2 General differential geometry

The mathematical language of GR is differential geometry, enriched by geometric analysis.

2.1 Manifolds

We start by reviewing the key definitions underlying the concept of a manifold.[128] First, a ***space*** means a topological space, assumed *Hausdorff*. The topology of M (i.e. the set of its open sets) is denoted by $\mathscr{O}(M)$, so that $U \in \mathscr{O}(M)$ means that $U \subset M$ and U is open. Since otherwise it cannot support a Lorentzian metric, in GR we may assume that M is also *metrizable*.[129] If M is a (topological) manifold, this is equivalent to M being *second countable* as well as *paracompact*.

Definition 2.1 1. *An n-dimensional* **(topological) manifold** *is a space M such that any $x \in M$ has a nbhd (= neighbourhood) $U \in \mathscr{O}(M)$ that is homeomorphic to some $V \in \mathscr{O}(\mathbb{R}^n)$. Equivalently, one may require V to be $\mathbb{R}^n$ itself, or some open ball in $\mathbb{R}^n$.*

2. *A* **chart** *on M is a pair (U,φ) where $U \in \mathscr{O}(M)$ and $\varphi : U \to \mathbb{R}^n$ is a homeomorphism onto its image $V = \varphi(U)$. A chart (U,φ) gives a* **coordinate system** *on U, in that the* **coordinates** *$(x^1,\ldots,x^n)$ of $x \in U$ of x are $x^i = \varphi^i(x)$, where one writes $\varphi : U \to \mathbb{R}^n$ as $(\varphi^1,\ldots,\varphi^n)$, where $\varphi^i : U \to \mathbb{R}$ in terms of the standard basis of $\mathbb{R}^n$ ($i = 1,\ldots,n$).*

3. *A C^k-***atlas** *on M (where $k \in \mathbb{N} \cup \{\infty\}$) is a collection of charts $(U_\alpha,\varphi_\alpha)$, where $M = \cup_\alpha U_\alpha$ (i.e. the U_α form an open cover of M), and, whenever $U_{\alpha\beta} = U_\alpha \cap U_\beta$ is not empty, writing $V_{\alpha\beta} = \varphi_\alpha(U_{\alpha\beta})$, the map $\varphi_\beta \circ \varphi_\alpha^{-1} : V_{\alpha\beta} \to \mathbb{R}^n$ is C^k (since $V_{\alpha\beta} \subset \mathbb{R}^n$ this is well defined).*

4. *Two C^k-atlases $(U_\alpha,\varphi_\alpha)$ and $(U'_{\alpha'},\varphi'_{\alpha'})$ on a topological manifold M are* **equivalent** *if their union is a C^k-atlas, i.e., if all transition functions $\varphi'_{\beta'} \circ \varphi_\alpha^{-1}$ and $\varphi_\beta \circ (\varphi'_{\alpha'})^{-1}$ (if defined) are C^k; this is indeed an equivalence relation. A C^k***-structure** *on M is an equivalence class of C^k atlases on M. A C^k-***manifold** *is a manifold with a C^k structure. A* **smooth manifold** *is a manifold with a C^∞ structure, that is, a C^∞-manifold.*

5. *A function $f : M \to \mathbb{R}$ on a smooth manifold is* **smooth**, *written $f \in C^\infty(M)$, if for some fixed atlas (within its equivalence class), each map $f \circ \varphi_\alpha^{-1} : V_\alpha \to \mathbb{R}$ is smooth.*[130]

6. *For two smooth manifolds M,N, a map $\psi : M \to N$ is* **smooth** *if for each $f \in C^\infty(N)$ the pullback $\psi^* f \equiv f \circ \psi$ is in $C^\infty(M)$. Equivalently, in terms of the manifolds: for any chart (U,φ) on M and chart $(\tilde{U},\tilde{\varphi})$ on N such that $U' = \psi(U) \cap \tilde{U} \neq \emptyset$, the function $\tilde{\varphi} \circ \psi \circ \varphi^{-1} : V' \to \tilde{V}$ is smooth (in the calculus sense), where $V' = \varphi(\psi^{-1}(U')) \subset V$.*

7. *A* **diffeomorphism** *of M is an invertible smooth map $\psi : M \to M$* with smooth inverse. *Under the obvious operations, such maps form the* **diffeomorphism group** $\mathrm{Diff}(M)$ *of M.*

Unless the contrary is stated, we henceforth assume that M is a smooth manifold equipped with some C^∞ atlas $(U_\alpha,\varphi_\alpha)$, and that all maps between smooth mathematical objects are smooth.

[128] See §2.6 for manifolds with boundary. References for this chapter are Choquet-Bruhat & DeWitt-Morette (1982), Abraham & Marsden (1985), Kriele (1999), Frankel (2004), Lee (2012), and Mărcuț (2016).

[129] See e.g. Palomo & Romero (2006), §1.1, or Minguzzi (2019), §1.8.

[130] This is then true for any atlas. Conversely, *M as a manifold* can be reconstructed from $C^\infty(M)$ *as a commutative algebra* via homomorphisms $\mathrm{ev} : C^\infty(M) \to \mathbb{R}$. See e.g. Navarro González & Sancho de Salas (2003), chapter 2.

2.2 Tangent bundle

The differential geometry relevant to GR comes from the tangent bundle, which generates the entire tensor calculus. Of the many roads to this bundle we prefer an (initially) algebraic construction in terms of derivations on $C^\infty(M)$, from which the geometric picture emerges.[131] Readers who are mainly interested in *using* tangent bundles can move straight to Definition 2.4.

Definition 2.2 *1. A* **derivation** *of an algebra A (over $\mathbb{R}$) is a linear map $\delta : A \to A$ satisfying*

$$\delta(ab) = \delta(a)b + a\delta(b). \tag{2.1}$$

2. For any smooth manifold M, a **point derivation** *at $x \in M$ is a linear map*

$$\delta_x : C^\infty(M) \to \mathbb{R} \tag{2.2}$$

that satisfies the **Leibniz rule**

$$\delta_x(fg) = \delta_x(f)g(x) + f(x)\delta_x(g). \tag{2.3}$$

In no. 2, $A = C^\infty(M)$ is seen as a (commutative) algebra with respect to pointwise operations.

The set $\mathrm{Der}(A)$ of all derivations of A is a vector space (again over $\mathbb{R}$). If A is associative and commutative, as is the case for $A = C^\infty(M)$, then $\mathrm{Der}(A)$ is also an A-module under the natural action $(a\delta)(b) = a\delta(b)$. In addition, $\mathrm{Der}(A)$ is a Lie algebra under the bracket

$$[\delta_1, \delta_2] := \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1. \tag{2.4}$$

For $M = \mathbb{R}^n$, taking $X^i = \delta(x^i)$ it follows that each derivation δ of $C^\infty(\mathbb{R}^n)$ assumes the form

$$\delta(f)(x) = \sum_{j=1}^{n} X^j(x) \frac{\partial f(x)}{\partial x^j}, \tag{2.5}$$

where $X \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, henceforth called $\mathfrak{X}(\mathbb{R}^n)$, is an "old-fashioned vector field" on $\mathbb{R}^n$, i.e. a field of arrows. Conversely, X defines a derivation $\delta \equiv \delta_X$ by reading (2.5) as a definition of δ. This gives a bijection $X \leftrightarrow \delta_X$ between the set $\mathfrak{X}(\mathbb{R}^n)$ of all *vector fields* on $\mathbb{R}^n$ and the set $\mathrm{Der}(C^\infty(\mathbb{R}^n))$ of all *derivations* on $C^\infty(\mathbb{R}^n)$. We further pass to point derivations by defining

$$\delta_x(f) := \delta(f)(x), \tag{2.6}$$

where $\delta \in \mathrm{Der}(C^\infty(\mathbb{R}^n))$. Conversely, Definition 2.2 implies that a family of point derivations $x \mapsto \delta_x$, defined for all $x \in \mathbb{R}^n$, comes from a single derivation δ via (2.6), and hence from a vector field X via $\delta = \delta_X$, iff the map $x \mapsto \delta_x(f)$ is smooth from $\mathbb{R}^n$ to $\mathbb{R}$ for each $f \in C^\infty(\mathbb{R}^n)$.

Eq. (2.4) also has a match for vector fields: $\mathfrak{X}(\mathbb{R}^n)$ is a Lie algebra under the ***commutator***

$$[X,Y](f) := X(Y(f)) - Y(X(f)). \tag{2.7}$$

[131] An ***algebra*** A (here always defined over $\mathbb{R}$) is a real vector space equipped with a bilinear map $A \times A \to A$, usually written $(a,b) \mapsto ab$. Many algebras are ***associative*** in that $(ab)c = a(bc)$ for all $a,b,c \in A$, as well as ***commutative***, i.e. $ab = ba$ for all $a,b \in A$. ***Lie algebras*** are neither: here one writes $(a,b) \mapsto [a,b]$, with axioms $[a,b] = -[b,a]$ as well as the ***Jacobi identity*** $[a,[b,c]] + [c,[a,b]] + [b,[c,a]] = 0$. A ***module*** over an algebra A is a vector space V with a bilinear map $A \times V \to V$, written $(a,v) \mapsto av$, such that $a(bv) = (ab)v$ (or $a(bv) = [a,b]v$).

In coordinates, where we use components $X = \sum_i X^i \partial_i$ and $Y = \sum_j Y^j \partial_j$, we have

$$[X,Y] = \sum_i [X,Y]^i \partial_i; \qquad [X,Y]^i = \sum_j (X^j \partial_j Y^i - Y^j \partial_j X^i). \tag{2.8}$$

Relative to (2.7) and (2.4), the bijection $X \leftrightarrow \delta_X$ is promoted to an isomorphism of Lie algebras.

Finally, if $\mathfrak{X}(\mathbb{R}^n)$ carries the $C^\infty(\mathbb{R}^n)$ action given by $(fX)^j(x) = f(x)X^j(x)$, then $X \leftrightarrow \delta_X$ is in addition an isomorphism of $C^\infty(\mathbb{R}^n)$ modules. Since $X : \mathbb{R}^n \to \mathbb{R}^n$ is given by its components $X^k : \mathbb{R}^n \to \mathbb{R}$, as a $C^\infty(\mathbb{R}^n)$ module $\mathfrak{X}(\mathbb{R}^n)$ decomposes as a direct sum of copies of $A = C^\infty(\mathbb{R}^n)$. By definition, this makes $\mathfrak{X}(\mathbb{R}^n)$ a ***free module*** over $C^\infty(\mathbb{R}^n)$. Of course, the same is then true for $\mathrm{Der}(C^\infty(\mathbb{R}^n))$. In sum, looking at a vector field X as the corresponding derivation δ_X, we often identify $\mathrm{Der}(C^\infty(\mathbb{R}^n))$ with $\mathfrak{X}(\mathbb{R}^n)$, and this identification preserves all relevant structure.

We now generalize this story to arbitrary manifolds M. On the algebraic side, we have the derivations $\mathrm{Der}(C^\infty(M))$. We are going to define vector fields geometrically as sections of the ***tangent bundle*** TM to M, whose construction is best understood in a more general form.

Definition 2.3 *A (real, locally trivial) k-dimensional* **vector bundle** *over M is an open surjective map $\pi : E \to M$, where E is a manifold, such that:*

1. *For each $x \in M$, the* **fiber** *$E_x := \pi^{-1}(x)$ is a k-dimensional (real) vector space, i.e. $E_x \cong \mathbb{R}^k$ (where k is independent of x). This is the main point. More technically:*
2. *M has an open cover (U_i) with diffeomorphisms $\Phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$ such that:*
   (a) *Each restriction $\Phi_i : E_x \to \{x\} \times \mathbb{R}^k$ is an isomorphism of vector spaces ($x \in U_i$);*
   (b) *If $U_{ij} \equiv U_i \cap U_j \neq \emptyset$, then $\Phi_{ij} \equiv \Phi_i \circ \Phi_j^{-1} : U_{ij} \times \mathbb{R}^k \to U_{ij} \times \mathbb{R}^k$ is the identity on the first coordinate and a vector space isomorphism on the second one.*

A **vector bundle map** *from $\pi_1 : E \to M$ to $\pi_2 : F \to N$ is a* pair *of maps $\varphi_f : E \to F$ and $\varphi_b : M \to N$ such that $\pi_2 \circ \varphi_f = \varphi_b \circ \pi_1$, and each "fiber" map $\varphi_f : E_x \to F_{\varphi_b(x)}$ is linear.*

The simplest k-dimensional vector bundle over M is $E = M \times \mathbb{R}^k$ with π given by projection on the first coordinate; this is called a ***trivial bundle***. A ***(cross-)section*** of E is a map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$, i.e., $\pi(s(x)) = x$ for each $x \in M$. The set of smooth sections of E is denoted by $\Gamma(E)$ or $\Gamma(M,E)$. This is a vector space. Under the natural action

$$C^\infty(M) \times \Gamma(E) \to \Gamma(E); \qquad (fs)(x) := f(x)s(x), \tag{2.9}$$

the vector space $\Gamma(E)$ is a finitely generated projective (f.g.p.) module over $C^\infty(M)$.[132]

Sections s of the trivial bundle $E = M \times \mathbb{R}^k \to M$ bijectively correspond to maps $\tilde{s} : M \to \mathbb{R}^k$ via $s(x) = (x, \tilde{s}(x))$. Hence we obtain, as an isomorphism of f.g.p. $C^\infty(M)$-modules,

$$\Gamma(M \times \mathbb{R}^k) \cong C^\infty(M, \mathbb{R}^k). \tag{2.10}$$

The ***Serre–Swan Theorem*** provides an isomorphism between f.g.p. modules $\mathscr{E}$ over $C^\infty(M)$ and vector bundles $E \to M$ over M, in such a way that $\mathscr{E} \cong \Gamma(E)$. We first define E as a set by

$$E := \sqcup_{x \in M} E_x; \qquad E_x := \mathscr{E}/\sim_x = \mathscr{E}/(C_x^\infty(M) \cdot \mathscr{E}). \tag{2.11}$$

[132] An A-module $\mathscr{E}$ is called ***finitely generated projective*** if there exists an A-module $\mathscr{F}$ such that $\mathscr{E} \oplus \mathscr{F}$ is free, i.e. isomorphic to a finite direct sum $\oplus^k A$. Equivalently, $\mathscr{E} \cong p(\oplus^k A)$ for some idempotent $p \in M_k(A)$ (i.e. $p^2 = p$).

I.e. $s_1 \sim_x s_2$ iff $s_1 - s_2 \in C_x^\infty(M)\cdot\mathscr{E}$, defined as the linear span in $\mathscr{E}$ of all fs, where $s \in \mathscr{E}$ and

$$f \in C_x^\infty(M) := \{f \in C^\infty(M) \mid f(x) = 0\}. \tag{2.12}$$

Then each fiber E_x of E is a vector space under the linear structure inherited from $\mathscr{E}$, that is,

$$\lambda[s_1]_x + \mu[s_2]_x := [\lambda s_1 + \mu s_2]_x; \qquad 0 := [0]_x, \tag{2.13}$$

where $[s]_x$ is the equivalence class of s with respect to $\sim_x$, and $\lambda, \mu \in \mathbb{R}$. Subsequently, define

$$\mathscr{E} \to \Gamma(E); \qquad \hat{s} \to s; \qquad s(x) = [\hat{s}]_x, \tag{2.14}$$

so that $s \in E_x$ and hence $s : M \to E$ is a cross-section of E. Then there is a unique smooth structure on E such that (2.14) is an isomorphism of $C^\infty(M)$ modules. This isomorphism maps $C_x^\infty(M)\cdot\mathscr{E}$ to $\Gamma(E;x) := \{s \in \Gamma(E) \mid s(x) = 0\}$, so that the mirror of (2.11) under the isomorphism (2.14) is

$$\Gamma(E)/\Gamma(E;x) \cong E_x. \tag{2.15}$$

We apply this to the $C^\infty(M)$-module $\mathscr{E} = \mathrm{Der}(C^\infty(M))$, and notice that we have an isomorphism

$$\mathrm{Der}(C^\infty(M))/\sim_x \stackrel{\cong}{\to} \mathrm{Der}_x(C^\infty(M)); \qquad [\delta]_x \mapsto \delta_x, \tag{2.16}$$

where $\mathrm{Der}_x(C^\infty(M))$ is the vector space of all point derivations δ_x of M, cf. (2.2) - (2.3). Although $\mathrm{Der}(C^\infty(M))$ may no longer be free (as in $M = \mathbb{R}^n$), using charts one can show that it is finitely generated projective, so that the above procedure for defining a vector bundle E is applicable.

Definition 2.4 *The* **tangent bundle** $\pi : TM \to M$ *is the vector bundle* E *constructed from*

$$\mathscr{E} = \mathrm{Der}(C^\infty(M)) \tag{2.17}$$

as in the above procedure, replacing (2.11) *by* (2.16)*. That is, the total space and fibers are*

$$TM := \sqcup_{x\in M} T_xM; \qquad T_xM := \mathrm{Der}_x(C^\infty(M)), \tag{2.18}$$

and the smooth structure of TM *is (uniquely) defined by the property that the map*

$$\mathrm{Der}(C^\infty(M)) \to \mathfrak{X}(M) := \Gamma(TM); \qquad \delta \mapsto (x \mapsto \delta_x), \tag{2.19}$$

where δ_x *is defined by* (2.6)*, is an isomorphism. A* **vector field** *on* M *is a cross-section of* TM*.*

In a local chart $\varphi : U \to \mathbb{R}^n$, for $x \in U$ we define the symbol ∂_i as an element of T_xM by

$$\partial_i f(x) := \frac{\partial(f\circ\varphi^{-1})}{\partial x^i}(\varphi(x)), \tag{2.20}$$

where $f \in C^\infty(U)$ and φ^{-1} is the inverse of $\varphi : U \to V = \varphi(U)$. With $V \subset \mathbb{R}^n$, the function $f\circ\varphi^{-1} : V \to \mathbb{R}$ is the coordinate expression $f(x^1,\dots,x^n)$ of f, so that ∂_i in (2.20) may be taken literally. This also shows that $(\partial_i,\dots,\partial_n)$ is a basis of T_xM, so that we may expand $X_x \in T_xM$ as

$$X_x = \sum_{i=1}^n X_x^i \partial_i; \qquad X_x^i = X\varphi^i(x), \tag{2.21}$$

where $\varphi = (\varphi^1,\dots,\varphi^n) : U \to \mathbb{R}^n$. Thus TM is an n-dimensional vector bundle over M.

In conclusion, a *vector field* on M, written $X \in \mathfrak{X}(M)$, is a map $x \mapsto X_x$, also written as $x \mapsto X(x)$, where $x \in M$ and $X_x \in T_xM$ as defined by (2.18). A *derivation* on M is a map $\delta : C^\infty(M) \to C^\infty(M)$ that satisfies (2.1). These concepts are related by (2.6) with $\delta_x = X_x$. We think of a vector field $X \in \mathfrak{X}(M)$ as the *collection* of all "tangent vectors" $X_x \in T_xM$, whereas we think of the corresponding derivation δ as a *single* global operation on $C^\infty(M)$.

- *Point* derivations ***push forward*** under maps $\psi : M \to N$: for $x \in M$ we have linear maps

$$T_x\psi \equiv \psi'_x : T_xM \to T_{\psi(x)}N; \qquad (\psi'_x\delta_x)(g) = \delta_x(\psi^*g) \;\; (g \in C^\infty(N)), \tag{2.22}$$

  where $\psi^*g := g \circ \psi$ is the ***pullback*** of g. Collecting these maps gives a vector bundle map

$$T\psi \equiv \psi_* \equiv \psi' : TM \to TN. \tag{2.23}$$

- However, *derivations* (or vector fields) push forward only if $\psi : M \to N$ is a *diffeomorphism*: the map $\psi_* : \mathrm{Der}(C^\infty(M)) \to \mathrm{Der}(C^\infty(N))$, or $\psi_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$, is given by

$$\psi_*(\delta) = (\psi^{-1})^* \circ \delta \circ \psi^*. \tag{2.24}$$

  One needs $(\psi^{-1})^*$ even if $N = M$, since $\delta \circ \psi^*$ fails to be a derivation of $C^\infty(M)$. Check!

So far, tangent vectors $X_x \in T_xM$ were defined *algebraically* as point derivations, i.e. as linear maps $\delta_x : C^\infty(M) \to \mathbb{R}$ satisfying (2.3). *Geometrically*, each tangent vector (*nomen est omen!*) is tangent to some ***curve*** γ through x, i.e., a map $\gamma : I \to M$, where $I \subset \mathbb{R}$ is some interval we always assume to contain 0, such that $\gamma(0) = x$ (see below for the existence of γ). In other words,

$$X_x(f) = \frac{d}{dt} f(\gamma(t))_{|t=0}, \tag{2.25}$$

which *symbolically* may be written as $X_x = \dot{\gamma} \equiv d\gamma/dt$, or even as $X_x = d/dt$, with γ understood. This description gives a geometric perspective on the pushforward of T_xM just described:

- If $X = d\gamma/dt$ is tangent to γ, then $\psi'X = d(\psi \circ \gamma)/dt$ is tangent to $\psi(\gamma)$.

In a chart $\varphi = (\varphi^1, \ldots, \varphi^n) : U \to \mathbb{R}^n$, with $x \in U$, the components X^i_φ of X_x are given by

$$X^i_\varphi = X\varphi^i(x) = \frac{d}{dt}\varphi^i(\gamma(t))_{|t=0} = \frac{d}{dt}\gamma^i(t)_{|t=0}, \tag{2.26}$$

where $\gamma^i(t) = \varphi^i(\gamma(t))$. This also shows that γ exists, given X_x, since it just has to satisfy (2.26). Of course, γ is far from unique. Eq. (2.26) gives the traditional transformation rule for vectors under a change of charts (i.e. of coordinates). If $x \in U_\alpha \cap U_\beta$, then (2.25) and (2.26) imply

$$X^i_\beta = \sum_j \frac{\partial x^i_\beta}{\partial x^j_\alpha} X^j_\alpha, \tag{2.27}$$

where $X^i_\beta \equiv X^i_{\varphi_\beta}$ etc., and each coordinate $x^i_\beta = \varphi^i_\beta(x)$ of x with respect to φ_β is seen as a function of all coordinates $x^i_\alpha = \varphi_\alpha(x)$ of x with respect to φ_α via the identity $\varphi^i_\beta = \varphi^i_\beta \circ \varphi_\alpha^{-1} \circ \varphi_\alpha$, i.e.

$$x^i_\beta(x_\alpha) = \varphi^i_\beta \circ \varphi_\alpha^{-1}(x_\alpha). \tag{2.28}$$

It is important to distinguish (2.27), which is a ***change of coordinates formula*** for a given tangent vector, from the ***pushforward of a tangent vector*** under a map $\psi : M \to M$. With $\varphi : U \to V \subset \mathbb{R}^n$, suppose for simplicity that $x \in U$ and also $\psi(x) \in U$. Then, writing $X^i_\varphi \equiv X^i$ as above, as well as $\psi^i = \varphi^i \circ \psi \circ \varphi^{-1}$ (which near x is a function from V to $\mathbb{R}$), we have

$$(\psi' X)^i = \sum_j \frac{\partial \psi^i}{\partial x^j} X^j. \tag{2.29}$$

A curve $\gamma : I \to M$ ***integrates*** a vector *field* X if $X_{\gamma(t)} = d\gamma(t)/dt$ for all $t \in I$, i.e., in coordinates,

$$\frac{d\gamma^j(t)}{dt} = X^j(\gamma^1(t), \ldots, \gamma^n(t)), \; (j = 1, \ldots, n). \tag{2.30}$$

The theory of ODEs shows that for each $x \in M$ there exists an open interval $I \subset \mathbb{R}$ (with $0 \in I$) and a curve $\gamma : I \to M$ on which (2.30) holds with $\gamma(0) = x$. This solution is unique in the sense that if $\gamma_1 : I_1 \to M$ and $\gamma_2 : I_2 \to M$ both satisfy (2.30) with $\gamma_1(0) = \gamma_2(0) = x_0$, then $\gamma_1 = \gamma_2$ on $I_1 \cap I_2$. Taking unions, it follows that there exists a maximal interval I on which γ is defined.

If for any $x \in M$ there is a curve $\gamma : \mathbb{R} \to M$ satisfying (2.30) with $\gamma(0) = x$, we say that $X \in \mathfrak{X}(M)$ is ***complete***.[133] In that case, all integrating curves γ can be assembled into the ***flow*** of X. This is a smooth map $\psi : \mathbb{R} \times M \to M$, written $\psi_t(x) \equiv \psi(t, x)$, that satisfies

$$\psi_0(x) = x; \tag{2.31}$$

$$X_{\psi_t(x)} f = \frac{d}{dt} f(\psi_t(x)) \tag{2.32}$$

for all $x \in M$, $t \in \mathbb{R}$, and $f \in C^\infty(M)$. Thus the flow ψ of X gives "the" integral curve γ of X through x_0 by $\gamma(t) = \psi_t(x_0)$. Any complete vector field has a unique flow. Uniqueness implies that M is a disjoint union of the integral curves of X (which can never cross each other because of the uniqueness of the solution), and also implies the composition rule

$$\psi_s \circ \psi_t = \psi_{s+t}. \tag{2.33}$$

From a group-theoretic point of view, a flow is therefore an action of $\mathbb{R}$ (as an additive group) on *M that in addition integrates* X in the sense of (2.32). In particular, (2.33) implies $\psi_{-t} = \psi_t^{-1}$, so that each $\psi_t : M \to M$ is automatically a diffeomorphism of M.

If X is not complete (a case that will be of great interest to GR!), we first define the ***domain*** $D_X \subset \mathbb{R} \times M$ of ψ as the set of all $(t, x) \in \mathbb{R} \times M$ for which there exists an open interval $I \subset \mathbb{R}$ containing 0 and t, as well as a (necessarily unique) curve $\gamma : I \to M$ that satisfies (2.30) with initial condition $\gamma(0) = x$. Obviously $\{0\} \times M \subset D_X$, and (less trivially) it turns out that D_X is open. Then a flow of X is a map $\psi : D_X \to M$ that satisfies (2.31) for all x and (2.32) for all $(t, x) \in D_X$. Eq. (2.33) then holds if the left-hand side (and hence the right-hand side) is defined.

As a first application of flows, let us define the ***Lie derivative*** $\mathscr{L}_X Y$ of some vector field $Y \in \mathfrak{X}(M)$ with respect to another vector field $X \in \mathfrak{X}(M)$ by

$$\mathscr{L}_X Y(x) = \lim_{t \to 0} \frac{Y_{\psi_t(x)} - \psi'_t(Y_x)}{t} = \lim_{t \to 0} \frac{\psi'_{-t}(Y_{\psi_t(x)}) - Y_x}{t} \tag{2.34}$$

where ψ is the flow of X. Note that $Y_{\psi_t(x)} - Y_x$ would be undefined, since $Y_{\psi_t(x)} \in T_{\psi_t(x)}M$ whilst $Y_x \in T_x M$ and these are different vector spaces; the pushforward ψ'_t serves to move Y_x to $T_{\psi_t(x)}M$. A simple computation then yields the extremely useful result

$$\mathscr{L}_X Y = [X, Y]. \tag{2.35}$$

[133] If X has *compact support*, then it is complete. So if M is compact, then every vector field on M is complete.

2.3 Dual vector spaces, metrics, and tensor products

In order to defined tensors we need some linear algebra. Let V be a finite-dimensional real vector space, with $\dim(V) = n$, which in GR will be $V = T_xM$. The ***dual*** $V^* = \mathrm{Hom}(V, \mathbb{R})$ consists of all linear maps from V to $\mathbb{R}$. This is a real vector space in its own right under pointwise constructions. It is isomorphic to V (as a vector space), but not canonically so: one needs to specify a basis $(e_1, \dots, e_n)$ of V, with corresponding dual basis $(\omega^1, \dots, \omega^n)$ defined by $\omega^a(e_b) = \delta^a_b$, upon which the ugly map $\sum_a v^a e_a \mapsto \sum_a v^a \omega^a$ from V to V^* is an isomorphism (which obviously depends on the chosen basis). However, we do have a canonical isomorphism

$$V \cong V^{**}; \qquad v \mapsto \hat{v}; \qquad \hat{v}(\theta) = \theta(v) \tag{2.36}$$

where $\hat{v} \in V^{**} = \mathrm{Hom}(V^*, \mathbb{R})$. This map is injective for any V, but it is surjective (and hence an isomorphism) iff V is finite-dimensional. One often writes $\langle \theta, v \rangle$ for both $\theta(v)$ and $\hat{v}(\theta)$.

The naturality of the isomorphism $V^* \cong V$ improves markedly in the presence of a *metric*.

Definition 2.5 *A* **metric** *g on V is a bilinear map $g : V \times V \to \mathbb{R}$ that is:*

- symmetric, *in that $g(v, w) = g(w, v)$ for all $v, w \in V$;*
- nondegenerate, *i.e. for each nonzero vector $v \in V$ there is $w \in V$ such that $g(v, w) \neq 0$.*

A metric g yields two maps that are mutually inverse and hence are isomorphisms $V^* \cong V$:

$$\flat : V \to V^*, \qquad \flat(v) \equiv v^\flat; \qquad v^\flat(w) := g(v, w); \tag{2.37}$$

$$\sharp : V^* \to V, \qquad \sharp(\theta) \equiv \theta_\sharp; \qquad g(\theta_\sharp, v) := \theta(v). \tag{2.38}$$

Any metric g can be diagonalized, i.e. V has an ***orthonormal*** basis $(e_a) \equiv (e_1, \dots, e_n)$, in which

$$g(e_a, e_b) = \varepsilon_a \delta_{ab}; \qquad \varepsilon_a = \pm 1. \tag{2.39}$$

The pair (n_-, n_+), where n_-/n_+ is the number of negative/positive numbers ε_a, is independent of the basis and hence is an intrinsic property of a metric g, called its ***signature***. Especially in relativity, the signature is often written as $(-\cdots-+\cdots+)$, with n_-/n_+ minus/plus signs.

We now turn to the tensor product. In the following proposition, V and W are real but not necessarily finite-dimensional (and the same construction works over any field, typically $\mathbb{C}$).

Proposition 2.6 *Let V and W be real vector spaces. There is a real vector space called $V \otimes W$, in words the* **tensor product** *of V and W (over $\mathbb{R}$), and a map*

$$p : V \times W \to V \otimes W; \qquad p(v, w) \equiv v \otimes w, \tag{2.40}$$

such that for any vector space X and any bilinear map $\beta : V \times W \to X$, there is a unique linear map $\beta' : V \otimes W \to X$ such that $\beta = \beta' \circ p$. In other words, the following diagram commutes:

$$\begin{array}{ccc} V \times W & \xrightarrow{\;p\;} & V \otimes W \\ & \searrow^{\beta} & \downarrow {\scriptstyle \exists! \beta'} \\ & & X \end{array} \tag{2.41}$$

Moreover, this so-called universal property implies that $V \otimes W$ is unique up to isomorphism.

We will not prove this here in general, but do show existence of $V\otimes W$ if V and W are finite-dimensional.[134] We first assume that $V=Y^*$ and $W=Z^*$, in which case we define

$$Y^*\otimes Z^* := \mathrm{Hom}(Y\times Z,\mathbb{R}); \tag{2.42}$$
$$(\sigma\otimes\rho)(y,z) := \sigma(y)\rho(z), \tag{2.43}$$

where $\mathrm{Hom}(Y\times Z,\mathbb{R})$ is the space of bilinear maps from $Y\times Z$ to $\mathbb{R}$, and of course $\sigma\in Y^*$, $\rho\in Z^*$, $y\in Y$, $z\in Z$. Then $\beta'(\sigma\otimes\rho)=\beta(\sigma,\rho)$ by construction, and this uniquely extends to a linear map $\beta':\mathrm{Hom}(Y\times Z,\mathbb{R})\to X$, since $\mathrm{Hom}(Y\times Z,\mathbb{R})$ is the linear span of all $\sigma\otimes\rho$.

This also covers V and W themselves, at least up to isomorphism, since in finite dimension we have the isomorphism (2.36), so that, identifying V with V^{**} etc., we obtain

$$V\otimes W\cong V^{**}\otimes W^{**}=\mathrm{Hom}(V^*\times W^*,\mathbb{R}); \tag{2.44}$$
$$(v\otimes w)(\theta,\tau)=\theta(v)\tau(w), \tag{2.45}$$

where this time $v\in V$, $w\in W$, $\theta\in V^*$, and $\tau\in W^*$. Once again, $\beta':\mathrm{Hom}(V^*\times W^*,\mathbb{R})\to X$ is uniquely defined by linear extension of $\beta'(v\otimes w)=\beta(v,w)$, since the linear span of all $v\otimes w$ equals $\mathrm{Hom}(V^*\times W^*,\mathbb{R})$. We have effectively identified v with $\hat{v}$ and w with $\hat{w}$, cf. (2.36), and this shows up: although (2.42) gives $Y^*\otimes\mathbb{R}=Y^*$ as expected, eq. (2.44) has the consequence

$$V\otimes\mathbb{R}=\mathrm{Hom}(V^*,\mathbb{R})=V^{**}, \tag{2.46}$$

where one would prefer to see V. But although no one would criticize the realization $V\otimes\mathbb{R}=V$, eq. (2.46) reconfirms that tensor products are merely defined *up to isomorphism*, cf. (2.36). Similarly, instead of $V^*\otimes V=\mathrm{Hom}(V\times V^*,\mathbb{R})$, as suggested by (2.42) and (2.44), we may take

$$V^*\otimes V\cong\mathrm{Hom}(V,V), \tag{2.47}$$

since one has an isomorphism $\mathrm{Hom}(V\times V^*,\mathbb{R})\to\mathrm{Hom}(V,V)$, given by linear extension of

$$w\otimes\theta\mapsto(v\mapsto\theta(v)w). \tag{2.48}$$

With $v\in V$ and $\theta\in V^*$ as before, the inverse of the map (2.48) is given by

$$\mathrm{Hom}(V,V)\to\mathrm{Hom}(V\times V^*,\mathbb{R});\qquad \varphi\mapsto\hat{\varphi};\qquad \hat{\varphi}(v,\theta)=\theta(\varphi(v)). \tag{2.49}$$

In connection with the Riemann tensor we will have occasion to use the induced isomorphism

$$V^*\otimes V^*\otimes W^*\otimes W\cong\mathrm{Hom}(V\times V,\mathrm{Hom}(W,W)); \tag{2.50}$$
$$\theta_1\otimes\theta_2\otimes\eta\otimes w_1\mapsto((v_1,v_2)\mapsto(w_2\mapsto\theta_1(v_1)\theta_2(v_2)\eta(w_2)w_1). \tag{2.51}$$

To describe the inverse of this map we combine (2.42) and (2.44) to pick the realization

$$V^*\otimes V^*\otimes W^*\otimes W=\mathrm{Hom}(V\times V\times W\times W^*,\mathbb{R}). \tag{2.52}$$

The image $\hat{\varphi}\in\mathrm{Hom}(V\times V\times W\times W^*,\mathbb{R})$ of $\varphi\in\mathrm{Hom}(V\otimes V,\mathrm{Hom}(W,W))$ is then given by

$$\hat{\varphi}(v_1,v_2,w,\eta)=\eta(\varphi(v_1,v_2)(w)). \tag{2.53}$$

[134] The construction applies in general if we define $V\otimes W$ as the finite linear span of all $a\otimes b$ in $\mathrm{Hom}(V^*\times W^*,\mathbb{R})$.

2.4 Cotangent bundle

Now that we have the tangent bundle TM and the constructions in §2.3, all relevant vector bundles on M that are relevant for GR follow. First, the ***cotangent bundle*** T^*M is defined as

$$T^*M := \sqcup_{x\in M} T^*_x M; \qquad T^*_x M \equiv (T_x M)^* := \mathrm{Hom}(T_x M, \mathbb{R}), \tag{2.54}$$

i.e. $T_x M$ is the dual of the vector space $T_x M$, consisting of all linear maps $\theta_x : T_x M \to \mathbb{R}$. The smooth structure of T^*M is the unique one such that elements $\theta \in \Gamma(T^*M) \equiv \Omega^1(M) \equiv \Omega(M)$, called ***covectors*** (or ***1-forms***), consist of those maps $x \mapsto \theta_x$ for which the function $x \mapsto \theta_x(X_x)$ from M to $\mathbb{R}$ is smooth for each vector field $X \in \mathfrak{X}(M)$. Since $T_x M \cong \mathbb{R}^n$ we also have $T^*_x M \cong \mathbb{R}^n$, so that, like the tangent bundle TM, also the cotangent bundle T^*M is an n-dimensional vector bundle over M. In a coordinate systems (x^i) defined by some chart, $T^*_x M$ has basis $(dx^1, \dots, dx^n)$ defined by $dx^i(\partial_j) = \delta^i_j$, which is dual to the basis $(\partial_1, \dots, \partial_n)$ of $T_x M$ defined in (2.20). Thus

$$\theta = \sum_i \theta_i dx^i; \qquad \theta_i = \theta(\partial_i). \tag{2.55}$$

For an equivalent view of dx^i, one may define the ***exterior derivative*** $d : C^\infty(M) \to \Omega(M)$ by

$$df(X) := X(f). \tag{2.56}$$

Then dx^i coincides with $d\varphi^i$, where $x^i = \varphi^i(x)$ as usual, and in coordinates (2.56) simply reads

$$df = \sum_i \left(\frac{\partial f}{\partial x^i}\right) dx^i. \tag{2.57}$$

More generally, let (e_a) be a basis of $T_x M$, with dual basis (ω^a) of $T^*_x M$ (i.e. $\omega^a(e_b) = \delta^a_b$). Once again, if we expand $\theta = \sum_a \theta_a \omega^a$, we have $\theta_a = \theta(e_a)$. This may be done at a single point, but bases like $(\partial_1, \dots, \partial_n)$ and $(dx^1, \dots, dx^n)$ are defined at each $x \in U$ on which the coordinates $x^i = \varphi^i(x)$ are defined. Similarly, some basis (e_a) may be defined at each $x \in U$, where $U \in \mathscr{O}(M)$ need not even be the domain of a chart. In that case (e_a) is called a (moving) ***frame*** or an ***n-bein*** (so that in GR one has a ***vierbein*** or ***tetrad***). Abstractly, if $E \to M$ is a k-dimensional vector bundle, one may locally find k linearly independent cross-sections $(u_1, \dots, u_k)$ of E and expand any $s \in \Gamma(E)$ by $s(x) = \sum_j s_j(x) u_j(x)$, where $s_j \in C^\infty(M)$ and $u_j \in \Gamma(E)$.

Whereas tangent vectors *push forward* from M to N under maps $\psi : M \to N$, covectors ***pull back*** from N to M, like functions: besides the pull-back $\psi^* : C^\infty(N) \to C^\infty(M)$ on functions, any (smooth) map ψ induces a pullback $\psi^* : \Omega(N) \to \Omega(M)$ on 1-forms by

$$(\psi^*\theta)_x(X_x) = \theta_{\psi(x)}(\psi'_x X_x), \tag{2.58}$$

where $\theta \in \Omega(N)$ and $X_x \in T_x M$. For any $f \in C^\infty(N)$ with $df \in \Omega(N)$, this yields

$$\psi^*(df) = d(\psi^* f). \tag{2.59}$$

A decent vector bundle map $\psi^* : T^*N \to T^*M$ is defined only if ψ is a diffeomorphism: for $\theta_y \in T^*_y N$ ($y \in N$), we need $x = \psi^{-1}(y) \in M$, so that the pullback $\psi^*_y(\theta_y) \in T^*_x M$ is defined by

$$(\psi^*_y \theta_y)(X_x) = \theta_y(\psi'_x X_x). \tag{2.60}$$

If ψ is merely injective, then we still obtain a map $\psi^* : T^*(\psi(M)) \to T^*M$ in this way.

2.5 Tensor bundles

For $(k,l) \in \mathbb{N} \times \mathbb{N}$ we define a vector bundle $T^{(k,l)}M$ over M in the usual way via its fibers

$$T_x^{(k,l)}M := \mathrm{Hom}((T_xM)^k \times (T_x^*M)^l, \mathbb{R}), \tag{2.61}$$

i.e. the vector space of $(k+l)$-fold multilinear maps from $(T_xM)^k \times (T_x^*M)^l$ to $\mathbb{R}$. These fibers comprise the total space of the bundle as a disjoint union

$$T^{(k,l)}M := \sqcup_{x\in M} T_x^{(k,l)}M, \tag{2.62}$$

whose manifold structure will be defined below (by defining the smooth sections). We then have

$$T^{(0,0)}M = M \times \mathbb{R}; \qquad T^{(1,0)}M = T^*M; \qquad T^{(0,1)}M \cong TM, \tag{2.63}$$

where in the last entry we used (2.36). Repeatedly using Proposition 2.6, taking (2.44) as a realization of "the" tensor product, and once again using (2.36), we obtain the realization

$$T_x^{(k,l)}M \cong (\otimes^k T_x^*M) \otimes (\otimes^l T_xM), \tag{2.64}$$

where $\otimes^l V$ is the l times iterated tensor product of V with itself. According to (2.64), the fiber $T_x^{(k,l)}M$ consists of finite sums of elementary tensors $\alpha_1 \otimes \cdots \otimes \alpha_k \otimes v_1 \otimes \cdots \otimes v_l$, defined for

$$\alpha_i \in T_x^*M (i = 1,\ldots,k); \qquad v_j \in T_xM (j = 1,\ldots,l).$$

In terms of (2.61), one has

$$\alpha_1 \otimes \cdots \otimes \alpha_k \otimes v_1 \otimes \cdots \otimes v_l(X_1,\ldots,X_k;\theta^1,\ldots,\theta^l) = \alpha_1(X_1)\cdots\alpha_k(X_k)v_1(\theta^1)\cdots v_l(\theta^l),$$

where each $X_i \in T_xM$ and each $\theta^j \in T_x^*M$. We then define $\Gamma(T^{(k,l)}M)$ as the set of all cross-sections $x \mapsto \tau_x$ from M to $T^{(k,l)}M$ (i.e. maps such that $\tau_x \in T_x^{(k,l)}M$) for which the map

$$x \mapsto \tau_x(X_1(x),\ldots,X_k(x);\theta^1(x),\ldots,\theta^l(x))$$

from M to $\mathbb{R}$ is smooth for each $(X_1,\ldots,X_k;\theta^1,\ldots,\theta^l)$ with $X_i \in \mathfrak{X}(M)$ and $\theta^j \in \Omega(M)$. This equips the vector bundles $T^{(k,l)}M$ with a manifold structure, in that we declare $\Gamma(T^{(k,l)}M)$ to be the space of smooth cross-sections of $T^{(k,l)}M$. Elements of $\Gamma(T^{(k,l)}M)$ are called ***tensors*** (or ***tensor fields*** (if τ_x is regarded as a tensor). In GR, $T^{(2,0)}M$ and $T^{(3,1)}M$ will be very important.

All this can be rewritten in terms of ***indices***. In terms of the (coordinate) basis $(\partial_1,\ldots,\partial_n)$ of T_xM with dual basis $(dx^1,\ldots,dx^n)$ of T_x^*M, the fiber $T_x^{(k,l)}M$ then has a basis

$$(dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes \partial_{j_1} \otimes \cdots \otimes \partial_{j_l}), \tag{2.65}$$

where all indices run from 1 to n. *Thus $T^{(k,l)}M$ is an n^{k+l}-dimensional vector bundle.* Like vectors, tensors at x may be specified by their components with respect to some basis of T_xM and associated dual basis of T_x^*M. In the usual coordinate basis (∂_i) we have

$$\tau_x = \tau_{i_1\cdots i_k}^{j_1\cdots j_l}(x)\, dx^{i_1} \otimes \cdots dx^{i_k} \otimes \partial_{j_1} \otimes \cdots \otimes \partial_{j_l}; \tag{2.66}$$

$$\tau_{i_1\cdots i_k}^{j_1\cdots j_l}(x) = \tau_x(\partial_{i_1},\ldots,\partial_{i_k};dx^{j_1},\ldots,dx^{j_l}), \tag{2.67}$$

where we use the ***Einstein summation convention: repeated indices are summed over***.

That is, the right-hand side of (2.66) should really be preceded by $\sum_{i_1,\dots,i_k,j_1,\dots,j_l=1}^n$.

Similarly, in an arbitrary basis (e_a) of T_xM with dual basis (θ^a) of T_x^*M one has

$$\tau_x = \tau_{a_1\cdots a_k}^{b_1\cdots b_l}(x)\,\theta^{a_1}\otimes\cdots\theta^{a_k}\otimes e_{b_1}\otimes\cdots\otimes e_{b_l}; \tag{2.68}$$

$$\tau_{a_1\cdots a_k}^{b_1\cdots b_l}(x) = \tau_x(e_{a_1},\dots,e_{a_k};\theta^{b_1},\dots,\theta^{b_l}). \tag{2.69}$$

We write $\mathfrak{X}^{(k,l)}(M)$ for the space of cross-sections $\Gamma(T^{(k,l)}M)$ of $T^{(k,l)}M$, so that

$$\mathfrak{X}^{(0,0)}(M) = C^\infty(M); \qquad \mathfrak{X}^{(0,1)}(M) = \mathfrak{X}(M); \qquad \mathfrak{X}^{(1,0)}(M) = \Omega(M). \tag{2.70}$$

A tensor $\tau\in\mathfrak{X}^{(k,l)}(M)$ of type (k,l) maps k vector fields $(X_1,\dots,X_k)$ and l covector fields $(\theta^1,\dots,\theta^l)$ to a smooth function on M by pointwise evaluation, i.e.

$$\tau : \mathfrak{X}(M)^k\times\Omega(M)^l \to C^\infty(M); \tag{2.71}$$

$$\tau(X_1,\dots,X_k,\theta^1,\dots,\theta^l) : x\mapsto \tau_x(X_1(x),\dots X_k(x);\theta^1(x),\dots,\theta^l(x)). \tag{2.72}$$

This map is evidently $k+l$-multilinear linear over $C^\infty(M)$, in the sense that e.g.

$$\tau(f_1X_1,\dots,f_kX_k,g_1\theta^1,\dots,g_l\theta^l) = f_1\cdots f_k\cdot g_1\cdots g_l\cdot\tau(X_1,\dots,X_k;\theta^1,\dots,\theta^l), \tag{2.73}$$

for all $f_i,g_j\in C^\infty(M)$; here we use the fact that $\mathfrak{X}(M)$ and $\Omega(M)$ are $C^\infty(M)$ modules.

Proposition 2.7 (tensoriality test) *A map*

$$\tau : \mathfrak{X}(M)^k\times\Omega(M)^l\to C^\infty(M) \tag{2.74}$$

is given by a tensor

$$\tau\in\mathfrak{X}^{(k,l)}(M) \tag{2.75}$$

through (2.72) iff τ satisfies (2.73), i.e., iff it is $C^\infty(M)$-multilinear in all entries.

Proof. The proof is easy in local coordinates, where (2.73) yields

$$\begin{aligned}\tau(X_1,\dots,X_k,\theta^1,\dots,\theta^l) &= \tau(X_1^{i_1}\partial_{i_1},\dots,X_k^{i_k}\partial_{i_k};\theta^1_{j_1}dx^{j_1},\dots\theta^l_{j_l}dx^{j_l})\\ &= X_1^{i_1}\cdots X_k^{i_k}\cdot\theta^1_{j_1}\cdots\theta^l_{j_l}\,\tau(\partial_{i_1},\dots,\partial_{i_k};dx^{j_1},\dots dx^{j_l}),\end{aligned} \tag{2.76}$$

so if we define the components $\tau_{i_1\cdots i_k}^{j_1\cdots j_l}(x)$ of τ_x by (2.67) and subsequently define τ_x itself by (2.66), we have found the desired tensor that via (2.72) reproduces the given map τ. □

Eqs. (2.66) - (2.67) imply the transformation properties of tensors under changes of coordinates (i.e. charts), which historically even *defined* tensors: in the situation of (2.27),

$$(\tau_\beta)_{i_1\cdots i_k}^{j_1\cdots j_l}(x_\beta) = \frac{\partial x_\beta^{j_1}}{\partial x_\alpha^{j'_1}}\cdots\frac{\partial x_\beta^{j_l}}{\partial x_\alpha^{j'_l}}\cdot\frac{\partial x_\alpha^{i'_1}}{\partial x_\beta^{i_1}}\cdots\frac{\partial x_\alpha^{i'_k}}{\partial x_\beta^{i_k}}\cdot(\tau_\alpha)_{i'_1\cdots i'_k}^{j'_1\cdots j'_l}(x_\alpha), \tag{2.77}$$

where the "new" coordinates $(x_\beta)=(x_\beta^1,\dots,x_\beta^n)$ are functions of the "old" coordinates $(x_\alpha)=(x_\alpha^1,\dots,x_\alpha^n)$, cf. (2.28), and hence the matrix $(\partial x_\alpha^{i'_1}/\partial x_\beta^{i_1})$ is defined as the inverse of the matrix $(\partial x_\beta^{i_1}/\partial x_\alpha^{i'_1})$, *both* seen as functions of the (x_α^i). Note that the argument x_β in (2.77) refers to the *same point* $x \in M$ as the argument x_α (but in *different coordinates*). Conversely, from a "tensor" (in the original historical sense of the word) $\tau^{j_1\cdots j_l}_{i_1\cdots i_k}(x)$ we obtain a map τ of the kind (2.71) by

$$\tau(X_1,\ldots,X_k,\theta^1,\ldots,\theta^l) : x \mapsto \tau^{j_1\cdots j_l}_{i_1\cdots i_k}(x)X_1^{i_1}(x)\cdots X_k^{i_k}(x)\cdot\theta^1_{j_1}(x)\cdots\theta^l_{j_l}(x). \tag{2.78}$$

It then follows from (2.77) that τ is well defined in being coordinate-independent. It is also $k+l$-multilinear linear over $C^\infty(M)$ by construction, so that we recover (2.71) - (2.73).

A smooth map

$$\psi : M \to N \tag{2.79}$$

induces a (vector bundle) map

$$\psi_*^{(0,l)} : T^{(0,l)}M \to T^{(0,l)}N \tag{2.80}$$

via the obvious pointwise maps

$$\psi_x^{(0,l)} : T_x^{(0,l)}M \to T_{\psi(x)}^{(0,l)}N. \tag{2.81}$$

However, to extend this to a map

$$\psi_*^{(k,l)} : T^{(k,l)}M \to T^{(k,l)}N, \tag{2.82}$$

we need ψ to be *invertible* (with smooth inverse), in which case we may as well take $N = M$ and assume that $\psi : M \to M$ is a diffeomorphism. In that case, we have

$$(\psi_*^{(k,l)}(\tau_x))(X_1(\psi(x)),\ldots,\theta^l(\psi(x))) := \tau_x(\psi_*^{-1}(X_1(\psi(x))),\ldots,\psi^*(\theta^l(\psi(x)))); \tag{2.83}$$

$$\psi_*^{(k,l)}(\tau_x) = \tau^{j_1\cdots j_l}_{i_1\cdots i_k}(x)\cdot(\psi^{-1})_x^*(dx^{i_1})\otimes\cdots\otimes\psi_x'(\partial_{j_l}). \tag{2.84}$$

This can also be done with ψ replaced by ψ^{-1}, giving maps

$$\psi^*_{(k,l)} : T^{(k,l)}M \to T^{(k,l)}N, \tag{2.85}$$

which in turn induce maps on the sections

$$\psi^*_{(k,l)} : \mathfrak{X}^{(k,l)}(M) \to \mathfrak{X}^{(k,l)}(M), \tag{2.86}$$

often just called ψ^*, via

$$(\psi^*_{(k,l)}\tau)_x(X_1(x),\ldots,\theta^l(x)) = \tau_{\psi(x)}(\psi_*(X_1(x)),\ldots,(\psi^{-1})^*(\theta^l(x))). \tag{2.87}$$

In particular, $\psi^*_{(1,0)}$ is the map ψ^* from (2.58), whereas $\psi^*_{(0,1)} = \psi_*^{-1}$ (recall that $\psi_* \equiv \psi'$).

A natural operation on tensors, which is often used in GR, is ***tensoring***: if

$$\tau_1 \in \mathfrak{X}^{(k_1,l_1)}(M) \qquad \text{and} \qquad \tau_2 \in \mathfrak{X}^{(k_2,l_2)}(M), \tag{2.88}$$

then

$$\tau_1 \otimes \tau_2 \in \mathfrak{X}^{(k_1+k_2,l_1+l_2)}(M) \tag{2.89}$$

is defined by ***concatenation***, i.e.

$$\tau_1 \otimes \tau_2(X_1,\ldots,X_{k_1},Y_1,\ldots Y_{k_2};\theta^1,\ldots,\theta^{l_1},\rho^1,\ldots,\rho^{l_2}) := \tau_1(X_1,\ldots,X_{k_1};\theta^1,\ldots,\theta^{l_1})\cdot\tau_2(Y_1,\ldots Y_{k_2};\rho^1,\ldots,\rho^{l_2}). \tag{2.90}$$

Indeed, $\mathfrak{X}^{(k,l)}(M)$ itself arose in this way by tensoring copies of $\mathfrak{X}^{(1,0)}(M)$ and $\mathfrak{X}^{(0,1)}(M)$.

Another important operation for GR is ***(index) contraction***: If $k>0$ and $l>0$, then a tensor $\tau \in \mathfrak{X}^{(k,l)}(M)$ may be contracted along one fixed upper and one lower index, say i and j (the result depends on this choice) so as to obtain a tensor $\sigma \in \mathfrak{X}^{(k-1,l-1)}(M)$ with two indices less. Let (e_a) be a basis of T_xM, with ***dual basis*** (ω^a) of T_x^*M (i.e. $\omega^a(e_b)=\delta^a_b$); in local coordinates one could take the (∂_i) basis, with dual (dx^i). Then

$$\sigma^{b_1,\ldots,\hat{b}_i,\ldots,b_l}_{a_1,\ldots,\hat{a}_j,\ldots,a_k}(x) := \tau^{b_1,\ldots,a,\ldots,b_l}_{a_1,\ldots,a,\ldots,a_k}(x), \tag{2.91}$$

where, according to the Einstein summation convention, a is summed over, and a hat means that the given index is omitted. This is easily seen to be independent of the basis.

Finally, the ***Lie derivative*** $\mathscr{L}_X$, so far only defined on vector fields, may be extended to a linear (and "$C^\infty(M)$-Leibnizian") map

$$\mathscr{L}_X^{(k,l)} : \mathfrak{X}^{(k,l)}(M) \to \mathfrak{X}^{(k,l)}(M) \tag{2.92}$$

in two equivalent ways:

- *Concretely*, writing $\mathscr{L}_X$ for $\mathscr{L}_X^{(k,l)}$ for simplicity, one may define

$$\mathscr{L}_X\tau := \lim_{t\to 0}(\psi_t^*(\tau)-\tau)/t \qquad (\tau \in \mathfrak{X}^{(k,l)}M), \tag{2.93}$$

  cf. (2.34). In local coordinates, this gives the following explicit formula:

$$(\mathscr{L}_X\tau)^{j_1\cdots j_l}_{i_1\cdots i_k} = X^i\partial_i\tau^{j_1\cdots j_l}_{i_1\cdots i_k} + (\partial_{i_1}X^i)\tau^{j_1\cdots j_l}_{i\cdots i_k} + \cdots + (\partial_{i_n}X^i)\tau^{j_1\cdots j_l}_{i_1\cdots i} - (\partial_jX^{j_1})\tau^{j\cdots j_l}_{i_1\cdots i_k} - \cdots - (\partial_jX^{j_l})\tau^{j_1\cdots j}_{i_1\cdots i_k}, \tag{2.94}$$

  of which (2.8) is clearly a special case.

- *Axiomatically*, one may define the $\mathscr{L}_X$ as the unique linear maps satisfying the rules:

  1. $\mathscr{L}_X^{(0,0)}f = Xf$ for functions $f \in C^\infty(M) \equiv \mathfrak{X}^{(0,0)}M$;
  2. $\mathscr{L}_X^{(0,1)}Y = [X,Y]$ for vector fields $Y \in \mathfrak{X}(M) \equiv \mathfrak{X}^{(0,1)}M$;
  3. $(\mathscr{L}_X^{(1,0)}\theta)(Y) = \mathscr{L}_X(\theta(Y)) - \theta(\mathscr{L}_XY)$ for covector fields $\theta \in \Omega(M) \equiv \mathfrak{X}^{(1,0)}M$;
  4. $\mathscr{L}_X^{(k,l)}(\sigma\otimes\tau) = (\mathscr{L}_X\sigma)\otimes\tau + \sigma\otimes\mathscr{L}_X\tau$ for all higher-order tensors (***Leibniz rule***).

It follows from either (a)–(d) or (2.94) that for all cases $\mathscr{L}_X^{(k,l)} \equiv \mathscr{L}_X$ one has the lovely rule

$$[\mathscr{L}_X,\mathscr{L}_Y] = \mathscr{L}_{[X,Y]}. \tag{2.95}$$

2.6 Manifolds with boundaries and corners

For the action principle in GR as well as for things like Penrose diagrams or Cauchy horizons we will need an extension of the manifold concept defined in §2.1 so as to incorporate (smooth) boundaries and, sometimes, corners (which lead to non-smooth boundaries).[135] For $n \geq 1$, let

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x^n \geq 0\}; \qquad \tilde{\mathbb{R}}^n_+ := \{x \in \mathbb{R}^n \mid x^1 \geq 0, \ldots, x^n \geq 0\}. \tag{2.96}$$

Definition 2.8 1. *A* C^k**-manifold with boundary** *M is defined in the same way as a manifold (cf. §2.1), except that one replaces $\mathbb{R}^n$ by $\mathbb{R}^n_+$ throughout the definition. In particular, each point $x \in M$ has a nbhd $U \in \mathcal{O}(M)$ that is homeomorphic to some $V \in \mathcal{O}(\mathbb{R}^n_+)$.*

2. *Similarly, a* **manifold with corners** *is defined from the model space $\tilde{\mathbb{R}}^n_+$ instead of $\mathbb{R}^n$.*

3. *In these definitions, C^k-regularity of the transition functions $\varphi_\beta \circ \varphi_\alpha^{-1}$ (see Definition 2.1.4 in §2.1) is defined by declaring $F : V \to \mathbb{R}^m$, where $V \in \mathcal{O}(\mathbb{R}^n_+)$ or $V \in \mathcal{O}(\tilde{\mathbb{R}}^n_+)$, to be C^k, $0 \leq k \leq \infty$, iff F can be extended to a C^k map on some open nbhd of V in $\mathbb{R}^n$.*[136]

4. *In both cases a map $f : M \to \mathbb{R}$ is C^k iff the map $f \circ \varphi_\alpha^{-1} : V_\alpha \to \mathbb{R}$ is C^k for each α.*

5. *The* **boundary** *∂M of a manifold M with boundary or corners is the set of all $x \in M$ whose image $\varphi(x)$ in some chart (U, φ) with $x \in U$ lies on the (topological) boundary $\partial\varphi(U)$ of $\varphi(U)$ in $\mathbb{R}^n$ (this is independent of the chart).*[137] *In addition, a boundary point of a manifold with corners is a* **corner point** *if at least two of the coordinates of $\varphi(x)$ vanish.*

6. *The* **interior** int(M) *is defined as $M \backslash \partial M$.*

7. *For $k = \infty$, the* **tangent bundle** *is defined exactly as in Definition 2.4. In particular, for any $x \in M$, the tangent space $T_x M$ is the space of all point derivations (2.2) of $C^\infty(M)$.*

The boundary of a manifold *with boundary* is itself a manifold (without boundary or corners), in the same class C^k as M itself, of dimension $n-1$ (i.e. one less than M). This should be clear for $\mathbb{R}^n_+$ itself, where $\partial\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x^n = 0\}$, which is clearly $\cong \mathbb{R}^{n-1}$. However, corner points typically ruin C^k regularity of the boundary; removing them leaves a disconnected C^k boundary. On the other hand, in both cases int(M) is again a "plain", n-dimensional manifold.[138]

Surprisingly, for $M = \mathbb{R}^n_+$ the tangent space is just $T_x\mathbb{R}^n_+ = T_x\mathbb{R}^n$ even at $x \in \partial\mathbb{R}^n_+$, and also for general M the fibers T_xM are vector spaces with a coordinate basis $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$ at any $x \in M$. This makes it possible to define tensors and (semi) Riemannian metrics as usual.

To recover the intuition that tangent vectors at boundary points $x \in \partial M$ should be directed inwards (at least without corners), note that the set-theoretic complement $T_xM \backslash T_x\partial M$ of $T_x\partial M$ is the disjoint union of two open half-spaces of which one, call it T_x^iM, consisting of inward tangent vectors, is distinguished by the property that for any $X \in T_x^iM$ there exists a smooth (or C^k) curve $c : [0, \varepsilon) \to M$ for which $c(0, \varepsilon) \in \text{int}(M)$ and $Xf(x) = df(c(t))/dt_{|t=0}$, as usual.

[135] See Lee (2012) for both boundaries and corners, and Gallot, Hulin, & Lafontaine (1990) for boundaries. Manifolds with corners are usually studied using the b-calculus of Melrose (1996).

[136] For $k = \infty$, ***Seeley's extension theorem*** states that this is equivalent to all derivatives of F being bounded on all bounded subsets of the (topological) interior int(V) of V (Seeley, 1964). See also Grieser (2000).

[137] Either $x \in M$ has an open nbhd $U \cong V \in \mathcal{O}(\mathbb{R}^n)$, in which case $x \notin \partial M$, or it doesn't, in which case $x \in \partial M$.

[138] A basic result is the ***collar neighbourhood theorem***, which states that if M is a smooth manifold with boundary, then ∂M has an open nbhd in M that is diffeomorphic to $\partial M \times [0, 1)$. See e.g. Schultz (undated).

2.7 Summary

- Differential geometry gets going as soon as we define the space $C^\infty(M)$ of smooth (real-valued) functions on a manifold M; this is done through local charts $\varphi : U \to \mathbb{R}^n$ $(U \subset M)$.
- The ***coordinates*** $(x^1,\dots,x^n)$ of $x \in U$ with respect to $\varphi = (\varphi^1,\dots,\varphi^n)$ are $x^i = \varphi^i(x)$.
- The ***tangent bundle*** TM to M is the union $TM = \sqcup_{x\in M} T_xM$, where T_xM is the space of ***point derivations*** at x, defined as linear maps $\delta_x : C^\infty(M) \to \mathbb{R}$ that satisfy the Leibniz rule $\delta_x(fg) = \delta_x(f)g(x) + f(x)\delta_x(g)$. Each δ_x takes the form $\delta_x(f) = \frac{d}{dt} f(\gamma(t))_{|t=0}$, where $\gamma : (-\varepsilon,\varepsilon) \to M$ is some curve through $x = \gamma(0)$; then δ_x is called a ***tangent vector*** X_x.
- A smooth section $x \mapsto \delta_x$ of TM corresponds to a ***derivation*** $\delta : C^\infty(M) \to C^\infty(M)$, i.e. a linear map satisfying $\delta(fg) = \delta(f)g + f\delta(g)$. Conversely, each derivation defines point derivations $\delta_x(f) = \delta(f)(x)$. Seen as $x \mapsto X_x$, a derivation $\delta \equiv X$ is a ***vector field*** on M. The set of all vector fields on M is denoted by $\mathfrak{X}(M)$. It is naturally a $C^\infty(M)$ module.
- The coordinates of $X_x \in T_xM$ with respect to φ are $X^i = X_x\varphi^i$ (where $X_x = \delta_x$ is restricted to U), and one has $X_x = X^i\partial_i$, where $\partial_i = \partial/\partial x^i$, and $(\partial_1,\dots,\partial_n)$ form a basis of T_xM.
- The ***cotangent bundle*** T^*M to M is the union $T^*M = \sqcup_{x\in M} T_x^*M$, where T_x^*M is the linear dual $\mathrm{Hom}(T_xM,\mathbb{R})$. Each $C^\infty(M)$-linear map $\theta : \mathfrak{X}(M) \to C^\infty(M)$, called a ***1-form***, comes from a cross-section $x \mapsto \theta_x$ with $\theta_x \in T_x^*M$. The set of all 1-forms on M is called $\Omega(M)$.
- The ***exterior derivative*** $d : C^\infty(M) \to \Omega(M)$ is canonically defined by $df(X) := X(f)$.
- The coordinates of $\theta_x \in T_x^*M$ with respect to φ are $\theta_i = \theta(\partial_i)$, and then $\theta = \theta_i dx^i$.
- The vector bundle $T^{(k,l)}M = \cup_x T_x^{(k,l)}M$ of ***tensors of type*** (k,l) over M is defined by

$$T_x^{(k,l)}M := \mathrm{Hom}((T_xM)^k \times (T_x^*M)^l,\mathbb{R}) \cong (\otimes^k T_x^*M) \otimes (\otimes^l T_xM).$$

  The cross-sections $x \mapsto \tau_x \in T_x^{(k,l)}M)$ are the maps $\tau : \mathfrak{X}(M)^k \times \Omega(M)^l \to C^\infty(M)$ that are $k+l$-multilinear linear over $C^\infty(M)$. These maps, also called tensors, form $\mathfrak{X}^{(k,l)}(M)$.
- Important special cases are: $T^{(1,0)} = T^*M$, so that $\mathfrak{X}^{(1,0)}(M) = \Omega(M)$, and $T^{(0,1)} = TM$, so that $\mathfrak{X}^{(0,1)}(M) = \mathfrak{X}(M)$. Furthermore, the metric tensor g of GR will be in $\mathfrak{X}^{(2,0)}(M)$.
- The coordinates $\tau_{i_1\cdots i_k}^{j_1\cdots j_l}$ of $\tau_x \in T_x^{(k,l)}M$ are given by $\tau_x(\partial_{i_1},\dots,\partial_{i_k}; dx^{j_1}\dots,dx^{j_l})$, and we have $\tau_x = \tau_{i_1\cdots i_k}^{j_1\cdots j_l}(x)\, dx^{i_1} \otimes \cdots dx^{i_k} \otimes \partial_{j_1} \otimes \cdots \otimes \partial_{j_l}$. For the metric, this gives $g_{ij} = g(\partial_i,\partial_j)$.
- For each vector field $X \in \mathfrak{X}(M)$, the ***Lie derivative*** $\mathscr{L}_X : \mathfrak{X}^{(k,l)}(M) \to \mathfrak{X}^{(k,l)}(M)$ is a linear map that satisfies $\mathscr{L}_X(f\tau) = (Xf)\tau + f\mathscr{L}_X(\tau)$ for each $f \in C^\infty(M)$ and $\tau \in \mathfrak{X}^{(k,l)}(M)$.
- The Lie derivative satisfies three important properties: for vector fields $Y \in \mathfrak{X}(M)$ one has $[\mathscr{L}_X,\mathscr{L}_Y] = \mathscr{L}_{[X,Y]}$ as well as $\mathscr{L}_X Y = [X,Y]$, whilst $\mathscr{L}_X f = Xf$ on functions $f \in C^\infty(M)$.
- Unless stated otherwise, all maps between smooth objects are required to be smooth.
- The ***Einstein summation convention*** holds: *repeated (diagonal) indices are summed over*.