text{if} \quad P(\sigma_h^t,\, \kappa_h^k) \;\leq\; 0, \\ D\triangle\varepsilon - \gamma_R\,\hat{n} & \text{if} \quad P(\sigma_h^t,\, \kappa_h^k) \;>\; 0, \end{cases} \tag{32}$$

γ_R is a *return parameter* and $\hat{n} = Dp$.

Example 1 (continuation):

$$p = \sqrt{\frac{3}{2}}\,\frac{s}{\parallel s \parallel}, \quad s = \mathrm{dev}(\sigma), \quad z = 1,$$

Implicit stress computation: we take $\sigma = \sigma_h^{k+1}$, $s = s_h^{k+1}$. Then

$$\sigma_h^{k+1} = \sigma_h^k + \triangle\sigma_h^e - \gamma Dp = \sigma_h^t - \gamma 2\mu p,$$

$$s_h^{k+1}\,[1 + \gamma 2\mu\sqrt{\frac{3}{2}} \parallel s_h^{k+1} \parallel^{-1}] = \mathrm{dev}(\sigma_h^t).$$

Thus, we are able to express the direction p in terms of the trial stress,

$$p = \sqrt{\frac{3}{2}}\;\hat{n}, \quad \hat{n} \;= \parallel s_h^t \parallel^{-1}\; s_h^t, \quad s_h^t \;=\; \mathrm{dev}(\sigma_h^t),$$

$$\triangle\sigma_h = \;\triangle\sigma_h^e - \gamma_R\,\hat{n}, \quad \gamma_R \;=\; \sqrt{\frac{3}{2}}\,\gamma\,2\mu.$$

It remains to determine the return parameter γ_R. Using the expression for $\triangle\kappa_h$,

$$\triangle\kappa_h \;=\; \sqrt{\frac{2}{3}} \parallel \triangle\varepsilon^p \parallel = \mid \gamma\, z \mid = \;\gamma\; = \sqrt{\frac{2}{3}}\,\frac{1}{2\mu}\,\gamma_R$$

and substituting it into (31), we obtain

$$\begin{aligned} 0 &= \sqrt{\frac{3}{2}} \parallel s_h^{k+1} \parallel - H(\kappa_h^{k+1}) \\ &= \sqrt{\frac{3}{2}} \parallel s_h^t - \gamma_R \parallel s_h^t \parallel^{-1} \; s_h^t \parallel - H(\kappa_h^k + \sqrt{\frac{2}{3}} \frac{\gamma_R}{2\mu}) \\ &= \sqrt{\frac{2}{3}} (\parallel s_h^t \parallel - \gamma_R) - H(\kappa_h^k + \sqrt{\frac{2}{3}} \frac{\gamma_R}{2\mu}). \end{aligned}$$

Hence the return mapping parameter γ_R can be computed from the, in general, nonlinear equation

$$\parallel s_h^t \parallel - \gamma_R = \sqrt{\frac{2}{3}} \, H(\kappa_h^k + \sqrt{\frac{2}{3}} \frac{\gamma_R}{2\mu}). \tag{33}$$

Note that in the frequently used case of linear hardening function, the above equation becomes linear.

Example 2 (continuation):

$$p = \sqrt{\frac{3}{2}} \frac{s - \kappa}{\parallel s - \kappa \parallel}, \quad z = c_0 p.$$

Implicit stress computation: we take $s = s_h^{k+1}$, $\kappa = \kappa_h^{k+1}$. Then

$$\begin{aligned} \sigma_h^{k+1} &= \sigma_h^k + \triangle\sigma_h^e - \gamma D p = \sigma_h^t - \gamma 2\mu p, \\ \kappa_h^{k+1} &= \kappa_h^k + \gamma z = \kappa_h^k + \gamma c_0 p. \end{aligned}$$

Thus,

$$s_h^{k+1} - \kappa_h^{k+1} = s_h^t - \kappa_h^k - \gamma(2\mu + c_o) p,$$

$$p = \sqrt{\frac{3}{2}}\,\hat{n}, \quad \hat{n} = \frac{s_h^t - \kappa_h^k}{\| s_h^t - \kappa_h^k \|},$$

$$\triangle\sigma_h = \triangle\sigma_h^e - \gamma_R\,\hat{n}, \quad \triangle\kappa_h = \frac{c_0}{2\mu}\,\gamma_R\hat{n}.$$

The consistency condition (31) gives

$$0 = \sqrt{\frac{3}{2}}\ \| s_h^{k+1} - \kappa_h^k \| - H_0 =$$

$$= \sqrt{\frac{3}{2}}\, \| s_h^t - \kappa_h^k - \gamma_R\,(1 + \frac{c_0}{2\mu})\,\hat{n}\ \| - H_0.$$

Hence,

$$\| s_h^t - \kappa_h^k \ \| - (1 + \frac{c_0}{2\mu})\,\gamma_R = \sqrt{\frac{2}{3}}\,H_0. \tag{34}$$

Note: In the case of the von Mises plasticity the yield surface is a sphere in the space of the stress deviators and the return to this sphere is made in direction of the proper radius to the sphere, frequently referred to as “radial return”.

Finally, we mention some properties of the RM stress operator T_k^{RM} which determine properties of the nonlinear system (9), (10).

The first important property is the continuity of T_k^{RM}. It is an important difference compared with the explicit stress computation. Further properties are differentiability in $S \setminus S_k^i$, $S_k^i = \{\varepsilon \in S : P(\sigma_h^k + D\varepsilon, \kappa_h^k) = 0\}$ and one-sided differentiability in S_k^i. For the derivative $T_k^{'}(\varepsilon)$, $\varepsilon \in S \setminus S_k^i$ we can prove symmetry, uniform boundedness, local uniform positive definiteness and local Lipschitz continuity. For a proof, see [7].

6 Iterative solution of the nonlinear systems arising from RM algorithms

In each load step of the incremental finite element algorithm, we have to solve the nonlinear system (9) and (10). As in Section 4, we can rewrite this system into an algebraic form

$$\mathcal{F}_k(\triangle u) \;=\; \triangle f^k, \quad \triangle u, \triangle f^k \,\epsilon R^n \tag{35}$$

where the nonlinear mapping $\mathcal{F}_k$ is defined variationally by the identity

$$< \mathcal{F}_k(u), v >= \int_\Omega < T_k(\varepsilon(u_h)), \, \varepsilon(v_h) \; > \; dx \tag{36}$$

which should be valid for all $u, v \epsilon R^n$.

Using the return mapping stress computation procedure, i.e. using $T_k = T_k^{RM}$, we obtain the nonlinear mapping $\mathcal{F}_k$ which is continuous and, with some later mentioned assumption, differentiable with the Frechet derivative $\mathcal{F}'_k(u)$ given variationally by

$$< \mathcal{F}'_k(u)w, v >= \int_\Omega \; < T'_k(\varepsilon(u_h))\varepsilon(w_h), \varepsilon(v_h) \; > \; dx. \tag{37}$$

For the von Mises plasticity with isotropic hardening, the derivative of the stress operator $T_k \;=\; T_k^{RM}$ is the following (see [16],[7]).

$$T'_k(\varepsilon)\,\eta \;=\; D\eta \qquad \text{if } \varepsilon \,\epsilon\, S_k^e, \tag{38}$$

$$\;=\; D\eta - 2\mu\bar{\beta}\text{dev}(\eta) - 2\mu\bar{\gamma}\hat{n}\hat{n}^T\eta \qquad \text{if } \varepsilon \,\epsilon\, S_k^p, \tag{39}$$

$$T'_k(\varepsilon)\eta+ \;=\; D\,\eta \qquad \text{if } \varepsilon \,\epsilon\, S_k^i, \;\; \hat{n}^T\eta \;\leq\; 0, \tag{40}$$

$$\;=\; D\eta - 2\mu\tilde{\gamma}\hat{n}\hat{n}^T\eta \qquad \text{if } \varepsilon \,\epsilon\, S_k^i, \;\; \hat{n}^T\eta \;>\; 0. \tag{41}$$

where

$$S_k^e \;=\; \{\, \varepsilon\epsilon S \;:\; P(\sigma_h^k + D\varepsilon, \, \kappa_h^k) \;<\; 0 \,\},$$
$$S_k^p \;=\; \{\, \varepsilon\epsilon S \;:\; P(\sigma_h^k + D\varepsilon, \, \kappa_h^k) \;>\; 0 \,\},$$
$$S_k^i \;=\; \{\, \varepsilon\epsilon S \;:\; P(\sigma_h^k + D\varepsilon, \, \kappa_h^k) \;=\; 0 \,\},$$

$$\hat{n} = \hat{n}(\varepsilon) = \| s^t \|^{-1} s^t, \quad s^t = \mathrm{dev}(\sigma_h^k + D\varepsilon),$$

$$\bar{\beta} = \gamma_R(\varepsilon) \| s^t \|^{-1}, \quad \bar{\gamma} = \tilde{\gamma} - \bar{\beta}, \tag{42}$$

$$\tilde{\gamma} = \frac{3\mu}{3\mu + H'_\varepsilon}, \quad H'_\varepsilon = H' \left(\kappa_h^k + \sqrt{\frac{2}{3}} \frac{\gamma_R(\varepsilon)}{2\mu} \right). \tag{43}$$

Note that for $\varepsilon \in S_k^i$ the derivative T'_k does not exist. We can take only one–sided derivatives $T'_k(\varepsilon)\eta + = \lim_{\theta \to 0+} \theta^{-1}[T_k(\varepsilon + \theta\eta) - T_k(\varepsilon)]$.

Thus, the operator $\mathcal{F}'_k(u)$ given by (37) will be the the Frechet's derivative only if $\varepsilon(u_h) \notin S_k^i$ almost everywhere in Ω.

The nonlinear system (35) can be solved again by the *initial stiffness method* . For this method, we have

$$\triangle u^{i+1} = \triangle u^i + \delta^i \tag{44}$$

$$A_e \delta^i = r^{k,i} = \triangle f^k - \mathcal{F}_k(\triangle u^i) \tag{45}$$

where A_e is the elasticity stiffness matrix.

Assuming the existence of $\mathcal{F}'_k$, we can also use the *Newton method*, given by

$$\triangle u^{i+1} = \triangle u^i + \delta^i, \tag{46}$$

$$\mathcal{F}'_k(\triangle u^i)\, \delta^i = r^{k,i} \tag{47}$$

As in Section 4, it is also useful to consider *inexact* initial stiffness and *inexact* Newton methods with some lower solution accuracy of the auxiliary linear problems (45) and (47).

The convergence of the initial stiffness and inexact Newton methods for the case of von Mises plasticity with isotropic hardening is studied in the following two theorems.

Theorem 3. Let the load increments be sufficiently small. Then the initial stiffness method converges to the solution of (35). Moreover, there is a constant $\zeta < 1$ such that

$$\| r^{k,i+1} \|_{E^{-1}} \leq \zeta \| r^{k,i+1} \|_{E^{-1}}$$

for the norm defined by (22).

Inexact initial stiffness method, obtained by replacing δ^i by $\tilde{\delta}^i$ which fulfils the condition

$$\| A_e \tilde{\delta}^i - r^{k,i} \|_{E^{-1}} \leq \eta \| r^{k,i} \|_{E^{-1}},$$

will converge if $\zeta + \eta + \zeta\eta < 1$.

The proof of the theorem can be based again on the approximation condition

$$\| \mathcal{F}_k(u) - \mathcal{F}_k(v) - A_e(u-v) \|_{E^{-1}} \leq \xi \| A_e(u-v) \|_{E^{-1}} .$$

For the details, see [7].

Theorem 4. Let $\triangle u$ be the solution of (35) and let us assume that

- $\triangle u^0$ be sufficiently good initial guess of $\triangle u$, i.e. let $r^{k,0} = \triangle f^k - \mathcal{F}_k(\triangle u^0)$ be sufficiently small,
- it is possible to construct the Newton sequence $\triangle u^0, \triangle u^1, \cdots$ from (46) and (47),
- any segment connecting the trial stress states $\sigma_h^k + D\varepsilon(\triangle u_h^i)$ and $\sigma_h^k + D\varepsilon(\triangle u_h)$ does not cross the yield surface, i.e. for all $\theta \epsilon (0,1)$ it holds:

$$P(\sigma_h^k + \theta D\varepsilon(\triangle u_h^i) + (1-\theta) D\,\varepsilon(\triangle u_h),\, \kappa_h^k) \neq 0.$$

Then the Newton method converges quadratically to $\triangle u$, i.e., there is a constant ξ such that

$$\| \triangle u^{i+1} - \triangle u \| \leq \xi \| \triangle u^i - \triangle u \|^2 .$$

For the inexact Newton method, obtained by replacing δ^i by $\tilde{\delta}^i$ which fulfils the condition

$$\| \mathcal{F}_k'(\triangle u^i)\tilde{\delta}^i - r^{k,i} \| \leq \eta_i \| r^{k,i} \|,$$

the superlinear convergence occurs if $\eta_i \to 0$.

To obtain the quadratic convergence, we can take e.g.(see [1])

$$\eta_i \leq C_A \parallel r^{k,i} \parallel .$$

The proof of this theorem is a modification of the proof of the well-known theorem on convergence of the inexact Newton method, see e.g. [1], It utilizes Lipschitz continuity and uniform boundedness of the inverse of $\mathcal{F}'_k$, see [7].

In practice, we need a fast and globally convergent method. It can be obtained by combination of the inexact initial stiffness method (at the beginning phase of iterative process) and the inexact Newton method (at the final phase). We can also use globally convergent Newton method with damping in the form

$$\triangle u^{i+1} = \triangle u^{i} + \omega_i \tilde{\delta}^{i} \text{ with } \omega_i < 1$$

at the initial phase of the iterative process, cf. [1], [17].

Other, recently analysed techniques involve the use of inexact Newton type methods with extended subspaces, see [12], [4]. Another type of methods are based on two level meshes, a coarse mesh (with mesh parameter H) on which a solution is computed to provide initial approximation for the solution of the nonlinear differential equation on a (much) finer mesh (with parameter h). Using just a single Newton step on this mesh, it was shown in [5] and [23], that if a relation $H = O(h^{1/2})$ or, in special cases, even $H = O(h^{1/4})$ holds, then $\parallel u - u_h \parallel_E = O(\inf_{v_h \in V_h} \parallel u - v_h \parallel_E)$. Here u_h is the approximate solution in the finite element space V_h, computed by the above two–level method and u is the solution to the differential equation. Hence the coarse mesh can be taken extremely coarse. However, the above relation assumes a sufficient smoothness of the solution, which may not be satisfied for plasticity problems. Therefore, the above relations may be useless in practice and one must still solve the fine mesh problem using a sufficient number of Newton steps or incremental finite element steps, as suggested in this paper.

As a conclusion, we see that some further research is still necessary for developing efficient and robust iterative solver for the return mapping algorithms.

7 Numerical experiments

The numerical experiments to be presented concern only the incremental finite element method with explicit stress computation using adaptive substepping according to size of violating of the consistency condition, see [15].

The experiments are performed with the inexact tangential stiffness method with the following aims:

1. to show the effect of inexact solution of the auxiliary linear problems,
2. to compare the efficiency of several kinds of preconditioners for the solution of auxiliary problems,
3. to demonstrate the increase of the computational work in the case of refinement of
   (a) the size of the finite element mesh,
   (b) the size of the load increments.

For the experiments, we choose a model problem (square footing) defined in a domain which is a parallelepiped of 40x40x10 meters. A square of 2x2 metres in the center of the top side represents flexible footing on plastic soil. The footing is modelled by a pressure of 5 MPa.

The material of the domains has an elasticity modulus E = 130 MPa, Poisson ratio $\nu = 0.3$, von Mises plasticity condition with isotropic hardening modulus $H^{'} = 10.83$ MPa and a yield limit for uniaxial loading $\sigma_y = 0.58$ MPa.

Due to symmetry, only one quarter of the domain is modelled and zero normal displacements are prescribed on all sides of the domain with exception of the top side where the pressure 5 MPa is applied on the square of 1x1 meter (one quarter of footing).

The problem is discretized in a 33x33x17 node grid, locally refined under the footing, see Figure 1. The domain is divided into bricks which are further divided each into six tetrahedra. The dimension of the arising nonlinear systems is 55 539.

The nonlinear iterations are terminated when the next condition holds:

$$\| r^{k,i} \| \leq \varepsilon \| \triangle f^k \| \ \text{ with } \varepsilon = 0.001.$$

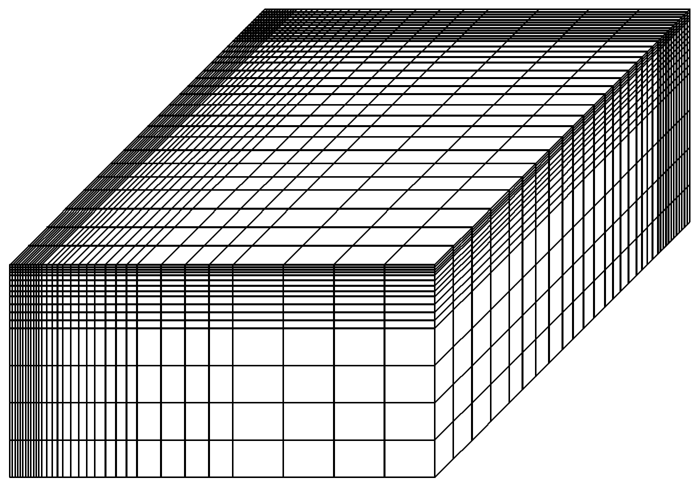

Figure 1: Square footing problem

Table 1: Various choices of accuracy for inner iterations. Grid 33x33x17, load increments 1/5, DiD-IF preconditioning.

$\eta =$	0.001	0.010	0.100	ADAPT
NeIT	13	13	13	14
InIT	625	472	278	223
TIME	17250	15497	13889	13559

Table 2: Comparison of preconditioners. Grid 33x33x17, load increments 1/5, $\eta = 0.1$.

preconditioning:	NO	DS	SSOR	DiD-IF
InIT	20294	1793	402	278

The auxiliary systems (17) are solved with the relative accuracy η,

$$\| A_{k,i} \tilde{\delta}^i - r^{k,i} \| \le \eta \| r^{k,i} \| .$$

The adaptive technique (ADAPT) choses η close to the reduction factor of the nonlinear iterations, e.g.,

$$\eta = \min \{0.5 ; 0.9 \| r^{k,i} \| / \| r^{k,i-1} \| \} .$$

For the first load increment covering the whole elastic behaviour, we take $\eta = \varepsilon = 0.001$.

The effect of inexact solution of auxiliary problems can be seen from Table 1. Here DiD-IF is an abbreviation for preconditioning by displacement decomposition-incomplete factorization technique, see [6]. In addition to this preconditioning, we have also tested use of no preconditioning (NO), diagonal scaling (DS) and SSOR preconditioning, cf. [3].The comparison of these preconditioners can be seen from Table 2. Here NeIT denotes the number of nonlinear (Newton-like) iterations required in all load steps, InIT is the total number of inner (PCG) iterations and TIME is the computer time (in seconds) of the whole computation when the software GEM32 (developed at Institute of Geonics AS CR, Ostrava) and the computer IBM RS/6000 m.320 with 24MB RAM is used.

Table 3 shows the numbers of nonlinear iterations and the numbers of PCG iterations for solving the model plasticity problem on the grids with 9x9x5, 17x17x9 and 33x33x17 nodes, i.e. on grids with gradually halved mesh size.

Table 4 shows again the numbers of nonlinear and PCG iterations but now we use only the grid with 33x33x17 nodes and change the size of load increments to be at most 1/5, 1/10 and 1/20 of the total load.

Table 3: Effect of the grid size on the number of nonlinear and inner iterations . Load increments 1/20, $\eta = 0.1$, DiD-IF preconditioning.

GRID	9x9x5		17x17x9		33x33x17	
load step	NeIT	InIT	NeIT	InIT	NeIT	InIT
1	0	13	0	25	0	45
2	1	3	1	5	2	21
3	1	3	2	12	2	22
4	1	4	2	14	2	22
5	2	8	2	16	2	28
6	1	4	2	16	2	28
7	1	4	1	7	2	28
8	1	4	2	18	2	29
9	1	4	1	7	2	30
10	1	5	1	7	1	11
11	1	5	1	9	1	11
12	1	5	1	7	1	12
13	1	4	1	7	1	11
14	1	5	1	7	1	11
15			1	7	1	11
16			1	7	1	11
TOTAL	14	71	20	171	23	335

Table 4: Effect of the size of load increments. Grid 33x33x17, $\eta = 0.1$, DiD-IF preconditioning.

load increments	1/5		1/10		1/20	
load step	NeIT	InIT	NeIT	InIT	NeIT	InIT
1	0	45	0	45	0	45
2	3	47	2	25	2	21
3	4	70	3	47	2	22
4	3	58	3	46	2	22
5	3	58	2	31	2	28
6			2	33	2	28
7			2	32	2	28
8			2	30	2	29
9			2	29	2	30
10					1	11
11					1	11
12					1	12
13					1	11
14					1	11
15					1	11
16					1	11
TOTAL	13	278	18	318	23	335

8 Conclusions

Based on the presented results, we can make the following conclusions concerning solution of large–scale plasticity problems.

1. Some *efficient linear solver* is needed. In our numerical experiments the most efficient technique was the displacement decomposition - incomplete factorization technique. Nevertheless there are many other possibilities which can be examined, see eg. [3].

2. *Inexact* (lower accuracy) solution of the auxiliary linear systems enhances apparently the efficiency of the solution.

3. The exploited *stress computation procedure* plays an important role for the efficiency of the incremental finite element method. The substepping procedure seems to be too expensive.

4. It is important to choose efficient Newton–type *nonlinear solver* . Some new ideas have been mentioned in this paper.

From the point of view of last two conclusions the return mapping algorithm with superlinearly convergent Newton–type nonlinear solver seems to be a very promising technique.

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