Available online at www.sciencedirect.com
ScienceDirect
Indagationes Mathematicae 32 (2021) 1005–1082
indagationes mathematicae
www.elsevier.com/locate/indag

Special issue to the memory of T.A. Springer

Affine Hecke algebras and their representations

Maarten Solleveld[1]

IMAPP, Radboud Universiteit Nijmegen, Heyendaalseweg 135, 6525AJ Nijmegen, The Netherlands

Abstract

This is a survey paper about affine Hecke algebras. We start from scratch and discuss some algebraic aspects of their representation theory, referring to the literature for proofs. We aim in particular at the classification of irreducible representations.

Only at the end we establish a new result: a natural bijection between the set of irreducible representations of an affine Hecke algebra with parameters in $\mathbb{R}_{\geq 1}$, and the set of irreducible representations of the affine Weyl group underlying the algebra. This can be regarded as a generalized Springer correspondence with affine Hecke algebras.

Keywords: Hecke algebras; Weyl groups; Representation theory; Discrete series

0. Introduction

Affine Hecke algebras typically arise in two ways. Firstly, they are deformations of the group algebra of a Coxeter system (W, S) of affine type. Namely, keep the braid relations of W, but replace every quadratic relation $s^2 = 1$ $(s \in S)$ by

$$(s - q_s)(s + 1) = 0, \tag{0.1}$$

where q_s is a parameter in some field (usually we take the field $\mathbb{C}$). That gives rise to an associative algebra $\mathcal{H}(W, q)$.

Secondly, affine Hecke algebras occur in the representation theory of reductive groups G over p-adic fields. They can be isomorphic to the algebra of G-endomorphisms of a suitable G-representation. The classical example is the convolution algebra of compactly supported functions on G that are bi-invariant with respect to an Iwahori subgroup. In this way the representation theory of affine Hecke algebras is related to that of reductive p-adic groups.

E-mail address: m.solleveld@science.ru.nl.

[1] The author is supported by a NWO Vidi grant “A Hecke algebra approach to the local Langlands correspondence” (nr. 639.032.528).

https://doi.org/10.1016/j.indag.2021.01.005

The interpretation of (affine) Hecke algebras as deformations of group algebras links them with quantum groups, knot theory and noncommutative geometry. Further, their relation with reductive groups makes them highly relevant in the representation theory of such groups and in the local Langlands program.

Both views of affine Hecke algebras build upon simpler objects: the Hecke algebras of finite Weyl groups. On the one hand such a finite dimensional Hecke algebra is a deformation of a group algebra, again with relations (0.1). On the other hand it appears naturally in the representation theory of reductive groups over finite fields. However, there is a crucial difference between the finite and affine cases: most aspects of finite dimensional Hecke algebras are easily described in terms of a Coxeter group, but that is far from true for affine Hecke algebras.

Of course there exists an extensive body of literature on affine Hecke algebras and their representations, see the references to this paper for a part of it. The theory is in a good state, and (in our opinion) most of the important questions that one can ask about affine Hecke algebras have been answered.

Unfortunately, the accessibility of this literature is rather limited. A first difficulty is that several slightly different algebras are involved, with several presentations, and sometimes their connection is not clear. Further, a plethora of techniques has been applied to affine Hecke algebras: algebraic, analytic, geometric or combinatoric to various degrees. Finally, to the best of our knowledge no textbook treats affine Hecke algebras any further than their presentations.

With this survey paper we intend to fill a part of that gap. Our aim is an introduction to affine Hecke algebras and their representations, of which the larger part is readable for mathematicians without previous experience with the subject. To ease the presentation, we will hardly prove anything, and we will provide many examples. As a consequence, our treatment is almost entirely algebraic — the deep analytic or geometric arguments behind some important results are largely suppressed.

Let us discuss the contents in a nutshell. In the first section we work out the various presentations of (affine) Hecke algebras over $\mathbb{C}$, and we compare them. Section 2 consists of explicit examples: we look at the most frequent affine Hecke algebras and provide an overview of their irreducible representations.

The third section is the core of the paper, here we build up the abstract representation theory of an affine Hecke algebra $\mathcal{H}$. This is done in the spirit of Harish-Chandra's analytic approach to representations of reductive groups: we put the emphasis on parabolic induction, the discrete series and the large commutative subalgebra of $\mathcal{H}$. To make this work well, we need to assume that the parameters q_s of $\mathcal{H}$ lie in $\mathbb{R}_{>0}$. With these techniques one can divide the set of irreducible representations $\mathrm{Irr}(\mathcal{H})$ into L-packets, like in the local Langlands program. To achieve more, it pays off to reduce from affine Hecke algebras to a simpler kind of algebras called graded Hecke algebras. This is like reducing questions about a Lie group to its Lie algebra.

More advanced techniques to classify $\mathrm{Irr}(\mathcal{H})$ are discussed in Section 4. In principle this achieves a complete classification, but in practice some computations remain to be done in examples.

In Section 5 we report on algebro–geometric approaches to affine Hecke algebras and graded Hecke algebras, largely due to Lusztig. In many cases, these methods yield beautiful constructions and parametrizations of all irreducible representations. Although we treat the involved (co)homology theories as a black box, we do provide a comparison between these constructions and the setup inspired by Harish-Chandra.

The final section of the paper is quite different from the rest, here we do actually prove some new results. The topic is the relation between an affine Hecke algebra $\mathcal{H}$ with parameters $q_s \in \mathbb{R}_{\geq 1}$ and its version with parameters $q_s = 1$. The latter algebra is of the form $\mathbb{C}[X \rtimes W]$, where X is a lattice and W is the (finite) Weyl group of a root system R in X. We note that $X \rtimes W$ contains the affine Weyl/Coxeter group $\mathbb{Z}R \rtimes W$. There is a natural way to regard any finite dimensional $\mathcal{H}$-representation π as a representation of $\mathbb{C}[W]$, we call that the W-type of π.

Theorem A (*See Theorem* 6.11)**.** *Let $\mathcal{H}$ be an affine Hecke algebra with parameters in $\mathbb{R}_{\geq 1}$ (and a mild condition when R has components of type F_4). There exists a natural bijection*

$$\zeta'_{\mathcal{H}} : \mathrm{Irr}(\mathcal{H}) \longrightarrow \mathrm{Irr}(X \rtimes W)$$

based on the W-types of irreducible tempered $\mathcal{H}$-representations.

In the case where all the parameters q_s are equal, Theorem A is closely related to the Kazhdan–Lusztig parametrization of $\mathrm{Irr}(\mathcal{H})$ [44], and shown already in [50].

It is interesting to restrict Theorem A to irreducible $X \rtimes W$-representations on which X acts trivially (those can clearly be identified with irreducible W-representations). The inverse image of these under $\zeta'_{\mathcal{H}}$ is the set $\mathrm{Irr}_0(\mathcal{H})$ of irreducible tempered $\mathcal{H}$-representations "with real central character", see Section 6.1. When $\mathcal{H}$ is of geometric origin, $\mathrm{Irr}_0(\mathcal{H})$ is naturally parametrized by data as in Lusztig's generalization of the Springer correspondence with intersection homology, see Section 5.2. In that case the bijection

$$\zeta'_{\mathcal{H}} : \mathrm{Irr}_0(\mathcal{H}) \to \mathrm{Irr}(W)$$

becomes an instance of the generalized Springer correspondence. Moreover, for such geometric affine Hecke algebras the whole of $\mathrm{Irr}(\mathcal{H})$ admits a natural parametrization in terms of data that are variations on Kazhdan–Lusztig parameters, see Section 5.3. Via this parametrization, $\zeta'_{\mathcal{H}}$ becomes a generalized Springer correspondence for the (extended) affine Weyl group $X \rtimes W$.

With that in mind we can regard Theorem A, for any eligible $\mathcal{H}$, as a "generalization of the Springer correspondence with affine Hecke algebras". This applies both to the finite Weyl group W and the (extended) affine Weyl group $X \rtimes W$.

In the final paragraph we use (the proof of) Theorem A to derive some properties of affine Hecke algebras $\mathcal{H}$ of type B_n/C_n with three independent positive q-parameters. When these parameters are generic, we provide an explicit, effective classification of $\mathrm{Irr}(\mathcal{H})$.

Of course the selection of topics in any survey is to a considerable extent the taste of the author. To preserve a reasonable size, we felt forced to omit many interesting aspects of affine Hecke algebras: the Kazhdan–Lusztig basis [43], asymptotic Hecke algebras [50,56], unitary representations [7,17], the Schwartz and C^*-completions of affine Hecke algebras [22,63], homological algebra [66,81], formal degrees of representations [18,67], spectral transfer morphisms [64,65] and so on. We apologize for these and other omissions and refer the reader to the literature.

1. Definitions and first properties

1.1. Finite dimensional Hecke algebras

Let (W, S) be a finite Coxeter system — so W is a finite group generated by a set S of elements of order 2. Moreover, W has a presentation

$$W = \langle S \mid (ss')^{m(s,s')} = e \ \forall s, s' \in S \rangle,$$

where $m(s, s') \in \mathbb{Z}_{\geq 1}$ is the order of ss' in W. The equalities $s^2 = e$ ($s \in S$) are called the quadratic relations, while $(ss')^{m(s,s')} = e$, or equivalently

$$\underbrace{s\, s'\, s\, s' \cdots}_{m(s,s') \text{ terms}} = \underbrace{s'\, s\, s'\, s \cdots}_{m(s,s') \text{ terms}}$$

is known as a braid relation. Examples to keep in mind are

- $W = S_n$, $S = \{(12), (23), \ldots, (n-1\, n)\}$ – type A_{n-1};
- $W = S_n \ltimes \{\pm 1\}^n$, $S = \{(12), (23), \ldots, (n-1\, n), (\mathrm{id}, (1, \ldots, 1, -1))\}$ – type B_n or C_n.

In the group algebra $\mathbb{C}[W]$ the quadratic relations are equivalent with

$$(s+1)(s-1) = 0 \qquad s \in S. \tag{1.1}$$

Now we choose, for every $s \in S$, a complex number q_s, such that

$$q_s = q_{s'} \text{ if } s \text{ and } s' \text{ are conjugate in } W. \tag{1.2}$$

Let $q : S \to \mathbb{C}$ be the function $s \mapsto q_s$. We define a new $\mathbb{C}$-algebra $\mathcal{H}(W, q)$ which has a vector space basis $\{T_w : w \in W\}$. Here T_e is the unit element and there are quadratic relations

$$(T_s + 1)(T_s - q_s) = 0 \qquad s \in S$$

and braid relations

$$\underbrace{T_s\, T_{s'}\, T_s \cdots}_{m(s,s') \text{ terms}} = \underbrace{T_{s'}\, T_s\, T_{s'} \cdots}_{m(s,s') \text{ terms}}. \tag{1.3}$$

Equivalent versions of these quadratic relations are

$$T_s^2 = (q_s - 1)T_s + q_s T_e \quad \text{and} \quad T_s^{-1} = q_s^{-1} T_s + (q_s^{-1} - 1)T_e \tag{1.4}$$

(the latter only when $q_s \neq 0$). For $q = 1$, the relations (1.3) become the defining relations of the Coxeter system (W, S), so $\mathcal{H}(W, 1) = \mathbb{C}[W]$. We say that $\mathcal{H}(W, q)$ has equal parameters if $q_s = q_{s'}$ for all $s, s' \in S$.

The condition (1.2) is necessary and sufficient for the existence of an associative unital algebra $\mathcal{H}(W, q)$ with these properties [35, §7.1–7.3]. It is known as a generic algebra or a Hecke algebra. Such algebras appear for instance in the representation theory of reductive groups over finite fields [33,34,36], in knot theory [39] and in combinatorics [31,32].

Further $\mathcal{H}(W, q)$ has the structure of a symmetric algebra: it carries a trace $\tau(T_w) = \delta_{w,e}$, an involution $T_w^* = T_{w^{-1}}$ and a bilinear form $(x, y) = \tau(x^* y)$. These also give rise to interesting properties [26], which however fall outside the scope of this survey.

Without the trace τ, the representation theory of finite dimensional Hecke algebras is quite easy. To explain this, we consider a more general situation.

Let G be any finite group. By Maschke's theorem the group algebra $\mathbb{C}[G]$ is semisimple. Let $\{T_g : g \in G\}$ be its canonical basis, and $\mathbf{k} = \mathbb{C}[x_1, \ldots, x_r]$ a polynomial ring over $\mathbb{C}$. Let A be a $\mathbf{k}$-algebra whose underlying $\mathbf{k}$-module is $\mathbf{k}[G]$ and whose multiplication is defined by

$$T_g \cdot T_h = \sum_{w \in G} a_{g,h,w} T_w \tag{1.5}$$

for certain $a_{g,h,w} \in \mathbf{k}$. For any point $q \in \mathbb{C}^r$ we can endow the vector space $\mathbb{C}[G]$ with the structure of an associative algebra by

$$T_g \cdot_q T_h = \sum_{w \in G} a_{g,h,w}(q) T_w \tag{1.6}$$

We denote the resulting algebra by $\mathcal{H}(G, q)$. It is isomorphic to the tensor product $A \otimes_{\mathbf{k}} \mathbb{C}$ where $\mathbb{C}$ has the $\mathbf{k}$-module structure obtained from evaluation at q. Assume moreover that there exists a $q^0 \in \mathbb{C}^r$ such that

$$\mathcal{H}(G, q^0) = \mathbb{C}[G]$$

We express the rigidity of finite dimensional semisimple algebras by the following special case of Tits' deformation theorem, see [12, p. 357 - 359] or [37, Appendix].

Theorem 1.1. *There exists a polynomial $P \in \mathbf{k}$ such that the following are equivalent:*

- $P(q) \neq 0$,
- $\mathcal{H}(G, q)$ *is semisimple,*
- $\mathcal{H}(G, q) \cong \mathbb{C}[G]$.

In other words: when $\mathcal{H}(G, q)$ is semisimple it is isomorphic to $\mathbb{C}[G]$, and otherwise the algebra $\mathcal{H}(W, q)$ has nilpotent ideals and looks very different from $\mathbb{C}[G]$. The simplest case where the latter occurs is already $G = \{e, s\}$. Namely, for $q = -1$ we get the Hecke algebra

$$\mathcal{H}(G, -1) = \mathbb{C}[T_s + 1]/(T_s + 1)^2. \tag{1.7}$$

Let us discuss how it works out for an arbitrary finite Coxeter system (W, S) and a parameter function q as before. For an element $w \in W$ with a reduced expression $s_1 s_2 \cdots s_r$ we put

$$q(w) = q_{s_1} q_{s_2} \cdots q_{s_r}. \tag{1.8}$$

This is well-defined by the presentation of W and condition (1.2). We want to know under which conditions the algebra $\mathcal{H}(W, q)$ is semisimple. For such groups the polynomials $P(q)$ of Theorem 1.1 have been determined explicitly. If we are in the equal label case $q(s) = q \; \forall s \in S$ then we may take

$$P(q) = q \sum_{w \in W} q^{\ell(w)} \tag{1.9}$$

except that we must omit the factor q if W is of type $(A_1)^n$, see [29]. More generally, suppose that (W, S) is irreducible and S consists of two conjugacy classes, with parameters q_1 and q_2. Gyoja [28, p. 569] showed that in most of these cases we may take

$$\begin{aligned} P(q_1, q_2) &= q_1^{|W|} q_2 W(q_1, q_2) W(q_1^{-1}, q_2) \\ W(q_1, q_2) &= \textstyle\sum_{w \in W} q(w) \end{aligned} \tag{1.10}$$

So generically there is an isomorphism

$$\mathcal{H}(W, q) \cong \mathbb{C}[W]. \tag{1.11}$$

In particular, for almost all q the representation theory of $\mathcal{H}(W, q)$ is just that of W (over $\mathbb{C}$).

When $q_1 q_2 \neq 0$ and $W(q_1, q_2) = 0$, the subgroup of $\mathbb{C}^\times$ generated by q_1 and q_2 contains a root of unity different from 1. Hence the non-semisimple algebras $\mathcal{H}(W, q)$ are those with (at least) one q_s equal to 0 and some for which q involves nontrivial roots of unity. Although these algebras can have an interesting combinatorial structure, their behaviour is quite different from that of affine Hecke algebras.

1.2. Iwahori–Hecke algebras

Let (W, S) be any Coxeter system. As in the previous paragraph, we can assign complex numbers q_s to the elements of S. When (1.2) is fulfilled, we can construct an algebra $\mathcal{H}(W, q)$ exactly as before. For $q = 1$ this is the group algebra of W, and hence, for $q \neq 1$, $\mathcal{H}(W, q)$ can be regarded as some deformation of $\mathbb{C}[W]$. However, the situation is much more complicated than when W is finite. Tits' deformation theorem does not work here, and in general many different q's can lead to mutually non-isomorphic algebras [87].

To get some grip on the situation, we restrict our scope from general Coxeter groups to affine Weyl groups. By that we mean Coxeter systems $(W_{\text{aff}}, S_{\text{aff}})$ such that every irreducible component S_i of S_{aff} generates a Coxeter group W_i of affine type. The affineness condition is equivalent to: the Cartan matrix of (W_i, S_i) is positive semidefinite but not positive definite [35, §2.5,§4.7, §6.5]. Thus irreducible affine Weyl groups are classified by the Dynkin diagrams $\tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{F}_4, \tilde{G}_2$.

In the simplest case $\tilde{A}_1$, W_{aff} is an infinite dihedral group, freely generated by two elements of order 2.

Definition 1.2. An Iwahori–Hecke algebra is an algebra of the form $\mathcal{H}(W, q)$, where (W, S) is any Coxeter system. We say that $\mathcal{H}(W, q)$ is of affine type if (W, S) is an affine Weyl group.

Iwahori–Hecke algebras of affine type were discovered first by Matsumoto and Iwahori [37,38,58], in the context of reductive p-adic groups. For instance, let G be a split, simply connected, semisimple group over $\mathbb{Q}_p$ and let I be an Iwahori subgroup of G. Then the convolution algebra $C_c(I\backslash G/I)$ is isomorphic to $\mathcal{H}(W_{\text{aff}}, q)$, where $(W_{\text{aff}}, S_{\text{aff}})$ is derived from G and $q_s = p$ for all $s \in S$. This is called the Iwahori-spherical Hecke algebra of G, because its modules classify G-representations with I-fixed vectors.

The basic structure of an affine Weyl group W_{aff} is described in [35, §4.2] and [10, §VI.2]. The set of elements whose conjugacy class is finite forms a finite index normal subgroup of W_{aff}, isomorphic to a lattice. That lattice is spanned by an integral root system R, and W_{aff} is the semidirect product of $\mathbb{Z}R$ and the Weyl group of R. In particular $\mathcal{H}(W_{\text{aff}}, q)$ contains $\mathcal{H}(W(R), q)$ as a subalgebra. However, the embedding of $W(R)$ in W_{aff} is in general not unique.

1.3. Affine Hecke algebras

Affine Hecke algebras generalize Iwahori–Hecke algebras of affine type. Instead of affine Weyl groups, we allow more general groups which are semidirect products of lattices and finite Weyl groups. The best way to do the bookkeeping is with root data, for which our standard references are [10,35].

Consider a quadruple $\mathcal{R} = (X, R, Y, R^\vee)$, where

- X and Y are lattices of finite rank, with a perfect pairing $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{Z}$,
- R is a root system in X,
- $R^\vee \subset Y$ is the dual root system, and a bijection $R \to R^\vee, \alpha \mapsto \alpha^\vee$ with $\langle \alpha, \alpha^\vee \rangle = 2$ is given,

- for every $\alpha \in R$, the reflection

$$s_\alpha : X \to X, \; s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$$

stabilizes R,
- for every $\alpha^\vee \in R^\vee$, the reflection

$$s_\alpha^\vee : Y \to Y, \; s_\alpha^\vee(y) = y - \langle \alpha, y \rangle \alpha^\vee$$

stabilizes $R^\vee$.

If all these conditions are met, we call $\mathcal{R}$ a root datum. It comes with a finite Weyl group $W = W(R)$ and an infinite group $W(\mathcal{R}) = X \rtimes W(R)$, the extended affine Weyl group of $\mathcal{R}$. Often we will add a base of R to $\mathcal{R}$, and speak of a based root datum.

Example 1.3. Take $X = Y = \mathbb{Z}$, $R = \{\pm 1\}$ and $R^\vee = \{\pm 2\}$. Then

$$W(\mathcal{R}) = \mathbb{Z} \rtimes S_2, \text{ an infinite dihedral group.} \tag{1.12}$$

We stress that we do not require R to span the vector space $X \otimes_\mathbb{Z} \mathbb{R}$. We say that $\mathcal{R}$ is semisimple if R does span $X \otimes_\mathbb{Z} \mathbb{R}$. For non-semisimple root data R and $R^\vee$ may even be empty. For instance, the root datum $\mathcal{R} = (\mathbb{Z}^n, \emptyset, \mathbb{Z}^n, \emptyset)$ has infinite group $W(\mathcal{R}) = \mathbb{Z}^n$, but no reflections.

More examples of root data (actually all) come from reductive groups, see [85]. Suppose that $\mathcal{G}$ is a reductive algebraic group, $\mathcal{T}$ is a maximal torus in $\mathcal{G}$ and $R(\mathcal{G}, \mathcal{T})$ is the associated root system. Denote the character lattice of $\mathcal{T}$ by $X^*(\mathcal{T})$ and its cocharacter lattice by $X_*(\mathcal{T})$. Then

$$\mathcal{R}(\mathcal{G}, \mathcal{T}) := \big(X^*(\mathcal{T}), R(\mathcal{G}, \mathcal{T}), X_*(\mathcal{T}), R(\mathcal{G}, \mathcal{T})^\vee \big). \tag{1.13}$$

is a root datum. We note that $\mathcal{R}(\mathcal{G}, \mathcal{T})$ is semisimple if and only if $\mathcal{G}$ is semisimple.

The group $W(\mathcal{R}) = X \rtimes W$ acts naturally on the vector space $X \otimes_\mathbb{Z} \mathbb{R}$, X by translations and W by linear extension of its action on X. The collection of hyperplanes

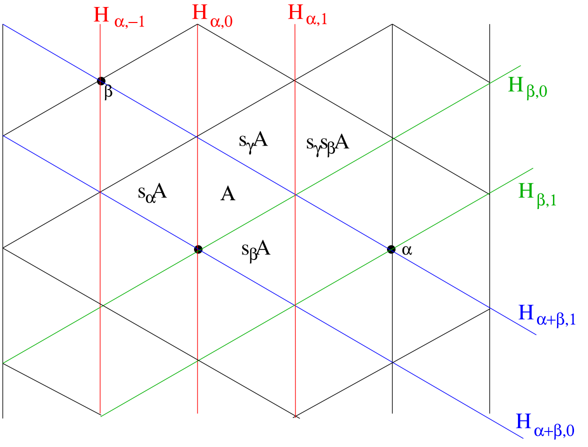
$$H_{\alpha,n} = \{x \in X \otimes_\mathbb{Z} \mathbb{R} : \alpha^\vee(x) = n\} \qquad \alpha \in R, n \in \mathbb{Z}$$

is $W(\mathcal{R})$-stable and divides $X \otimes_\mathbb{Z} \mathbb{R}$ in open subsets called alcoves. Let W_{aff} be the subgroup of $W(\mathcal{R})$ generated by the (affine) reflections in the hyperplanes $H_{\alpha,n}$. This is an affine Weyl group, and $X \otimes_\mathbb{Z} \mathbb{R}$ with this hyperplane arrangement is its Coxeter complex.

To construct Hecke algebras from root data, we need to specify a set of Coxeter generators of W_{aff}. In this setting they will be affine reflections. Let Δ be a base of R. As is well-known, it yields a set of simple reflections $S = \{s_\alpha : \alpha \in \Delta\}$, and (W, S) is a finite Coxeter system. (For a root datum of the form $\mathcal{R}(\mathcal{G}, \mathcal{T})$ as in (1.12), the choice of Δ is equivalent to the choice of a Borel subgroup $\mathcal{B} \subset \mathcal{G}$ containing $\mathcal{T}$.)

The base Δ determines a "fundamental alcove" A_0 in $X \otimes_\mathbb{Z} \mathbb{R}$, namely the unique alcove contained in the positive Weyl chamber (with respect to Δ), such that $0 \in \overline{A_0}$. The reflections in those walls of A_0 that contain 0 constitute precisely S. The set S_{aff} of (affine) reflections with respect to all walls of A_0 forms the required collection of Coxeter generators of W_{aff}.

Example 1.4. A part of the hyperplane arrangement for an affine Weyl group of type $\tilde{A}_2$, with $A = A_0$, $\Delta = \{\alpha, \beta\}$ and $S_{\mathrm{aff}} = \{s_\alpha, s_\beta, s_\gamma\}$.



We can make S_{aff} more explicit. Let $R^\vee_{\max}$ be the set of maximal elements of $R^\vee$, with respect to the base $\Delta^\vee$. It contains one element for every irreducible component of $R^\vee$. For $\alpha^\vee \in R^\vee_{\max}$, define

$$s'_\alpha : X \to X,\; s'_\alpha(x) = x + \alpha - \langle x, \alpha^\vee \rangle \alpha.$$

This is the reflection of $X \otimes_{\mathbb{Z}} \mathbb{R}$ in the hyperplane $H_{\alpha,1}$, a wall of A_0. Then

$$S_{\mathrm{aff}} = S \cup \{ s'_\alpha : \alpha^\vee \in R^\vee_{\max} \}.$$

We denote the based root datum $(X, R, Y, R^\vee, \Delta)$ also by $\mathcal{R}$. Thus the sets S, S_{aff} and the subgroup $W_{\mathrm{aff}} \subset W(\mathcal{R})$ are determined by $\mathcal{R}$.

Example 1.5. A couple of important instances, coming from the reductive groups PGL_2 and GL_n:

- $X = Y = \mathbb{Z}$, $R = \{\pm 2\}$, $R^\vee = \{\pm 1\}$, $\Delta = \{\alpha = 2\}$. Here $S_{\mathrm{aff}} = \{s_\alpha, s'_\alpha : x \mapsto 2 - x\}$ and $W_{\mathrm{aff}} = 2\mathbb{Z} \rtimes S_2$, a proper subgroup of $W(\mathcal{R}) = \mathbb{Z} \rtimes S_2$. This is the same $W(\mathcal{R})$ as in (1.12), but with a different set of simple affine reflections.
- $X = Y = \mathbb{Z}^n$, $R = R^\vee = A_{n-1} = \{e_i - e_j : 1 \leq i, j \leq n, i \neq j\}$, $\Delta = \{e_i - e_{i+1} : i = 1, \ldots, n-1\}$. In this case

$$S_{\mathrm{aff}} = \{s_i = s_{e_i - e_{i+1}} : i = 1, \ldots, n-1\} \cup \{s_0 : x \mapsto x + (1 - \langle x, e_1 - e_n \rangle)(e_1 - e_n)\},$$
$$W_{\mathrm{aff}} = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 + \cdots + x_n = 0\} \rtimes S_n. \tag{1.14}$$

Like Iwahori–Hecke algebras, affine Hecke algebras involve q-parameters. Fix $\mathbf{q} \in \mathbb{R}_{>1}$ and let $\lambda, \lambda^* : R \to \mathbb{C}$ be functions such that

- if $\alpha, \beta \in R$ are W-associate, then $\lambda(\alpha) = \lambda(\beta)$ and $\lambda^*(\alpha) = \lambda^*(\beta)$,
- if $\alpha^\vee \notin 2Y$, then $\lambda^*(\alpha) = \lambda(\alpha)$.

We note that $\alpha^\vee \in 2Y$ is only possible for short roots α in a type B component of R. For $\alpha \in R$ we write

$$q_{s_\alpha} = \mathbf{q}^{\lambda(\alpha)} \quad \text{and} \quad (\text{if } \alpha^\vee \in R^\vee_{\max}) \quad q_{s'_\alpha} = \mathbf{q}^{\lambda^*(\alpha)}. \tag{1.15}$$

Recall that $\mathcal{H}(W, q)$ is the Iwahori–Hecke algebra of $W = W(R)$ and let $\{\theta_x : x \in X\}$ be the standard basis of $\mathbb{C}[X]$.

Definition 1.6. The affine Hecke algebra $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})$ is the vector space $\mathbb{C}[X] \otimes_{\mathbb{C}} \mathcal{H}(W, q)$ with the multiplication rules:

- $\mathbb{C}[X]$ and $\mathcal{H}(W, q)$ are embedded as subalgebras,
- for $\alpha \in \Delta$ and $x \in X$:

$$\theta_x T_{s_\alpha} - T_{s_\alpha} \theta_{s_\alpha(x)} = \Big((\mathbf{q}^{\lambda(\alpha)} - 1) + \theta_{-\alpha} \big(\mathbf{q}^{(\lambda(\alpha)+\lambda^*(\alpha))/2} - \mathbf{q}^{(\lambda(\alpha)-\lambda^*(\alpha))/2} \big) \Big) \frac{\theta_x - \theta_{s_\alpha(x)}}{\theta_0 - \theta_{-2\alpha}}.$$

When $\alpha^\vee \notin 2Y$, the cross relation simplifies to

$$\theta_x T_{s_\alpha} - T_{s_\alpha} \theta_{s_\alpha(x)} = (\mathbf{q}^{\lambda(\alpha)} - 1)(\theta_x - \theta_{s_\alpha(x)})(\theta_0 - \theta_{-\alpha})^{-1}.$$

Notice that here the right hand side lies in $\mathbb{C}[X]$ because

$$\theta_x - \theta_{s_\alpha(x)} = \theta_x - \theta_{x - \langle x, \alpha^\vee \rangle \alpha} = \theta_x (\theta_0 - \theta_{-\langle x, \alpha^\vee \rangle \alpha})$$

is divisible by $\theta_0 - \theta_{-\alpha}$. It follows from [52, §3] that $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})$ is really an associative algebra with unit element $\theta_0 \otimes T_e$, and that the multiplication map

$$\begin{array}{ccc} \mathcal{H}(W, q) \otimes_{\mathbb{C}} \mathbb{C}[X] & \to & \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q}) \\ h \otimes f & \mapsto & h \cdot f \end{array} \tag{1.16}$$

is bijective. We say that $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})$ has equal parameters if

$$\lambda(\alpha) = \lambda(\beta) = \lambda^*(\alpha) = \lambda^*(\beta) \quad \text{for all } \alpha, \beta \in R.$$

When $\lambda = \lambda^* = 1$, we omit them from the notation and write simply $\mathcal{H}(\mathcal{R}, \mathbf{q})$. In that setting we will often allow $\mathbf{q}$ to be any element of $\mathbb{C}^\times$. We note that for $\lambda = \lambda^* = 0$ or $\mathbf{q} = 1$ we recover the group algebra $\mathbb{C}[X \rtimes W]$. In particular, the only affine Hecke algebra associated to $(X, \emptyset, Y, \emptyset)$ is $\mathbb{C}[X]$.

Affine Hecke algebras appear foremostly in the representation theory of reductive p-adic groups, see [2,11,61,73,84]. They also have strong ties to orthogonal polynomials [45,57], which often run via double affine Hecke algebras [15]. This has lead to a whole family of Hecke algebras, with adjectives like degenerate, cyclotomic, rational, graded and (double) affine. We will only discuss one further member of this family, in Section 1.5.

As already explained, Tits' deformation theorem does not apply to affine Hecke algebras. But there is a substitute for the semisimplicity part of Theorem 1.1. Recall that a finite dimensional algebra A is semisimple (i.e. a direct sum of simple algebras) if and only if its Jacobson radical $\mathrm{Jac}(A)$ (the intersection of the kernels of all simple modules) is zero. In that sense Theorem 1.1 admits a partial generalization, see [77, Lemma 3.4] and [59, (3.4.5)]:

Lemma 1.7. *The Jacobson radical of an affine Hecke algebra $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})$ is zero.*

1.4. Presentations of affine Hecke algebras

We will make the relations between the algebras in Sections 1.2 and 1.3 explicit. We start with the Bernstein presentation of an Iwahori–Hecke algebra of affine type.

Let W_{aff} be an affine Weyl group with Coxeter generators S_{aff}, and write it as $W_{\mathrm{aff}} = \mathbb{Z}R \rtimes W(R)$. Put

$$\mathcal{R} = \big(\mathbb{Z}R, R, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}R, \mathbb{Z}), R^\vee, \Delta \big),$$

where $S_{\mathrm{aff}} \cap W(\mathcal{R}) = \{ s_\alpha : \alpha \in \Delta \}$. Consider an Iwahori–Hecke algebra $\mathcal{H}(W_{\mathrm{aff}}, q)$ with parameters $q_s \in \mathbb{C}$.

Theorem 1.8 (*Bernstein, see [52, §3]*). *Suppose that $q_s \neq 0$ for all $s \in S_{\mathrm{aff}}$. Pick $\lambda(\alpha), \lambda^*(\alpha)$ such that $q_{s_\alpha} = \mathbf{q}^{\lambda(\alpha)}$ for all $\alpha \in R$ and $q_{s'_\alpha} = \mathbf{q}^{\lambda^*(\alpha)}$ when $\alpha^\vee \in R^\vee_{\max}$. Then there exists a unique algebra isomorphism $\mathcal{H}(W_{\mathrm{aff}}, q) \to \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})$ such that:*

- *it is the identity on $\mathcal{H}(W, q)$,*
- *for $x \in \mathbb{Z}R$ with $\langle x, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Delta$, it sends $q(x)^{-1/2} T_x$ to θ_x.*

Here $q(x)^{1/2}$ is defined via (1.8) and $q_{s_\alpha}^{1/2} = \mathbf{q}^{\lambda(\alpha)/2}$, $q_{s'_\alpha}^{1/2} = \mathbf{q}^{\lambda^*(\alpha)/2}$.

Not every affine Hecke algebra is isomorphic to an Iwahori–Hecke algebra, for instance $\mathbb{C}[X]$ is not. To be precise, an isomorphism as in Theorem 1.8 exists if and only if the root datum $\mathcal{R}$ is that of an adjoint semisimple group. To compensate for this difference in scope, we take another look at the structure of $W(\mathcal{R})$ for an arbitrary based root datum $\mathcal{R}$.

We know from [35, §4.4] that the length function ℓ of $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ satisfies

$$\ell(w) = \text{ number of hyperplanes } H_{\alpha,n} \text{ that separate } w(A_0) \text{ from } A_0. \tag{1.17}$$

We extend ℓ to a function $W(\mathcal{R}) \to \mathbb{Z}_{\geq 0}$, by decreeing that (1.17) is valid for all $w \in W(\mathcal{R})$. Then

$$\Omega := \{w \in W(\mathcal{R}) : \ell(w) = 0\} = \text{ stabilizer of } A_0 \text{ in } W(\mathcal{R})$$

is a subgroup of $W(\mathcal{R})$. The group Ω acts by conjugation on W_{aff} and that action stabilizes S_{aff} (the set of reflections with respect to the walls of A_0). Moreover, since W_{aff} acts simply transitively on the set of alcoves in $X \otimes_{\mathbb{Z}} \mathbb{R}$:

$$W(\mathcal{R}) = W_{\mathrm{aff}} \rtimes \Omega.$$

Example 1.9. Let $\mathcal{R}$ be of type GL_n, as in (1.14). Then Ω is isomorphic to $\mathbb{Z}$, generated by $\omega = e_1(1\,2 \cdots n)$. The action of ω on $S_{\mathrm{aff}} = \{s_0, s_1, \ldots, s_{n-1}\}$ is $\omega s_i \omega^{-1} = s_{i+1}$ (where s_n means s_0).

Let $q : S_{\mathrm{aff}} \to \mathbb{C}^\times$ be the parameter function determined by $\mathbf{q}, \lambda, \lambda^*$. The conditions on λ and λ^* (before Definition 1.6) ensure that q is Ω-invariant. Hence the formula

$$\omega(T_w) = T_{\omega w \omega^{-1}}$$

defines an algebra automorphism of $\mathcal{H}(W_{\mathrm{aff}}, q)$. This gives a group action of Ω on $\mathcal{H}(W_{\mathrm{aff}}, q)$. Recall that the crossed product algebra $\mathcal{H}(W_{\mathrm{aff}}, q) \rtimes \Omega$ is the vector space $\mathcal{H}(W_{\mathrm{aff}}, q) \otimes_{\mathbb{C}} \mathbb{C}[\Omega]$ with multiplication defined by

$$\omega \cdot T_w \cdot \omega^{-1} = \omega(T_w).$$

For $w \in W_{\mathrm{aff}}, \omega \in \Omega$ we write $T_{w\omega} = T_w \omega \in \mathcal{H}(W_{\mathrm{aff}}, q) \rtimes \Omega$. The multiplication relations in $\mathcal{H}(W_{\mathrm{aff}}, q) \rtimes \Omega$ become

$$\begin{array}{ll} T_v T_w = T_{vw} & \text{if } \ell(vw) = \ell(v) + \ell(w), \\ (T_s + 1)(T_s - q_s) = 0 & \text{for } s \in S_{\mathrm{aff}}. \end{array} \tag{1.18}$$

Now we can formulate the counterpart of Theorem 1.8.

Theorem 1.10 (*Bernstein, see [52, §3]*). *There is a unique algebra isomorphism $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q}) \to \mathcal{H}(W_{\mathrm{aff}}, q) \rtimes \Omega$ such that*

- *it is the identity on $\mathcal{H}(W, q)$,*

- *for $x \in \mathbb{Z}R$ with $\langle x, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Delta$, it sends θ_x to $q(x)^{-1/2} T_x$.*

The algebra $\mathcal{H}(W_{\mathrm{aff}}, q) \rtimes \Omega$, with the multiplication rules (1.18), is called the Iwahori–Matsumoto presentation of $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})$.

We conclude this paragraph with yet another, more geometric, presentation of affine Hecke algebras. Let T be the complex algebraic torus $\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{C}^\times)$. By duality $\mathrm{Hom}(T, \mathbb{C}^\times) = X$, and the ring of regular functions $\mathcal{O}(T)$ is the group algebra $\mathbb{C}[X]$. The group W acts naturally on X, and that induces actions on $\mathbb{C}[X]$ and on T.

Definition 1.11. The algebra $\mathcal{H}(T, \lambda, \lambda^*, \mathbf{q})$ is the vector space $\mathcal{O}(T) \otimes_{\mathbb{C}} \mathcal{H}(W, q)$ with the multiplication rules

- $\mathcal{O}(T)$ and $\mathcal{H}(W, q)$ are embedded as subalgebras,
- for $\alpha \in \Delta$ and $f \in \mathcal{O}(T)$:

$$f T_{s_\alpha} - T_{s_\alpha}(s_\alpha \cdot f) = \Big((\mathbf{q}^{\lambda(\alpha)} - 1) + \theta_{-\alpha} \big(\mathbf{q}^{(\lambda(\alpha)+\lambda^*(\alpha))/2} - \mathbf{q}^{(\lambda(\alpha)-\lambda^*(\alpha))/2} \big) \Big) \frac{f - s_\alpha(f)}{\theta_0 - \theta_{-2\alpha}}.$$

Clearly the identification $\mathcal{O}(T) \cong \mathbb{C}[X]$ induces an algebra isomorphism

$$\mathcal{H}(T, \lambda, \lambda^*, \mathbf{q}) \cong \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q}).$$

From the above presentation it is easy to find the centre of these algebras [52]:

$$Z\big(\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})\big) \cong Z\big(\mathcal{H}(T, \lambda, \lambda^*, \mathbf{q})\big) = \mathcal{O}(T)^W = \mathcal{O}(T/W) \cong \mathbb{C}[X]^W. \tag{1.19}$$

An advantage of Definition 1.11 is that this presentation can also be used if T is just known as an algebraic variety, without a group structure. In that situation $\mathcal{H}(T, \lambda, \lambda^*, \mathbf{q})$ can be studied without fixing a basepoint of T, it suffices to have the W-action and the elements $\theta_{-\alpha} \in \mathcal{O}(T)^\times$ for $\alpha \in \Delta$. This is particularly handy for affine Hecke algebras arising from Bernstein components for reductive p-adic groups.

Even more flexibly, the above presentation applies when $\mathcal{O}(T)$ is replaced by a reasonable algebra of differentiable functions on T, like rational functions, analytic functions or smooth functions.

We already observed that Tits' deformation theorem fails for affine Hecke algebras. Nevertheless, apart from Lemma 1.7 there is another analogue of Theorem 1.1, obtained by replacing the centre of an affine Hecke algebra by its quotient field.

Let $\mathbb{C}(X)$ be the quotient field of $\mathbb{C}[X]$, that is, the field of rational functions on the complex algebraic variety T. The action of W on T gives rise to the crossed product algebra $\mathbb{C}(X) \rtimes W$. The quotient field of $Z\big(\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})\big) \cong \mathbb{C}[X]^W$ is $\mathbb{C}(X)^W$, which is also the centre of $\mathbb{C}(X) \rtimes W$. We construct the algebra

$$\mathbb{C}(X)^W \otimes_{\mathbb{C}[X]^W} \mathcal{H}(T, \lambda, \lambda^*, \mathbf{q}) \cong \mathbb{C}(X) \otimes_{\mathbb{C}[X]} \mathcal{H}(T, \lambda, \lambda^*, \mathbf{q}) \cong \mathbb{C}(X) \otimes_{\mathbb{C}} \mathcal{H}(W, q). \tag{1.20}$$

Here the multiplication comes from the description on the left, and it is the same algebra as obtained from Definition 1.11 by substituting $\mathbb{C}(X)$ for $\mathcal{O}(T)$.

For $\alpha \in \Delta$ we define an element $\imath^\circ_{s_\alpha}$ of (1.20) by

$$\imath^\circ_{s_\alpha} + 1 = \frac{\mathbf{q}^{-\lambda(\alpha)} (\theta_\alpha - 1)(\theta_\alpha + 1)}{(\theta_\alpha - \mathbf{q}^{(\lambda(\alpha)+\lambda^*(\alpha))/2})(\theta_\alpha + \mathbf{q}^{(\lambda(\alpha)-\lambda^*(\alpha))/2})} (1 + T_{s_\alpha}).$$

Proposition 1.12 (*[52, §5]*).

(a) The map $s_\alpha \mapsto \imath^\circ_{s_\alpha}$ *extends to a group homomorphism*

$$W \to \big(\mathbb{C}(X)^W \otimes_{\mathbb{C}[X]^W} \mathcal{H}(T,\lambda,\lambda^*,\mathbf{q})\big)^\times : w \mapsto \imath^\circ_w .$$

(b) Using the description (1.20) we define a map

$$\mathbb{C}(X) \rtimes W \to \mathbb{C}(X)^W \otimes_{\mathbb{C}[X]^W} \mathcal{H}(T,\lambda,\lambda^*,\mathbf{q}) : f \otimes w \mapsto f \imath^\circ_w .$$

This is an algebra isomorphism, and in particular

$$\imath^\circ_w f \imath^\circ_{w^{-1}} = w(f) \qquad f \in \mathbb{C}(X), w \in W.$$

1.5. Graded Hecke algebras

Graded (affine) Hecke algebras were discovered in [24,52]. They are simplified versions of affine Hecke algebras, more or less in the same way that a Lie algebra is a simplification of a Lie group.

Let $\mathfrak{a}$ be a finite dimensional Euclidean space and let W be a finite Coxeter group acting isometrically on $\mathfrak{a}$ (and hence also on $\mathfrak{a}^*$). Let $R \subset \mathfrak{a}^*$ be a reduced root system, stable under the action of W, such that the reflections s_α with $\alpha \in R$ generate W. These conditions imply that W acts trivially on the orthogonal complement of $\mathbb{R}R$ in $\mathfrak{a}^*$.

In contrast with the previous paragraphs, R does not have to be integral and W does not have to be crystallographic. Thus we are dealing with root systems as in [35, §1.2]: just reduced and W-stable, no further condition. In particular the upcoming construction applies equally well to Coxeter groups of type H_3, H_4 or $I_2^{(m)}$.

Write $\mathfrak{t} = \mathfrak{a} \otimes_\mathbb{R} \mathbb{C}$ and let $S(\mathfrak{t}^*) = \mathcal{O}(\mathfrak{t})$ be the algebra of polynomial functions on $\mathfrak{t}$. We choose a W-invariant parameter function $k : R \to \mathbb{C}$ and we let $\mathbf{r}$ be a formal variable. We also fix a base Δ of R.

Definition 1.13. The graded Hecke algebra $\mathbb{H}(\mathfrak{t}, W, k, \mathbf{r})$ is the vector space $\mathbb{C}[W] \otimes_\mathbb{C} S(\mathfrak{t}^*) \otimes_\mathbb{C} \mathbb{C}[\mathbf{r}]$ with the multiplication rules

- $\mathbb{C}[W]$ and $S(\mathfrak{t}^*) \otimes_\mathbb{C} \mathbb{C}[\mathbf{r}] \cong \mathcal{O}(\mathfrak{t} \oplus \mathbb{C})$ are embedded as subalgebras,
- $\mathbb{C}[\mathbf{r}]$ is central,
- the cross relation for $\alpha \in \Delta$ and $\xi \in S(\mathfrak{t}^*)$:

$$\xi \cdot s_\alpha - s_\alpha \cdot s_\alpha(\xi) = k(\alpha)\mathbf{r}\frac{\xi - s_\alpha(\xi)}{\alpha}.$$

The grading is given by $\deg(w) = 0$ for $w \in W$ and $\deg(x) = \deg(\mathbf{r}) = 2$ for $x \in \mathfrak{t}^* \setminus \{0\}$.

The grading is motivated by a construction of graded Hecke algebras with equivariant (co)homology, which we will discuss in Section 5.2. Notice that for $k = 0$ Definition 1.13 yields the crossed product algebra

$$\mathbb{H}(\mathfrak{t}, W, 0, \mathbf{r}) = \mathbb{C}[\mathbf{r}] \otimes_\mathbb{C} S(\mathfrak{t}^*) \rtimes W.$$

Let $R^\vee \subset \mathfrak{a}$ be the coroot system of R, so with $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in R$. For $x \in \mathfrak{t}^*$ the following relation holds in $\mathbb{H}(\mathfrak{t}, W, k, \mathbf{r})$:

$$s_\alpha \cdot x - s_\alpha(x) \cdot s_\alpha = k(\alpha)\mathbf{r}\langle x, \alpha^\vee \rangle. \tag{1.21}$$

In fact (1.21) can be substituted for the cross relation in Definition 1.13, that suffices to determine the algebra structure uniquely.

With such a simple presentation, it is no surprise that the graded Hecke algebras $\mathbb{H}(\mathfrak{t}, W, k)$ have more diverse applications than affine Hecke algebras. They appear in the representation theory of reductive groups over local fields, both in the p-adic case [7,8,53] and in the real case [19,20]. Further, these graded Hecke algebras can be realized with Dunkl operators [14,62], which enables them to act on many interesting function spaces.

The above constructions can be modified a little, and still produce the same algebra. Namely, fix $\alpha \in R$ and $\epsilon \in \mathbb{R}_{>0}$. For all $w \in W$ we replace $k(w\alpha)$ by $\epsilon k(w\alpha)$ and $w\alpha$ by $\epsilon w\alpha$–that again gives a root system in the sense of [35, §1.2]. This operation preserves the cross relation in Definition 1.13, so does not change the algebra. Further, we can allow R to be non-reduced, as long as we impose in addition that $k(\epsilon\alpha) = \epsilon k(\alpha)$ whenever $\epsilon > 0$ and $\alpha, \epsilon\alpha \in R$ – that still gives the same graded Hecke algebras.

Similarly, we can scale all parameters $k(\alpha)$ simultaneously. Namely, scalar multiplication with $z \in \mathbb{C}^\times$ defines a bijection $m_z : \mathfrak{t}^* \to \mathfrak{t}^*$, which clearly extends to an algebra automorphism of $S(\mathfrak{t}^*)$. From Definition 1.13 we see that it extends even further, to an algebra isomorphism

$$m_z : \mathbb{H}(\mathfrak{t}, W, zk, \mathbf{r}) \to \mathbb{H}(\mathfrak{t}, W, k, \mathbf{r}) \tag{1.22}$$

which is the identity on $\mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{r}]$. Notice that for $z = 0$ the map m_z is well-defined, but no longer bijective. It is the canonical surjection

$$\mathbb{H}(\mathfrak{t}, W, 0, \mathbf{r}) \to \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{r}].$$

Algebras like $\mathbb{H}(\mathfrak{t}, W, k, \mathbf{r})$ are degenerations of affine Hecke algebras (the version where $\mathbf{q}$ is a formal variable) and arise in the study of cuspidal local systems on unipotent orbits in complex reductive Lie algebras [5,51,54,55].

More often one encounters versions of $\mathbb{H}(\mathfrak{t}, W, k, \mathbf{r})$ with $\mathbf{r}$ specialized to a nonzero complex number. In view of (1.22) it hardly matters which specialization, so it suffices to look at $\mathbf{r} \mapsto 1$. The resulting algebra $\mathbb{H}(\mathfrak{t}, W, k)$ has underlying vector space $\mathbb{C}[W] \otimes_{\mathbb{C}} S(\mathfrak{t}^*)$ and cross relations

$$\xi \cdot s_\alpha - s_\alpha \cdot s_\alpha(\xi) = k(\alpha)(\xi - s_\alpha(\xi))/\alpha \qquad \alpha \in \Delta, \xi \in S(\mathfrak{t}^*). \tag{1.23}$$

Like for affine Hecke algebras, we see from (1.23) that the centre of $\mathbb{H}(\mathfrak{t}, W, k)$ is

$$Z(\mathbb{H}(\mathfrak{t}, W, k)) = S(\mathfrak{t}^*)^W = \mathcal{O}(\mathfrak{t}/W). \tag{1.24}$$

As a vector space, $\mathbb{H}(\mathfrak{t}, W, k)$ is still graded by $\deg(w) = 0$ for $w \in W$ and $\deg(x) = 2$ for $x \in \mathfrak{t}^* \setminus \{0\}$. However, it is not a graded algebra any more, because (1.23) is not homogeneous in the case $\xi = \alpha$. Instead, the above grading merely makes $\mathbb{H}(\mathfrak{t}, W, k)$ into a filtered algebra.

The graded algebra obtained from this filtration is obtained by setting the right hand side of (1.23) equal to 0. In other words, the associated graded of $\mathbb{H}(\mathfrak{t}, W, k)$ is the crossed product algebra

$$\mathbb{H}(\mathfrak{t}, W, 0) = S(\mathfrak{t}^*) \rtimes W.$$

The algebras $\mathbb{H}(\mathfrak{t}, W, 0)$ and $\mathbb{H}(\mathfrak{t}, W, k)$ with $k \neq 0$ are usually not isomorphic. But there is an analogue of Tits' deformation theorem, similar to Proposition 1.12. Let $Q(S(\mathfrak{t}^*))$ be the quotient field of $S(\mathfrak{t}^*)$, that is, the field of rational functions on $\mathfrak{t}$. It admits a natural W-action, and the centre of the crossed product $Q(S(\mathfrak{t}^*)) \rtimes W$ is $Q(S(\mathfrak{t}^*))^W$. Using (1.24) we construct the algebra $Q(S(\mathfrak{t}^*))^W \otimes_{S(\mathfrak{t}^*)^W} \mathbb{H}(\mathfrak{t}, W, k)$, which as vector space equals

$$Q(S(\mathfrak{t}^*)) \otimes_{S(\mathfrak{t}^*)} \mathbb{H}(\mathfrak{t}, W, k) = Q(S(\mathfrak{t}^*)) \otimes_{\mathbb{C}} \mathbb{C}[W]. \tag{1.25}$$

In there we have elements

$$\tilde{\imath}_{s_\alpha} = \frac{\alpha}{\alpha + k(\alpha)}(1 + s_\alpha) - 1 = \frac{\alpha}{\alpha + k(\alpha)} s_\alpha - \frac{k(\alpha)}{\alpha + k(\alpha)} \qquad \alpha \in \Delta.$$

Proposition 1.14 (*[52, §5]*).

(a) The map $s_\alpha \mapsto \tilde{\imath}_{s_\alpha}$ *extends to a group homomorphism*

$$W \to \big(Q(S(\mathfrak{t}^*))^W \otimes_{S(\mathfrak{t}^*)^W} \mathbb{H}(\mathfrak{t}, W, k) \big)^\times : w \mapsto \tilde{\imath}_w .$$

(b) With the description (1.25) we define a map

$$Q(S(\mathfrak{t}^*)) \rtimes W \to Q(S(\mathfrak{t}^*))^W \otimes_{S(\mathfrak{t}^*)^W} \mathbb{H}(\mathfrak{t}, W, k) : f \otimes w \mapsto f \tilde{\imath}_w .$$

This is an algebra isomorphism, and in particular

$$\tilde{\imath}_w f \tilde{\imath}_{w^{-1}} = w(f) \qquad f \in Q(S(\mathfrak{t}^*)), w \in W.$$

Graded Hecke algebras can be decomposed like root systems and reductive Lie algebras. Let $R_1, \ldots, R_d$ be the irreducible components of R. Write $\mathfrak{a}_i^* = \mathrm{span}(R_i) \subset \mathfrak{a}^*$, $\mathfrak{t}_i = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}_i^*, \mathbb{C})$ and $\mathfrak{z} = R^\perp \subset \mathfrak{t}$. Then

$$\mathfrak{t} = \mathfrak{t}_1 \oplus \cdots \oplus \mathfrak{t}_d \oplus \mathfrak{z}.$$

The inclusions $W(R_i) \to W(R)$, $\mathfrak{t}_i^* \to \mathfrak{t}^*$ and $\mathfrak{z}^* \to \mathfrak{t}^*$ induce an algebra isomorphism

$$\mathbb{H}(\mathfrak{t}_1, W(R_1), k) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{H}(\mathfrak{t}_d, W(R_d), k) \otimes_{\mathbb{C}} \mathcal{O}(\mathfrak{z}) \longrightarrow \mathbb{H}(\mathfrak{t}, W, k). \tag{1.26}$$

Hence the representation theory of $\mathbb{H}(\mathfrak{t}, W, k)$ is more or less the product of the representation theories of the tensor factors in (1.26). The commutative algebra $\mathcal{O}(\mathfrak{z}) \cong S(\mathfrak{z}^*)$ is of course very simple, so the study of graded Hecke algebra can be reduced to the case where the root system R is irreducible.

2. Irreducible representations in special cases

The most elementary instance of an affine Hecke algebra is with empty root system R. The algebra associated to the root datum $(X, \emptyset, Y, \emptyset)$ is just the group algebra $\mathbb{C}[X]$ of the lattice X. Its space of irreducible complex representations is

$$\mathrm{Irr}(\mathbb{C}[X]) = \mathrm{Irr}(X) = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{C}^\times) = T. \tag{2.1}$$

Since $\mathbb{C}[X]$ is a subalgebra of $\mathcal{H}((X, R, Y, R^\vee), \lambda, \lambda^*, \mathbf{q})$ for any additional data R, λ, λ^* and $\mathbf{q}$, (2.1) is a good starting point for the representation theory of any affine Hecke algebra.

We will discuss the affine Hecke algebras that appear most often, and we construct and classify all their irreducible representations.

2.1. Affine Hecke algebras with $q = 1$

Let $\mathcal{R} = (X, R, Y, R^\vee, \Delta)$ be an arbitrary based root datum and take the parameters $\lambda = \lambda^* = 0$. Then $q_s = 1$ for all $s \in S_{\mathrm{aff}}$, and

$$\mathcal{H}(\mathcal{R}, 0, 0, \mathbf{q}) = \mathcal{H}(\mathcal{R}, 1) = \mathbb{C}[X] \rtimes W.$$

We denote the onedimensional representation of $\mathbb{C}[X]$ associated to $t \in T$ by $\mathbb{C}_t$. Then $\mathrm{ind}_{\mathbb{C}[X]}^{\mathbb{C}[X] \rtimes W}(\mathbb{C}_t)$ is a $|W|$-dimensional representation of $\mathbb{C}[X] \rtimes W$. Its restriction to $\mathbb{C}[X]$ is

$$\mathrm{Res}_{\mathbb{C}[X]}^{\mathbb{C}[X] \rtimes W} \mathrm{ind}_{\mathbb{C}[X]}^{\mathbb{C}[X] \rtimes W}(\mathbb{C}_t) = \bigoplus_{w \in W} w\mathbb{C}_t \cong \bigoplus_{w \in W} \mathbb{C}_{w(t)}.$$

More generally, consider any $\mathbb{C}[X] \rtimes W$-representation (π, V) that is generated by the subspace

$$V_t := \{v \in V : \pi(\theta_x)v = x(t)v \; \forall x \in X\}.$$

Then $V = \sum_{w \in W} \pi(w)V_t$ and $\pi(w)V_t = V_{w(t)}$. As $V_t \cap V_{t'} = \{0\}$ for $t \neq t'$,

$$V = \mathrm{ind}_{\mathbb{C}[X] \rtimes W_t}^{\mathbb{C}[X] \rtimes W}(V_t), \quad \text{where } W_t = \{w \in W : w(t) = t\}.$$

By Frobenius reciprocity

$$\mathrm{End}_{\mathbb{C}[X] \rtimes W}(V) \cong \mathrm{Hom}_{\mathbb{C}[X] \rtimes W_t}(V_t, V) = \mathrm{Hom}_{\mathbb{C}[X] \rtimes W_t}\big(V_t, \sum_{w \in W/W_t} V_{w(t)}\big) = \mathrm{End}_{\mathbb{C}[X] \rtimes W_t}(V_t).$$

Corollary 2.1. *The functor* $\mathrm{ind}_{\mathbb{C}[X] \rtimes W_t}^{\mathbb{C}[X] \rtimes W}$ *induces an equivalence between the following categories:*

- $\mathbb{C}[X] \rtimes W_t$*-representations on which* $\mathbb{C}[X]$ *acts via the character* t*,*
- $\mathbb{C}[X] \rtimes W$*-representations* V *that are generated by* V_t*.*

The first category in Corollary 2.1 is naturally equivalent with the category of W_t-representations. We conclude that, for every irreducible W_t-representation (ρ, V_ρ), the $\mathbb{C}[X] \rtimes W$-representation

$$\pi(t, \rho) := \mathrm{ind}_{\mathbb{C}[X] \rtimes W_t}^{\mathbb{C}[X] \rtimes W}(\mathbb{C}_t \otimes V_\rho)$$

is irreducible. For comparison with later results we point out that $\pi(t, \rho)$ is a direct summand of the induced representation

$$\mathrm{ind}_{\mathbb{C}[X]}^{\mathbb{C}[X] \rtimes W}(\mathbb{C}_t) = \mathrm{ind}_{\mathbb{C}[X] \rtimes W_t}^{\mathbb{C}[X] \rtimes W}(\mathbb{C}_t \otimes \mathbb{C}[W_t])$$

and that

$$\mathrm{Res}_{\mathbb{C}[X]}^{\mathbb{C}[X] \rtimes W} \pi(t, \rho) = \bigoplus\nolimits_{w \in W/W_t} \mathbb{C}_{w(t)}^{\dim(V_\rho)}.$$

The next result goes back to Frobenius and Clifford, see [68, Appendix] for a modern account.

Theorem 2.2. *Every irreducible* $\mathbb{C}[X] \rtimes W$*-representation is of the form* $\pi(t, \rho)$ *for a* $t \in T$ *and a* $\rho \in \mathrm{Irr}(W_t)$*. Two such representations* $\pi(t, \rho)$ *and* $\pi(t', \rho')$ *are equivalent if and only if there exists a* $w \in W$ *with* $t' = w(t)$ *and* $\rho' = w \cdot \rho$*.*

Here $w \cdot \rho = \rho \circ \mathrm{Ad}(w)^{-1} : wW_tw^{-1} \to \mathrm{Aut}_{\mathbb{C}}(V_\rho)$. Theorem 2.2 involves a group action of W on the set

$$\tilde{T} = \{(t, \rho) : t \in T, \rho \in \mathrm{Irr}(W_t)\}.$$

We call

$$T//W := \tilde{T}/W$$

the extended quotient of T by W. Thus Theorem 2.2 gives a canonical bijection

$$T//W \longleftrightarrow \mathrm{Irr}(\mathbb{C}[X] \rtimes W) = \mathrm{Irr}\big(\mathcal{H}(\mathcal{R}, 1)\big). \tag{2.2}$$

On $\mathrm{Irr}(\mathbb{C}[X] \rtimes W)$ we have the Jacobson topology, whose closed sets are

$$\{\pi \in \mathrm{Irr}(\mathbb{C}[X] \rtimes W) : S \subset \ker \pi\} \quad \text{for } S \subset \mathbb{C}[X] \rtimes W.$$

Via (2.2) we transfer this to a topology on $T//W$. Then the natural maps

$$\begin{array}{ccccccc} T/W & \to & T//W & T//W & \to & T/W \\ Wt & \mapsto & [t, \mathrm{triv}] & [t, \rho] & \mapsto & Wt \end{array}$$

are continuous. The composition of (2.2) with $T//W \to T/W$ is just the restriction of an irreducible $\mathbb{C}[X] \rtimes W$-representation to $\mathbb{C}[X]^W \cong \mathcal{O}(T/W)$, in other words, it is the central character map.

In the same terms we can analyse graded Hecke algebras with $k = 0$. As in Section 1.5, we let W be a finite Coxeter group acting isometrically on a finite dimensional Euclidean space $\mathfrak{a}$. When $k = 0$, we do not need a root system $R \subset \mathfrak{a}^*$ to construct the algebra

$$\mathbb{H}(\mathfrak{t}, W, 0) = \mathcal{O}(\mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}) \rtimes W = S(\mathfrak{t}^*) \rtimes W.$$

Considerations with Clifford theory, exactly as for $\mathcal{O}(T) \rtimes W$, lead to:

Theorem 2.3. *Every irreducible representation of $\mathcal{O}(\mathfrak{t}) \rtimes W$ is of the form*

$$\pi(\nu, \rho) = \mathrm{ind}_{\mathcal{O}(\mathfrak{t}) \rtimes W_\nu}^{\mathcal{O}(\mathfrak{t}) \rtimes W} (\mathbb{C}_\nu \otimes V_\rho) \text{ for some } \nu \in \mathfrak{t}, (\rho, V_\rho) \in \mathrm{Irr}(W_\nu).$$

Two such representations $\pi(\nu, \rho)$ and $\pi(\nu', \rho')$ are equivalent if and only if there exists a $w \in W$ with $\nu' = w(\nu)$ and $\rho' = w \cdot \rho$.

Like for affine Hecke algebras with $q = 1$, $\pi(\nu, \rho)$ is a direct summand of

$$\mathrm{ind}_{\mathcal{O}(\mathfrak{t})}^{\mathcal{O}(\mathfrak{t}) \rtimes W} (\mathbb{C}_\nu) = \mathrm{ind}_{\mathcal{O}(\mathfrak{t}) \rtimes W_\nu}^{\mathcal{O}(\mathfrak{t}) \rtimes W} (\mathbb{C}_t \otimes \mathbb{C}[W]).$$

Also, there is a canonical bijection

$$\mathfrak{t}//W \longleftrightarrow \mathrm{Irr}(\mathcal{O}(\mathfrak{t}) \rtimes W) = \mathrm{Irr}\big(\mathbb{H}(\mathfrak{t}, W, 0)\big).$$

2.2. *Iwahori–Hecke algebras of type $\widetilde{A_1}$*

Consider the based root datum

$$\mathcal{R} = \big(X = \mathbb{Z}, R = \{\pm 1\}, Y = \mathbb{Z}, R^\vee = \{\pm 2\}, \Delta = \{\alpha\} = \{1\}\big).$$

It has an affine Weyl group $W_{\mathrm{aff}} = W(\mathcal{R})$ of type $\widetilde{A_1}$, with Coxeter generators $S_{\mathrm{aff}} = \{s_\alpha, s'_\alpha : x \mapsto 1 - x\}$. Affine Hecke algebras of the form $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})$, for various parameters $\lambda(\alpha)$ and $\lambda^*(\alpha)$, appear often in the representation theory of classical p-adic groups [27,60]. We will work out the irreducible representations of $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})$ in detail. To avoid singular cases, we assume throughout this paragraph that

$$\mathbf{q}^{\lambda(\alpha)} \neq -1, \mathbf{q}^{\lambda^*(\alpha)} \neq -1, \mathbf{q}^{\lambda(\alpha)+\lambda^*(\alpha)} \neq 1. \tag{2.3}$$

Recall that the case $\lambda(\alpha) = \lambda^*(\alpha) = 0$ was already discussed in Section 2.1. We abbreviate

$$q_1^{1/2} = \mathbf{q}^{\lambda(\alpha)/2}, \; q_0^{1/2} = \mathbf{q}^{\lambda^*(\alpha)/2} \quad \text{and} \quad \mathcal{H} = \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q}).$$

It is not difficult to see [58] that every irreducible $\mathcal{H}$-representation is a quotient of $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$ for some $t \in T$. By (1.16) this induced representation has dimension two. Using the Iwahori–Matsumoto presentation $\mathcal{H}(W_{\mathrm{aff}}, q)$, two onedimensional representations can be written down immediately. Firstly the trivial representation, given by

$$\mathrm{triv}(T_{s_\alpha}) = \mathbf{q}^{\lambda(\alpha)} = q_1, \ \mathrm{triv}(T_{s'_\alpha}) = \mathbf{q}^{\lambda^*(\alpha)} = q_0,$$

and secondly the Steinberg representation, defined by

$$\mathrm{St}(T_{s_\alpha}) = -1, \ \mathrm{St}(T_{s'_\alpha}) = -1.$$

When $\mathcal{H}$ is the Iwahori-spherical Hecke algebra of SL_2 over a p-adic field F, these two representations correspond to the trivial and the Steinberg representations of $SL_2(F)$ — as their names already suggested.

Via evaluation at $1 \in \mathbb{Z}$, we identify $T = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{C}^\times)$ with $\mathbb{C}^\times$. By Theorem 1.8 $\theta_1 = q_0^{-1/2} q_1^{-1/2} T_{s'_\alpha} T_{s_\alpha}$. For the trivial and Steinberg representations that means

$$\mathrm{triv}(\theta_1) = q_0^{-1/2} q_1^{-1/2} \mathrm{triv}(T_{s'_\alpha} T_{s_\alpha}) = q_0^{1/2} q_1^{1/2},$$
$$\mathrm{St}(\theta_1) = q_0^{-1/2} q_1^{-1/2} \mathrm{St}(T_{s'_\alpha} T_{s_\alpha}) = q_0^{-1/2} q_1^{-1/2}.$$

Therefore, as $\mathbb{C}[X]$-representations:

$$\mathrm{triv}|_{\mathbb{C}[X]} = \mathbb{C}_{q_0^{1/2} q_1^{1/2}} \text{ and } \mathrm{St}|_{\mathbb{C}[X]} = \mathbb{C}_{q_0^{-1/2} q_1^{-1/2}}. \tag{2.4}$$

Theorem 2.4.

(a) The $\mathcal{H}$-representation $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$ is irreducible for all

$$t \in \mathbb{C}^\times \setminus \{ q_0^{1/2} q_1^{1/2}, q_0^{-1/2} q_1^{-1/2}, -q_0^{1/2} q_1^{-1/2}, -q_0^{-1/2} q_1^{1/2} \}.$$

(b) For t as in part (a), $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$ is isomorphic with $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_{t^{-1}})$. There are no further relations between the irreducible representations $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$.

(c) The algebra $\mathcal{H}$ has precisely four other irreducible representations: triv, St *and two that we call* $\pi(-1, \mathrm{triv})$, $\pi(-1, \mathrm{St})$. *They have dimension one and fit in short exact sequences*

$$\begin{array}{lcccl} 0 \to & \mathrm{St} & \to \mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_{q_0^{1/2} q_1^{1/2}}) \to & \mathrm{triv} & \to 0, \\ 0 \to & \mathrm{triv} & \to \mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_{q_0^{-1/2} q_1^{-1/2}}) \to & \mathrm{St} & \to 0, \\ 0 \to & \pi(-1, \mathrm{St}) & \to \mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_{-q_0^{-1/2} q_1^{1/2}}) \to & \pi(-1, \mathrm{triv}) & \to 0, \\ 0 \to & \pi(-1, \mathrm{triv}) & \to \mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_{-q_0^{1/2} q_1^{-1/2}}) \to & \pi(-1, \mathrm{St}) & \to 0. \end{array}$$

Remark. By the conditions (2.3), the four special values of t are all different, except that the last two coincide if $q_1 = q_0$.

Proof. (a) By (1.16) $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t) = \mathcal{H}(W, q)$ as $\mathcal{H}(W, q)$-module. Since $q_1 \neq -1$, the algebra $\mathcal{H}(W, q)$ is semisimple, and isomorphic with $\mathbb{C}[W] \cong \mathbb{C} \oplus \mathbb{C}$. The quadratic relation (1.1) points us to the minimal central idempotents in $\mathcal{H}(W, q)$:

$$p_+ := (T_{s_\alpha} + T_e)(1 + q_1)^{-1} \quad \text{and} \quad p_- := (T_{s_\alpha} - q_1 T_e)(1 + q_1)^{-1}.$$

Then $\mathbb{C}p_+$ and $\mathbb{C}p_-$ are the only nontrivial $\mathcal{H}(W, q)$-submodules of $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$. It follows that, whenever $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$ is irreducible as $\mathcal{H}$-module, $\mathbb{C}p_+$ or $\mathbb{C}p_-$ is an $\mathcal{H}$-submodule. We test for which $t \in T$ this happens. The cross relation in $\mathcal{H}$ gives

$$\begin{aligned}&\theta_1(T_{s_\alpha}+T_e) = \\ &\theta_1 + T_{s_\alpha}\theta_{-1} + \Big((\mathbf{q}^{\lambda(\alpha)}-1) + \theta_{-\alpha}\big(\mathbf{q}^{(\lambda(\alpha)+\lambda^*(\alpha))/2} - \mathbf{q}^{(\lambda(\alpha)-\lambda^*(\alpha))/2}\big)\Big)\frac{\theta_1-\theta_{-1}}{\theta_0-\theta_{-2}} = \\ &T_{s_\alpha}\theta_{-1} + \theta_1\big((q_1-1)+\theta_{-1}(q_1^{1/2}q_0^{1/2} - q_1^{1/2}q_0^{-1/2})\big)\theta_1 = \\ &T_{s_\alpha}\theta_{-1} + q_1\theta_1 + q_1^{1/2}(q_0^{1/2}-q_0^{-1/2}).\end{aligned}$$

In $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$ we get

$$\begin{aligned}\theta_1(1+q_1)p_+ &= \theta_1(T_{s_\alpha}+T_e) = T_{s_\alpha}\cdot t^{-1} + q_1 t + q_1^{1/2}(q_0^{1/2}-q_0^{-1/2}) \\ &= t^{-1}(T_{s_\alpha}+T_e) + \big(tq_1 - t^{-1} + q_1^{1/2}(q_0^{1/2}-q_0^{-1/2})\big)T_e \\ &= t^{-1}(T_{s_\alpha}+T_e) + q_1^{1/2}(tq_1^{1/2} - t^{-1}q_1^{-1/2} + q_0^{1/2} - q_0^{-1/2})T_e.\end{aligned} \tag{2.5}$$

This can only be a scalar multiple of p_+ if $tq_1^{1/2} - t^{-1}q_1^{-1/2} + q_0^{1/2} - q_0^{-1/2} = 0$, and that happens only if $q_0^{1/2} = -tq_1^{1/2}$ or $q_0^{1/2} = t^{-1}q_1^{-1/2}$.

Similarly we compute in $\mathcal{H}$:

$$\theta_1(T_{s_\alpha} - q_1T_e) = T_{s_\alpha}\theta_{-1} - \theta_1 + q_1^{1/2}(q_0^{1/2}-q_0^{-1/2}).$$

In $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$ that leads to

$$\begin{aligned}\theta_1(1+q_1)p_- &= \theta_1(T_{s_\alpha}-q_1T_e) = T_{s_\alpha}\cdot t^{-1} - t + q_1^{1/2}(q_0^{1/2}-q_0^{-1/2}) \\ &= t^{-1}(T_{s_\alpha}-q_1T_e) + \big(q_1t^{-1} - t + q_1^{1/2}(q_0^{1/2}-q_0^{-1/2})\big)T_e \\ &= t^{-1}(T_{s_\alpha}-q_1T_e) + q_1^{1/2}(t^{-1}q_1^{1/2} - tq_1^{-1/2} + q_0^{1/2} - q_0^{-1/2})T_e.\end{aligned} \tag{2.6}$$

If this is a scalar multiple of p_-, then $t^{-1}q_1^{1/2} - tq_1^{-1/2} + q_0^{1/2} - q_0^{-1/2} = 0$, which means that $q_0^{1/2} = -t^{-1}q_1^{1/2}$ or $q_0^{1/2} = tq_1^{-1/2}$.

We conclude that for

$$t \in \mathbb{C}^\times \setminus \big\{q_0^{1/2}q_1^{1/2}, q_0^{-1/2}q_1^{-1/2}, -q_0^{1/2}q_1^{-1/2}, -q_0^{-1/2}q_1^{1/2}\big\}$$

neither $\mathbb{C}p_+$ nor $\mathbb{C}p_-$ is an $\mathcal{H}$-submodule of $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$, so that $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$ is irreducible. On the other hand, when t equals one of these special values, the above calculations in combination with the fact that θ_1 generates $\mathbb{C}[X]$ imply that $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$ does have a onedimensional $\mathcal{H}$-submodule.

(b) Consider the element

$$f_\alpha := \frac{\theta_1(q_1-1) + q_1^{1/2}(q_0^{1/2}-q_0^{-1/2})}{\theta_1 - \theta_{-1}}$$

of the quotient field $\mathbb{C}(X)$ of $\mathbb{C}[X] = \mathcal{O}(T)$. It lies in the version of $\mathcal{H} = \mathcal{H}(T,\lambda,\lambda^*,\mathbf{q})$ obtained from Definition 1.13 by replacing $\mathcal{O}(T)$ with $\mathbb{C}(X)$. By direct calculation in that algebra:

$$\theta_x(T_{s_\alpha} - f_\alpha) = (T_{s_\alpha} - f_\alpha)\theta_{-x} \quad \text{for all } x \in X = \mathbb{Z}.$$

Although $f_\alpha \notin \mathbb{C}[X]$, it has a well-defined action on $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$, provided that $(\theta_1 - \theta_{-1})(t) \neq 0$, or equivalently $t \notin \{1,-1\}$. For $v \in \mathbb{C}_t \setminus \{0\}$, the element

$$(T_{s_\alpha} - f_\alpha)v \in \mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t) \setminus \{0\}$$

satisfies

$$\theta_x(T_{s_\alpha} - f_\alpha)v = (T_{s_\alpha} - f_\alpha)\theta_{-x}v = (T_{s_\alpha} - f_\alpha)\theta_{-x}(t)v = t^{-1}(x)(T_{s_\alpha} - f_\alpha)v.$$

Hence as $\mathbb{C}[X]$-modules

$$\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t) = \mathbb{C}_t \oplus \mathbb{C}_{t^{-1}} \quad \text{for all } t \in \mathbb{C}^\times \setminus \{1, -1\}. \tag{2.7}$$

By Frobenius reciprocity, for such t:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{H}}\big(\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_{t^{-1}}), \mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)\big) &\cong \mathrm{Hom}_{\mathbb{C}[X]}\big(\mathbb{C}_{t^{-1}}, \mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)\big)\\ &= \mathrm{Hom}_{\mathbb{C}[X]}\big(\mathbb{C}_{t^{-1}}, \mathbb{C}_t \oplus \mathbb{C}_{t^{-1}}\big) \cong \mathbb{C}.\end{aligned}$$

In particular this shows that

$$\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_{t^{-1}}) \cong \mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$$

whenever these representations are irreducible (for $t = \pm 1$ this isomorphism is tautological).
(c) From (2.5) we know that $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_{q_0^{1/2}q_1^{1/2}})$ has a subrepresentation $\mathbb{C}p_-$ with $\theta_1 p_- = q_0^{-1/2}q_1^{-1/2}p_-$. Further, by (1.1)

$$(T_{s_\alpha} + T_e)p_- = 0, \quad \text{so } T_{s_\alpha} p_- = -p_-. \tag{2.8}$$

As T_{s_α} and θ_1 generate $\mathcal{H}$, it follows that here $\mathbb{C}p_-$ is the Steinberg representation. By (2.7)

$$\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_{q_0^{1/2}q_1^{1/2}})/\mathbb{C}p_- \cong \mathbb{C}_{q_0^{1/2}q_1^{1/2}}$$

as $\mathbb{C}[X]$-representation. Also

$$\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_{q_0^{1/2}q_1^{1/2}})/\mathbb{C}p_- \cong \mathcal{H}(W, q)/\mathbb{C}p_- \cong \mathbb{C}p_+$$

as $\mathcal{H}(W, q)$-representation. From $(T_{s_\alpha} - q_1 T_e)p_+ = 0$ we see that $T_{s_\alpha} p_+ = q_1 p_+$. Again, since θ_1 and T_{s_α} generate $\mathcal{H}$, we can conclude that $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_{q_0^{1/2}q_1^{1/2}})/\mathbb{C}p_-$ is the trivial representation of $\mathcal{H}$.

The calculations around (2.6) show that $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_{-q_0^{-1/2}q_1^{1/2}})$ contains a subrepresentation $\pi(-1, \mathrm{St}) = \mathbb{C}p_-$ with

$$\theta_1 p_- = -q_0^{1/2}q_1^{-1/2}p_-. \tag{2.9}$$

By (2.8) it has the same restriction to $\mathcal{H}(W, q)$ as the Steinberg representation, which explains our notation $\pi(-1, \mathrm{St})$ for this $\mathbb{C}p_-$. The quotient

$$\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_{-q_0^{-1/2}q_1^{1/2}})/\mathbb{C}p_- \text{ equals } \mathbb{C}_{-q_0^{-1/2}q_1^{1/2}} \text{ as } \mathbb{C}[X]\text{-representation.} \tag{2.10}$$

As $\mathcal{H}(W, q)$-representation it is isomorphic to $\mathbb{C}p_+ \cong \mathrm{triv}$, and therefore we write

$$\pi(-1, \mathrm{triv}) = \mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_{-q_0^{1/2}q_1^{-1/2}})/\mathbb{C}p_-.$$

Analogous considerations apply to $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_{q_0^{-1/2}q_1^{-1/2}})$ and $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_{-q_0^{1/2}q_1^{-1/2}})$. □

An important special case is $q_1 = q_0 = \mathbf{q}$. When F is a non-archimedean local field with residue field of order $\mathbf{q}$, $\mathcal{H}(W_{\mathrm{aff}}, \mathbf{q}) = \mathcal{H}(\mathcal{R}, \mathbf{q})$ arises as the Iwahori-spherical Hecke algebra of $SL_2(F)$.

The algebra $\mathcal{H}(\mathcal{R}, \mathbf{q})$ is the simplest example of a true affine Hecke algebra, not isomorphic to some more elementary kind of algebra. For this algebra Theorem 2.4 says:

- The $\mathcal{H}(\mathcal{R},\mathbf{q})$-representation $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}(\mathcal{R},\mathbf{q})}(\mathbb{C}_t)$ is irreducible for all $t \in \mathbb{C}^\times \setminus \{\mathbf{q},\mathbf{q}^{-1},-1\}$.
- $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}(\mathcal{R},\mathbf{q})}(\mathbb{C}_t) \cong \mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}(\mathcal{R},\mathbf{q})}(\mathbb{C}_{t^{-1}})$ for all $t \in \mathbb{C}^\times \setminus \{\mathbf{q},\mathbf{q}^{-1}\}$.
- $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}(\mathcal{R},\mathbf{q})}(\mathbb{C}_{-1}) = \pi(-1,\mathrm{triv}) \oplus \pi(-1,\mathrm{St})$, with $\pi(-1,\mathrm{triv})$ and $\pi(-1,\mathrm{St})$ irreducible and inequivalent.
- There are only two other irreducible $\mathcal{H}(\mathcal{R},\mathbf{q})$-representations, triv and St, which both occur as subquotients of $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}(\mathcal{R},\mathbf{q})}(\mathbb{C}_\mathbf{q})$ and of $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}(\mathcal{R},\mathbf{q})}(\mathbb{C}_{\mathbf{q}^{-1}})$.

This classification works for almost all $\mathbf{q} \in \mathbb{C}^\times$, only 1 and -1 are exceptional. (In view of (1.7), that is hardly surprising.) For $\mathcal{H}(\mathcal{R},-1)$ the trivial representation coincides with $\pi(-1,\mathrm{triv})$ and the Steinberg representation coincides with $\pi(-1,\mathrm{St})$. Consequently $\mathcal{H}(\mathcal{R},-1)$ has only two onedimensional representations. Apart from that, the above statements are valid when $\mathbf{q} = -1$.

Although the definition and formulas for $\mathcal{H}(\mathcal{R},\mathbf{q})$ look considerably simpler than those for $\mathcal{H}(\mathcal{R},\lambda,\lambda^*,\mathbf{q})$, in the end the latter is hardly more difficult. In terms of the induced representations $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$, the only differences occur when $t \in \{-1, -q_0^{1/2}q_1^{-1/2}, -q_0^{-1/2}q_1^{1/2}\}$.

2.3. *Affine Hecke algebras of type GL_n*

The root datum of type GL_n is

$$\mathcal{R}_n = (\mathbb{Z}^n, A_{n-1}, \mathbb{Z}^n, A_{n-1}, \Delta_{n-1})$$

with $A_{n-1} = \{e_i - e_j : 1 \le i,j \le n, i \neq j\}$ and $\Delta_{n-1} = \{e_i - e_{i+1} : i = 1,2,\ldots,n-1\}$. Via $t \mapsto (t(e_1),\ldots,t(e_n))$ we identify T with $(\mathbb{C}^\times)^n$. We saw in Section 1 that

$$W_{\mathrm{aff}} = \{x \in \mathbb{Z}^n : x_1 + \cdots + x_n = 0\} \rtimes S_n,$$
$$S_{\mathrm{aff}} = \{s_i = s_{e_i - e_{i+1}} : i = 1,\ldots,n-1\} \cup \{s_0\},$$
$$\Omega = \langle \omega \rangle \cong \mathbb{Z}, \quad \omega(x) = e_1 + (1\,2\cdots n)x,$$

where $s_0(x) = x + (1 - \langle x, e_1 - e_n\rangle)(e_1 - e_n)$. All the simple affine reflections from S_{aff} are Ω-conjugate, so $q_s = q_{s'}$ for $s,s' \in S_{\mathrm{aff}}$. Call this parameter $\mathbf{q}$ and consider the affine Hecke algebra

$$\mathcal{H}_n(\mathbf{q}) := \mathcal{H}(\mathcal{R}_n,\mathbf{q})$$

of type GL_n. The primary importance of such algebras is that they describe every Bernstein block in the representation theory of $GL_n(F)$ for a non-archimedean local field F [11]. In all those cases $\mathbf{q}$ is a power of a prime number, but just as affine Hecke algebra that is not necessary. We do not even have to require that $\mathbf{q} \in \mathbb{R}_{>1}$, the representation theory of $\mathcal{H}_n(\mathbf{q})$ looks the same for every $\mathbf{q} \in \mathbb{C}^\times$ which is not a root of unity.

The irreducible representations of $GL_n(F)$ were classified (in terms of supercuspidal representations) by Zelevinsky [9,88]. That classification follows a combinatorial pattern involving certain "segments", and the irreducible representations of $\mathcal{H}_n(\mathbf{q})$ exhibit the same pattern. We formulate their classification in terms intrinsic to Hecke algebras. The algebra $\mathcal{H}_1(\mathbf{q}) = \mathbb{C}[\mathbb{Z}]$ has already been discussed, so we may assume that $n \ge 2$ (which ensures that S_{aff} is nonempty).

Like in Section 2.2 we start with the trivial and the Steinberg representations. By definition

$$\mathrm{triv}(T_s) = \mathbf{q}, \ \mathrm{St}(T_s) = -1 \quad \text{for all } s \in S_{\mathrm{aff}}. \tag{2.11}$$

This does not yet determine the values of these representations on T_ω, to fix that one requires in addition

$$\begin{array}{rclcl} \mathrm{triv}|_{\mathbb{C}[X]} & = & \mathbb{C}_{t_+}, & t_+ & = & \big(\mathbf{q}^{(n-1)/2}, \mathbf{q}^{(n-3)/2}, \ldots, \mathbf{q}^{(1-n)/2}\big), \\ \mathrm{St}|_{\mathbb{C}[X]} & = & \mathbb{C}_{t_-}, & t_- & = & \big(\mathbf{q}^{(1-n)/2}, \mathbf{q}^{(3-n)/2}, \ldots, \mathbf{q}^{(n-1)/2}\big). \end{array}$$

The terminology is motivated by Iwahori-spherical representation of GL_n over a p-adic field F: triv and St correspond to the epynomous representations of $GL_n(F)$.

In contrast with the Iwahori–Hecke algebra of type $\widetilde{A_1}$, the trivial and Steinberg representations of $\mathcal{H}_n(\mathbf{q})$ come in families of representations, parametrized by $\mathbb{C}^\times$. This is implemented as follows. For $z \in \mathbb{C}^\times$ we put $t_z = (z, z, \ldots, z) \in T^{S_n}$. We define the $\mathcal{H}_n(\mathbf{q})$-representation $\mathrm{triv} \otimes t_z$ by (2.11) and

$$(\mathrm{triv} \otimes t_z)|_{\mathbb{C}[X]} = \mathbb{C}_{t_+ t_z}. \tag{2.12}$$

This is possible because $\alpha(t_z) = 1$ for all $\alpha \in R$, so that $t_z \in \mathrm{Hom}(X, \mathbb{C}^\times)$ is trivial on $X \cap W_{\mathrm{aff}}$. Similarly we define the onedimensional $\mathcal{H}_n(\mathbf{q})$-representation $\mathrm{St} \otimes t_z$, by requiring (2.11) and

$$(\mathrm{St} \otimes t_z)|_{\mathbb{C}[X]} = \mathbb{C}_{t_- t_z}. \tag{2.13}$$

For $\mathcal{H}_2(\mathbf{q})$ the above onedimensional representations, in combination with the induced representations $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}_2(\mathbf{q})}(\mathbb{C}_t)$, already exhaust $\mathrm{Irr}(\mathcal{H}_2(\mathbf{q}))$. With arguments very similar to those in Section 2.2 one can show:

Theorem 2.5. *The $\mathcal{H}_2(\mathbf{q})$-representation $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}_2(\mathbf{q})}(\mathbb{C}_t)$ is irreducible for all $t = (t_1, t_2) \in T$ that are not of the form $t_+ t_z = (\mathbf{q}^{1/2} z, \mathbf{q}^{-1/2} z)$ or $t_- t_z = (\mathbf{q}^{-1/2} z, \mathbf{q}^{1/2} z)$.*

This representation is isomorphic to $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}_2(\mathbf{q})}(\mathbb{C}_{(t_2, t_1)})$, but apart from that there are no relations between the irreducible representations of the form $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}_2(\mathbf{q})}(\mathbb{C}_t)$.

The only other irreducible $\mathcal{H}_2(\mathbf{q})$-representations are $\mathrm{triv} \otimes t_z$ and $\mathrm{St} \otimes t_z$ with $z \in \mathbb{C}^\times$. They are mutually inequivalent and fit in short exact sequences

$$\begin{array}{l} 0 \to \ \mathrm{St} \otimes t_z \ \to \mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}_2(\mathbf{q})}(\mathbb{C}_{(\mathbf{q}^{1/2} z, \mathbf{q}^{-1/2} z)}) \to \mathrm{triv} \otimes t_z \to 0, \\ 0 \to \mathrm{triv} \otimes t_z \to \mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}_2(\mathbf{q})}(\mathbb{C}_{(\mathbf{q}^{-1/2} z, \mathbf{q}^{1/2} z)}) \to \ \mathrm{St} \otimes t_z \ \to 0. \end{array}$$

The irreducible representations from Theorem 2.5 come in three kinds, but there is a natural way to gather them in two families:

- The twists $\mathrm{St} \otimes t_z$ of the Steinberg representation, parametrized by $t_z \in T^{S_2}$.
- A family parametrized by T/S_2, which for almost all $Wt \in T/W$ has the member $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}_2(\mathbf{q})}(\mathbb{C}_t)$. When that representation happens to be reducible, we take the unique representative t of Wt with $|\alpha(t)| \geq 1$ and we assign to Wt the unique irreducible quotient of $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}_2(\mathbf{q})}(\mathbb{C}_t)$.

The irreducible representations of $\mathcal{H}_n(\mathbf{q})$ with $n \geq 2$ can be parametrized with a longer list of similar families. We sketch the construction and classification, as translated from [88], stepwise.

- Choose a partition $\vec{n} = (n_1, n_2, \ldots, n_d)$ of n, where $n_i \geq 1$ (but the sequence (n_i) need not be monotone).
- The algebra

$$\bigotimes_{i=1}^d \mathcal{H}_{n_i}(\mathbf{q}) = \bigotimes_{i=1}^d \mathcal{H}\big((\mathbb{Z}^{n_i}, A_{n_i-1}, \mathbb{Z}^{n_i}, A_{n_i-1}, \Delta_{n_i-1}), \mathbf{q}\big)$$

is naturally a subalgebra of $\mathcal{H}_n(\mathbf{q})$, with the same commutative subalgebra $\mathbb{C}[X] = \mathbb{C}[\mathbb{Z}^n] = \otimes_{i=1}^n \mathbb{C}[\mathbb{Z}^{n_i}]$.

- For every i we pick

$$z_i \in \mathrm{Irr}(\mathbb{C}[\mathbb{Z}^{n_i}])^{S_{n_i}} \cong \big((\mathbb{C}^\times)^{n_i}\big)^{S_{n_i}} \cong \mathbb{C}^\times,$$

and we construct the irreducible $\mathcal{H}_{n_i}(\mathbf{q})$-representation $\mathrm{St} \otimes z_i$. (It corresponds to a segment in Zelevinsky's setup.)

- Then $\boxtimes_{i=1}^d (\mathrm{St} \otimes z_i)$ is an irreducible representation of $\bigotimes_{i=1}^d \mathcal{H}_{n_i}(\mathbf{q})$ and

$$\pi(\vec{n}, \vec{z}) := \mathrm{ind}_{\bigotimes_{i=1}^d \mathcal{H}_{n_i}(\mathbf{q})}^{\mathcal{H}_n(\mathbf{q})} \Big(\boxtimes_{i=1}^d (\mathrm{St} \otimes z_i) \Big)$$

is an $\mathcal{H}_n(\mathbf{q})$-representation.

- For almost all $\vec{z} = (z_i)_{i=1}^d$, $\pi(\vec{n}, \vec{z})$ is irreducible and depends only on the orbit of $\vec{z}$ under $N_{S_n}\big(\prod_{i=1}^d S_{n_i}\big)$. Here $\prod_{i=1}^d S_{n_i}$ fixes $(z_i)_{i=1}^d$ and

$$N_{S_n}\Big(\prod_{i=1}^d S_{n_i}\Big) \Big/ \prod_{i=1}^d S_{n_i} \cong \prod_{m \geq 1} S(\{i : n_i = m\}).$$

- When $\pi(\vec{n}, \vec{z})$ is reducible, we pick a representative $\vec{z}$ for $N_{S_n}(\prod_{i=1}^d S_{n_i})\vec{z}$ such that $|z_i| \geq |z_j|$ whenever $n_i = n_j$ and $i \leq j$.
- For such $\vec{z}$ it follows from the Langlands classification that $\pi(\vec{n}, \vec{z})$ has a unique irreducible quotient. That is the irreducible $\mathcal{H}_n(\mathbf{q})$-representation associated with $(\vec{n}, \vec{z})$.
- The irreducible $\mathcal{H}_n(\mathbf{q})$-representations assigned to $(\vec{n}, \vec{z})$ and $(\vec{n}', \vec{z}')$ are equivalent if and only if $(\vec{n}, \vec{z})$ and $(\vec{n}', \vec{z}')$ are S_n-associate.

In the above parametrization $z_i \in \big((\mathbb{C}^\times)^{n_i}\big)^{S_{n_i}}$, so $\vec{z}$ can be regarded as a diagonal matrix in $GL_n(\mathbb{C})$. The partition $\vec{n}$ determines a Levi subgroup

$$M = GL_{n_1}(\mathbb{C}) \times \cdots \times GL_{n_d}(\mathbb{C}) \quad \text{of} \quad GL_n(\mathbb{C}),$$

and $\vec{z} \in Z(M)$. Let u_m be a regular unipotent element of $GL_m(\mathbb{C})$ (it is unique up to conjugation). Then

$$\vec{u} := (u_{n_1}, u_{n_2}, \ldots, u_{n_d})$$

is a regular unipotent element of M, and $\vec{z}\vec{u}$ is the Jordan decomposition of an element of $GL_n(\mathbb{C})$. Up to conjugacy, every element of $GL_n(\mathbb{C})$ has this shape, for some $\vec{z}$ (unique up to S_n-association) and $\vec{u}$. This setup leads to:

Theorem 2.6. *There exists a canonical bijection between the following sets:*

- *conjugacy classes in $GL_n(\mathbb{C})$,*
- *S_n-association classes of data $(\vec{n}, \vec{z})$, where $\vec{n} = (n_i)$ is a partition of n and $z_i \in \big((\mathbb{C}^\times)^{n_i}\big)^{S_{n_i}}$,*
- *$\mathrm{Irr}(\mathcal{H}_n(\mathbf{q}))$.*

3. Representation theory

3.1. Parabolic induction

In the representation theory of reductive groups, a pivotal role is played by parabolic induction. An analogous operation exists for affine Hecke algebras, and it will be crucial in large parts of this paper.

Given a based root datum $\mathcal{R} = (X, R, Y, R^\vee, \Delta)$ and a subset $P \subset \Delta$, we can form the based root datum

$$\mathcal{R}^P = (X, R_P, Y, R_P^\vee, P).$$

Here $R_P = \mathbb{Q}P \cap R$ is a standard parabolic root subsystem of R, with dual root system $R_P^\vee = \mathbb{Q}P^\vee \cap R^\vee$. We record the special cases

$$\mathcal{R}^\Delta = \mathcal{R} \quad \text{and} \quad \mathcal{R}^\emptyset = (X, \emptyset, Y, \emptyset, \emptyset).$$

Let W_P be the Weyl group of R_P. Any parameter functions λ, λ^* for $\mathcal{R}$ restrict to parameter functions for $\mathcal{R}^P$, and

$$\mathcal{H}^P = \mathcal{H}(\mathcal{R}^P, \lambda, \lambda^*, \mathbf{q})$$

is a subalgebra of $\mathcal{H} = \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})$. This corresponds to the notion of a parabolic subgroup $\mathcal{P}$ of a reductive group $\mathcal{G}$, and simultaneously to the notion of a Levi factor of $\mathcal{P}$– for affine Hecke algebras the unipotent radical of $\mathcal{P}$ is more or less automatically divided out.

Notice that $\mathcal{H}$ and $\mathcal{H}^P$ share the same commutative subalgebra $\mathbb{C}[X] = \mathcal{O}(T)$. By (1.16), as vector spaces

$$\mathcal{H} = \mathcal{H}(W, q) \underset{\mathcal{H}(W_P, q)}{\otimes} \mathcal{H}(W_P, q) \otimes_{\mathbb{C}} \mathbb{C}[X] = \mathcal{H}(W, q) \underset{\mathcal{H}(W_P, q)}{\otimes} \mathcal{H}^P. \tag{3.1}$$

Parabolic induction for representations of affine Hecke algebras is the functor

$$\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}} : \mathrm{Mod}(\mathcal{H}^P) \to \mathrm{Mod}(\mathcal{H}),$$

for any parabolic subalgebra $\mathcal{H}^P$ of $\mathcal{H}$. We also have a "parabolic restriction" functor, that is just restriction of $\mathcal{H}$-modules to $\mathcal{H}^P$.

The link with parabolic induction for reductive p-adic groups is made precise in [83, §4.1]. In that setting parabolic restriction for Hecke algebras corresponds to the Jacquet restriction functor (but with respect to a parabolic subgroup opposite to the one used for induction).

In practice we often precompose parabolic induction with inflations of representations from a quotient algebra of $\mathcal{H}^P$. To define it, we write

$$\begin{array}{ll} X_P = X \big/ \big(X \cap (P^\vee)^\perp\big) & Y^P = Y \cap P^\perp \\ X^P = X/(X \cap \mathbb{Q}P) & Y_P = Y \cap \mathbb{Q}P^\vee \\ \mathcal{R}_P = (X_P, R_P, Y_P, R_P^\vee, P). & \end{array}$$

In terms of reductive groups, the semisimple root datum $\mathcal{R}_P$ corresponds to the maximal semisimple quotient of a parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$. The parameter functions λ and λ^* remain well-defined for $\mathcal{R}_P$, so there is an affine Hecke algebra

$$\mathcal{H}_P = \mathcal{H}(\mathcal{R}_P, \lambda, \lambda^*, \mathbf{q}).$$

In particular we have

$$\mathcal{H}_\emptyset = \mathcal{H}(0, \emptyset, 0, \emptyset, \emptyset, \lambda, \lambda^*, \mathbf{q}) = \mathbb{C} \quad \text{and} \quad \mathcal{H}_\Delta = \mathcal{H}(X_\Delta, R, Y_\Delta, R^\vee, \Delta, \lambda, \lambda^*, \mathbf{q}).$$

From Definition 1.6 we see that the quotient map

$$X \to X_P : x \mapsto x_P$$

induces a surjective algebra homomorphism

$$\mathcal{H}^P \to \mathcal{H}_P : \theta_x T_w \mapsto \theta_{x_P} T_w. \tag{3.2}$$

Via this quotient map we will often (implicitly) inflate $\mathcal{H}_P$-representations to $\mathcal{H}^P$-representations. To incorporate representations of $\mathcal{H}^P$ that are nontrivial on $\{\theta_x : x \in X \cap (P^\vee)^\perp\}$, we need more flexibility. Write

$$T_P = \mathrm{Hom}_{\mathbb{Z}}(X_P, \mathbb{C}^\times), \qquad T^P = \mathrm{Hom}_{\mathbb{Z}}(X^P, \mathbb{C}^\times),$$

so that $T_P T^P = T$ and $T_P \cap T^P$ is finite. Every $t \in T^P$ gives rise to an algebra automorphism

$$\begin{array}{llll} \psi_t : & \mathcal{H}^P & \to & \mathcal{H}^P \\ & \theta_x T_w & \mapsto & x(t)\theta_x T_w \end{array}.$$

This operation corresponds to twisting representations of a reductive group by an unramified character. For any $\mathcal{H}_P$-representation (π, V) and any $t \in T^P$, we can form the $\mathcal{H}_P$-representation

$$\psi_t^* \mathrm{inf}^P(\pi) = \pi \circ \psi_t : \theta_x T_w \mapsto \pi(x(t)\theta_{x_P} T_w).$$

Definition 3.1. The parabolically induced representation associated to $P \subset \Delta$, $(\sigma, V_\sigma) \in \mathrm{Mod}(\mathcal{H}_P)$ and $t \in T^P$ is

$$\pi(P, \sigma, t) = \mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\sigma \circ \psi_t).$$

By (3.1) the vector space underlying $\pi(P, \sigma, t)$ is

$$\mathcal{H}(W, q) \underset{\mathcal{H}(W_P, q)}{\otimes} V_\sigma,$$

and it has dimension $[W : W_P]\dim(V_\sigma)$. Let us discuss how parabolic induction works out for the subalgebra $\mathbb{C}[X]$ of $\mathcal{H}$.

Definition 3.2. Let (π, V) be an $\mathcal{H}$-representation. For $t \in T$ we write

$$V_t^{gen} = \left\{ v \in V \mid \exists N \in \mathbb{N} : (\pi(\theta_x) - x(t))^N v = 0 \right\}.$$

When $V_t^{gen} \neq 0$, we call t a $\mathbb{C}[X]$-weight of π, and V_t^{gen} its generalized weight space. We denote the set of $\mathbb{C}[X]$-weights of (π, V) by $\mathrm{Wt}(\pi)$ or $\mathrm{Wt}(V)$.

For all $t \in \mathrm{Wt}(\pi)$ the space V_t^{gen} contains a vector $v' \neq 0$ with

$$\pi(\theta_x)v' = x(t)v' \qquad \forall x \in X,$$

so $0 \neq V_t \subset V_t^{gen}$. If V has finite dimension, then we can triangularize the commuting operators $\pi(\theta_x)$ simultaneously, and we find

$$V = \bigoplus\nolimits_{t \in T} V_t^{gen}. \tag{3.3}$$

An infinite dimensional $\mathcal{H}$-representation does not necessarily have any $\mathbb{C}[X]$-weight.

Recall that the centre of $\mathcal{H}$ is $Z(\mathcal{H}) = \mathbb{C}[X]^W = \mathcal{O}(T/W)$. Hence, whenever π admits a central character $\mathrm{cc}(\pi)$, we have

$$\mathrm{cc}(\pi) = W\mathrm{Wt}(\pi) \in T/W.$$

The set of $\mathbb{C}[X]$-weights of a representation behaves well under parabolic induction. To describe the effect, let W^P be the set of shortest length representatives for W/W_P.

Lemma 3.3.

(a) *Let π be a finite dimensional $\mathcal{H}^P$-representation. Then the $\mathbb{C}[X]$-weights of* $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi)$ *are the elements $w(t)$ with $t \in \mathrm{Wt}(\pi)$ and $w \in W^P$.*

(b) *Let σ be a finite dimensional $\mathcal{H}_P$-representation and let $s \in T^P$. Then*

$$\mathrm{Wt}(\pi(P,\sigma,s)) = \{w(st) : t \in \mathrm{Wt}(\sigma), w \in W^P\}.$$

Proof. (a) is a consequence of the proof of [63, Proposition 4.20].
(b) Clearly $\mathrm{Wt}(\sigma \circ \psi_s) = s\mathrm{Wt}(\sigma)$. Combine with part (a). □

Since $\mathcal{H}$ has finite rank as a module over its centre $\mathbb{C}[X]^W$, every irreducible $\mathcal{H}$-representation has finite dimension. Hence π admits at least one $\mathbb{C}[X]$-weight, say t. By Frobenius reciprocity

$$\mathrm{Hom}_{\mathcal{H}}(\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t),\pi) \cong \mathrm{Hom}_{\mathbb{C}[X]}(\mathbb{C}_t,\pi) \neq 0.$$

We conclude that:

Corollary 3.4. *Every irreducible $\mathcal{H}$-representation is a quotient of* $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$ *for some $t \in T$.*

Most of the time $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$ is itself irreducible. To make that precise, consider the following rational functions on T:

$$c_\alpha = \frac{\big(\theta_\alpha - \mathbf{q}^{(-\lambda^*(\alpha)-\lambda(\alpha))/2}\big)\big(\theta_\alpha + \mathbf{q}^{(\lambda^*(\alpha)-\lambda(\alpha))/2}\big)}{(\theta_\alpha - 1)(\theta_\alpha + 1)} \qquad \alpha \in R. \tag{3.4}$$

There are a few ways in which c_α can simplify:

- if $\lambda(\alpha) = \lambda^*(\alpha) = 0$, then $c_\alpha = 1$,
- if $\lambda(\alpha) = \lambda^*(\alpha) \neq 0$, then $c_\alpha = (\theta_\alpha - \mathbf{q}^{-\lambda(\alpha)})(\theta_\alpha - 1)^{-1}$.

Theorem 3.5 (*[40, Theorem 2.2]*)**.** *Let $t \in T$. The $\mathcal{H}$-representation* $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$ *is irreducible if and only if*

- *$c_\alpha(t) \neq 0$ for all $\alpha \in R$ and*
- *W_t is generated by $\{s_\alpha : \alpha \in R, s_\alpha(t) = t, c_\alpha^{-1}(t) = 0\}$.*

The parabolically induced representations

$$\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t) = \pi(\emptyset, \mathrm{triv}, t) \qquad t \in T \tag{3.5}$$

can all be realized on the same vector space $\mathcal{H}(W,q)$. In fact, they are isomorphic to $\mathcal{H}(W,q)$ as $\mathcal{H}(W,q)$-modules. In principle, the entire representation theory of $\mathcal{H}$ can be uncovered by analysing the family of representations (3.5). We already did that successfully for W_{aff} of type $\widetilde{A_1}$ in Section 2.2. However, this direct approach is very difficult in general. Indeed, while the irreducible representations of $\mathcal{H}$ have been classified in several ways, the finer structure of $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$ (e.g. a Jordan–Hölder sequence or the multiplicity with which irreducible representations appear) is not always known.

3.2. Tempered representations

An admissible representation of a reductive group G over a local field is tempered if all its matrix coefficients have moderate growth on G, see [86, §III.2] and [46, §VII.11]. This notion has several uses:

- the irreducible tempered G-representations form precisely the support of the Plancherel measure of G,
- the Langlands classification of irreducible admissible G-representations in terms of irreducible tempered representations of Levi subgroups of G,
- for general harmonic analysis on G, e.g. the Plancherel isomorphism.

Analogous of all these well-known results have been established for affine Hecke algebras, see [22,63]. In this paragraph we will discuss the second of the above three items.

Recall that every (finite dimensional) $\mathcal{H}$-module (π, V) has a set of $\mathbb{C}[X]$-weights $\mathrm{Wt}(\pi) \subset T = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{C}^\times)$. To formulate the condition for temperedness in terms of weights, it will be convenient to abbreviate

$$\mathfrak{a} = Y \otimes_{\mathbb{Z}} \mathbb{R},\ \mathfrak{t} = Y \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{Lie}(T),\ \mathfrak{a}^* = X \otimes_{\mathbb{Z}} \mathbb{R},\ \mathfrak{t}^* = X \otimes_{\mathbb{Z}} \mathbb{C}.$$

The complex torus T admits a polar decomposition

$$T = \mathrm{Hom}_{\mathbb{Z}}(X, S^1) \times \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{R}_{>0}) = T_{\mathrm{un}} \times \exp(\mathfrak{a}). \tag{3.6}$$

Here the unitary part T_{un} is the maximal compact subgroup of T and the positive part $\exp(\mathfrak{a})$ is the identity component of the maximal real split subtorus of T. Notice that

$$\mathrm{Lie}(T_{\mathrm{un}}) = i\mathfrak{a} = Y \otimes_{\mathbb{Z}} i\mathbb{R} \subset \mathfrak{t} = \mathrm{Lie}(T).$$

For any $t \in T$ we write $|t|$ for the homomorphism $x \mapsto |t(x)|$. Then $t = t\,|t|^{-1}\,|t|$ is the polar decomposition of t.

The acute positive cones in $\mathfrak{a}$ are

$$\begin{aligned} \mathfrak{a}^+ &= \{\nu \in \mathfrak{a} : \langle \alpha, \nu \rangle \geq 0\ \forall \alpha \in \Delta\}, \\ \mathfrak{a}^{++} &= \{\nu \in \mathfrak{a} : \langle \alpha, \nu \rangle > 0\ \forall \alpha \in \Delta\}. \end{aligned}$$

We define $\mathfrak{a}^{*,+}$ and $\mathfrak{a}^{*,++}$ similarly, so in particular $X^+ = X \cap \mathfrak{a}^{*,+}$. Next we have the obtuse negative cones in $\mathfrak{a}$:

$$\begin{aligned} \mathfrak{a}^- &= \{\nu \in \mathfrak{a} : \langle \delta, \nu \rangle \leq 0\ \forall \delta \in \mathfrak{a}^{*,+}\} = \{\textstyle\sum_{\alpha \in \Delta} x_\alpha \alpha^\vee : x_\alpha \leq 0\}, \\ \mathfrak{a}^{--} &= \{\nu \in \mathfrak{a} : \langle \delta, \nu \rangle < 0\ \forall \delta \in \mathfrak{a}^{*,+} \setminus \{0\}\}. \end{aligned} \tag{3.7}$$

Via the exponential map $\exp : \mathfrak{t} \to T$ we get

$$T^+ = \exp(\mathfrak{a}^+),\ T^{++} = \exp(\mathfrak{a}^{++}),\ T^- = \exp(\mathfrak{a}^-),\ T^{--} = \exp(\mathfrak{a}^{--}).$$

Definition 3.6. A finite dimensional $\mathcal{H}$-representation (π, V) is tempered if the following equivalent conditions are satisfied:

- $|t(x)| \leq 1$ for all $t \in \mathrm{Wt}(\pi), x \in X^+$,
- $\mathrm{Wt}(V) \subset T_{\mathrm{un}} T^-$,
- $|\mathrm{Wt}(V)| \subset T^-$.

Example 3.7.

- Consider the root datum $\mathcal{R}_n$ of type GL_n. Then

$$\begin{aligned} &\mathfrak{a} = \mathfrak{a}^* = \mathbb{R}^n, \qquad \mathfrak{a}^+ = \mathfrak{a}^{*,+} = \{\nu \in \mathbb{R}^n : \nu_1 \geq \nu_2 \geq \cdots \geq \nu_n\}, \\ &\mathfrak{a}^- = \{\nu \in \mathbb{R}^n : \nu_1 \leq 0, \nu_1 + \nu_2 \leq 0, \ldots, \nu_1 + \nu_2 + \cdots \\ &\quad + \nu_{n-1} \leq 0, \nu_1 + \nu_2 + \cdots + \nu_n = 0\}. \end{aligned}$$

The Steinberg representation has $\big(\mathbf{q}^{(1-n)/2}, \mathbf{q}^{(3-n)/2}, \dots, \mathbf{q}^{(n-1)/2}\big)$ as its only $\mathbb{C}[X]$-weight, so it is tempered when $\mathbf{q} \geq 1$. On the other hand, the trivial $\mathcal{H}_n(\mathbf{q})$-representation satisfies $\mathrm{Wt}(\mathrm{triv}) = \big\{(\mathbf{q}^{(n-1)/2}, \mathbf{q}^{(n-3)/2}, \dots, \mathbf{q}^{(1-n)/2})\big\}$, so it is tempered when $\mathbf{q} \leq 1$.
- For $q = 1$, the $\mathcal{O}(T)$-weights of any irreducible representation of $\mathcal{H}(\mathcal{R}, 1) = \mathbb{C}[X \rtimes W]$ form a full W-orbit in T, see Section 2.1. The W-orbit of $\log|t| \in \mathfrak{a}$ can only be contained in $\mathfrak{a}^-$ if $\log|t| = 0$, that is, $|t| = 1$. Hence the irreducible tempered representations of $\mathbb{C}[X \rtimes W]$ are precisely the irreducible constituents of the modules $\mathrm{ind}_{\mathbb{C}[X]}^{\mathbb{C}[X \rtimes W]} \mathbb{C}_t$ with $t \in T_{\mathrm{un}}$.

With Lemma 3.3 one can show (see [79, Lemma 3.1.1] and [6, Lemma 2.4.c]):

Proposition 3.8. *Let (π, V) be a finite dimensional representation of $\mathcal{H}^P$, for some $P \subset \Delta$. Then*

$$\pi \text{ is tempered } \Longleftrightarrow \mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi) \text{ is tempered.}$$

For $P = \emptyset$ we have $R = \emptyset$ and $T^+ = T^- = \{1\}$. Then Proposition 3.8 says that $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$ is tempered if and only if $t \in T_{\mathrm{un}}$.

In the Langlands classification we need a somewhat more general kind of $\mathcal{H}$-representation, for which we merely require that it becomes tempered upon restriction to $\mathcal{H}(W_{\mathrm{aff}}, q)$. This can also be formulated with a more relaxed condition on the weights. It involves the Lie subgroup T^{Δ} of T, whose Lie algebra is identified with $\mathfrak{t}^{\Delta} := R^{\perp} \subset \mathfrak{t}$.

Definition 3.9. A finite dimensional $\mathcal{H}$-representation (π, V) is essentially tempered if the following equivalent conditions are satisfied:

- $|t(x)| \leq 1$ for all $t \in \mathrm{Wt}(\pi)$, $x \in X^+ \cap W_{\mathrm{aff}}$,
- $\mathrm{Wt}(V) \subset T^{\Delta} T_{\mathrm{un}} T^-$,
- $|\mathrm{Wt}(V)| \subset T^{\Delta} T^-$.

When the root datum $\mathcal{R}$ is semisimple, essentially tempered is the same as tempered.

Lemma 3.10 (*See [77, Lemma 3.5] and [83, Lemma 2.3]*). *For any irreducible essentially tempered $\mathcal{H}$-representation π, there exists $t \in T^{\Delta}$ such that $\pi \circ \psi_t$ is tempered and arises by inflation from a representation of $\mathcal{H}_{\Delta}$.*

Example 3.11. Consider $\mathcal{R}_n$ and $\mathcal{H}_n(\mathbf{q})$ with $\mathbf{q} > 1$. Then

$$X^+ \cap W_{\mathrm{aff}} = \{x \in \mathbb{Z}^n : x_1 \geq x_2 \geq \dots \geq x_n, x_1 + x_2 + \dots + x_n = 0\}.$$

For $z \in \mathbb{C}^\times$ the $\mathcal{H}_n(\mathbf{q})$-representation $\mathrm{St} \otimes t_z$ from (2.13) has a unique $\mathbb{C}[X]$-weight

$$\big(z\mathbf{q}^{(1-n)/2}, z\mathbf{q}^{(3-n)/2}, \dots, z\mathbf{q}^{(n-1)/2}\big) \in T^{\Delta} T_{\mathrm{un}} T^-.$$

It is essentially tempered for all $z \in \mathbb{C}$, and tempered if and only if $|z| = 1$.

In the Langlands classification we employ irreducible representations of $\mathcal{H}^P$, where $P \subset \Delta$. We need some further notations:

$$\begin{aligned}
&\mathfrak{t}^P = Y^P \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{Lie}(T^P), \qquad \mathfrak{a}^P = Y^P \otimes_{\mathbb{Z}} i\mathbb{R} = \mathrm{Lie}(T^P_{\mathrm{un}}),\\
&\mathfrak{a}^{P,+} = \{\nu \in \mathfrak{a}^P : \langle \alpha, \nu \rangle \geq 0 \ \forall \alpha \in \Delta \setminus P\},\\
&T^{P,+} = \exp(\mathfrak{a}^{P,+}) = \{t \in T^P \cap \exp(\mathfrak{a}) : |t(\alpha)| \geq 1 \ \forall \alpha \in \Delta \setminus P\},\\
&\mathfrak{a}^{P,++} = \{\nu \in \mathfrak{a}^P : \langle \alpha, \nu \rangle > 0 \ \forall \Delta \setminus P\} \qquad T^{P,++} = \exp(\mathfrak{a}^{P,++}).
\end{aligned}$$

We say that $(\pi, V) \in \mathrm{Irr}(\mathcal{H}^P)$ is in positive position if $\mathrm{cc}(\pi) = tW_P r$ with $t \in T^{P,++}T_{\mathrm{un}}$ and $r \in T_P$. By Lemma 3.10 this is equivalent to requiring that $\pi = \pi' \circ \psi_t$ for some $t \in T^{P,++}T^P_{\mathrm{un}}$ and $\pi' \in \mathrm{Irr}(\mathcal{H}_P)$.

Definition 3.12. A Langlands datum for $\mathcal{H}$ consists of a subset $P \subset \Delta$ and an irreducible essentially tempered representation σ of $\mathcal{H}^P$ in positive position.

Equivalently, it can be given by a triple (P, τ, t), where $P \subset \Delta$, $\tau \in \mathrm{Irr}(\mathcal{H}_P)$ is tempered and $t \in T^{P,++}T^{P,\mathrm{un}}$. Then the associated $\mathcal{H}^P$-representation is $\sigma = \tau \circ \psi_t$.

The $\mathcal{H}$-representations $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\sigma)$ and $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\tau \circ \psi_t)$ are called standard.

By Lemma 3.3 every standard $\mathcal{H}$-module admits a central character.

Now we can finally state the Langlands classification for affine Hecke algebras.

Theorem 3.13 ([79, Theorem 2.4.4]). *Let (P, σ) be a Langlands datum for $\mathcal{H}$.*

(a) The $\mathcal{H}$-representation $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\sigma)$ has a unique irreducible quotient, which we call $L(P, \sigma)$.
(b) For every irreducible $\mathcal{H}$-representation π there exists a Langlands datum (P, σ) with $L(P, \sigma) \cong \pi$.
(c) If (P', σ') is another Langlands datum and $L(P', \sigma') \cong L(P, \sigma)$, then $P' = P$ and the $\mathcal{H}^P$-representations σ' and σ are equivalent.

Some consequences can be drawn immediately:

- $L(P, \sigma)$ is tempered if and only if $P = \Delta$ and $\sigma \in \mathrm{Irr}(\mathcal{H})$ is tempered (because $L(\Delta, \sigma) = \sigma$ and Langlands data are unique).
- In terms of Langlands data (P, τ, t) and (P', τ', t'), the irreducible $\mathcal{H}$-representations $L(P, \tau, t) = L(P, \tau \circ \psi_t)$ and $L(P', \tau', t') = L(P', \tau' \circ \psi_{t'})$ are equivalent if and only if $P = P'$ and $\tau \circ t \cong \tau' \circ t'$. (The latter does not imply $t = t'$, for t and t' may still differ by an element of $T^P \cap T_P$.)
- For $q = 1$, Corollary 2.1 shows that every standard module is irreducible. Hence the notions of irreducible representations and standard modules coincide for $\mathbb{C}[X \rtimes W]$.

Example 3.14. We work out the Langlands classification for $\mathcal{H}(\mathcal{R}, \mathbf{q})$ with $\mathcal{R}$ of type $\widetilde{A_1}$. Its irreducible representations were already listed in Section 2.2. Recall that $\mathbf{q} > 1$. The irreducible tempered representations are:

- the Steinberg representation,
- the parabolically induced representations $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$ with $t \in T_{\mathrm{un}} = S^1$ but $t \neq -1$ (where we recall that $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_{t^{-1}}) \cong \mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$),
- the two representations $\pi(-1, \mathrm{triv})$ and $\pi(-1, \mathrm{St})$ which sum to $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_{-1})$.

These representations exhaust the Langlands data (P, σ) with $P = \Delta = \{\alpha\}$. For $P = \emptyset$ we have $\mathcal{H}_\emptyset = \mathbb{C}[X]$. Its irreducible essentially tempered representations in positive position are the $\mathbb{C}_t$ with $t \in T^{++}T_{\mathrm{un}} = \{z \in \mathbb{C}^\times : |z| > 1\}$. We have

- $L(\emptyset, \mathbb{C}_t) = \mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$ unless $t = \mathbf{q}$,
- $L(\emptyset, \mathbb{C}_{\mathbf{q}}) = \mathrm{triv}$.

In this way we obtain every element of $\mathrm{Irr}(\mathcal{H}(\mathcal{R}, \mathbf{q}))$–as listed at the end of Section 2.2–exactly once, because $T^{++}T_{\mathrm{un}}$ is a fundamental domain for the action of $W = \langle s_\alpha \rangle$ on $T \setminus T_{\mathrm{un}}$.

Theorem 3.13 provides a quick and beautiful way to classify $\mathrm{Irr}(\mathcal{H})$ in terms of the irreducible tempered representations of its parabolic subquotient algebras $\mathcal{H}_P$.

On the downside, it conceals the topological structure of $\mathrm{Irr}(\mathcal{H})$. For instance, with $\mathcal{R}$ of type $\widetilde{A_1}$ as discussed above, there is a family of $\mathcal{H}$-representations $\pi(\emptyset, \mathrm{triv}, t) = \mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$ with $t \in T$, irreducible for almost all t. The Langlands classification breaks it into two families, one with $t \in T_{\mathrm{un}}$ and one with $t \in T \setminus T_{\mathrm{un}}$. In the next paragraph will see how this can be improved.

We end this paragraph with some useful extras about the Langlands classification. The central character of an irreducible $\mathcal{H}_P$-representation τ is an element $\mathrm{cc}(\tau)$ of T_P/W_P. Its absolute value $|\mathrm{cc}(\tau)|$, with respect to the polar decomposition (3.6), lies in $\exp(\mathfrak{a}_P)/W_P$, and $\log|\mathrm{cc}(\tau)| \in \mathfrak{a}_P/W_P$. We fix a W-invariant inner product on $\mathfrak{a}$. The norm of $\log|t|$ is the same for all representatives $t \in T_P$ of $\mathrm{cc}(\tau)$. That enables us to write

$$\|\mathrm{cc}(\tau)\| = \|\log|t|\| \qquad \text{for any } t \in T_P \text{ with } W_P t = \mathrm{cc}(\tau).$$

Lemma 3.15 (*[79, Lemma 2.2.6]*)**.** *Let (P, τ, t) be a Langlands datum.*

(a) $\mathrm{End}_{\mathcal{H}}(\pi(P, \tau, t)) = \mathbb{C}\,\mathrm{id}$.
(b) The representation $L(P, \tau, t)$ appears with multiplicity one in $\pi(P, \tau, t) = \mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\tau \circ \psi_t)$. All other constituents $L(P', \tau', t')$ of $\pi(P, \tau, t)$ are larger, in the sense that $\|\mathrm{cc}(\tau')\| > \|\mathrm{cc}(\tau)\|$.
(c) Let $Ws \in T/W$. Both $\{\pi \in \mathrm{Irr}(\mathcal{H}) : \mathrm{cc}(\pi) = Ws\}$ and

$$\{\pi(P, \tau, t) : (P, \tau, t) \text{ Langlands datum with } \mathrm{cc}(\tau)t \subset Ws\}$$

are bases of the Grothendieck group of the category of finite dimensional $\mathcal{H}$-modules all whose $\mathcal{O}(T)$-weights are in Ws. With respect to a total ordering that extends the partial ordering defined by part (b), the transition matrix between these two bases is unipotent and upper triangular.

3.3. Discrete series representations

In the representation theory of a reductive group G over a local field, Harish-Chandra showed that every irreducible tempered G-representation τ can be obtained from an irreducible square-integrable modulo centre representation δ of a Levi subgroup M of G, see [46, Theorem 8.5.3] and [86, Proposition III.4.1.i]. More precisely, τ is a direct summand of the (normalized) parabolic induction of δ. When the centre of M is compact, δ is an isolated point of the space of irreducible tempered M-representations, and it is called a discrete series representation. Then it is a subrepresentation of $L^2(M)$.

Like for the Langlands classification, these results can be formulated and proven for affine Hecke algebras as well. For these purposes it is essential that $q_s \in \mathbb{R}_{>0}$ for all $s \in S_{\mathrm{aff}}$. To achieve that, we require not only that $\mathbf{q} \in \mathbb{R}_{>1}$ (as we already did), but also that

$$\lambda(\alpha) \in \mathbb{R}, \lambda^*(\alpha) \in \mathbb{R} \qquad \forall \alpha \in \mathbb{R}. \tag{3.8}$$

This condition will be in force in the remainder of this paragraph.

We note that the space $\mathfrak{a}^{--}$ from (3.7) is empty unless R spans $\mathfrak{a}^*$, and in that case

$$\mathfrak{a}^{--} = \{\textstyle\sum_{\alpha \in \Delta} x_\alpha \alpha^\vee : x_\alpha < 0\}.$$

For $P \subset \Delta$ we have

$$\mathfrak{a}_P = Y_P \otimes_{\mathbb{Z}} \mathbb{R}, \quad \mathfrak{t}_P = Y_P \otimes_{\mathbb{Z}} \mathbb{C},$$

and that gives rise to $\mathfrak{a}_P^+, \mathfrak{a}_P^-, T_P^+, T_P^-$. Let

$$\mathfrak{a}_P^{--} = \left\{ \nu \in \mathfrak{a}_P : \langle \delta, \nu \rangle < 0 \; \forall \delta \in \mathfrak{a}_P^{*+} \setminus \{0\} \right\}$$

and $T_P^{--} = \exp(\mathfrak{a}_P^{--})$ be the versions of $\mathfrak{a}^{--}$ and $T^{--} = \exp(\mathfrak{a}^{--})$ for $\mathcal{R}_P$.

Definition 3.16. Let (π, V) be a finite dimensional $\mathcal{H}$-representation. We say that π belongs to the discrete series if the following equivalent conditions hold:

- $|x(t)| < 1$ for all $t \in \mathrm{Wt}(V)$ and all $x \in X^+ \setminus \{0\}$,
- $|\mathrm{Wt}(V)| \subset T^{--}$,
- $\mathrm{Wt}(V) \subset T_{\mathrm{un}} T^{--}$.

Further, we call π essentially discrete series if the following equivalent conditions hold:

- $|x(t)| < 1$ for all $t \in \mathrm{Wt}(V)$ and all $x \in W_{\mathrm{aff}} X^+ \setminus \{0\}$,
- $|\mathrm{Wt}(V)| \subset T^{\Delta} T^{--}$,
- $\mathrm{Wt}(V) \subset T^{\Delta} T_{\mathrm{un}} T^{--}$.

Here UT^{--} (for any $U \subset T$) is considered as empty if $T^{--} = \emptyset$.

Example 3.17.

- Let $\mathcal{R}$ be of type $\widetilde{A_1}$ and consider $\mathcal{H} = \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})$ with $\lambda(\alpha) > 0$ and $\lambda^*(\alpha) > 0$. Here $\mathfrak{a}^{--} = \mathbb{R}_{<0}$ and $T^{--} = (0, 1)$. The only weight of the Steinberg representation is $q^{(\lambda(\alpha)+\lambda^*(\alpha))/2}$, so it is discrete series. The trivial representation and all the twodimensional irreducible representations of $\mathcal{H}$ are not discrete series. From the classification in Section 2.2, especially (2.9) and (2.10), we see that St is the only irreducible discrete series $\mathcal{H}$-representation if $\lambda(\alpha) = \lambda^*(\alpha)$. When $\lambda(\alpha) > \lambda^*(\alpha)$, $\pi(-1, \mathrm{St})$ is the only other irreducible discrete series representation, while $\pi(-1, \mathrm{triv})$ is discrete series if $\lambda(\alpha) < \lambda^*(\alpha)$.
- The affine Hecke algebra $\mathcal{H}_n(\mathbf{q})$ of type GL_n has no discrete series, because its root datum $\mathcal{R}_n$ is not semisimple. Here

$$\mathfrak{a}_{\Delta}^{--} = \{\nu \in \mathbb{R}^n : \nu_1 < 0, \nu_1 + \nu_2 < 0, \dots, \nu_1 + \dots + \nu_{n-1} < 0, \nu_1 + \dots + \nu_{n-1} + \nu_n = 0\}.$$

  For any $z \in \mathbb{C}^\times$, the twist $\mathrm{St} \otimes t_z = \mathrm{St} \circ \psi_{t_z}$ of the Steinberg representation is essentially discrete series. In fact these are all irreducible essentially discrete series representations of $\mathcal{H}_n(\mathbf{q})$ (recall that $\mathbf{q} > 1$).
- An affine Hecke algebra with $\lambda = \lambda^* = 0$ does not have discrete series representations, apart from the case $\mathcal{R} = (0, \emptyset, 0, \emptyset)$, when the trivial representation of $\mathcal{H} = \mathbb{C}$ is regarded as discrete series.

The relation between Definition 3.16 and representations of reductive p-adic groups goes via Casselman's criterium for square-integrability, see [13, Theorem 4.4.6] and [72, §VII.1.2]. Opdam [63, Lemma 2.22] translated this to various criteria for $\mathcal{H}$-representations, which are equivalent with Definition 3.16.

We condition (3.8) at hand, we can define a Hermitian inner product on $\mathcal{H}$ by declaring that $\{q(w)^{-1/2} T_w : w \in W(\mathcal{R})\}$ is an orthonormal basis. Let $L^2(W(\mathcal{R}), q)$ be the Hilbert space completion of $\mathcal{H}$ with respect to this inner product—it is canonically isomorphic to $L^2(W(\mathcal{R}))$. By [63, Lemma 2.22], every irreducible discrete series representation of $\mathcal{H}$ is isomorphic to a subrepresentation of the regular representation of $\mathcal{H}$ on $L^2(W(\mathcal{R}), q)$.

In terms of representations of a reductive group G, "essentially discrete series" means that a representation has finite length and that its restriction to the derived group of G is square-integrable. An essentially discrete series representation is tempered if and only if $\{\theta_x : x \in X \cap (\Delta^\vee)^\perp\}$ acts on it by characters from T_{un}^{Δ}.

Theorem 3.18 (*[22, Theorem 3.22] and [83, Lemma 1.3]*)**.**

(a) Let π' be an irreducible tempered $\mathcal{H}$-representation. There exist $P \subset \Delta$ and a tempered essentially discrete series representation δ' of $\mathcal{H}^P$ such that π' is a direct summand of $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\delta')$.

(b) Let π be an irreducible essentially discrete series $\mathcal{H}^P$-representation. There exist $t \in T^P$ and a discrete series $\mathcal{H}_P$-representation δ such that $\pi \cong \delta \circ \psi_t$.

Clearly, a combination of Theorems 3.13 and 3.18 yields some description of $\mathrm{Irr}(\mathcal{H})$ in terms of parabolic induction and the discrete series of the subquotient algebras $\mathcal{H}_P$ with $P \subset \Delta$. We work this out in detail.

Definition 3.19. An induction datum for $\mathcal{H}$ is a triple $\xi = (P, \delta, t)$, where

- $P \subset \Delta$,
- δ is an irreducible discrete series representation of $\mathcal{H}_P$,
- $t \in T^P$.

We regard two triples ξ and $\xi' = (P', \delta', t')$ as isomorphic (notation $\xi \cong \xi'$) if $P = P', t = t'$ and $\delta \cong \delta'$. Let Ξ be the space of such induction data, topologized by regarding P and δ as discrete variables and T^P as a complex analytic variety. We say that $\xi = (P, \delta, t)$ is positive, written $\xi \in \Xi^+$, when $|t| \in T^{P+}$.

We already associated to such an induction datum the parabolically induced representation

$$\pi(P, \delta, t) = \mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\delta \circ \psi_t).$$

By Proposition 3.8

$$\pi(P, \delta, t) \text{ is tempered } \iff t \in T_{\mathrm{un}}^P. \tag{3.9}$$

For $\xi \in \Xi^+$ we write

$$P(\xi) = \{\alpha \in \Delta : |t(\alpha)| = 1\}.$$

Then $P \subset P(\xi)$. This set of simple roots is useful because it enables us to break the process of parabolic induction in two steps: the first dealing only with essentially tempered representations and the second similar to the Langlands classification. More concretely, by Proposition 3.8 $\mathrm{ind}_{(\mathcal{H}_{P(\xi)})^P}^{\mathcal{H}_{P(\xi)}}(\delta \circ \psi_{|t|})$ is tempered, while

$$\pi^{P(\xi)}(\xi) := \mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}^{P(\xi)}}(\delta \circ \psi_t)$$

is essentially tempered.

Proposition 3.20 (*[79, Proposition 3.1.4]*)**.** *Let $\xi = (P, \delta, t) \in \Xi^+$ and pick $t^{P(\xi)} \in T^{P(\xi)}$ such that $t^{P(\xi)} t^{-1} \in T_{P(\xi)}$.*

(a) The $\mathcal{H}^{P(\xi)}$-representation $\pi^{P(\xi)}(\xi)$ is completely reducible and $\pi^{P(\xi)}(\xi) \circ \psi^{-1}_{t^{P(\xi)}}$ is tempered.

(b) Every irreducible summand of $\pi^{P(\xi)}(\xi)$ is of the form $\pi^{P(\xi)}(P(\xi), \tau, t^{P(\xi)})$, where $(P(\xi), \tau, t^{P(\xi)})$ is a Langlands datum for $\mathcal{H}$.

(c) The irreducible quotients of $\pi(\xi)$ are the representations $L(P(\xi), \tau, t^{P(\xi)})$, with $(P(\xi), \tau, t^{P(\xi)})$ coming from part (b).

(d) Every irreducible $\mathcal{H}$-representation is of the form described in part (c).

(e) The functor $\mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}^{P(\xi)}}$ induces an isomorphism

$$\mathrm{End}_{\mathcal{H}^{P(\xi)}}\big(\pi^{P(\xi)}(\xi)\big) \xrightarrow{\sim} \mathrm{End}_{\mathcal{H}}(\pi(\xi)).$$

For a given $\pi \in \mathrm{Irr}(\mathcal{H})$, there are in general several induction data $\xi \in \Xi^+$ such that π is a quotient of $\pi(\xi)$. So, in contrast with the Langlands classification, Proposition 3.20 does not provide an actual parametrization of $\mathrm{Irr}(\mathcal{H})$. To bring that goal closer, one has to analyse the relations between the various representations $\pi(\xi)$ with $\xi \in \Xi$.

For u in the finite group

$$T^P \cap T_P = \mathrm{Hom}_{\mathbb{Z}}\big(X/(X \cap \mathbb{Q}P) \oplus (X \cap (P^\vee)^\perp), \mathbb{C}^\times\big),$$

we have an automorphism ψ_u of $\mathcal{H}^P$ and a similar automorphism $\psi_{P,u}$ of $\mathcal{H}_P$, given by

$$\psi_{P,u}(\theta_{x_P} T_w) = u(x_P)\theta_{x_P} T_w \qquad x_P \in X_P, w \in W_P.$$

Then $(\delta \circ \psi^{-1}_{P,u}) \circ \psi_{ut} = \delta \circ \psi_t$, so

$$\pi(P, \delta \circ \psi^{-1}_{P,u}, ut) = \pi(P, \delta, t). \tag{3.10}$$

Suppose that $w \in W$ and $w(P) = P' \subset \Delta$. Then

$$\begin{array}{cccc} \psi_w : & \mathcal{H}^P & \to & \mathcal{H}^{P'} \\ & \theta_x T_{w'} & \mapsto & \theta_{w(x)} T_{ww'w^{-1}} \end{array} \tag{3.11}$$

is an algebra isomorphism, and it descends to an algebra isomorphism $\psi_w : \mathcal{H}_P \to \mathcal{H}_{P'}$. Moreover, ψ_w can be implemented as conjugation by the element

$$\imath^\circ_w \in \mathbb{C}(X)^W \otimes_{\mathbb{C}[X]^W} \mathcal{H} \tag{3.12}$$

from Proposition 1.12, see [79, (3.124)]. There is a bijection

$$\begin{array}{cccc} I_w : & \big(\mathbb{C}(X)^W \otimes_{\mathbb{C}[X]^W} \mathcal{H}\big) \otimes_{\mathcal{H}^P} V_\delta & \to & \big(\mathbb{C}(X)^W \otimes_{\mathbb{C}[X]^W} \mathcal{H}\big) \otimes_{\mathcal{H}^{P'}} V_\delta \\ & h \otimes v & \mapsto & h\imath^\circ_{w^{-1}} \otimes v \end{array}. \tag{3.13}$$

By [63, Theorem 4.33] it is an isomorphism between the $\mathcal{H}$-representations

$$\mathrm{ind}^{\mathbb{C}(X)^W \otimes_{\mathbb{C}[X]^W} \mathcal{H}}_{\mathcal{H}^P}(\delta \circ \psi_t) \quad \text{and} \quad \mathrm{ind}^{\mathbb{C}(X)^W \otimes_{\mathbb{C}[X]^W} \mathcal{H}}_{\mathcal{H}^{P'}}(\delta \circ \psi^{-1}_w \circ \psi_{w(t)}).$$

For t in a Zariski-open dense subset of T^P, I_w specializes to an $\mathcal{H}$-isomorphism

$$\pi(P, \delta, t) \xrightarrow{\sim} \pi(w(P), \delta \circ \psi^{-1}_w, w(t)). \tag{3.14}$$

As such, $I_w : \mathcal{H} \otimes_{\mathcal{H}^P} V_\delta \to \mathcal{H} \otimes_{\mathcal{H}_{P'}} V_\delta$ is a rational map in the variable $t \in T^P$ (possibly with poles for some $t \in T^P$).

Lemma 3.21. *For all $t \in T^P$ and all $w \in W$ with $w(P) \subset \Delta$, the representations $\pi(P, \delta, t)$ and $\pi(w(P), \delta \circ \psi^{-1}_w, w(t))$ have the same irreducible constituents, with the same multiplicities.*

Proof. The above implies that, for t in a Zariski-open subset of T^P, these two $\mathcal{H}$-representations have the same character. The vector space $\mathcal{H} \otimes_{\mathcal{H}^P} V_\delta$ does not depend on t and the character of $\pi(P, \delta, t)$ depends algebraically on $t \in T^P$, so in fact $\pi(P, \delta, t)$ and $\pi(w(P), \delta \circ \psi_w^{-1}, w(t))$ have the same character for all $t \in T^P$.

Since $\mathcal{H}$ is of finite rank as module over its centre, the Frobenius–Schur theorem [21, Theorem 27.8] applies, and says that the characters of inequivalent irreducible $\mathcal{H}$-representations are linearly independent functionals on $\mathcal{H}$. As the character of $\pi(P, \delta, t)$ determines that representation up to semisimplification, it carries enough information to determine the multiplicities with which the irreducible representations appear in $\pi(P, \delta, t)$. □

The next results are much deeper, for their proofs involve a study of topological completions of affine Hecke algebras [22,63]. For a systematic bookkeeping of the $\mathcal{H}$-isomorphisms (3.10) and (3.14), we put them in a groupoid $\mathcal{W}_\Xi$. Its base space is the power set of Δ, the collection of morphisms from P to P' is

$$\mathcal{W}_{\Xi,PP'} = \{(w, u) \in W \times (T^P \cap T_P) : w(P) = P'\},$$

and the composition comes from the group $T \rtimes W$. This groupoid acts on the space of induction data Ξ as

$$(w, u)(P, \delta, t) = w \cdot (P, \delta \circ \psi_u^{-1}, ut) = (w(P), \delta \circ \psi_u^{-1} \circ \psi_w^{-1}, w(ut)).$$

Theorem 3.22. *Let $\xi = (P, \delta, t), \xi' = (P', \delta', t') \in \Xi^+$. The $\mathcal{H}$-representations $\pi(\xi)$ and $\pi(\xi')$ have a common irreducible quotient if and only if there exists a $(w, u) \in \mathcal{W}_\Xi$ with $(w, u)\xi = \xi'$.*

When $t, t' \in T_{\mathrm{un}}$, Proposition 3.20.a says that $\pi(\xi)$ and $\pi(\xi')$ are completely reducible. Then the statement of the theorem becomes: $\pi(\xi)$ and $\pi(\xi')$ have a common irreducible subquotient if and only if $\xi' \in \mathcal{W}_\Xi \xi$. This is an analogue of Langlands' disjointness theorem (see [46, Theorem 14.90] for real reductive groups and [86, Proposition 4.1.ii] for p-adic reductive groups), and it was proven in [22, Corollary 5.6]. The generalization to $\xi, \xi' \in \Xi^+$ was established in [79, Theorem 3.3.1.a].

Every element $(w, u) \in \mathcal{W}_\Xi$ gives rise to an intertwining operator

$$\pi((w, u), P, \delta, t) : \pi(P, \delta, t) \to \pi(w(P), \delta \circ \psi_u^{-1} \circ \psi_w^{-1}, w(ut)). \tag{3.15}$$

It is rational as a function of $t \in T^P$ and regular for almost all $t \in T^P$. Namely, the isomorphisms (3.14) come from (3.13), while (3.10) is just the identity on the underlying vector space. For isomorphic induction data (P, δ, t) and (P, σ, t) we also need an $\mathcal{H}$-isomorphism

$$\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\delta \circ \psi_t) \to \mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\sigma \circ \psi_t).$$

To that end we pick (independently of t) an $\mathcal{H}_P$-isomorphism $\delta \cong \sigma$ (which anyway is unique up to scalars) and apply $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}$ to that. In this way we get an intertwining operator

$$\pi((w, u), \xi) : \pi(\xi) \to \pi(\xi')$$

whenever $(w, u)\xi$ and ξ' are isomorphic. This operator is unique up to scalars and

$$\pi((w', u'), (w, u)\xi) \circ \pi((w, u), \xi) = \natural((w', u'), (w, u))\, \pi((w', u')(w, u), \xi) \tag{3.16}$$

for some $\natural((w', u'), (w, u)) \in \mathbb{C}^\times$.

Theorem 3.23. *Let $\xi, \xi' \in \Xi^+$. The operators*

$$\{\pi((w,u),\xi) : (w,u) \in \mathcal{W}_\Xi, (w,u)\xi \cong \xi'\}$$

are regular and invertible, and they span $\mathrm{Hom}_{\mathcal{H}}(\pi(\xi), \pi(\xi'))$.

In case the coordinates t, t' of ξ, ξ' lie in T_{un}, this is shown in [22, Corollary 5.4]. That version is an analogue of Harish-Chandra's completeness theorem, see [46, Theorem 14.31] and [74, Theorem 5.5.3.2]. For the version with arbitrary $\xi, \xi' \in \Xi^+$ we refer to [79, Theorem 3.3.1.b].

The relations between parabolically induced representations $\pi(\xi)$ and $\pi(\xi')$ with $\xi, \xi' \in \Xi$ $\mathcal{W}_\Xi$-associate but not positive are more complicated, and not understood well. For the principal series $\pi(\emptyset, \mathrm{triv}, t)$ this issue was investigated in [69].

Finally, everything is in place to formulate an extension of the Langlands classification that incorporates results about discrete series representations. The outcome is similar to L-packets in the local Langlands correspondence. Recall that (3.8) is in force.

Theorem 3.24 (*[79, Theorem 3.3.2]*). *Let π be an irreducible $\mathcal{H}$-representation. There exists a unique $\mathcal{W}_\Xi$-association class $(P, \delta, t) \in \Xi / \mathcal{W}_\Xi$ such that the following equivalent statements hold:*

(a) π is isomorphic to an irreducible quotient of $\pi(\xi^+)$, for some $\xi^+ \in \Xi^+ \cap \mathcal{W}_\Xi(P, \delta, t)$;
(b) π is a constituent of $\pi(P, \delta, t)$, and $\|cc(\delta)\|$ is maximal for this property.

Further, π is tempered if and only if $t \in T_{\mathrm{un}}$.

Theorem 3.24 associates to every irreducible $\mathcal{H}$-representation π an essentially unique positive induction datum (P, δ, t). If $\xi' = (P', \delta', t')$ is another positive induction datum associated to π, then ξ' $\mathcal{W}_\Xi$-associate to (P, δ, t) and Theorem 3.23 entails that $\pi(\xi') \cong \pi(P, \delta, t)$. Hence the parabolically induced representation $\pi(P, \delta, t)$ is uniquely determined by π, up to isomorphism of $\mathcal{H}$-representations.

Conversely, however, $\pi(P, \delta, t)$ can have more than one irreducible quotient. Like an L-packet for a reductive group can have more than one element, (P, δ, t) might be associated (in Theorem 3.24) to several irreducible $\mathcal{H}$-representations.

Example 3.25.

- Consider $\mathcal{R}$ of type $\widetilde{A_1}$, $\mathcal{H} = \mathcal{H}(\mathcal{R}, \mathbf{q})$. Then $\mathcal{W}_{\Xi,\emptyset\emptyset} = \{1, s_\alpha\}, \mathcal{W}_{\Xi,\Delta\Delta} = \{1\}$ and Theorem 3.24 works out as follows, where we take t or t^{-1} depending on whether $|t| \geq 1$ or $|t| \leq 1$:

$\mathrm{Irr}(\mathcal{H})$		induction data
St	$\mapsto$	$(\Delta = \{\alpha\}, \mathrm{St}, 1)$
triv	$\mapsto$	$(\emptyset, \mathrm{triv}, \mathbf{q})$
$\pi(-1, \mathrm{St}), \pi(-1, \mathrm{triv})$	$\mapsto$	$(\emptyset, \mathrm{triv}, -1)$
$\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t) \quad t \notin \{-1, \mathbf{q}, \mathbf{q}^{-1}\}$	$\mapsto$	$(\emptyset, \mathrm{triv}, t^{\pm 1})$

- Keep $\mathcal{R}$ of type $\widetilde{A_1}$, but consider $\mathcal{H} = \mathcal{H}(\mathcal{R}, 1) = \mathbb{C}[W_{\mathrm{aff}}]$. In this case $\mathcal{H}$ does not have any discrete series representations, and the effect of Theorem 3.24 is:

$\mathrm{Irr}(\mathcal{H})$		induction data
St, triv	$\mapsto$	$(\emptyset, \mathrm{triv}, 1)$
$\pi(-1, \mathrm{St}), \pi(-1, \mathrm{triv})$	$\mapsto$	$(\emptyset, \mathrm{triv}, -1)$
$\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t) \quad t \notin \{1, -1\}$	$\mapsto$	$(\emptyset, \mathrm{triv}, t^{\pm 1})$

- For $\mathcal{R}$ of type GL_n, $\mathcal{H} = \mathcal{H}_n(\mathbf{q})$, and $P \subset \Delta$ given by a partition $\vec{n} = (n_1, n_2, \ldots, n_d)$ of n, we have
$$\mathcal{W}_{\Xi,PP}/(T^P \cap T_P) = \prod_{m \geq 1} S(\{i : n_i = m\}) \cong N_{S_n}\Big(\prod_{i=1}^d S_{n_i}\Big) \Big/ \prod_{i=1}^d S_{n_i}.$$
The only irreducible discrete series representations of $\mathcal{H}(\mathcal{R}_{n,P}, \mathbf{q})$ are $\mathrm{St} \otimes t_z = \mathrm{St} \circ \psi_{P,t_z}$ with $t_z \in T^P \cap T_P$. Recall that
$$T^P \cong \prod\nolimits_{i=1}^d ((\mathbb{C}^\times)^{n_i})^{S_{n_i}} \cong (\mathbb{C}^\times)^d.$$
An induction datum $(P, \mathrm{St} \otimes t_z, \vec{z})$ with $\vec{z} = (z_1, \ldots, z_d) \in T^P$ is positive if and only if $|z_1| \geq |z_2| \cdots \geq |z_n|$, a condition which is preserved by the action of $T^P \cap T_P$ on such induction data. Up to $\mathcal{W}_\Xi$-association, it suffices to consider only positive induction data (P, St, t)–so with $t_z = 1$.
It is easily checked that every pair $(\vec{n}, \vec{z})$ as in Theorem 2.6.b is S_n-associate to a pair $(\vec{n}, \vec{z})$ which is positive in the above sense, and that the latter is unique up to $\mathcal{W}_\Xi$-association. The irreducible $\mathcal{H}_n(\mathbf{q})$-representation attached to $(\vec{n}, \vec{z})$ in Section 2.3 is a quotient of $\pi(P, \mathrm{St}, \vec{z}) = \pi(\vec{n}, \vec{z})$, just as in Theorem 3.24. Thus, for $\mathcal{H}_n(\mathbf{q})$ Theorem 3.24 recovers Theorem 2.6: both parametrize $\mathrm{Irr}(\mathcal{H}_n(\mathbf{q}))$ bijectively with basically the same data.

3.4. Lusztig's reduction theorems

In Section 1.5 we already hinted at a link between affine Hecke algebras and graded Hecke algebras. The connection is established with two reduction steps, whose original versions are due to Lusztig [52]. These simplifications do not work in the same way for all representations, they depend on the central characters. The first reduction step limits the set of central characters that has to be considered to understand all finite length $\mathcal{H}$-modules.

By [52, Lemma 3.15], for $t \in T$ and $\alpha \in R$:
$$s_\alpha(t) = t \iff \alpha(t) = \begin{cases} 1 & \alpha^\vee \notin 2Y \\ \pm 1 & \alpha^\vee \in 2Y \end{cases}. \tag{3.17}$$
Fix $t \in T$. We will exhibit an algebra, almost a subalgebra of $\mathcal{H}$, that captures the behaviour of $\mathcal{H}$-representations with central character close to $Wt \in T/W$. We consider the root system
$$R'_t = \{\alpha \in R : s_\alpha(t) = t\}.$$
It fits in a root datum $\mathcal{R}'_t = (X, R'_t, Y, R'^\vee_t)$, and λ and λ^* restrict to parameter functions for $\mathcal{R}'_t$. The affine Hecke algebra $\mathcal{H}(\mathcal{R}'_t, \lambda, \lambda^*, \mathbf{q})$ naturally embeds in $\mathcal{H} = \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})$.

But we have to be careful, this construction is not suitable when $c_\alpha(t) = 0$ for some $\alpha \in R$. For instance, when $\mathcal{R}$ is of type $\widetilde{A_1}$ and $t = \mathbf{q} \neq 1$, the algebra $\mathcal{H}(\mathcal{R}'_t, \lambda, \lambda^*, \mathbf{q})$ is just $\mathbb{C}[X]$. That is hardly helpful to describe all $\mathcal{H}$-modules with weights in $W\mathbf{q}$ (e.g. $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_\mathbf{q})$, triv and St).

When $c_\alpha(t) \neq 0$ for all $\alpha \in R$, $\mathcal{H}(\mathcal{R}'_t, \lambda, \lambda^*, \mathbf{q})$ detects most of what can happen with $\mathcal{H}$-representations with central character Wt, but still not everything.

Example 3.26. Consider the root datum $\mathcal{R}$ of type $\widetilde{A_2}$, with
$$X = \{x \in \mathbb{Z}^3 : x_1 + x_2 + x_3 = 0\}, R = \{e_i - e_j : i \neq j\} \cong A_2, \Delta = \{e_1 - e_2, e_2 - e_3\}.$$

The point $t \in T$ with $t(e_1 - e_2) = t(e_2 - e_3) = e^{2\pi i/3}$ satisfies $R_t = \emptyset$ but $W_t = \{\text{id}, (123), (132)\}$. We have $\mathcal{H}(\mathcal{R}'_t, \lambda, \lambda^*, \mathbf{q}) = \mathbb{C}[X]$, which ignores W_t and possesses only one irreducible representation with central character t. On the other hand, Theorem 4.2 will entail that $\text{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)$ splits into three inequivalent irreducible $\mathcal{H}$-representations, all with central character Wt.

The set

$$R_t = \{\alpha \in R : s_\alpha(t) = t, c_\alpha(t) \neq 0\} \tag{3.18}$$

is a root system, because λ and λ^* are W-invariant. When $t \in T_{\text{un}}$ and (3.8) holds, R_t coincides with R'_t because $c_\alpha(t)$ cannot be 0. We define

$$\mathcal{R}_t = (X, R_t, Y, R_t^\vee).$$

Let $R_t^+ = R^+ \cap R_t$ be the set of positive roots, and let Δ_t be the unique basis of R_t contained in R_t^+. We warn that Δ_t need not be a subset of Δ. The group

$$\Gamma_t = \{w \in W_t : w(R_t^+) = R_t^+\},$$

satisfies $W_t = W(R_t) \rtimes \Gamma_t$. For every $w \in \Gamma_t$ there exists an algebra automorphism like (3.11)

$$\begin{array}{cccc} \psi_w : & \mathcal{H}(\mathcal{R}_t, \lambda, \lambda^*, \mathbf{q}) & \to & \mathcal{H}(\mathcal{R}_t, \lambda, \lambda^*, \mathbf{q}) \\ & \theta_x T_{w'} & \mapsto & \theta_{w(x)} T_{ww'w^{-1}} \end{array}.$$

This is a group action of Γ_t, so we can form the crossed product $\mathcal{H}(\mathcal{R}_t, \lambda, \lambda^*, \mathbf{q}) \rtimes \Gamma_t$. We recall that this means $\mathcal{H}(\mathcal{R}_t, \lambda, \lambda^*, \mathbf{q}) \otimes \mathbb{C}[\Gamma_t]$ as vector spaces, with multiplication rule

$$(h \otimes \gamma)(h' \otimes \gamma') = h\psi_\gamma(h') \otimes \gamma\gamma'.$$

The group Γ_t can be embedded in $\big(\mathbb{C}(X)^W \otimes_{\mathbb{C}[X]^W} \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})\big)^\times$ with the elements ι_w° from Proposition 1.12. In view of (3.11) and (3.12), this realizes $\mathcal{H}(\mathcal{R}_t, \lambda, \lambda^*, \mathbf{q}) \rtimes \Gamma_t$ as a subalgebra of $\mathbb{C}(X)^W \otimes_{\mathbb{C}[X]^W} \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})$.

Unfortunately tensoring with $\mathbb{C}(X)^W$ kills many interesting representations, so the above does not yet suffice to relate the module categories of $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})$, of $\mathcal{H}(\mathcal{R}_t, \lambda, \lambda^*, \mathbf{q}) \rtimes \Gamma_t$ and of a graded Hecke algebra. To achieve that, some technicalities are needed.

For an (analytically) open W-stable subset U of T, let $C^{an}(U)$ be the algebra of complex analytic functions on U. As $Z(\mathcal{H})$ injects in $C^{an}(U)^W$, we can form the algebra

$$\mathcal{H}^{an}(U) := C^{an}(U)^W \otimes_{\mathbb{C}[X]^W} \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q}).$$

It can also be obtained from Definition 1.11 by using $C^{an}(U)$ instead of $\mathcal{O}(T)$, and as a vector space it is just $C^{an}(U) \otimes_{\mathbb{C}} \mathcal{H}(W, q)$.

Let $\text{Mod}_{f,U}$ be the category of finite length $\mathcal{H}$-modules, all whose $\mathcal{O}(T)$-weights lie in U.

Proposition 3.27. *The inclusion $\mathcal{H} \to \mathcal{H}^{an}(U)$ induces an equivalence of categories*

$$\text{Mod}_f(\mathcal{H}^{an}(U)) \to \text{Mod}_{f,U}(\mathcal{H}).$$

Let U_t be a "sufficiently small" W_t-invariant open neighbourhood of t in T and put $U = WU_t$. Then

$$C^{an}(U) = \bigoplus_{w \in W/W_t} C^{an}(wU_t)$$

and there is a well-defined group homomorphism

$$\Gamma_t \to \mathcal{H}^{an}(U)^\times : w \mapsto 1_{U_t} \iota_w^\circ.$$

This realizes $\mathcal{H}(\mathcal{R}_t, \lambda, \lambda^*, \mathbf{q}) \rtimes \Gamma_t$ as a subalgebra of $\mathcal{H}^{an}(U)$. Our version of Lusztig's first reduction theorem [52, §8] is:

Theorem 3.28. *Assume that (3.8) holds and that $t \in T$ satisfies $W_{|t|^{-1}t} = W_t$. Then there exist open W_t-stable neighbourhoods U_t of t with the following properties.*

(a) There is a natural embedding of $C^{an}(U)^W$-algebras

$$\mathcal{H}(\mathcal{R}_t, \lambda, \lambda^*, \mathbf{q})^{an}(U_t) \rtimes \Gamma_t \to \mathcal{H}^{an}(U).$$

(b) Part (a) and Proposition 3.27 induce equivalences of categories

$$\begin{aligned}\mathrm{Mod}_{f,U_t}(\mathcal{H}(\mathcal{R}_t, \lambda, \lambda^*, \mathbf{q}) \rtimes \Gamma_t) &\cong \mathrm{Mod}_f(\mathcal{H}(\mathcal{R}_t, \lambda, \lambda^*, \mathbf{q})^{an}(U_t) \rtimes \Gamma_t)\\ &\cong \mathrm{Mod}_f(\mathcal{H}^{an}(WU_t)) \cong \mathrm{Mod}_{f,WU_t}(\mathcal{H}).\end{aligned}$$

(c) When $t \in T_{\mathrm{un}}$, we may take U_t of the form $U_t' \times \exp(\mathfrak{a})$, where $U_t' \subset T_{\mathrm{un}}$ is a small open W_t-stable ball around t. In that case the equivalences of categories between the outer terms in part (b) preserve temperedness and (essentially) discrete series.

Parts (a) and (b) of Theorem 3.28 can be found in [8, Theorem 3.3] and [79, Theorem 2.1.2]. For part (c) we refer to [6, Proposition 2.7].

The conditions in Theorem 3.28 avoid possible unpleasantness caused by non-invertible intertwining operators. Algebras of the form $\mathcal{H}(\mathcal{R}_t, \lambda, \lambda^*, \mathbf{q}) \rtimes \Gamma_t$ behave almost the same as affine Hecke algebras. The difference can be handled with Clifford theory, as in [68, Appendix]. In fact everything we said so far in this paper can be generalized to such crossed product algebras, see [6,79]. For simplicity, we prefer to keep the finite groups Γ_t out of our presentation.

An advantage of Theorem 3.28 is that it reduces the study of $\mathrm{Mod}_f(\mathcal{H})$ to those modules of $\mathcal{H}(\mathcal{R}_t, \lambda, \lambda^*, \mathbf{q}) \rtimes \Gamma_t$ all whose $\mathbb{C}[X]$-weights belong to a small neighbourhood of the point t, which is fixed by $W(R_t) \rtimes \Gamma_t$. We will use this in a loose sense, suppressing Γ_t. Then Theorem 3.28 says that it suffices to consider those finite dimensional modules of an affine Hecke algebra $\mathcal{H}$, all whose $\mathbb{C}[X]$-weights lie in a small neighbourhood of a W-fixed point $t \in T$.

The second reduction theorem will transfer such $\mathcal{H}$-representations to representations of graded Hecke algebras. In the remainder of this paragraph we assume that u is fixed by W. We define a parameter function k_u for the root system R_u by

$$k_u(\alpha) = (\lambda(\alpha) + \alpha(u)\lambda^*(\alpha)) \log(\mathbf{q})/2. \tag{3.19}$$

By (3.17) $\alpha(u) \in \{\pm 1\}$, so k_u is real-valued whenever the positivity condition (3.8) for q holds. This gives a graded Hecke algebra $\mathbb{H}(\mathfrak{t}, W, k_u)$.

Let $V \subset \mathfrak{t}$ be an analytically open W-stable subset. We can form the algebra

$$\mathbb{H}(\mathfrak{t}, W, k_u)^{an}(V) = C^{an}(V)^W \otimes_{\mathcal{O}(\mathfrak{t})^W} \mathbb{H}(\mathfrak{t}, W, k_u),$$

which as vector space is just

$$C^{an}(V) \otimes_{\mathcal{O}(\mathfrak{t})} \mathbb{H}(\mathfrak{t}, W, k_u) = C^{an}(V) \otimes_{\mathbb{C}} \mathbb{C}[W].$$

Let $\mathrm{Mod}_{f,V}(\mathbb{H}(\mathfrak{t}, W, k_u))$ be the category of those finite dimensional $\mathbb{H}(\mathfrak{t}, W, k_u)$-modules, all whose $\mathcal{O}(\mathfrak{t})$-weights lie in V. An analogue of Proposition 3.27 says that the inclusion

$$\mathbb{H}(\mathfrak{t}, W, k_u) \to \mathbb{H}(\mathfrak{t}, W, k_u)^{an}(V)$$

induces an equivalence of categories

$$\mathrm{Mod}_f(\mathbb{H}(\mathfrak{t}, W, k_u)^{an}(V)) \xrightarrow{\sim} \mathrm{Mod}_{f,V}(\mathbb{H}(\mathfrak{t}, W, k_u)). \tag{3.20}$$

The analytic map

$$\exp_u : \mathfrak{t} \to T, \quad \lambda \mapsto u \exp(\lambda)$$

is W-equivariant, because u is fixed by W. It gives rise to algebra homomorphisms

$$\begin{array}{llll} \exp_u^* : & C^{an}(\exp_u(V)) & \to & C^{an}(V) \\ & f & \mapsto & f \circ \exp_u \\ \Phi_u : & \mathbb{C}(X)^W \otimes_{\mathbb{C}[X]^W} \mathcal{H}^{an}(\exp_u(V)) & \to & Q(S(\mathfrak{t}^*))^W \otimes_{S(\mathfrak{t}^*)^W} \mathbb{H}(\mathfrak{t}, W, k_u)^{an}(V) \\ & f \imath_w^\circ & \mapsto & (f \circ \exp_u)\tilde{\imath}_w \end{array}$$

In good circumstances, this works already without involving rational (non-regular) functions.

Theorem 3.29 (*[52, Theorem 9.3], [8, §4] and [79, Theorem 2.1.4]*). *Let $V \subset \mathfrak{t}$ be an (analytically) open subset such that*

- *V is W-stable,*
- *$\exp_u$ is injective on V,*
- *for all $\alpha \in R, \lambda \in V$ the numbers $\langle \alpha, \lambda \rangle, \langle \alpha, \lambda \rangle + k_u(\alpha)$ do not lie in $\pi i \mathbb{Z} \setminus \{0\}$.*

(a) $\exp_u^* : C^{an}(\exp_u(V)) \to C^{an}(V)$ is a W-equivariant algebra isomorphism, and makes $\mathcal{H}^{an}(\exp_u(V))$ into a $C^{an}(V)^W$-algebra.

(b) Φ_u restricts to an isomorphism of $C^{an}(V)^W$-algebras

$$\mathcal{H}^{an}(\exp_u(V)) \to \mathbb{H}(\mathfrak{t}, W, k_u)^{an}(V).$$

Notice that the conditions in Theorem 3.29 always hold when V is a small neighbourhood of 0 in $\mathfrak{t}$. When k_u is real-valued, for instance whenever (3.8) holds, these conditions also hold for V of the form $\mathfrak{a} + V'$ with $V' \subset i\mathfrak{a}$ a small ball around 0.

Combining Theorems 3.28, 3.29 and Proposition 3.27, we can draw important consequences for the category of finite dimensional $\mathcal{H}$-modules

Corollary 3.30 (*[79, Corollary 2.15]*). *Assume that (3.8) holds and that $u \in T_{\mathrm{un}}$.*

(a) For $\lambda \in \mathfrak{a}$ the categories $\mathrm{Mod}_{f, Wu\exp(\lambda)}(\mathcal{H})$ and $\mathrm{Mod}_{f, W_u\lambda}(\mathbb{H}(\mathfrak{t}, W(R_u), k_u) \rtimes \Gamma_u)$ are naturally equivalent.

(b) The categories $\mathrm{Mod}_{f, Wu\exp(\mathfrak{a})}(\mathcal{H})$ and $\mathrm{Mod}_{f,\mathfrak{a}}(\mathbb{H}(\mathfrak{t}, W(R_u), k_u) \rtimes \Gamma_u)$ are naturally equivalent.

(c) The equivalences of categories from parts (a) and (b) are compatible with parabolic induction.

Sometimes it is not enough that k_u is real-valued, we want to have parameters in $\mathbb{R}_{\geq 0}$. Suppose that $\lambda, \lambda^* : R \to \mathbb{R}_{\geq 0}$ are parameter functions. Then $q_{s_\alpha} = \mathbf{q}^{\lambda(\alpha)}$ and $q_{s'_\alpha} = \mathbf{q}^{\lambda^*(\alpha)}$ belong to $\mathbb{R}_{\geq 1}$ for all $\alpha \in R$, but $\lambda(\alpha) - \lambda^*(\alpha)$ could be negative. In that case Theorem 3.29 could produce a graded Hecke algebra with a negative parameter $k_u(\alpha)$, which could lead to unnecessary complications.

Lemma 3.31. *Let $\mathcal{H}$ be an affine Hecke algebra constructed from a root datum $\mathcal{R}$ and a parameter function $q : S_{\mathrm{aff}} \to \mathbb{R}_{\geq 1}$. Then $\mathcal{H}$ admits a presentation $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})$ with $\lambda(\alpha), \lambda^*(\alpha), \lambda(\alpha) - \lambda^*(\alpha) \in \mathbb{R}_{\geq 0}$ for all $\alpha \in R$.*

Proof. Choose $\lambda, \lambda^* : R \to \mathbb{R}_{\geq 0}$ as in (1.15), so that $\mathcal{H} \cong \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})$.

When $\lambda^*(\alpha) > \lambda(\alpha)$, α must be a short root in a type B_n component R_i of R and $\mathbb{R}R_i^\vee \cap Y = C_n$. Use the presentation from Definition 1.11. Replace the basepoint 1 of $T = Y \otimes_{\mathbb{Z}} \mathbb{C}^\times$ by $\sum_{i=1}^n e_i \otimes -1 \in T^W$. This produces a new torus T', and the algebra $\mathcal{H}$ can be presented as $\mathcal{H}(T', \lambda', \lambda^{*'}, \mathbf{q})$. Here $\lambda'(\alpha) = \lambda^*(\alpha), \lambda^{*'}(\alpha) = \lambda(\alpha)$ and $\lambda' = \lambda, \lambda^{*'} = \lambda^*$ on $R \setminus W\alpha$. This translates to a Bernstein presentation $\mathcal{H}(\mathcal{R}, \lambda', \lambda^{*'}, \mathbf{q})$ of $\mathcal{H}$, with $\lambda^{*'}(\beta) > \lambda'(\beta)$ for fewer $\beta \in R$ than before. Repeating the procedure, we can achieve $\lambda(\alpha) \geq \lambda^*(\alpha)$ for all $\alpha \in R$. $\square$

3.5. Analogues for graded Hecke algebras

Motivated by Corollary 3.30 we investigate all finite dimensional representations of graded Hecke algebras. The representation theory of graded Hecke algebras has been developed together with that of affine Hecke algebras, and they are very similar. As far as the topics in this section are concerned, these two kinds of algebras behave analogously. Instead of translating the Sections 3.1–3.4 to graded Hecke algebras, we will be more sketchy here, just providing the necessary definitions and pointing out the analogies. For more background and proofs we refer to [8,25,47,78,80].

Compared to Section 1.5, we assume in addition that W is a crystallographic Weyl group. Equivalently, the data $\mathfrak{t}, \mathfrak{a}, R, W, S$ for a graded Hecke algebra now come from a based root datum $\mathcal{R} = (X, R, Y, R^\vee, \Delta)$. Let $k : R \to \mathbb{C}$ be a W-invariant parameter function and consider the algebra $\mathbb{H} = \mathbb{H}(\mathfrak{t}, W, k)$.

A parabolic subalgebra of $\mathbb{H}$ is by definition of the form $\mathbb{H}^P = \mathbb{H}(\mathfrak{t}, W_P, k)$ for a subset $P \subset \Delta$. The associated "semisimple" quotient algebra is $\mathbb{H}_P = \mathbb{H}(\mathfrak{t}_P, W_P, k)$. The relation between these two subquotients of $\mathbb{H}$ is simple:

$$\mathbb{H}^P = \mathbb{H}_P \otimes_{\mathbb{C}} S(\mathfrak{t}^{P*}) = \mathbb{H}_P \otimes_{\mathbb{C}} \mathcal{O}(\mathfrak{t}^P).$$

In particular every irreducible representation of $\mathbb{H}^P$ is of the form $\pi_P \otimes \lambda$ for unique $\pi_P \in \mathrm{Irr}(\mathbb{H}_P)$ and $\lambda \in \mathfrak{t}^P$. Parabolic induction for $\mathbb{H}$ is the functor $\mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H}}$.

Lemma 3.32 (*[6, Theorems 2.5.b and 2.11.b]*)**.** *The equivalences of categories from Theorems 3.28 and 3.29 and Corollary 3.30 are compatible with parabolic induction.*

Finite dimensional $\mathbb{H}$-modules have weights with respect to the commutative subalgebra $S(\mathfrak{t}^*) = \mathcal{O}(\mathfrak{t})$. With Theorem 3.29 one can translate many notions for $\mathcal{H}$-modules to $\mathbb{H}$-modules. In terms of weights, this boils down to replacing a subset $U \subset T$ by $\exp^{-1}(U) \subset \mathfrak{t}$. More concretely, we say that a finite dimensional $\mathbb{H}$-representation π is

- tempered if $\mathrm{Wt}(\pi) \subset i\mathfrak{a} + \mathfrak{a}^-$,
- essentially tempered if $\mathrm{Wt}(\pi) \subset \mathfrak{t}^\Delta + i\mathfrak{a} + \mathfrak{a}^-$,
- discrete series if $\mathrm{Wt}(\pi) \subset i\mathfrak{a} + \mathfrak{a}^{--}$,
- essentially discrete series if $\mathrm{Wt}(\pi) \subset \mathfrak{t}^\Delta + i\mathfrak{a} + \mathfrak{a}^{--}$.

Here $V + \mathfrak{a}^{--}$ (for any $V \subset \mathfrak{t}$) is considered as empty when $\mathfrak{a}^{--} = \emptyset$.

Then a Langlands datum for $\mathbb{H}$ is a triple (P, τ, λ), where $P \subset \Delta$, $\tau \in \mathrm{Irr}(\mathbb{H}_P)$ is tempered and $\lambda \in i\mathfrak{a}^P + \mathfrak{a}^{P++}$. Then $\mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H}}(\tau \otimes \lambda)$ is called a standard $\mathbb{H}$-module. With these definitions, the Langlands classification (as in Theorem 3.13 and Lemma 3.15) holds true for graded Hecke algebras [25].

As expected, and proven in [6, Theorem 2.11.d], the equivalence of categories between $\mathrm{Mod}_{f,V}(\mathbb{H}(\mathfrak{t}, W, k_u))$ and $\mathrm{Mod}_{f,\exp_u(V)}(\mathcal{H})$ resulting from Theorem 3.29 and Proposition 3.27 preserves temperedness and (essentially) discrete series. Combining that with Theorem 3.28, we find:

Lemma 3.33. *Assume that (3.8) holds. The equivalence of categories between* $\mathrm{Mod}_{f,Wu\exp(\mathfrak{a})}(\mathcal{H})$ *and* $\mathrm{Mod}_{f,\mathfrak{a}}(\mathbb{H}(\mathfrak{t}, W(R_u), k_u) \rtimes \Gamma_u)$ *from Corollary 3.30 preserves temperedness and (essentially) discrete series.*

From now on we assume that $k : R \to \mathbb{C}$ has values in $\mathbb{R}$, so that $\mathbb{H}$ is related to an affine Hecke algebra with parameters in $\mathbb{R}_{>0}$. As induction data for $\mathbb{H}$ we take triples $\tilde{\xi} = (P, \delta, \lambda)$ where $P \subset \Delta$, $(\delta, V_\delta) \in \mathrm{Irr}(\mathbb{H}_P)$ is discrete series and $\lambda \in \mathfrak{t}^P$. The space of such triples (with δ considered up to isomorphism) is denoted $\tilde{\Xi}$. The parabolically induced representation attached to an induction datum is

$$\pi(P, \delta, \lambda) = \mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H}}(\delta \otimes \lambda).$$

Lemma 3.34. *Let* $\tilde{\xi} = (P, \delta, \lambda) \in \tilde{\Xi}$.

(a) $\pi(\tilde{\xi})$ *is tempered if and only if* $\lambda \in i\mathfrak{a}^P$. *In that case* $\pi(\tilde{\xi})$ *is completely reducible.*
(b) Let $W_P\mathrm{cc}(\delta)$ *be the central character of* δ. *The central character of* $\pi(\tilde{\xi})$ *is* $W(\mathrm{cc}(\delta)+\lambda)$. *It lies in* $\mathfrak{a}/W$ *if and only if* $\lambda \in \mathfrak{a}^P$.

Proof. We give the arguments for these statements in the terms of affine Hecke algebras. From there they can be translated to graded Hecke algebras with Section 3.4.
(a) follows from Propositions 3.8 and 3.20.a.
(b) The expression for the central character comes from Lemma 3.3. Since k is real-valued and δ is discrete series, $W_P\mathrm{cc}(\delta)$ lies in $\mathfrak{a}_P/W_P$ [76, Lemma 2.13]. These two facts imply the second statement. □

The collection of intertwining operators between the representation $\pi(\tilde{\xi})$ with $\tilde{\xi} \in \tilde{\Xi}$ is simpler than for affine Hecke algebras, because $\mathfrak{t}^P \cap \mathfrak{t}_P = \{0\}$ does not contribute to it. There is a groupoid $\mathcal{W}_{\tilde{\Xi}}$ over Δ, with

$$\mathcal{W}_{\tilde{\Xi},PP'} = \{w \in W : w(P) = P'\}.$$

To every $w \in \mathcal{W}_{\tilde{\Xi},PP'}$ one can associate an algebra isomorphism

$$\begin{array}{llll} \psi_w : & \mathbb{H}^P & \to & \mathbb{H}^{w(P)} \\ & x \otimes w' & \mapsto & w(x) \otimes ww'w^{-1} \qquad x \in \mathfrak{t}^*, w' \in W_P. \end{array}$$

Then $w(\tilde{\xi}) = (w(P), \delta \circ \psi_w^{-1}, w(\lambda))$ is another induction datum, and there is an intertwining operator

$$I_w : \pi(P, \delta, \lambda) \to \pi(w(P), \delta \circ \psi_w^{-1}, w(\lambda)).$$

The latter is rational as a function of $\lambda \in \mathfrak{t}^P$ and comes from

$$\begin{array}{ccc} \big(Q(S(\mathfrak{t}^*))^W \otimes_{S(\mathfrak{t}^*)^W} \mathbb{H}\big) \otimes_{\mathbb{H}^P} V_\delta & \to & \big(Q(S(\mathfrak{t}^*))^W \otimes_{S(\mathfrak{t}^*)^W} \mathbb{H}\big) \otimes_{\mathbb{H}^{w(P)}} V_\delta \\ h \otimes v & \mapsto & h\tilde{\imath}_w \otimes v \end{array},$$

where $\tilde{\imath}_w$ is as in Proposition 1.14. We call an induction datum $\tilde{\xi} = (P, \delta, \lambda)$ positive if $\lambda \in i\mathfrak{a}^P + \mathfrak{a}^{P+}$, and we define

$$P(\tilde{\xi}) = \{\alpha \in \Delta : \Re\langle \alpha, \lambda \rangle = 0\}.$$

Using these notions, the whole of Section 3.3 holds for graded Hecke algebras. see [80].

Example 3.35. Consider $\mathfrak{a} = \mathfrak{a}^* = \mathbb{R}, \mathfrak{t} = \mathfrak{t}^* = \mathbb{C}, R = \{\pm 1\}, \Delta = \{\alpha = 1\}, W = \langle s_\alpha \rangle$. The graded Hecke algebra $\mathbb{H} = \mathbb{H}(\mathfrak{t}, W, k)$ with $k(\alpha) = k > 0$ has a unique discrete series representation. It is the analogue of the Steinberg representation, given in this setting by $\mathrm{St}|_{\mathcal{O}(\mathfrak{t})} = \mathbb{C}_{-k}$ and $\mathrm{St}|_{\mathbb{C}[W]} = \mathrm{sign}$.

Apart from $(\emptyset, \mathrm{St}, 0)$, the positive induction data are $(\emptyset, \mathrm{triv}, \lambda)$ with $\lambda \in i\mathfrak{a} + \mathfrak{a}^+ = i\mathbb{R} + \mathbb{R}_{\geq 0}$. For $\lambda \neq k$, $\pi(\emptyset, \mathrm{triv}, \lambda) = \mathrm{ind}_{\mathcal{O}(\mathfrak{t})}^{\mathbb{H}}$ is irreducible, while $\pi(\emptyset, \mathrm{triv}, k)$ has the "trivial representation" as unique irreducible quotient. It is given by $\mathrm{triv}|_{\mathcal{O}(\mathfrak{t})} = \mathbb{C}_k$ and $\mathrm{triv}|_{\mathbb{C}[W]} = \mathrm{triv}$.

4. Classification of irreducible representations

As in Section 3.3, we work with an affine Hecke algebra $\mathcal{H} = \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})$ where $\mathbf{q} > 1$ and λ, λ^* are real-valued. We abbreviate $\mathbb{H} = \mathbb{H}(\mathfrak{t}, W, k)$ where $W = W(R), \mathfrak{t} = \mathrm{Lie}(T)$ and k is a real-valued parameter function.

In Theorem 3.24 we reduced the classification of irreducible $\mathcal{H}$-representations to a little combinatorics with a groupoid $\mathcal{W}_\Xi$ and two substantial subproblems:

- classify the irreducible discrete series representations δ of the parabolic subquotient algebras $\mathcal{H}_P$ (modulo the action of $T^P \cap T_P$ via the automorphisms $\psi_{P,u}$),
- determine the irreducible quotients of $\pi(\xi)$ for $\xi = (P, \delta, t) \in \Xi^+$.

In this section we address both these issues.

4.1. Analytic R-groups

By Proposition 3.20.a–c the second subproblem above is equivalent to classifying the irreducible summands of the completely reducible $\mathcal{H}^{P(\xi)}$-representation

$$\pi^{P(\xi)}(\xi) := \mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}^{P(\xi)}}(\delta \circ \psi_t).$$

For that we have to analyse $\mathrm{End}_{\mathcal{H}^{P(\xi)}}(\pi^{P(\xi)}(\xi))$, which by Proposition 3.20.e boils down to investigating $\mathrm{End}_{\mathcal{H}}(\pi(\xi))$.

For $\xi = (P, \delta, t) \in \Xi^+$ we let $\mathcal{W}_\xi$ be the subgroup of $\mathcal{W}_{\Xi,PP}$ that stabilizes ξ (up to isomorphism of induction data). From Theorem 3.23 we know that the intertwining operators $\pi(w, \xi)$ with $w \in \mathcal{W}_\xi$ span $\mathrm{End}_{\mathcal{H}}(\pi(\xi))$, but they need not be linearly independent. Knapp and Stein exhibited a subgroup $\mathfrak{R}_\xi$ of $\mathcal{W}_\xi$ such that the $\pi(w, \xi)$ with $w \in \mathfrak{R}_\xi$ do form a basis of $\mathrm{End}_{\mathcal{H}}(\pi(\xi))$.

Let R_P^+ be the set of positive roots in R_P, with respect to the basis P. Suppose that $\alpha \in R^+ \setminus R_P^+$ and that $P \cup \{\alpha\}$ is a basis of a parabolic root subsystem $R_{P \cup \{\alpha\}}$ of R. Then we put

$$\alpha^P = \alpha|_{\mathfrak{a}^{P*}} \qquad \text{and} \qquad c_\alpha^P = \prod\nolimits_{\beta \in R^+_{P \cup \{\alpha\}}} c_\beta \in \mathbb{C}(X).$$

We note that c_α^P is W_P-invariant because W_P stabilizes R_P and does not make positive roots outside R_P negative. Let $\delta \in \mathrm{Irr}(\mathcal{H}_P)$ be discrete series, with central character $\mathrm{cc}(\delta) = W_P r$.

Then $t \mapsto c_\alpha^P(rt)$ is a rational function on T^P, independent of the choice of the representative r for cc(δ). For $\xi = (P, \delta, t)$ we consider the following subset of $\mathfrak{a}^{P*}$:

$$R_\xi^+ = \big\{\alpha^P : \alpha \in R^+ \setminus R_P^+ \text{ as above, } c_\alpha^P \text{ has a non-removable pole at } rt\big\}.$$

The set $R_\xi = R_\xi^+ \cup -R_\xi^+$ generalizes the root system R_t from (3.18).

Proposition 4.1 (*[23, Proposition 4.5]*).

(a) R_ξ is a reduced root system in $\mathfrak{a}^{P*}$,
(b) the Weyl group $W(R_\xi)$ is naturally a normal subgroup of $\mathcal{W}_\xi$.

The group $\mathcal{W}_\xi$ acts naturally on $\mathfrak{a}^{P*}$ and stabilizes all the data used to construct R_ξ, so it also acts naturally on R_ξ. We define

$$\mathfrak{R}_\xi = \{w \in \mathcal{W}_\xi : w(R_\xi^+) = R_\xi^+\}. \tag{4.1}$$

As $W(R_\xi)$ acts simply transitively on the collection of positive systems of R_ξ:

$$\mathcal{W}_\xi = W(R_\xi) \rtimes \mathfrak{R}_\xi. \tag{4.2}$$

Theorem 4.2. *Let $\xi \in \Xi^+$.*

(a) For $w \in \mathcal{W}_\xi$, the intertwining operator $\pi(w, \xi)$ is scalar if and only if $w \in W(R_\xi)$.
(b) There exists a 2-cocycle $\natural_\xi : \mathfrak{R}_\xi \times \mathfrak{R}_\xi \to \mathbb{C}^\times$ (depending on the normalization of the operators $\pi(w, \xi)$ with $w \in \mathcal{W}_\xi$) such that

$$\mathrm{End}_{\mathcal{H}}(\pi(\xi)) = \mathrm{span}\{\pi(w, \xi) : w \in \mathfrak{R}_\xi\}$$

is isomorphic to the twisted group algebra $\mathbb{C}[\mathfrak{R}_\xi, \natural_\xi]$. The multiplication in $\mathbb{C}[\mathfrak{R}_\xi, \natural_\xi]$ is as in (3.16).
(c) Given the normalization of these intertwining operators, we write

$$\pi^{P(\xi)}(\xi, \rho) = \mathrm{Hom}_{\mathbb{C}[\mathfrak{R}_\xi, \natural_\xi]}\big(\rho, \pi^{P(\xi)}(\xi)\big).$$

There are bijections

$$\begin{array}{ccccc} \mathrm{Irr}(\mathbb{C}[\mathfrak{R}_\xi, \natural_\xi]) & \to & \{\textit{irreducible summands of} & \to & \{\textit{irreducible quotients of} \\ & & \pi^{P(\xi)}(\xi), \ \textit{up to isomorphism}\} & & \pi(\xi), \ \textit{up to isomorphism}\} \\ \rho & \mapsto & \pi^{P(\xi)}(\xi, \rho) & \mapsto & L\big(P(\xi), \pi^{P(\xi)}(\xi, \rho)\big) \end{array}$$

Proof. For $\xi = (P, \delta, t)$ with $t \in T_{\mathrm{un}}^P$, all this (and more) was shown in [23, Theorems 5.4 and 5.5]. Using Proposition 3.20.a, the same proofs work for $\pi^{P(\xi)}(P, \delta, t)$ with any $\xi \in \Xi^+$, they show the theorem on the level of $\mathcal{H}^{P(\xi)}$. Finally, we apply Proposition 3.20.c,e. □

Example 4.3.

- $\mathcal{R}$ of type $\widetilde{A_1}$, $\mathcal{H} = \mathcal{H}(\mathcal{R}, \mathbf{q})$. The root system R_ξ is nonempty only for $\xi = (\emptyset, \mathrm{triv}, 1)$, and in that case $R_\xi = R, \mathcal{W}_\xi = W = W(R_\xi), \mathfrak{R}_\xi = 1$ and $\pi(\xi) = \mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_1)$ is irreducible.
  The R-group R_ξ is nontrivial only for $\xi = (\emptyset, \mathrm{triv}, -1)$, and in that case $R_\xi = \emptyset, \mathcal{W}_\xi = \mathfrak{R}_\xi = W$. Further $\pi(\xi) = \mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_{-1})$ is reducible and $\mathrm{End}_{\mathcal{H}}(\pi(\xi)) \cong \mathbb{C}[\mathfrak{R}_\xi]$. With the appropriate normalization of the intertwining operator $\pi(s_\alpha, \xi)$, we have

$$\pi(\xi, \rho = \mathrm{triv}) = \pi(-1, \mathrm{triv}), \quad \pi(\xi, \rho = \mathrm{sign}) = \pi(-1, \mathrm{St}).$$

- $\mathcal{R} = \mathcal{R}_n, \mathcal{H} = \mathcal{H}_n(\mathbf{q})$. For all $\xi = (P, \delta, t) \in \Xi$,
$$\mathcal{W}_\xi = \{w \in S_n : w(P) = P, w(t) = t\} = W(R_\xi)$$
and $\mathfrak{R}_\xi = 1$. Hence
$$\mathrm{End}_{\mathcal{H}^{P(\xi)}}(\pi^{P(\xi)}(\xi)) \cong \mathrm{End}_{\mathcal{H}}(\pi(\xi)) \cong \mathbb{C}$$
and $\pi(\xi)$ has only one irreducible quotient (as we already saw in several ways).
- $\mathcal{R}$ arbitrary, $\lambda = \lambda^* = 0, \mathcal{H} = \mathcal{H}(\mathcal{R}, 1) = \mathbb{C}[X \rtimes W]$. The only discrete series representation of a parabolic subquotient algebra of this $\mathcal{H}$ is the trivial representation of $\mathcal{H}_\emptyset = \mathbb{C}$. Hence
$$\Xi = \{(\emptyset, \mathrm{triv}, t) : t \in T\}.$$
Further $c_\alpha = 1$ for all $\alpha \in R$, so R_ξ is empty for all $\xi \in \Xi$. As $T^\emptyset \cap T_\emptyset = T_\emptyset = \{1\}$,
$$\mathcal{W}_\xi = \mathcal{W}_{(\emptyset,\mathrm{triv},t)} = W_t = R_\xi .$$
Here $\mathrm{End}_{\mathcal{H}}(\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t)) \cong \mathbb{C}[W_t]$ acts on the vector space $\mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_t) \cong \mathbb{C}[W]$ as the induction, from W_t to W of the right regular representation of W_t, as can be inferred from (3.13).

The last example shows that R-groups can be as complicated as W itself. This is in sharp contrast with the situations for real reductive groups and for classical p-adic groups, where all R-groups are abelian 2-groups. In all examples that we are aware of, the 2-cocycle $\natural_\xi$ of $\mathfrak{R}_\xi$ is a coboundary, so that $\mathbb{C}[\mathfrak{R}_\xi, \natural_\xi]$ is isomorphic to $\mathbb{C}[\mathfrak{R}_\xi]$. It would be interesting to know whether or not this is always true for affine Hecke algebras.

By Proposition 3.20.b, the $\mathcal{H}$-module
$$\begin{aligned}\mathrm{ind}_{\mathcal{H}^{P(\xi)}}^{\mathcal{H}}(\pi^{P(\xi)}(\xi, \rho)) = \mathrm{ind}_{\mathcal{H}^{P(\xi)}}^{\mathcal{H}}\big(\mathrm{Hom}_{\mathbb{C}[\mathfrak{R}_\xi,\natural_\xi]}(\rho, \pi^{P(\xi)}(\xi))\big) = \\ \mathrm{Hom}_{\mathbb{C}[\mathfrak{R}_\xi,\natural_\xi]}\big(\rho, \mathrm{ind}_{\mathcal{H}^{P(\xi)}}^{\mathcal{H}}\pi^{P(\xi)}(\xi)\big) = \mathrm{Hom}_{\mathbb{C}[\mathfrak{R}_\xi,\natural_\xi]}\big(\rho, \pi(\xi)\big) = \pi(\xi, \rho)\end{aligned} \tag{4.3}$$
from Theorem 4.2.c is standard. Its unique irreducible quotient is $L(P(\xi), \pi^{P(\xi)}(\xi, \rho))$. By Proposition 3.20.c,e, every standard $\mathcal{H}$-module is of the form (4.3), for some $\xi \in \Xi^+$ and $\rho \in \mathrm{Irr}(\mathbb{C}[\mathfrak{R}_\xi, \natural_\xi])$.

When $\xi, \xi' \in \Xi^+, \rho \in \mathrm{Irr}(\mathbb{C}[\mathfrak{R}_\xi, \natural_\xi])$ and $w \in \mathcal{W}_\Xi$ with $w(\xi) \cong \xi'$, we can define $\rho' \in \mathrm{Irr}(\mathbb{C}[\mathfrak{R}_{w(\xi)}, \natural_{w(\xi)}])$ by
$$\rho'\big(\pi(w', \xi')\big) := \rho\big(\pi(w, \xi)^{-1}\pi(w', \xi')\pi(w, \xi)\big).$$
Although $\pi(w, \xi)$ is only defined up to a scalar, the formula for ρ' is independent of the choice of a normalization.

We denote this ρ' by $w(\rho)$, and we say that (ξ, ρ) and $(w(\xi), w(\rho))$ are $\mathcal{W}_\Xi$-associate. It is not clear whether this comes from a groupoid action on a set containing all (ξ, ρ) as above, because $\mathcal{W}_\Xi$ does not stabilize the set of positive induction data Ξ^+ and we did not define R-groups for non-positive induction data.

Let us summarize some properties of standard $\mathcal{H}$-modules.

Corollary 4.4. *Write $\Xi_e^+ = \{(\xi, \rho) : \xi \in \Xi^+, \rho \in \mathrm{Irr}(\mathbb{C}[\mathfrak{R}_\xi, \natural_\xi])\}$.*

(a) The set of standard $\mathcal{H}$-modules (up to isomorphism) is parametrized by Ξ_e^+ up to $\mathcal{W}_\Xi$-association.
(b) Every standard $\mathcal{H}$-module has a unique irreducible quotient.

(c) For every irreducible $\mathcal{H}$-representation π there is a unique (up to isomorphism) standard $\mathcal{H}$-module that has π as quotient.

(d) There are bijections

$$\begin{array}{ccccc} \Xi_e^+/\mathcal{W}_\Xi & \to & \{\text{standard } \mathcal{H}\text{-modules}\} & \to & \mathrm{Irr}(\mathcal{H}) \\ (\xi,\rho) & \mapsto & \pi(\xi,\rho) & \mapsto & L(P(\xi), \pi^{P(\xi)}(\xi,\rho)). \end{array}$$

Proof. (a) follows from Theorem 3.23.
(b) is already contained in Theorem 4.2.
(c) Recall from the remark after Theorem 3.24 that π determines a unique parabolically induced representation $\pi(\xi)$, with $\xi \in \Xi^+$, that has π as quotient.
(d) is a consequence of parts (a), (b) and (c). □

Corollary 4.4 provides a classification of $\mathrm{Irr}(\mathcal{H})$ in terms of induction data and R-groups. The role of the R-groups is quite subtle, firstly because it can be hard to determine them, secondly because a potentially non-trivial 2-cocycle can be involved in the $\mathcal{H}$-endomorphism algebra of a parabolically induced representation.

Sometimes it is easier to work with standard modules than with irreducible representations, as their structure is more predictable.

Example 4.5.

- For $\mathcal{H}(\mathcal{R}, \mathbf{q})$ with $\mathcal{R}$ of type $\widetilde{A_1}$, almost all standard modules are irreducible. The only reducible standard $\mathcal{H}$-module is $\pi(\emptyset, \mathrm{triv}, \mathbf{q}) = \mathrm{ind}_{\mathbb{C}[X]}^{\mathcal{H}}(\mathbb{C}_\mathbf{q})$, which has triv as irreducible quotient.
- For $\mathcal{H}_n(\mathbf{q})$ all the groups $\mathfrak{R}_\xi$ are trivial, so the standard modules are just the parabolically induced representations $\pi(\xi)$ with $\xi \in \Xi^+$.
- When $\mathcal{H} = \mathcal{H}(\mathcal{R}, 1) = \mathbb{C}[X \rtimes W]$ (any $\mathcal{R}$), there is a standard module $\pi(\emptyset, \mathrm{triv}, t, \rho)$ for every $t \in T$ and $\rho \in \mathrm{Irr}(W_t)$. In the notation from Section 2.1 it equals $\pi(t, \rho^*)$. In view of Theorem 2.2, these $\mathcal{H}$-representations are irreducible and there are no other standard modules.

To the best of our knowledge, the theory of R-groups for graded Hecke algebras has never been written down explicitly. It can be deduced readily from [23] and the algebra isomorphisms from Theorem 3.29. In this setting the c_α-function for one root $\alpha \in R$ becomes

$$\tilde{c_\alpha}(\lambda) = \frac{\langle \alpha, \lambda \rangle + k(\alpha)}{\langle \alpha, \lambda \rangle} \qquad \lambda \in \mathfrak{t}.$$

For every $\tilde{\xi} = (P, \delta, \lambda) \in \tilde{\Xi}^+$ we obtain an R-group $\mathfrak{R}_{\tilde{\xi}}$ and a 2-cocycle $\natural_{\tilde{\xi}}$ such that

$$\mathrm{End}_{\mathbb{H}}(\pi(\tilde{\xi})) \cong \mathbb{C}[\mathfrak{R}_{\tilde{\xi}}, \natural_{\tilde{\xi}}]. \tag{4.4}$$

We will see in Lemma 6.12 that $\natural_{\tilde{\xi}}$ is always trivial, except maybe for some instances with root systems of type F_4. However, this triviality does not automatically generalize to affine Hecke algebras (as in Theorem 4.2.b) because the reduction to graded Hecke algebras could pick up groups Γ_t in Theorem 3.28.

The standard module associated to $\tilde{\xi} \in \tilde{\Xi}^+$ and $\rho \in \mathrm{Irr}(\mathbb{C}[\mathfrak{R}_{\tilde{\xi}}, \natural_{\tilde{\xi}}])$ is

$$\pi(\tilde{\xi}, \rho) = \mathrm{Hom}_{\mathbb{C}[\mathfrak{R}_{\tilde{\xi}}, \natural_{\tilde{\xi}}]}(\rho, \pi(\tilde{\xi})). \tag{4.5}$$

Now Theorem 4.2 and Corollary 4.4 apply to $\mathbb{H}$, and they provide bijections

$$\begin{array}{ccccc} \big\{(\tilde\xi,\rho) : \tilde\xi \in \tilde\Xi^+, \rho \in \mathrm{Irr}(\mathbb{C}[\mathfrak{R}_{\tilde\xi}, \natural_{\tilde\xi}])\big\} / \mathcal{W}_{\tilde\Xi} & \to & \{\text{standard } \mathbb{H}\text{-modules}\} & \to & \mathrm{Irr}(\mathbb{H}) \\ (\tilde\xi,\rho) & \mapsto & \pi(\tilde\xi,\rho) & \mapsto & \text{irreducible quotient of } \pi(\tilde\xi,\rho). \end{array} \tag{4.6}$$

4.2. Residual cosets

The most significant step towards the classification of discrete series $\mathcal{H}$-representations is the determination of their central characters, which was achieved by Opdam [63]. Consider the following rational function on T:

$$c_R = \prod_{\alpha \in R} c_\alpha = \prod_{\alpha\in R} \frac{\big(\theta_\alpha - \mathbf{q}^{(-\lambda^*(\alpha)-\lambda(\alpha))/2}\big)\big(\theta_\alpha + \mathbf{q}^{(\lambda^*(\alpha)-\lambda(\alpha))/2}\big)}{(\theta_\alpha - 1)(\theta_\alpha + 1)}. \tag{4.7}$$

Its counterpart for $\mathbb{H}(\mathfrak{t}, W, k)$ is

$$\tilde c_R = \prod_{\alpha\in R} \tilde c_\alpha = \prod_{\alpha \in R} (\alpha + k(\alpha))\alpha^{-1}.$$

Definition 4.6. Let $L \subset T$ be a coset of a complex algebraic subtorus of T. We call L a residual coset (with respect to R and q) if the zero order of c_R along L is at least the (complex) codimension of L in T.

An affine subspace $\mathfrak{l} \subset \mathfrak{a}$ is called a residual subspace (with respect to R and k) if the zero order of $\tilde c_R$ along $\mathfrak{l}$ is at least the (real) codimension of $\mathfrak{l}$ in $\mathfrak{a}$.

A residual point is a residual coset/subspace of dimension zero.

For any L or $\mathfrak{l}$ as above, its zero order is always at most its codimension [63, Corollary A.12]. Hence we may replace "is at least" by "equals" in Definition 4.6. This also implies that residual points can only exist if R spans $\mathfrak{a}^*$. The collection of residual cosets/subspaces is stable under W, because R and q/k are so.

Example 4.7.

- T itself is always a residual coset, and $\mathfrak{a}$ itself is always a residual subspace.
- There are no residual points for $\mathcal{H}$ (resp. for $\mathbb{H}$) if $R \neq \emptyset$ and $\lambda = \lambda^* = 0$ (resp. $k = 0$).
- Consider $\mathfrak{a} = \mathfrak{a}^* = \{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$, $R = A_{n-1} = \{e_i - e_j : i \neq j\}$, $W = S_n$, $k(\alpha) = k \in \mathbb{R}^\times$. There is just one S_n-orbit of residual points for $\mathbb{H}$, and it contains

  $$(k(1-n)/2, k(3-n)/2, \ldots, k(n-1)/2).$$

  This point is the $\mathcal{O}(\mathfrak{t})$-character of the Steinberg representation of $\mathbb{H}$, which by definition is onedimensional and restricts on $\mathbb{C}[W]$ to the sign representation.
- Take $\mathfrak{a} = \mathfrak{a}^* = \mathbb{R}^2$, $R = B_2 = \{\pm e_1 \pm e_2\} \cup \{\pm e_i\}$, $W = W(B_2) \cong D_4$, $k(\pm e_1 \pm e_2) = k_1$, $k(\pm e_i) = k_2$. There are at most two W-orbits of residual points for $\mathbb{H}(\mathbb{C}^2, W(B_2), k)$, represented by $(k_1 + k_2, k_2)$ and $(k_1 - k_2, k_2)$. These points are indeed residual if

  $$k_1 k_2 (k_1 + 2k_2)(k_1 + k_2)(k_1 - 2k_2)(k_1 - k_2) \neq 0.$$

The crucial property of residual points is:

Theorem 4.8 (*[63, Lemma 3.31]*). *Let $\delta \in \mathrm{Irr}(\mathcal{H})$ be discrete series. Then all its $\mathbb{C}[X]$-weights are residual points for (R, q). Conversely, if $t \in T$ is a residual point for (R, q), then there exists a discrete series $\mathcal{H}$-representation with central character Wt.*

Example 4.9. Consider the root datum $\mathcal{R}$ of $PGL_n(\mathbb{C})$, with

$$X = \mathbb{Z}^n/\mathbb{Z}(1, 1, \ldots, 1), Y = \{y \in \mathbb{Z}^n : y_1 + \cdots + y_n = 0\},$$

$R = R^\vee = A_{n-1}$ and $W = S_n$. Notice that $T^W \cong \mathbb{Z}/n\mathbb{Z}$, generated by

$$\zeta_n : x \mapsto \exp(2\pi i(x_1 + \cdots + x_n)).$$

For $\mathbf{q} \neq 1$, $\mathcal{H}(\mathcal{R}, \mathbf{q})$ admits a unique $S_n \times T^W$-orbit of residual points, one such point being

$$t_{\mathbf{q}} = \big(\mathbf{q}^{(1-n)/2}, \mathbf{q}^{(3-n)/2}, \ldots, \mathbf{q}^{(n-1)/2}\big).$$

For $\mathbf{q} > 1$, this is the unique $\mathbb{C}[X]$-weight of the Steinberg representation of $\mathcal{H}(\mathcal{R}, \mathbf{q})$–which is defined just like St for $\mathcal{H}_n(\mathbf{q})$ in (2.11) and (2.13). Similarly $\zeta_n t_{\mathbf{q}}$ is the $\mathbb{C}[X]$-weight of the discrete series representation St$\otimes\zeta_n$.

There is a general method to construct residual cosets from residual points for subquotient algebras. Namely, let $P \subset \Delta$ and let $r \in T_P$ be a residual point for $\mathcal{H}_P$. Then $T^P r$ is a residual coset for $\mathcal{H}$ [63, Proposition A.4]. Up to the action of W, every residual coset is of this form.

From that, Theorem 4.8 and Lemma 3.3 we deduce: for any induction datum $(P, \delta, t) \in \Xi$, every weight of $\pi(P, \delta, t)$ lies in a residual coset of the same dimension as T^P.

Now we relate residual cosets to residual subspaces for graded Hecke algebras. By definition every residual coset for $\mathcal{H}$ can be written as $L = u \exp(\lambda) T_L$, where $u \in T_{\mathrm{un}}$, $\lambda \in \mathfrak{a}$ and $T_L \subset T$ is a complex algebraic subtorus.

Proposition 4.10 (*[63, Theorem A.7]*)**.**

(a) With the above notations, $\lambda + \log|T_L| \subset \mathfrak{a}$ is a residual subspace for $\mathbb{H}(\mathfrak{t}, W(R_u), k_u)$.
(b) Every residual subspace for $\mathbb{H}(\mathfrak{t}, W(R_u), k_u)$ arises in this way.
(c) When $T_L = T^P$ and $u \exp(\lambda)$ is a residual point for $\mathcal{H}_P$, $\lambda \in \mathfrak{a}_P$ is a residual point for $\mathbb{H}(\mathfrak{t}_P, W(R_{P,u}), k_u)$. In this case $R_{P,u}$ has the same rank as R_P, namely $|P|$.

Example 4.11. Take $X = Y = \mathbb{Z}^2$, $R = B_2$, $R^\vee = C_2 = \{\pm e_1 \pm e_2\} \cup \{\pm 2e_1, \pm 2e_2\}$ and $\Delta = \{e_1 - e_2, e_2\}$. We write

$$q_1 = \mathbf{q}^{\lambda(\pm e_1 \pm e_2)}, q_2 = \mathbf{q}^{\lambda(\pm e_i)}, q_0 = \mathbf{q}^{\lambda^*(\pm e_i)}.$$

The points $u \in T_{\mathrm{un}}$ with R_u of rank 2 are $(1, 1), (-1, -1), (1, -1)$ and $(-1, 1)$, the last two being W-associate. There are at most 5 W-orbits of residual points, summarized in the following table:

u	$(1, 1)$	$(-1, -1)$	$(1, -1)$
R_u	B_2	B_2	$A_1 \times A_1$
k_u	$\log(q_1), k_2 = \dfrac{\log(q_2 q_0)}{2}$	$\log(q_1), k_2 = \dfrac{\log(q_2 q_0^{-1})}{2}$	$\dfrac{\log(q_2 q_0)}{2}, \dfrac{\log(q_2 q_0^{-1})}{2}$
residual points	$(q_1^{-1} e^{-k_2}, e^{-k_2})$ $(q_1^{-1} e^{k_2}, e^{-k_2})$	$(-q_1^{-1} e^{-k_2}, -e^{-k_2})$ $(-q_1^{-1} e^{k_2}, -e^{-k_2})$	$((q_2 q_0)^{-1/2},$ $-q_2^{-1/2} q_0^{1/2})$

Here we give k_u in terms of its values on a basis of R_u. For generic parameters q_1, q_2, q_0, all the five points of T in the above table are indeed residual, and each of them represents the central character of a unique discrete series representation [79, §5.5].

As a consequence of Proposition 4.10, everything we said before about residual cosets for $\mathcal{H}$ can be translated to residual subspaces for graded Hecke algebras. Combining that with Lemma 3.33, we find that every $\mathcal{O}(\mathfrak{t})$-weight of a discrete series $\mathbb{H}$-representation is a residual point in $\mathfrak{a}$.

Example 4.12. We continue Example 4.11, but now for the graded Hecke algebra $\mathbb{H}$ built from $\mathfrak{a} = \mathfrak{a}^* = \mathbb{R}^2, \mathfrak{t} = \mathfrak{t}^* = \mathbb{C}^2$, $R = B_2, W = W(B_2)$, $\Delta = \{\alpha = e_1 - e_2, \beta = e_2\}$, $k(\pm e_1 \pm e_2) = k_1 > 0, k(\pm e_i) = k_2 > 0$. The residual point $(-k_1 - k_2, -k_2)$ is the $\mathcal{O}(\mathfrak{t})$-character of the Steinberg representation of $\mathbb{H}$, which is discrete series and restricts to the sign character of $\mathbb{C}[W]$. When $k_1 < 2k_2$, the residual point $(k_1 - k_2, -k_2)$ is the $\mathcal{O}(\mathfrak{t})$-character of a onedimensional discrete series $\mathbb{H}$-representation δ. Its restriction to $\mathbb{C}[W]$ is given by $\delta(s_{e_2}) = -1, \delta(s_{e_1 - e_2}) = 1$.

With Theorem 3.29 and Corollary 4.4 we can complete the classification of Irr($\mathbb{H}$). To this end we note that

$$\mathbb{H}_{\{\alpha\}} = \mathbb{H}(\mathbb{C}\alpha, \langle s_\alpha\rangle, k_1) \quad \text{and} \quad \mathbb{H}_{\{\beta\}} = \mathbb{H}(\mathbb{C}\beta, \langle s_\beta\rangle, k_2).$$

These algebras have a unique discrete series representation, namely St. Further

$$\mathfrak{t}^{\{\alpha\}} = \mathbb{C}(e_1 + e_2), \mathfrak{a}^{\{\alpha\}+} = \mathbb{R}_{\geq 0}(e_1 + e_2), \mathfrak{t}^{\{\beta\}} = \mathbb{C}e_1, \mathfrak{a}^{\{\beta\}+} = \mathbb{R}_{\geq 0}e_1$$

and $\mathfrak{a}^+ = \mathbb{R}_{\geq} e_1 + \mathbb{R}_{\geq 0}(e_1 + e_2)$. All the R-groups $\mathfrak{R}_{\tilde{\xi}}$ for $\mathbb{H}$ are trivial, so (4.6) provides a bijection from $\tilde{\Xi}^+/\mathcal{W}_{\tilde{\Xi}}$ to Irr($\mathbb{H}$), where

$$\begin{aligned}\tilde{\Xi}^+ &= \big\{(\emptyset, \text{triv}, \lambda) : \lambda \in i\mathfrak{a} + \mathfrak{a}^+\big\} \cup \big\{(\{\alpha\}, \text{St}, \lambda) : \lambda \in i\mathfrak{a}^{\{\alpha\}} + \mathfrak{a}^{\{\alpha\}+}\big\} \\ &\cup \big\{(\{\beta\}, \text{St}, \beta) : \lambda \in i\mathfrak{a}^{\{\beta\}} + \mathfrak{a}^{\{\beta\}+}\big\} \cup \{\text{St}, \delta\}.\end{aligned}$$

The groupoid $\mathcal{W}_{\tilde{\Xi}}$ consists of the groups

$$\mathcal{W}_{\tilde{\Xi},\emptyset\emptyset} = W, \mathcal{W}_{\tilde{\Xi},\{\alpha\}\{\alpha\}} = \langle s_{e_1+e_2}\rangle, \mathcal{W}_{\tilde{\Xi},\{\beta\}\{\beta\}} = \langle s_{e_1}\rangle, \mathcal{W}_{\tilde{\Xi},\Delta\Delta} = \{\text{id}\}.$$

The action of $\mathcal{W}_{\tilde{\Xi}}$ on $\tilde{\Xi}$ makes some of the $(P, \delta, \lambda) \in \tilde{\Xi}^+$ with $\Re(\lambda) \in \partial(\mathfrak{a}^{P+})$ $\mathcal{W}_{\tilde{\Xi}}$-associate, for instance $(\emptyset, \text{triv}, (1, i))$ and $(\emptyset, \text{triv}, (1, -i))$.

In general it is easy to classify all points $u \in T_{\text{un}}$ for which R_u has full rank in R, in terms of the affine Dynkin diagram of $\mathcal{R}$ [63, Lemma A.8]. Recall that the relation between the representations of $\mathbb{H}(\mathfrak{t}, W(R_u), k_u)$ and of $\mathbb{H}(\mathfrak{t}, W(R_u), k_u) \rtimes \Gamma_u$ is well-understood from Clifford theory. Thus the classification of discrete series $\mathcal{H}$-representation boils down to two tasks:

- classify all residual points for $(\mathfrak{t}, W, k)$, where k is any real-valued parameter function,
- for a given residual point $\lambda \in \mathfrak{a}$, classify the discrete series $\mathbb{H}$-representations with central character $W\lambda$.

In view of the isomorphism (1.26), it suffices to this when R is irreducible. The residual points for $\mathbb{H}(\mathfrak{t}, W, k)$ with such R and k have been classified completely in [30, §4]. They are always linear expressions $f(k)$ in the parameters $k(\alpha)$ for $\alpha \in R$. For a given such f, $f(k)$ is residual with respect to R and k for almost all $k : R \to \mathbb{R}$. We say that a parameter function k is generic if all potentially residual points $f(k)$, for the $\mathbb{H}(\mathfrak{t}, W(R_P), k)$ with $P \subset \Delta$, are really residual for this k, and are all different.

Theorem 4.13 (*[67, Theorems 3.4 and 7.1]*)**.** *Let $R \subset \mathfrak{a}^*$ be an irreducible root system that spans $\mathfrak{a}^*$.*

(a) *Let $k : R \to \mathbb{R}$ be a generic parameter function. The central character map gives a bijection from the set of irreducible discrete series representations of $\mathbb{H}(\mathfrak{t}, W(R), k)$ to the set of W-orbits of residual points for R and k (except when $R \cong F_4$, then one fibre of this map has two elements).*

(b) *For a non-generic parameter function $k' : R \to \mathbb{R}$ and a residual point $\lambda \in \mathfrak{a}$, consider the collection of generic residual points $\{f_i(k)\}_i$ that specialize to ξ at $k' = k$. For k close to k' in the space of all parameter functions $R \to \mathbb{R}$, there is a natural bijection between:*

- *the set of irreducible discrete series representations of $\mathbb{H}(\mathfrak{t}, W(R), k)$ with central character in $\{W f_i(k)\}_i$,*
- *the set of irreducible discrete series representation of $\mathbb{H}(\mathfrak{t}, W(R), k')$ with central character $W\lambda$.*

More explicitly, every $\mathbb{H}(\mathfrak{t}, W(R), k')$-representation of the indicated kind is of part of a unique continuous family of such representations, one representation for each k in some neighbourhood of k'. The above bijection matches the members of such a continuous family at k and at k'.

From Theorem 4.13 one obtains a complete classification of discrete series representations of affine Hecke algebras with positive parameters. However, we have to point out that this does not yet achieve an actual classification of all irreducible representations. The problem is that it can remain difficult to effectively compute the R-groups $\mathfrak{R}_{P,\delta,t}$ and their 2-cocycles $\natural_{P,\delta,t}$ from Section 4.1.

We conclude this section with a discussion of the residual points for $\mathbb{H}$ in the most intricate case, for root systems of type B_n. We take $\mathfrak{a} = \mathfrak{a}^* = \mathbb{R}^n, R = B_n = \{\pm e_i : i = , 1 \ldots, n\} \cup \{\pm e_i \pm e_j : i \neq j\}$ and we write $k(\pm e_i \pm e_j) = k_1, k(\pm e_i) = k_2$. For every partition $\vec{n} = (n_1, n_2, \ldots, n_d)$ of n we construct a point $\lambda(\vec{n}, k) \in \mathfrak{a}$, or rather a S_n-orbit in $\mathfrak{a}$, in the following way. Draw the Young diagram, with first column of n_1 boxes, second column of n_2 boxes and so on. Label the boxes from b_1 to b_n in some way (to does not matter how, for another labelling will produce a point in the same S_n-orbit in $\mathfrak{a}$). We define the height of a box b in column i and row j to be $h(b) = j - i$ and we write

$$\lambda(\vec{n}, k) = (h(b_1)k_1 + k_2, h(b_2)k_1 + k_2, \ldots, h(b_n)k_1 + k_2). \tag{4.8}$$

For example, when $n = 2$ we have

$$\lambda((2), k) = (k_1 + k_2, k_2), \quad \lambda((1, 1), k) = (-k_1 + k_2, k_2)$$

Every residual point for $\mathbb{H}(\mathbb{C}^n, W(B_n), k)$ is $W(B_n)$-associate to a $\lambda(\vec{n}, k)$ [30, §4]. For most parameters k_i indeed all these points of $\mathfrak{a}$ are residual, but not for all parameters. An extreme case is $k_1 = k_2 = 0$, then there are no residual points.

A parameter function $k : B_n \to \mathbb{R}$ is generic if

$$k_1 k_2 \prod_{j=1}^{2(n-1)} (jk_1 + 2k_2)(jk_1 - 2k_2) \neq 0. \tag{4.9}$$

For generic k, all the $\lambda(\vec{n}, k)$ are residual [30, Proposition 4.3] and they belong to different $W(B_n)$-orbits [67, proof of Theorem 7.1]. On the other hand, when k is not generic, some of the $\lambda(\vec{n}, k)$ are not residual, and some of them may belong to the same $W(B_n)$-orbit.

5. Geometric methods

We survey some of results on affine Hecke algebras obtained with methods from complex algebraic geometry. In many cases, these provide a complete classification of standard modules and of irreducible representations.

5.1. Equivariant K-theory

In this paragraph we discuss equal label affine Hecke algebras, that is, with a single parameter $\mathbf{q}$. Recall from Section 1.2 that these algebras are especially important because they classify representations of reductive p-adic groups with vectors fixed by an Iwahori subgroup. Lusztig [49] discovered that such affine Hecke algebras can be realized as the equivariant K-theory of a suitable complex algebraic variety. Then its representations can be analysed in algebro–geometric terms, and that leads to a beautiful construction and parametrization of all irreducible representations.

Let G be a connected complex reductive group with a maximal torus T, and let $\mathcal{R}(G,T)$ be the associated root datum. We define $\mathcal{H}(G,T)$ to be like $\mathcal{H}(\mathcal{R}(G,T),\mathbf{q})$, but with $\mathbf{q}$ replaced by an invertible formal variable $\mathbf{z}$. As vector spaces

$$\mathcal{H}(G,T) = \mathbb{C}[X^*(T)] \otimes_{\mathbb{C}} \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{z},\mathbf{z}^{-1}],$$

where $W = W(G,T)$. Let $\mathcal{B}$ be the variety of Borel subgroups of G, it is isomorphic to G/B for one Borel subgroup B.

The upcoming constructions work best when the derived group G_{der} of G is simply connected, so we assume that in this paragraph (unless explicitly mentioned otherwise). A main role is played by the Steinberg variety of G from [44, §3.3]:

$$\mathcal{Z} := \{(B,u,B') \in \mathcal{B} \times G \times \mathcal{B} : u \in B \cap B' \text{ unipotent}\}.$$

The group $G \times \mathbb{C}^\times$ acts on $\mathcal{Z}$ by

$$(g,\lambda)(B,u,B') = (gBg^{-1}, gu^{\lambda^{-1}}g^{-1}, gB'g^{-1}).$$

Note that $u^{\lambda^{-1}}$ is defined because u is unipotent. This $\mathbb{C}^\times$-action might appear ad hoc, but it is indispensable to obtain Hecke algebras. Without it, we could at best build the G-equivariant K-group $K^G(\mathcal{Z})$, which turns out be isomorphic to $\mathbb{C}[X] \rtimes W$ [16, Theorem 7.2]. According to [44, Theorem 3.5] and [16, Theorem 7.2.5], there is a natural isomorphism

$$K^{G\times\mathbb{C}^\times}(\mathcal{Z}) \cong \mathcal{H}(G,T). \tag{5.1}$$

The $\mathbf{z}$'s in $\mathcal{H}(G,T)$ are due to the $\mathbb{C}^\times$-action on $\mathcal{Z}$. The ring of regular class functions on G is

$$R(G) = \mathcal{O}(G)^G \cong \mathcal{O}(T/W).$$

When we regard $\mathbf{z}$ as the identity representation of $\mathbb{C}^\times$, we can write the rings of regular class functions on $\mathbb{C}^\times$ and on $G \times \mathbb{C}^\times$ as

$$R(\mathbb{C}^\times) = \mathbb{C}[\mathbf{z},\mathbf{z}^{-1}], \qquad R(G\times\mathbb{C}^\times) = \mathcal{O}(G)^G \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{z},\mathbf{z}^{-1}].$$

By construction [16, §5.2], $R(G\times\mathbb{C}^\times)$ acts naturally on $K^{G\times\mathbb{C}^\times}(\mathcal{Z})$. On the other hand, a variation on (1.19) shows that $R(G\times\mathbb{C}^\times)$ is also naturally isomorphic to the centre of $\mathcal{H}(G,T)$.

With these identifications the isomorphism (5.1) is $R(G \times \mathbb{C}^\times)$-linear. For any $\mathbf{q} \in \mathbb{C}^\times$ we can specialize (5.1) to an isomorphism

$$K^{G \times \mathbb{C}^\times}(\mathcal{Z}) \otimes_{\mathbb{C}[\mathbf{z},\mathbf{z}^{-1}]} \mathbb{C}_{\mathbf{q}} \cong \mathcal{H}(\mathcal{R}(G,T),\mathbf{q}). \tag{5.2}$$

Further, let $t \in G$ be a semisimple element and denote the associated onedimensional representation of $R(G \times \mathbb{C}^\times)$ by $\mathbb{C}_{t,\mathbf{q}}$. Let $\mathcal{Z}^{t,q}$ be the subvariety of $\mathcal{Z}$ fixed by $(t,\mathbf{q}) \in G \times \mathbb{C}^\times$. By [16, p. 414] there is an isomorphism

$$K^{G \times \mathbb{C}^\times}(\mathcal{Z}) \otimes_{R(G \times \mathbb{C}^\times)} \mathbb{C}_{t,\mathbf{q}} \cong K(\mathcal{Z}^{t,\mathbf{q}}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H_*(\mathcal{Z}^{t,\mathbf{q}},\mathbb{C}). \tag{5.3}$$

The construction of $K^{G \times \mathbb{C}^\times}(\mathcal{Z})$-modules is performed most naturally with Borel–Moore homology (that is equivalent to the constructions with equivariant K-theory in [44]).

Let $u \in G$ be a unipotent element such that $tut^{-1} = u^{\mathbf{q}}$ and let $\mathcal{B}^{t,u} \subset \mathcal{B}$ be the subvariety of Borel subgroups that contain t and u. The convolution product in Borel–Moore homology [16, Corollary 2.7.42] provides an action of $H_*(\mathcal{Z}^{t,\mathbf{q}},\mathbb{C})$ on $H_*(\mathcal{B}^{t,u},\mathbb{C})$. This and (5.3) make $H_*(\mathcal{B}^{t,u},\mathbb{C})$ into a $K^{G \times \mathbb{C}^\times}(\mathcal{Z})$-module, usually reducible.

By [16, Lemma 8.1.8] these constructions commute with the G-action, in the sense that

$$H_*(\mathrm{Ad}_g)^* : H_*(\mathcal{B}^{t,u},\mathbb{C}) \to H_*(\mathcal{B}^{gtg^{-1},gug^{-1}},\mathbb{C})$$

intertwines the $K^{G \times \mathbb{C}^\times}(\mathcal{Z})$-actions. In particular $Z_G(t,u)$, the centralizer of $\{t,u\}$ in G, acts on $H_*(\mathcal{B}^{t,u},\mathbb{C})$ by $K^{G \times \mathbb{C}^\times}(\mathcal{Z})$-intertwiners. The neutral component of $Z_G(t,u)$ acts trivially, so we may regard it as an action of the component group $\pi_0(Z_G(t,u))$. That can be used to decompose the module $H_*(\mathcal{B}^{t,u},\mathbb{C})$. Let ρ be an irreducible representation of $\pi_0(Z_G(t,u))$ which occurs in $H_*(\mathcal{B}^{t,u},\mathbb{C})$. Then

$$K_{t,u,\rho} := \mathrm{Hom}_{\pi_0(Z_G(t,u))}\big(\rho, H_*(\mathcal{B}^{t,u},\mathbb{C})\big)$$

is a nonzero $K^{G \times \mathbb{C}^\times}(\mathcal{Z})$-module, called standard in [44, 5.12] and [16, Definition 8.1.9]. Since the action factors via (5.3), $K_{t,u,\rho}$ can be regarded as a $\mathcal{H}(\mathcal{R}(G,T),\mathbf{q})$-representation with central character Wt. In view of their role in [44], data (t,u,ρ) with the above properties are usually called Kazhdan–Lusztig triples for $(G,\mathbf{q})$.

Theorem 5.1 (*[44, Theorem 7.12]*)**.** *Recall that G_{der} is simply connected and let $\mathbf{q} \in \mathbb{C}^\times$ be of infinite order.*

(a) For every Kazhdan–Lusztig triple (t,u,ρ), the $\mathcal{H}(\mathcal{R}(G,T),\mathbf{q})$-module $K_{t,u,\rho}$ has a unique irreducible quotient $L_{t,u,\rho}$.
(b) Every irreducible $\mathcal{H}(\mathcal{R}(G,T),\mathbf{q})$-module is of the form $L_{t,u,\rho}$, for a suitable Kazhdan–Lusztig triple.
(c) Let (t',u',ρ') be another Kazhdan–Lusztig triple. Then $L_{t,u,\rho} \cong L_{t',u',\rho'}$ if and only if there exists a $g \in G$ such that $t' = gtg^{-1}$, $u' = gug^{-1}$ and $\rho' = \rho \circ \mathrm{Ad}(g^{-1})$.

This major result, also shown later in a somewhat different way in [16, Theorem 8.1.16], comes with a lot of extras. Firstly, suppose that L is a standard Levi subgroup of G and that it contains $\{t,u\}$. Then

$$K^{L \times \mathbb{C}^\times}(\mathcal{Z}_L) \otimes_{\mathbb{C}[\mathbf{z},\mathbf{z}^{-1}]} \mathbb{C}_{\mathbf{q}} \cong \mathcal{H}(\mathcal{R}(L,T),\mathbf{q})$$

embeds naturally in $\mathcal{H}(\mathcal{R}(G,T),\mathbf{q})$. By [44, Theorem 6.2], provided a certain polynomial function of $(t,\mathbf{q})$ is nonzero:

$$H_*(\mathcal{B}^{t,u},\mathbb{C}) \cong \mathrm{ind}_{\mathcal{H}(\mathcal{R}(L,T),\mathbf{q})}^{\mathcal{H}(\mathcal{R}(G,T),\mathbf{q})} H_*(\mathcal{B}_L^{t,u},\mathbb{C}).$$

For the second extra we suppose that $\mathbf{q} \in \mathbb{R}_{>1}$. By [44, Theorem 8.3] and [3, Proposition 9.3] the $\mathcal{H}(\mathcal{R}(G,T),\mathbf{q})$-module $L_{t,u,\rho}$ is essentially discrete series if and only if $\{t,u\}$ is not contained in any Levi subgroup of any proper parabolic subgroup of G.

With these two extras at hand, we can compare $K_{t,u,\rho}$ with the more analytic approach from Sections 3 and 4. Let L be a Levi subgroup of G which contains $\{t,u\}$ and is minimal for that property. Upon conjugating everything by an element of G, we may assume that L is standard, that $t \in T$ and that $\log|t|$ is as positive as possible in its W-orbit. Let $\rho_L \in \mathrm{Irr}\big(\pi_0(Z_L(t,u))\big)$ be an irreducible constituent of $\rho|_{\pi_0(Z_L(t,u))}$. Then

$$\mathrm{Hom}_{\pi_0(Z_L(t,u))}\big(\rho_L, H_*(\mathcal{B}_L^{t,u},\mathbb{C})\big) \tag{5.4}$$

is an irreducible essentially discrete series $\mathcal{H}(\mathcal{R}(L,T),\mathbf{q})$-representation. By Theorem 3.18 it is of the form $\delta \circ \psi_{t'}$, where δ is discrete series. Then $\xi = (\Delta_L, \delta, t') \in \Xi_+$ by the assumption on t. The induction of (5.4) to $\mathcal{H}(\mathcal{R}(G,T),\mathbf{q})$ contains $K_{t,u,\rho}$ as a direct summand. In view of Theorem 4.2.b, this summand must be picked out by an irreducible representation ρ' of $\mathbb{C}[\mathfrak{R}_\xi, \natural_\xi]$. We conclude that $K_{t,u,\rho} \cong \pi(\xi,\rho')$, a standard module in the sense of Definition 3.12.

This argument also works in the opposite direction, and then it shows that the standard modules from Definition 3.12 are precisely the standard modules from [44] and [16].

Recall that in Theorem 5.1 the complex reductive group G has simply connected derived group G_{der}. Without that assumption on G, $K^{G \times \mathbb{C}^\times}$ behaves less well. Nevertheless, the parametrization of irreducible $\mathcal{H}(\mathcal{R}(G,T),\mathbf{q})$-representations obtained in Theorem 5.1 is valid for any complex reductive group G. This was shown by Reeder [71, Theorem 3.5.4], via reduction to the case with simply connected G_{der}.

When $\mathbf{q}$ is a root of unity, Theorem 5.1 can definitely fail, in particular when $\mathbf{q}$ is a zero of the Poincaré polynomial of W. On the other hand, if $\mathbf{q}$ is not a zero of the polynomials (1.9) for any finite reflection subgroup of $X \rtimes W$, then it seems likely that large parts of the K-theoretic approach are still valid.

A very special case arises when $\mathbf{q} = 1$. Then Theorem 5.1 holds, in a slightly different, simpler way [41]. In this case the action of

$$\mathcal{H}(\mathcal{R}(G,T),1) = \mathbb{C}[X^*(T)] \rtimes W$$

on $K_{t,u,\rho}$ preserves the homological degree, so $\mathrm{Hom}_{\pi_0(Z_G(t,u))}\big(\rho, H_d(\mathcal{B}^{t,u},\mathbb{C})\big)$ is a subrepresentation for any $d \in \mathbb{Z}_{\geq 0}$. Then $K_{t,u,\rho}$ may obviously have many irreducible quotients, so the previous definition of $L_{t,u,\rho}$ cannot be used anymore. Instead we define

$$L_{t,u,\rho} = \mathrm{Hom}_{\pi_0(Z_G(t,u))}\big(\rho, H_{\dim_{\mathbb{R}}(\mathcal{B}^{t,u})}(\mathcal{B}^{t,u},\mathbb{C})\big) \qquad \text{when } \mathbf{q} = 1, \tag{5.5}$$

that is, we only use the homology in the largest possible degree. It can be shown that (5.5) is the canonical irreducible quotient of $K_{t,u,\rho}$ [3, §12]. Thus modified, Theorem 5.1 becomes valid for $\mathbf{q} = 1$ without any restriction on G [41, Theorem 4.1]. Notice that here the triples (t,u,ρ) satisfy $tu = ut$, so u is a unipotent element of $Z_G(t)$. This classification of $\mathrm{Irr}(X \rtimes W)$ can be regarded as a Springer correspondence for affine Weyl groups. The proof is much shorter than that of Theorem 5.1, it mainly relies on the Springer correspondence for finite Weyl groups.

As already observed in [44, §2], there are several equivalent ways to present Kazhdan–Lusztig parameters (for arbitrary complex reductive groups G), when we consider them modulo G-conjugation (as in Theorem 5.1.c). One alternative shows the connection between different $\mathbf{q}$'s very nicely.

Let (t_1, u, ρ_1) be a Kazhdan–Lusztig triple for $(G,1)$. Pick an algebraic homomorphism $\phi : SL_2(\mathbb{C}) \to Z_G(t_1)$ with $\phi\big(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\big) = u$. By the Jacobson–Morozov theorem, such a ϕ exists and is unique up to conjugation by $Z_G(t_1, u)$. Assume that we have a preferred square root of $\mathbf{q}$. We put $t_{\mathbf{q}} = t_1 \phi(\begin{pmatrix} \mathbf{q}^{1/2} & 0 \\ 0 & \mathbf{q}^{-1/2} \end{pmatrix})$, so that $t_{\mathbf{q}} u t_{\mathbf{q}}^{-1} = u^{\mathbf{q}}$. Then $\mathcal{B}^{t_1,u}$ and $\mathcal{B}^{t_{\mathbf{q}},u}$ are homotopy equivalent and ρ_1 gives rise to a unique $\rho_{\mathbf{q}}$ [3, Lemma 6.1]. This provides a bijection between G-conjugacy classes of Kazhdan–Lusztig tripes for $(G, 1)$ and for $(G, \mathbf{q})$ [3, Lemma 7.1]. In combination with Theorem 5.1 we obtain:

Corollary 5.2. *Let G be a complex reductive group and let $\mathbf{q} \in \mathbb{C}^\times$ be either 1 or not a root of unity. There exists a canonical bijection*

$$\begin{array}{ccc} \{\textit{Kazhdan–Lusztig triples for } (G,1)\}/G & \longleftrightarrow & \mathrm{Irr}\big(\mathcal{H}(\mathcal{R}(G,T),\mathbf{q})\big) \\ (t_1, u, \rho_1) & \mapsto & L_{t_{\mathbf{q}},u,\rho_{\mathbf{q}}} \end{array}$$

One advantage of this parametrization is that the pair (t_1, u) is the Jordan decomposition of an arbitrary element of G, so that (t_1, u) up to G-conjugacy parametrizes the conjugacy classes of G.

Example 5.3. Let $\mathcal{R}$ be of type GL_n and consider $\mathcal{H}_n(\mathbf{q})$. As $Z_{GL_n(\mathbb{C})}(t_1, u) = Z_{GL_n(\mathbb{C})}(t_1 u)$ is always connected, ρ is necessarily trivial and may be ignored. Corollary 5.2 recovers the parametrization of $\mathrm{Irr}(\mathcal{H}_n(\mathbf{q}))$ summarized in Theorem 2.6.

In addition to the already discussed properties of these bijections, we mention that temperedness can be detected easily in Corollary 5.2. Namely, by [3, Proposition 9.3], for $\mathbf{q} \in \mathbb{R}_{\geq 1}$:

$$L_{t_{\mathbf{q}},u,\rho_{\mathbf{q}}} \text{ is tempered} \quad \Longleftrightarrow \quad t_1 \text{ lies in a compact subgroup of } G. \tag{5.6}$$

Finally, we mention an interesting extension of the method from [16] to an affine Hecke algebra $\mathcal{H}$ of type B_n/C_n with three unequal parameters [42]. Like in [44] it is shown that $\mathcal{H}$ is isomorphic to the equivariant K-theory of a certain algebraic variety, and that is used to parametrize $\mathrm{Irr}(\mathcal{H})$ when the parameters are generic (in a sense related to Theorem 4.13). We return to this in Section 6.3.

5.2. Equivariant homology

The material in the previous paragraph learns us a lot about equal label affine Hecke algebras, but very little about the cases with several parameters q_s. Faced with this problem, Lusztig discovered that Hecke algebras with multiple parameters can still be studied geometrically, if we accept two substantial modifications:

- replace affine Hecke algebras by their graded versions,
- replace equivariant K-theory by equivariant homology.

Not all combinations of q-parameters can be obtained in this way, but still a considerable number of them. Our treatment of this method is based on the papers [5,48,51,54,55].

Let G be a connected complex reductive group and let P be a parabolic subgroup of G with Levi factor L and unipotent radical U. We denote the Lie algebras of these groups by $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{l}$ and $\mathfrak{u}$. Let $v \in \mathfrak{l}$ be nilpotent and let $\mathcal{C}_v^L$ be its adjoint orbit. Let $\mathcal{L}$ be an irreducible L-equivariant cuspidal local system on $\mathcal{C}_v^L$. We refrain from explaining these notions here, instead we refer to [48], where cuspidal local systems are introduced and classified.

To the above data we will associate a graded Hecke algebra. We take $T = Z(L)^\circ$, a (not necessarily maximal) torus in G. According to [51, Proposition 2.5], the cuspidality of $\mathcal{L}$ implies that:

- the set of weights of T acting on $\mathfrak{g}$ is a (possibly nonreduced) root system $R(G,T)$ in $X^*(T)$,
- the Weyl group of $R(G,T)$ is $W_L := N_G(L)/L = N_G(T)/Z_G(T)$.

The parabolic subgroup P determines a basis Δ_L of $R(G,T)$. Let $\mathfrak{t} = X_*(T) \otimes_{\mathbb{Z}} \mathbb{C}$ be the Lie algebra of T, so that

$$R(G,T) \subset \mathfrak{t}^* = X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}.$$

The action of W_L on T induces actions on $\mathfrak{t}$ and $\mathfrak{t}^*$, which stabilize $R(G,T)$. The definition of the parameter function $k : R(G,T) \to \mathbb{Z}$ involves the nilpotent element v. Let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ be the root space and let $s_\alpha \in W_L$ be the reflection associated to $\alpha \in R(G,T)$. Since $v \in \mathfrak{l}$ commutes with $\mathfrak{t}$, $\mathrm{ad}(v)$ stabilizes each $\mathfrak{g}_\alpha$. For $\alpha \in \Delta_L$ one defines $k(\alpha) \in \mathbb{Z}_{\geq 2}$ by

$$\begin{array}{llcll} \mathrm{ad}(v)^{k(\alpha)-2}: & \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} & \to & \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} & \text{is nonzero,} \\ \mathrm{ad}(v)^{k(\alpha)-1}: & \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} & \to & \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} & \text{is zero.} \end{array}$$

Then $k(\alpha) = k(\beta)$ whenever $\alpha, \beta \in \Delta_L$ are W_L-associate. Now we can define

$$\mathbb{H}(G,L,\mathcal{L},\mathbf{r}) = \mathbb{H}(\mathfrak{t}, W_L, k, \mathbf{r}).$$

Suppose that G is an almost direct product of connected normal subgroups G_1 and G_2. Then $L, \mathcal{C}_v^L$ and $\mathcal{L}$ decompose accordingly and

$$\mathbb{H}(G,L,\mathcal{L},\mathbf{r}) = \mathbb{H}(G_1,L_1,\mathcal{L}_1,\mathbf{r}) \otimes_{\mathbb{C}[\mathbf{r}]} \mathbb{H}(G_2,L_2,\mathcal{L}_2,\mathbf{r}).$$

If G is a torus, then necessarily $L = T = G$ and $v = 0$. In that case $\mathcal{L}$ is trivial and $\mathbb{H}(T,T,\mathcal{L},\mathbf{r})$ is just $\mathcal{O}(\mathfrak{t}) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{r}]$. Hence the study of $\mathbb{H}(G,L,\mathcal{L},\mathbf{r})$ can be reduced to simple G. Then it becomes feasible to classify the data, and indeed this has been done in [51, 2.13]. We tabulate the possibilities for $\mathfrak{g}$, $\mathfrak{l}$, $R(G,T)$ and k:

$\mathfrak{g}$	$\mathfrak{l}$	$R(G,T)$	k	
simple	Cartan	irreducible	$k(\alpha) = 2$	
$\mathfrak{sl}_{(d+1)p}$	$\mathfrak{sl}_p^{d+1} \oplus \mathbb{C}^d$	A_d	$k(\alpha) = 2p$	
$\mathfrak{sp}_{2n+2d}, 2n = p(p+1)$	$\mathfrak{sp}_{2n} \oplus \mathbb{C}^d$	BC_d	$k(\alpha) = 2, k(\beta) = 2p+1$	
$\mathfrak{so}_{n+2d}, n = p^2$	$\mathfrak{so}_n \oplus \mathbb{C}^d$	B_d	$k(\alpha) = 2, k(\beta) = 2p$	(5.7)
$\mathfrak{so}_{n+4d}, n = p(2p-1)$	$\mathfrak{so}_n \oplus \mathfrak{sl}_2^d \oplus \mathbb{C}^d$	BC_d	$k(\alpha) = 4, k(\beta) = 4p-1$	
$\mathfrak{so}_{n+4d}, n = p(2p+1)$	$\mathfrak{so}_n \oplus \mathfrak{sl}_2^d \oplus \mathbb{C}^d$	BC_d	$k(\alpha) = 4, k(\beta) = 4p+1$	
E_6	$\mathfrak{sl}_3^2 \oplus \mathbb{C}^2$	G_2	$k(\alpha) = 2, k(\beta) = 6$	
E_7	$\mathfrak{sl}_2^3 \oplus \mathbb{C}^4$	F_4	$k(\alpha) = 2, k(\beta) = 4$	

In this table $d, p \in \mathbb{Z}_{>0}$ are arbitrary, $\alpha \in R(G,T)$ is a long root and $\beta \in R(G,T)$ is a short root. (For $R(G,T)$ of type BC_d we mean that $\alpha = \pm e_i \pm e_j$ and $\beta = \pm e_i$.)

Recall from (1.22) that we can simultaneously rescale all the $k(\alpha)$ without changing the algebra (up to isomorphism). When $R(G,T)$ has roots of different lengths, we can also adjust the parameters for roots of one length in a specific way:

Example 5.4. Let R be of type B_n, F_4 or G_2. Assume that R spans $\mathfrak{a}^*$. Any parameter function k for R has two independent values $k_1 = k(\alpha)$ and $k_2 = k(\beta)$, which can be chosen arbitrarily. We write $k = (k_1, k_2)$ and we consider the graded Hecke algebra $\mathbb{H}(\mathfrak{t}, W(R), k_1, k_2, \mathbf{r})$.

Take $\epsilon = 2$ for B_n or F_4, and $\epsilon = 3$ for G_2. The set

$$\{w\alpha : w \in W\} \cup \{\epsilon w\beta : w \in W\}$$

is a root system in $\mathfrak{a}^*$, of type (respectively) C_n, F_4 or G_2. Notice that now α is short and $\epsilon\beta$ is long. The identity map on the vector space underlying $\mathbb{H}(\mathfrak{t}, W(R), k_1, k_2, \mathbf{r})$ provides an algebra isomorphism

$$\mathbb{H}(\mathbb{C}^n, W(B_n), k_1, k_2, \mathbf{r}) \to \mathbb{H}(\mathbb{C}^n, W(C_n), 2k_2, k_1, \mathbf{r}) \tag{5.8}$$

$$\mathbb{H}(\mathbb{C}^4, W(F_4), k_1, k_2, \mathbf{r}) \to \mathbb{H}(\mathbb{C}^4, W(F_4), 2k_2, k_1, \mathbf{r}) \tag{5.9}$$

$$\mathbb{H}(\mathbb{C}^2, W(G_2), k_1, k_2, \mathbf{r}) \to \mathbb{H}(\mathbb{C}^2, W(G_2), 3k_2, k_1, \mathbf{r}) \tag{5.10}$$

In particular any graded Hecke algebra of type C_n is also a graded Hecke algebra of type B_n (but with different parameters).

We will call a parameter function obtained from the above table by a composition of the isomorphisms (1.22) and those from Example 5.4 geometric. Thus we have a large supply of geometric parameter functions for type B/C root systems.

Next we describe the geometric realization of $\mathbb{H}(G, L, \mathcal{L}, \mathbf{r})$. We need the varieties

$$\dot{\mathfrak{g}} = \{(x, gP) \in \mathfrak{g} \times G/P : \mathrm{Ad}(g^{-1})x \in \mathcal{C}_v^L + \mathfrak{t} + \mathfrak{u}\},$$
$$\ddot{\mathfrak{g}}_N = \{(x, gP, g'P) \in \mathfrak{g} \times (G/P)^2 : (x, gP) \in \dot{\mathfrak{g}}, (x, g'P) \in \dot{\mathfrak{g}}, x \text{ nilpotent}\}.$$

The first is a variation on the variety $\mathcal{B}$ of Borel subgroups of G, while the second has a flavour of the Steinberg variety of G. The group $G \times \mathbb{C}^\times$ acts on $\dot{\mathfrak{g}}$ by

$$(g_1, \lambda)(x, gP) = (\lambda^{-2}\mathrm{Ad}(g_1)x, g_1 gP),$$

and similarly on $\ddot{\mathfrak{g}}_N$. The $L \times \mathbb{C}^\times$-equivariant local system $\mathcal{L}$ on $\mathcal{C}_v^L$ yields a $G \times \mathbb{C}^\times$-equivariant local system $\dot{\mathcal{L}}$ on $\dot{\mathfrak{g}}$. The two projections $\ddot{\mathfrak{g}}_N \to \dot{\mathfrak{g}}$ give rise to an equivariant local system $\ddot{\mathcal{L}} = \dot{\mathcal{L}} \boxtimes \dot{\mathcal{L}}^*$ on $\ddot{\mathfrak{g}}_N$. Equivariant (co)homology with coefficients in a local system is defined in [51, §1]. In special cases this theory admits a convolution product [54, §2]. That makes $H_*^{G\times\mathbb{C}^\times}(\ddot{\mathfrak{g}}_N, \ddot{\mathcal{L}})$ into a graded algebra, see the proof of [54, Theorem 8.11].

Theorem 5.5 (*[51, Corollary 6.4] and [54, Theorem 8.11]*)**.** *There exists a canonical isomorphism of graded algebras*

$$\mathbb{H}(G, L, \mathcal{L}, \mathbf{r}) \longrightarrow H_*^{G\times\mathbb{C}^\times}(\ddot{\mathfrak{g}}_N, \ddot{\mathcal{L}}).$$

With equivariant homology one can construct many modules for $H_*^{G\times\mathbb{C}^\times}(\ddot{\mathfrak{g}}_N, \ddot{\mathcal{L}})$. Let $y \in \mathfrak{g}$ be nilpotent and define

$$\mathcal{P}_y = \{gP \in G/P : \mathrm{Ad}(g^{-1})y \in \mathcal{C}_v^L + \mathfrak{u}\}.$$

This is the appropriate analogue of the variety $\mathcal{B}^u$ of Borel subgroups containing u. The group

$$M(y) := \{(g_1, \lambda) \in G \times \mathbb{C}^\times : \mathrm{Ad}(g_1)y = \lambda^2 y\}$$

acts on $\mathcal{P}_y$ by $(g_1, \lambda)gP = g_1 gP$. The inclusion $\{y\} \times \mathcal{P}_y \to \dot{\mathfrak{g}}$ is $M(y)$-equivariant, which allows us to restrict $\dot{\mathcal{L}}$ to an equivariant local system on $\mathcal{P}_y$. With constructions in equivariant (co)homology [5, §3.1] one can define an action of $\mathbb{H}(G, L, \mathcal{L}, \mathbf{r})$ on $H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}})$. It commutes with the natural actions of $\pi_0(M(y))$ and of $H^*_{M(y)^\circ}(\{y\})$ on $H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}})$, which enables us to decompose it as $\mathbb{H}(G, L, \mathcal{L}, \mathbf{r})$-module. It is known that $H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}})$ is projective over $H^*_{M(y)^\circ}(\{y\})$. One can naturally identify

$$H^*_{M(y)^\circ}(\{y\}) = \mathcal{O}(\mathrm{Lie}(M(y)^\circ))^{M(y)^\circ},$$
$$\mathrm{Lie}(M(y)^\circ) = \{(\sigma, r) \in \mathfrak{g} \oplus \mathbb{C} : [\sigma, y] = 2ry\}.$$

In particular the characters of $H^*_{M(y)^\circ}(\{y\})$ are parametrized by semisimple adjoint orbits in $\mathrm{Lie}(M(y)^\circ)$. For a semisimple element $(\sigma, r) \in \mathrm{Lie}(M(y)^\circ)$ we have the $\mathbb{H}(G, L, \mathcal{L}, \mathbf{r})$-module

$$E_{y,\sigma,r} = \mathbb{C}_{\sigma,r} \otimes_{H^*_{M(y)^\circ}(\{y\})} H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}}). \tag{5.11}$$

By the projectivity of $H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}})$, $(\sigma, r) \mapsto E_{y,\sigma,r}$ is an algebraic family of $\mathbb{H}(G, L, \mathcal{L}, \mathbf{r})$-modules. In particular the restriction of $E_{y,\sigma,r}$ to the finite dimensional semisimple algebra $\mathbb{C}[W_L]$ does not depend on (σ, r). As usual, the isomorphism class of $E_{y,\sigma,r}$ depends only on (y, σ, r) up to the adjoint action of G (which fixes r). From the action of $\pi_0(M(y))$ on $H_*^{M(y)^\circ}(\mathcal{P}_y, \dot{\mathcal{L}})$, only the operators that stabilize the adjoint orbit $[\sigma]$ of σ in $\mathrm{Lie}(M(y)^\circ)$ act on $E_{y,\sigma,r}$. Hence irreducible representations ρ of $\pi_0(M)_{[\sigma]}$ can be used to decompose the $\mathbb{H}(G, L, \mathcal{L}, \mathbf{r})$-module $E_{y,\sigma,r}$ further. We define the $\mathbb{H}(G, L, \mathcal{L}, \mathbf{r})$-representation

$$E_{y,\sigma,r,\rho} = \mathrm{Hom}_{\pi_0(M(y))_{[\sigma]}}(\rho, E_{y,\sigma,r}).$$

We call this a standard module, provided it is nonzero.

Let us improve the bookkeeping for the parameters (y, σ, r, ρ) just obtained. With Jacobson–Morozov we pick an algebraic homomorphism $\gamma_y : SL_2(\mathbb{C}) \to G$ such that $\mathrm{d}\gamma_y(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)) = y$. Notice that the semisimple element

$$\sigma_0 = \sigma + \mathrm{d}\gamma_y(\left(\begin{smallmatrix} -r & 0 \\ 0 & r \end{smallmatrix}\right))$$

commutes with y. It is not difficult to see that $\pi_0(M(y))_{[\sigma]}$ is naturally isomorphic to $\pi_0(Z_G(\sigma_0, y))$. By [5, Lemma 3.6] these constructions provide a bijection between G-association classes of data (y, σ, ρ) as above (for a fixed $r \in \mathbb{C}$) and G-association classes of triples (y, σ_0, ρ), where

$$y \in \mathfrak{g} \text{ nilpotent}, \ \sigma_0 \in \mathfrak{g} \text{ semisimple}, \ [\sigma_0, y] = 0, \ \rho \in \mathrm{Irr}\big(\pi_0(Z_G(\sigma_0, y))\big). \tag{5.12}$$

In the previous paragraph we encountered a clear condition on the representation ρ of the component group: it should appear in the homology of a particular variety, otherwise the associated module would be 0.

In the current setting the condition on ρ is more subtle, because $\mathcal{P}_y$ can be empty and a local system $\mathcal{L}$ is involved. To formulate it we need the cuspidal support map Ψ_G from [48, 6.4]. It associates a cuspidal support $(L', \mathcal{C}_{v'}^{L'}, \mathcal{L}')$ to every pair (x, ρ') with x nilpotent and $\rho' \in \mathrm{Irr}\big(\pi_0(Z_G(x))\big)$. Giving such (x, ρ) is equivalent to giving a G-equivariant cuspidal local system on a nilpotent orbit in $\mathfrak{g}$ (which is also an equivariant perverse sheaf). The cuspidal support map can be expressed with a version of parabolic induction for equivariant perverse sheaves [1, §4.1]. According to [5, Proposition 3.7]:

$$E_{y,\sigma,r,\rho} \neq 0 \quad \Longleftrightarrow \quad \Psi_{Z_G(\sigma_0)}(y, \rho) \text{ is } G\text{-associate to } (L, \mathcal{C}_v^L, \mathcal{L}). \tag{5.13}$$

When $L = T$ is a maximal torus of G and $v = 0$, we have $\mathcal{P}_y = \mathcal{B}^{\exp y}$ and $\mathcal{L}$ is trivial. Then the condition (5.13) reduces to: ρ appears in $H_*(\mathcal{B}^{\exp y}_{Z_G(\sigma_0)}, \mathbb{C})$. That is equivalent to the condition on ρ in the Kazhdan–Lusztig triple $(\exp(\sigma_0), \exp(y), \rho)$ for $(G, 1)$.

Theorem 5.6 ([55, Theorem 1.15] and [5, Theorem 3.11]). *Let (y, σ_0, ρ) be as in (5.12), such that $\Psi_{Z_G(\sigma_0)}(y, \rho)$ is G-associate to $(L, \mathcal{C}_v^L, \mathcal{L})$. For $r \in \mathbb{C}$ we write $\sigma_r = \sigma_0 + \mathrm{d}\gamma_y(\begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix})$, where $\gamma_y : SL_2(\mathbb{C}) \to Z_G(\sigma_0)$ with $d\gamma_y(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = y$.*

(a) For $r \neq 0$, $E_{y,\sigma_r,r,\rho}$ has a unique irreducible quotient, which we call $M_{y,\sigma_r,r,\rho}$.
(b) For $r = 0$, $E_{y,\sigma_0,0,\rho}$ has a canonical irreducible quotient $M_{y,\sigma_0,0,\rho}$ (the direct summand in one specific homological degree).
(c) Parts (a) and (b) set up a bijection between $\mathrm{Irr}\big(\mathbb{H}(G, L, \mathcal{L}, \mathbf{r})/(\mathbf{r} - r)\big)$ and the G-association classes of triples (y, σ_0, ρ) as above.
(d) Every irreducible constituent of $E_{y,\sigma_r,r,\rho}$ different from $M_{y,\sigma_r,r,\rho}$ is isomorphic to $M_{y',\sigma',r,\rho'}$, for data (y', σ', ρ') as above that satisfy $\dim \mathcal{C}_y^G < \dim \mathcal{C}_{y'}^G$.

Lusztig investigated when the modules $E_{y,\sigma,r,\rho}$ are tempered or discrete series [55]. Unfortunately his notions differ from ours, and as a consequence the resulting properties are opposite to what we want. To reconcile it, we use the Iwahori–Matsumoto involution of $\mathbb{H}(G, L, \mathcal{L}, \mathbf{r})$. It is the algebra automorphism

$$
\begin{aligned}
&\mathrm{IM} : \mathbb{H}(G, L, \mathcal{L}, \mathbf{r}) \to \mathbb{H}(G, L, \mathcal{L}, \mathbf{r}) \\
&\mathrm{IM}(N_w) = \mathrm{sign}(w) N_w, \quad \mathrm{IM}(\mathbf{r}) = \mathbf{r}, \quad \mathrm{IM}(\xi) = -\xi \qquad w \in W_L, \xi \in \mathfrak{t}^*.
\end{aligned}
$$

Clearly composition with IM has the effect $x \mapsto -x$ on $\mathcal{O}(\mathfrak{t})$-weights of $\mathbb{H}(G, L, \mathcal{L}, \mathbf{r})$-representations, and similarly for central characters. Let y, σ_0, ρ and γ_y be as in Theorem 5.6 and (5.12). We define

$$
\begin{aligned}
\widetilde{E}_{y,\sigma_0,r,\rho} &= \mathrm{IM}^* E_{y,\mathrm{d}\gamma_y\left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) - \sigma_0, r, \rho} \\
\widetilde{M}_{y,\sigma_0,r,\rho} &= \mathrm{IM}^* M_{y,\mathrm{d}\gamma_y\left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) - \sigma_0, r, \rho}
\end{aligned}
\tag{5.14}
$$

The modules (5.14) enjoy the same properties as their ancestors without tildes in Theorem 5.6. By [54, Theorem 8.13] and [5, Theorem 3.29] all these four modules admit the same central character, namely

$$
(\mathrm{Ad}(G)(\sigma_r) \cap \mathfrak{t}, r) = (\mathrm{Ad}(G)(\sigma_0 - \mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right)) \cap \mathfrak{t}, r). \tag{5.15}
$$

Theorem 5.7. *Let (y, σ_0, ρ) be as in Theorem 5.6.*

(a) When $\Re(r) \geq 0$, the following are equivalent:

- *$\widetilde{E}_{y,\sigma_0,r,\rho}$ is tempered,*
- *$\widetilde{M}_{y,\sigma_0,r,\rho}$ is tempered,*
- *$\mathrm{Ad}(G)\sigma_0$ intersects $i\mathfrak{a} = i\mathbb{R} \otimes_{\mathbb{Z}} X_*(T)$.*

(b) When $\Re(r) > 0$, the following are equivalent:

- *$\widetilde{M}_{y,\sigma_0,r,\rho}$ is essentially discrete series,*
- *y is distinguished nilpotent in $\mathfrak{g}$, that is, not contained in any proper Levi subalgebra of $\mathfrak{g}$.*

Moreover in this case $\sigma_0 \in Z(\mathfrak{g})$ and $\widetilde{E}_{y,\sigma_0,r,\rho} = \widetilde{M}_{y,\sigma_0,r,\rho}$.

(c) When $r \in \mathbb{R}$, the central character of $\widetilde{E}_{y,\sigma_0,r,\rho}$ lies in $\mathfrak{a}/W$ if and only if $\sigma_0 \in \mathrm{Ad}(G)\mathfrak{a}$.

Proof. (a) and (b) See [5, (84) and (85)].
(c) Upon conjugating the parameters by a suitable element of G, we may assume that $\sigma_0, \mathrm{d}\gamma_y \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathfrak{t}$ [5, Proposition 3.5.c]. Then $\mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right)$ represents the central character of the module $\widetilde{E}_{y,0,r} = \mathrm{IM}^* E_{y,\mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right),r}$ for

$$\mathbb{H}\big(Z_G(\sigma_0), L, \mathcal{L}, \mathbf{r}\big)/(\mathbf{r} - r) \cong \mathbb{H}\big(Z_G(\sigma_0)_{\mathrm{der}}, L \cap Z_G(\sigma_0)_{\mathrm{der}}, \mathcal{L}, \mathbf{r}\big)/(\mathbf{r} - r) \otimes_{\mathbb{C}} \mathcal{O}(Z_{\mathfrak{g}}(\sigma_0)).$$

Part (b) tells us that the restriction of $\widetilde{E}_{y,0,r}$ to $\mathbb{H}(Z_G(\sigma_0)_{\mathrm{der}}, L \cap Z_G(\sigma_0)_{\mathrm{der}}, \mathcal{L}, \mathbf{r})$ is a direct sum of discrete series representations. By [76, Lemma 2.13] the central characters of these representations lie in $\mathbb{Z}_{\mathfrak{g}}(\sigma_0)_{\mathrm{der}} \cap \mathfrak{a}/W_{L \cap Z_G(\sigma)_{\mathrm{der}}}$. Hence $\mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) \in \mathfrak{a}$, which implies that

$$\sigma_0 \in \mathfrak{a} \quad \Longleftrightarrow \quad \sigma_0 - \mathrm{d}\gamma_y \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right) \in \mathfrak{a}.$$

Compare that with (5.15) □

The family of representations $E_{y,\sigma,r}$ is compatible with parabolic induction, under a mild condition that (σ, r) is not a zero of a certain polynomial function ϵ [55, Corollary 1.18]. Namely, let Q be a standard Levi subgroup of G containing L and suppose that $\{y, \sigma_0\} \subset \mathrm{Lie}(Q)$. When $\epsilon(\sigma, r) \neq 0$ (or $r = 0$, see [5, Theorem A.2]), there is a canonical isomorphism

$$\mathrm{ind}_{\mathbb{H}(Q,L,\mathcal{L},\mathbf{r})}^{\mathbb{H}(G,L,\mathcal{L},\mathbf{r})} E^Q_{y,\sigma,r} \to E_{y,\sigma,r}. \tag{5.16}$$

The Iwahori–Matsumoto involution commutes with parabolic induction, so (5.16) also holds for the family of representations $\widetilde{E}_{y,\sigma_0,r,\rho}$. With that and Theorem 5.6 one can show that (at least when $\Re(r) > 0$) $\widetilde{E}_{y,\sigma_0,r,\rho}$ is a standard module in the sense of Section 3.5, see [5, Proposition A.3]. (This refers to the last arXiv-version of [5], in which an appendix was added to deal with a mistake in the published version.) The argument for standardness is analogous to what we sketched for $K_{t,u,\rho}$ around (5.4).

Example 5.8. We illustrate the material in this section with an example of rank 1. Let $G = \mathrm{Sp}_4(\mathbb{C})$, $L = \mathrm{Sp}_2(\mathbb{C}) \times GL_1(\mathbb{C})$ and v regular nilpotent in $\mathfrak{l}$. The local system on $\mathcal{C}_v^L$ corresponding to the sign representation of $\pi_0(Z_L(v)) = \pi_0(Z(L)) = \{\pm 1\}$ is cuspidal. The root system with respect to $T = Z(L)^\circ \cong \mathbb{C}^\times$ is $R(G, T) = \{\pm\alpha, \pm 2\alpha\}$, of type BC_1. Its Weyl group is $W_L = \{1, s_\alpha\}$, with s_α acting on T by inversion. The parameter function by determined by $\mathcal{L}$ satisfies $k(\pm\alpha) = 3$ and $k(\pm 2\alpha) = 6$. The associated graded Hecke algebra is

$$\mathbb{H}(G, L, \mathcal{L}, \mathbf{r}) = \mathbb{H}(\mathfrak{t}, W_L, k, \mathbf{r}).$$

The irreducible representations on which $\mathbf{r}$ acts by a fixed $r \in \mathbb{C}^\times$ were classified in Example 3.35 and for $r = 0$ in Theorem 2.3.

Let us analyse the possibilities for the geometric parameters. It turns out that the condition on ρ can only be met for nilpotent elements y in two adjoint orbits in $\mathfrak{g}$: the orbits of v and of $v+v'$, where v' is regular nilpotent in $Z(\mathfrak{l}_{\mathrm{der}}) \cong \mathfrak{sp}_2(\mathbb{C})$. The cuspidal support condition becomes that the subgroup of $\pi_0(Z_G(\sigma_0, y))$ coming from $\pi_0(Z(L))$ must act via the sign representation.

We may assume that $\mathbb{C}v'$ is stable under the adjoint action of T. A complete list of representatives for the G-conjugacy classes of parameters for $\mathbb{H}(G, L, \mathcal{L}, \mathbf{r})$ is:

σ_r	$\mathrm{diag}(r, r, -r, -r)$	$\sigma_0 = 0, r = 0$	$\mathrm{diag}(\sigma, r, -r, -\sigma), \Re(\sigma) \geq 0$
y	$v + v'$	v	v
$\pi_0(Z_G(\sigma_0, y))$	$Z(\mathrm{Sp}_2(\mathbb{C})^2)$	$Z(\mathrm{Sp}_2(\mathbb{C}))$	$Z(\mathrm{Sp}_2(\mathbb{C}))$
ρ	sign ⊠ triv	sign	sign
$E_{y,\sigma_r,r,\rho}$	triv	St	$\mathrm{ind}_{\mathcal{O}(\mathfrak{t}) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{r}]}^{\mathbb{H}(G,L,\mathcal{L},\mathbf{r})} (\mathbb{C}_{\mathrm{diag}(-\sigma,r,-r,\sigma),r})$
$\widetilde{E}_{y,\sigma_r,r,\rho}$	St	triv	$\mathrm{ind}_{\mathcal{O}(\mathfrak{t}) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{r}]}^{\mathbb{H}(G,L,\mathcal{L},\mathbf{r})} (\mathbb{C}_{\mathrm{diag}(\sigma,r,-r,-\sigma),r})$

In the last column we exclude the case $\sigma_0 = 0, r = 0$. For almost all $\sigma \in \mathbb{C}$, the standard module $\mathrm{ind}_{\mathcal{O}(\mathfrak{t})\otimes_{\mathbb{C}}\mathbb{C}[\mathbf{r}]}^{\mathbb{H}(G,L,\mathcal{L},\mathbf{r})}(\mathbb{C}_{\mathrm{diag}(\sigma,r,-r,-\sigma),r})$ is irreducible, and hence equal to both $M_{y,\sigma_r,r,\rho}$ and $\widetilde{M}_{y,\sigma_r,r,\rho}$. The exceptions are $\sigma = \pm r$, then

$$M_{y,\mathrm{diag}(r,r,-r,-r),r,\rho} = \mathrm{St} \quad \text{and} \quad \widetilde{M}_{y,\mathrm{diag}(r,r,-r,-r),r,\rho} = \mathrm{triv}.$$

5.3. Affine Hecke algebras from cuspidal local systems

In the previous paragraph we associated a graded Hecke algebra to a cuspidal local system on a nilpotent orbit in a Levi algebra of $\mathfrak{g}$. With a process that is more or less inverse to the reduction theorems in Section 3.4, we can glue a suitable family of such algebras into one affine Hecke algebra.

Concretely, let $G, P, L, v, T, \mathcal{L}$ be as before. Let $R(G,T)^\vee_{\mathrm{red}} \subset X_*(T)$ be the dual of the reduced root system $R(G,T)_{\mathrm{red}} \subset X^*(T)$, and consider the root datum

$$\mathcal{R}(G,T) = (R(G,T)_{\mathrm{red}}, X^*(T), R(G,T)^\vee_{\mathrm{red}}, X_*(T), \Delta_L).$$

There are unique parameter functions $\lambda, \lambda^* : R(G,T)_{\mathrm{red}} \to \mathbb{Z}_{\geq 0}$ that meet the requirements sketched above, see [6, Proposition 2.1 and (28)]. With those, we define

$$\mathcal{H}(G,L,\mathcal{L}) = \mathcal{H}(\mathcal{R}(G,T), \lambda, \lambda^*, \mathbf{q}). \tag{5.17}$$

This is slightly different from [6, §2], where the algebras involved a formal variable $\mathbf{z}$ instead of $\mathbf{q}^{1/2}$. To compensate for the square root ($\mathbf{q}^{1/2}$ versus $\mathbf{q}$) we could replace λ, λ^* by $2\lambda, 2\lambda^*$–but that is not necessary, since we are at liberty to choose $\mathbf{q} \in \mathbb{R}_{>0}$ or $\mathbf{r} \in \mathbb{R}^\times$ as we like.

When $\lambda, \lambda^* : R \to \mathbb{R}$ arise from a cuspidal local system as in (5.17), we call them geometric parameter functions for $\mathcal{R}$. As we may choose any $\mathbf{b} \in \mathbb{R}^\times$, λ, λ^* may be scaled by any nonzero real factor, and remain geometric.

For a unitary $t \in T_{\mathrm{un}}$, we saw in Corollary 3.30 that there is an equivalence of categories

$$\mathrm{Mod}_{f, W_L t \exp(\mathfrak{a})}\big(\mathcal{H}(G,L,\mathcal{L})\big) \quad \cong \quad \mathrm{Mod}_{f,\mathfrak{a}}\big(\mathbb{H}(\mathfrak{t}, W(R_t), k_t) \rtimes \Gamma_t\big).$$

Let $\widetilde{Z_G}(t)$ be the subgroup of G generated by $Z_G(t)$ and the root subgroups U_α with $s_\alpha(t) = t$. By [6, Theorems 2.5 and 2.9] we may identify

$$\mathbb{H}(\mathfrak{t}, W(R_t), k_t) \rtimes \Gamma_t = \mathbb{H}(\widetilde{Z_G}(t), L, \mathcal{L}, \mathbf{r})/(\mathbf{r} - \log(\mathbf{q})/2).$$

Here $\widetilde{Z_G}(t)$ has component group Γ_t, so it can be disconnected. In that case these graded Hecke algebras are a little more general than in the previous paragraph, but that does not matter much.

Example 5.9. We continue Example 5.8. We identify T with $\mathbb{C}^\times$ by means of the map $\mathrm{diag}(z,1,1,z^{-1}) \mapsto z$. For $t \in T_{\mathrm{un}} \setminus \{1,-1\}$, $Z_G(t) = \widetilde{Z_G}(t) = L$ and

$$\mathbb{H}(\widetilde{Z_G}(t), L, \mathcal{L}, \mathbf{r}) = \mathbb{H}(T, T, \mathrm{triv}, \mathbf{r}) = \mathcal{O}(\mathfrak{t}) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{r}].$$

The most interesting element of T_{un} is $-1 = \mathrm{diag}(-1,1,1,-1)$. Then $Z_G(-1) = L$ but $\widetilde{Z_G}(-1) = \mathrm{Sp}_2(\mathbb{C}) \times \mathrm{Sp}_2(\mathbb{C})$. Now the root system is $R(\widetilde{Z_G}(-1), T) = \{\pm 2\alpha\}$, with parameter $k_{-1}(\pm 2\alpha) = 2$. The associated graded Hecke algebra is

$$\mathbb{H}(\widetilde{Z_G}(-1), L, \mathcal{L}, \mathbf{r}) = \mathbb{H}(\mathfrak{t}, W_L, k_{-1}, \mathbf{r}).$$

These data suffice to determine the parameters for $\mathcal{H}(G,L,\mathcal{L})$, they are

$$\lambda(\alpha) = k(\alpha) + k_{-1}(2\alpha)/2 = 4 \quad \text{and} \quad \lambda^*(\alpha) = k(\alpha) - k_{-1}(2\alpha)/2 = 2.$$

Thus $\mathcal{H}(G, L, \mathcal{L}) = \mathcal{H}(\mathcal{R}(G, T), \lambda, \lambda^*, \mathbf{q})$, an affine Hecke algebra of type $\widetilde{A}_1$ with unequal parameters.

With Theorems 3.28 and 3.29 we can reduce the classification of $\mathrm{Irr}(\mathcal{H}(G, L, \mathcal{L}))$ to Theorem 5.6. For better additional benefits, we prefer to use the modules $\widetilde{E}_{y,\sigma_0,r,\rho}$ and $\widetilde{M}_{y,\sigma_0,r,\rho}$. In this context it is more natural to use data from G than from $\mathfrak{g}$. The correct parameters turn out to be a variation on Kazhdan–Lusztig triples, namely triples (s, u, ρ) where

- $s \in G$ is semisimple,
- $u \in Z_G(s)$ is unipotent,
- $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(s, u))\big)$ such that the cuspidal support $\Psi_{Z_G(s)}(u, \rho)$ is $(L, \exp(\mathcal{C}_v^L), \exp_*(\mathcal{L}))$ modulo G-conjugacy.

When $Z_G(s)$ is disconnected, we have to use a generalization of the cuspidal support map, defined in [4, §4]. By conjugating with a suitable element of G, we may always assume that $s \in T$. Let $\overline{E}_{s,u,\rho}$ (resp. $\overline{M}_{s,u,\rho}$) be the $\mathcal{H}(G, L, \mathcal{L})$-module obtained from

$$\widetilde{E}_{\log u,\log|s|,\log(\mathbf{q})/2,\rho} \in \mathrm{Mod}\big(\mathbb{H}(\widetilde{Z_G}(s|s|^{-1}), L, \mathcal{L}, \mathbf{r})/(\mathbf{r} - \log(\mathbf{q})/2)\big)$$

(resp. $\widetilde{M}_{\log u,\log|s|,\log(\mathbf{q})/2,\rho}$) via Theorems 3.29 and 3.28, with respect to $s|s|^{-1} \in T$.

Theorem 5.10 *([6, Theorem 2.11]). Let $\mathbf{q} \in \mathbb{R}_{>0}$ and consider triples (s, u, ρ) as above.*

(a) The maps $(s, u, \rho) \mapsto \overline{E}_{s,u,\rho} \mapsto \overline{M}_{s,u,\rho}$ provide canonical bijections

$$\{\textit{triples as above}\}/G \longrightarrow \{\textit{standard } \mathcal{H}(G, L, \mathcal{L})\textit{-modules}\} \longrightarrow \mathrm{Irr}(\mathcal{H}(G, L, \mathcal{L})).$$

(b) Suppose that $s \in T$ and let $\gamma_u : SL_2(\mathbb{C}) \to Z_G(s)$ be an algebraic homomorphism with $\gamma_u(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)) = u$. Then $\overline{E}_{s,u,\rho}$ and $\overline{M}_{s,u,\rho}$ admit the central character $W_L s \gamma_u(\begin{pmatrix} \mathbf{q}^{1/2} & 0 \\ 0 & \mathbf{q}^{-1/2} \end{pmatrix})$.

(c) Suppose that $\mathbf{q} \geq 1$. The following are equivalent:

- *$\overline{E}_{s,u,\rho}$ is tempered,*
- *$\overline{M}_{s,u,\rho}$ is tempered,*
- *s lies in a compact subgroup of G.*

(d) Suppose that $\mathbf{q} > 1$. Then $\overline{M}_{s,u,\rho}$ is essentially discrete series if and only if u is distinguished unipotent in G (that is, not contained in any proper Levi subgroup of G).

Notice that $\mathbf{q} = 1$ is allowed here. For $\mathbf{q} = 1$, Theorem 5.10 provides a parametrization of $\mathrm{Irr}(X^*(T) \rtimes W(G, T))$ with G-association classes of triples (s, u, ρ) as above. That can be regarded as an affine version of the generalized Springer correspondence from [48].

Example 5.11. We work out the parametrization from Theorem 5.10 for $(G, L, \mathcal{L})$ as in Examples 5.8 and 5.9. Here $\mathcal{H}(G, L, \mathcal{L})$ is of type $\widetilde{A}_1$, with parameters $\lambda(\alpha) = 4$ and $\lambda^*(\alpha) = 2$. With the example at the end of Section 5.2 at hand, it is easy to determine all geometric parameters (s, u, ρ) for $\mathcal{H}(G, L, \mathcal{L})$, and the relevant modules can be found by applying Theorems 3.29 and 3.28.

s	1	$-1 = \mathrm{diag}(-1, 1, 1, -1)$	$s \in T \cong \mathbb{C}^\times, \lvert s\rvert \geq 1$
u	$\exp(v + v')$	$\exp(v + v')$	$\exp(v)$
$\pi_0(Z_G(s, u))$	$Z(\mathrm{Sp}_2(\mathbb{C})^2)$	$Z(\mathrm{Sp}_2(\mathbb{C})^2)$	$Z(\mathrm{Sp}_2(\mathbb{C}))$
ρ	sign $\boxtimes$ triv	sign $\boxtimes$ triv	sign
$\overline{E}_{s,u,\rho}$	St	$\pi(-1, \mathrm{St})$	$\mathrm{ind}_{\mathbb{C}[X^*(T)]}^{\mathcal{H}(G,L,\mathcal{L})}(\mathbb{C}_s)$

For almost all $s \in T$ the standard module $\mathrm{ind}_{\mathbb{C}[X^*(T)]}^{\mathcal{H}(G,L,\mathcal{L})}(\mathbb{C}_s)$ is irreducible, and hence equal to $\overline{M}_{s,1,\mathrm{triv}}$. The exceptions are

$$\overline{M}_{\mathrm{diag}(\mathbf{q}^{3/2},1,1,\mathbf{q}^{-3/2}),1,\mathrm{triv}} = \mathrm{triv} \quad \text{and} \quad \overline{M}_{\mathrm{diag}(-\mathbf{q}^{1/2},1,1,-\mathbf{q}^{-1/2}),1,\mathrm{triv}} = \pi(-1,\mathrm{triv}).$$

A comparison with Section 2.2 shows that we indeed found every irreducible $\mathcal{H}(G,L,\mathcal{L})$-representation once in this way.

We have associated to every complex reductive group G a family of affine Hecke algebras, one for every cuspidal local system on a nilpotent orbit for a Levi subgroup of G. Every nilpotent orbit admits only a few inequivalent cuspidal local systems, and G-conjugate data $(L, \mathcal{C}_v^L, \mathcal{L})$ yield isomorphic Hecke algebras. Thus we have a finite family of affine Hecke algebras associated to G.

The simplest member of this family arises when $L = T$, $v = 0$ and $\mathcal{L}$ is trivial. Then the graded Hecke algebras $\mathbb{H}(\widetilde{Z_G}(t), T, \mathcal{L} = \mathrm{triv})$ have parameters $k(\alpha) = 2$ for all $\alpha \in R(G,T)$. When we specialize $\mathbf{r}$ to $\log(\mathbf{q})/2$, (3.19) shows that $\lambda(\alpha) = \lambda^*(\alpha) = 1$ for all $\alpha \in R(G,T)$. In other words:

$$\mathcal{H}(G,T,\mathcal{L} = \mathrm{triv}) = \mathcal{H}(\mathcal{R}(G,T),\mathbf{q}).$$

For this algebra the cuspidal support condition on the triples (s,u,ρ) reduces to the condition on ρ in a Kazhdan–Lusztig triple for $(G,1)$. By [6, Proposition 2.18], the parametrization of standard and irreducible $\mathcal{H}(G,T,\mathcal{L} = \mathrm{triv})$-modules in Theorem 5.10 coincides with the Kazhdan–Lusztig parametrization from Theorem 5.1, modified as in Corollary 5.2. That is less obvious than it might seem though, the twist with the Iwahori–Matsumoto involution in (5.14) is necessary to achieve the agreement.

6. Comparison between different q-parameters

The aim of this section is a canonical bijection between the set of irreducible representations of an affine Hecke algebra with arbitrary parameters $q_s \in \mathbb{R}_{\geq 1}$, and the set of irreducible representations of the same algebra with parameters $q_s = 1$. This will be achieved in several steps of increasing generality.

6.1. W-types of irreducible tempered representations

Consider any graded Hecke algebra $\mathbb{H} = \mathbb{H}(\mathfrak{t}, W, k)$. The group algebra $\mathbb{C}[W]$ is embedded in $\mathbb{H}$, so every $\mathbb{H}$-representation can be restricted to a W-representation. For $k = 0$, the isomorphism

$$\mathbb{H}(\mathfrak{t}, W, 0)/(\mathfrak{t}^*) \cong \mathbb{C}[W] \tag{6.1}$$

shows that a representation on which $\mathcal{O}(\mathfrak{t})$ acts via evaluation at $0 \in \mathfrak{t}$ is the same as a $\mathbb{C}[W]$-representation. From Example 3.7 we know that the irreducible tempered representations of $\mathcal{H}(\mathcal{R},1)$ with central character in $\exp(\mathfrak{a})$ are precisely the irreducible representations which admit the $\mathcal{O}(T)$-character $1 \in T$. Via Corollary 3.30 this implies that the irreducible representations of (6.1) are precisely the irreducible tempered $\mathbb{H}(\mathfrak{t}, W, 0)$-representations whose central character is real, that is, lies in $\mathfrak{a}/W$.

That and the results of Section 3.4 indicate that we should focus on $\mathbb{H}$-representations with $\mathcal{O}(\mathfrak{t})$-weights in $\mathfrak{a} = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. We say that those have real weights. Let

$$\mathrm{Irr}_0(\mathbb{H}(\mathfrak{t}, W, k)) = \{\pi \in \mathrm{Mod}_{f,\mathfrak{a}}(\mathbb{H}(\mathfrak{t}, W, k)) : \pi \text{ is irreducible and tempered}\}$$

be the set of irreducible tempered representations with real central character. The above says that $\mathrm{Irr}_0(\mathbb{H}(\mathfrak{t}, W, 0))$ can be identified with $\mathrm{Irr}(W)$. Since $\mathbb{H}(\mathfrak{t}, W, k)$ is a deformation of $\mathbb{H}(\mathfrak{t}, W, 0)$, it can be expected that something similar holds for $\mathbb{H}(\mathfrak{t}, W, k)$. On closer inspection the parameters $k(\alpha)$ interact with the notion of temperedness, and it is natural to require that k takes real values.

For later applications to affine Hecke algebras, it will pay off to increase our generality. Let Γ be a finite group which acts on $(\mathfrak{a}^*, \mathfrak{z}^*, R, \Delta)$. That is, Γ acts $\mathbb{R}$-linearly on $\mathfrak{a}^*$, and that action stabilizes R, Δ and the decomposition $\mathfrak{a}^* = \mathbb{R}R \oplus \mathfrak{z}^*$. Suppose further that $k : R \to \mathbb{R}$ is constant on Γ-orbits. Then Γ acts on $\mathbb{H}(\mathfrak{t}, W, k)$ by the algebra automorphisms

$$\xi w \mapsto \gamma(\xi)(\gamma w \gamma^{-1}) \qquad \xi \in \mathfrak{t}^*, w \in W.$$

The crossed product algebra $\mathbb{H}(\mathfrak{t}, W, k) \rtimes \Gamma = \mathbb{H} \rtimes \Gamma$ is of the kind already encountered in Corollary 3.30. The Γ-action on $\mathbb{H}$ preserves all the available structure, so all the usual notions for $\mathbb{H}$ also make sense for $\mathbb{H} \rtimes \Gamma$.

We denote the restriction of any $\mathbb{H} \rtimes \Gamma$-representation π to the subalgebra $\mathbb{C}[W \rtimes \Gamma]$ by $\mathrm{Res}_{W \rtimes \Gamma}(\pi)$. An initial result in the direction sketched above is:

Theorem 6.1 (*[78, Theorem 6.5]*). *$\mathrm{Irr}_0(\mathbb{H} \rtimes \Gamma)$ and $\mathrm{Irr}(W \rtimes \Gamma)$ have the same cardinality, and the set $\mathrm{Res}_{W \rtimes \Gamma}(\mathrm{Irr}_0(\mathbb{H} \rtimes \Gamma))$ is linearly independent in the representation ring of $W \rtimes \Gamma$.*

In particular it is possible to choose a bijection $\mathrm{Irr}_0(\mathbb{H} \rtimes \Gamma) \to \mathrm{Irr}(W \rtimes \Gamma)$ such that the image of $\pi \in \mathrm{Irr}_0(\mathbb{H} \rtimes \Gamma)$ is always a constituent of $\mathrm{Res}_{W \rtimes \Gamma}(\pi)$. We will establish a much more precise version of Theorem 6.1, for almost all positive parameter functions k.

Theorem 6.2. *Let $k : R \to \mathbb{R}_{\geq 0}$ be a Γ-invariant parameter function whose restriction to any type F_4 component of R is geometric or has $k(\alpha) = 0$ for a root α in that component.*

(a) The set $\mathrm{Res}_{W \rtimes \Gamma}(\mathrm{Irr}_0(\mathbb{H} \rtimes \Gamma))$ is a $\mathbb{Z}$-basis of $\mathbb{Z}\,\mathrm{Irr}(W \rtimes \Gamma)$.

(b) There exist total orders on $\mathrm{Irr}_0(\mathbb{H} \rtimes \Gamma)$ and on $\mathrm{Irr}(W \rtimes \Gamma)$ such that the matrix of the $\mathbb{Z}$-linear map

$$\mathrm{Res}_{W \rtimes \Gamma} : \mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H} \rtimes \Gamma) \to \mathbb{Z}\,\mathrm{Irr}(W \rtimes \Gamma)$$

is upper triangular and unipotent.

(c) There exists a unique bijection

$$\zeta_{\mathbb{H} \rtimes \Gamma} : \mathrm{Irr}_0(\mathbb{H} \rtimes \Gamma) \to \mathrm{Irr}(W \rtimes \Gamma)$$

such that, for any $\pi \in \mathrm{Irr}_0(\mathbb{H} \rtimes \Gamma)$, $\zeta_{\mathbb{H} \rtimes \Gamma}(\pi)$ occurs in $\mathrm{Res}_{W \rtimes \Gamma}(\pi)$.

Part (a) is known from [82, Proposition 1.7] and part (c) is a direct consequence of part (b). Part (b) was already conjectured by Slooten [75, §1.4.5].

Recall from table (5.7) that the geometric parameter functions for $R = F_4$ are given by $(k(\alpha), k(\beta))$ equal to

$$(k, k), (2k, k), (k, 2k) \text{ or } (4k, k)$$

for any $k \in \mathbb{C}^\times$, where α is a short root and β is a long root. We expect that Theorem 6.2.b also holds for non-geometric parameter functions $k : F_4 \to \mathbb{R}_{>0}$, but found it too cumbersome to check.

The discussion after (6.1) shows that the theorem is trivial for $k = 0$, so we assume from now on that $k \neq 0$. Our proof of Theorem 6.2 will occupy the entire paragraph.

Lemma 6.3. *Theorem 6.2 holds for $\mathbb{H}(\mathfrak{t}, W, k)$ with $k : R \to \mathbb{R}_{>0}$ geometric.*

Proof. The only irreducible tempered representation with real central character of $\mathbb{H}(\mathfrak{t}, R = \emptyset, k) = \mathcal{O}(\mathfrak{t})$ is $\mathbb{C}_0$, so the result is trivially true for that algebra. In view of the decomposition of $\mathbb{H}$ according to the irreducible components of R (1.26), we may assume that R is irreducible and spans $\mathfrak{a}^*$.

The algebra isomorphisms (5.8)–(5.10) are the identity on $\mathcal{O}(\mathfrak{t})$, so they preserve the set $\mathrm{Irr}_0(\mathbb{H})$. The same holds for m_z from (1.22), when $z \in \mathbb{R}_{>0}$. Therefore we may just as well suppose that k is one of the parameter functions in the table (5.7), and that

$$\mathbb{H} = \mathbb{H}(G, L, \mathcal{L}, \mathbf{r})/(\mathbf{r} - r) \text{ for some } r \in \mathbb{R}_{>0}.$$

By Theorem 5.7 $\mathrm{Irr}_0(\mathbb{H})$ consists of the representations $\widetilde{M}_{y,0,r,\rho}$ with $y \in \mathfrak{g}$ nilpotent, $\sigma_0 = 0$, $\rho \in \mathrm{Irr}\big(\pi_0(Z_G(y))\big)$ and $\Psi_G(y, \rho) = (L, \mathcal{C}_v^L, \mathcal{L})$ up to G-conjugacy. By Lemma 3.34 all the irreducible constituents of $\widetilde{E}_{y,0,r,\rho}$ are tempered and have central character in $\mathfrak{a}/W$. For $r = 0$ all these representations admit the $\mathcal{O}(\mathfrak{t})$-character 0, so they can be identified with W-representations via (6.1).

Recall from (5.11) that

$$\mathrm{Res}_W \widetilde{E}_{y,0,r,\rho} = \mathrm{Res}_W \widetilde{E}_{y,0,0,\rho}.$$

Theorem 5.6.d for $r = 0$ says that all constituents of $\widetilde{E}_{y,0,0,\rho}$ different from $\widetilde{M}_{y,0,0,\rho}$ are of the form $\widetilde{M}_{y',0,0,\rho'}$ with $\dim \mathcal{C}_y^G < \dim \mathcal{C}_{y'}^G$. The numbers $\dim \mathcal{C}_y^G$ define a partial order on the set of eligible pairs (y, ρ), considered modulo G-conjugation. Refine that to a total order. We transfer that to a total ordering on $\mathrm{Irr}_0(\mathbb{H})$ (resp. $\mathrm{Irr}(W)$) via $(y, \rho) \mapsto \widetilde{M}_{y,0,r,\rho}$ (resp. $\widetilde{M}_{y,0,0,\rho}$). With respect to these orders, the matrix of

$$\mathrm{Res}_W : \mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}) \to \mathbb{Z}\,\mathrm{Irr}(W)$$

is unipotent and upper triangular. It follows that

$$\zeta_{\mathbb{H}} : \widetilde{M}_{y,0,r,\rho} \mapsto \widetilde{M}_{y,0,0,\rho}$$

is the unique map $\mathrm{Irr}_0(\mathbb{H}) \to \mathrm{Irr}(W)$ with the required properties. □

We remark that Lemma 6.3 with respect to Lusztig's alternative version of temperedness was proven in [17, §3].

When R is irreducible and all roots have the same length, k is determined by the single number $k(\alpha) > 0$, and it is geometric. So we just dealt with R of type A_n, D_n, E_6, E_7 or E_8. For R of type B_n, C_n, F_4 or G_2, let k_1 be the k-parameter of a long root and k_2 the k-parameter of a short root. We will consider the algebras $\mathbb{H}(\mathfrak{t}, W(R), k)$ with $k_1 = 0$ or $k_2 = 0$ in and after (6.12).

That settles $R = F_4$ for the moment, because we excluded non-geometric strictly positive parameters. For $R = G_2$ with strictly positive k, Theorem 6.2 was proven in [75, §1.4.4], by working it out completely for all possible cases.

Amongst the $\mathbb{H}$ with R irreducible, that leaves type B_n or C_n. In view of the isomorphism (5.8), it suffices to consider $R = B_n$. Lemma 6.3 proves Theorem 6.2 for the following geometric k (and any $p \in \mathbb{Z}_{>0}$):

k_2/k_1	$\mathfrak{g}$	$\mathfrak{l}$
$1/2$	$\mathfrak{sp}_{2n}$	Cartan
p	$\mathfrak{so}_{p^2+2n}$	$\mathfrak{so}_{p^2} \oplus \mathbb{C}^n$
$p+1/2$	$\mathfrak{sp}_{p(p+1)+2n}$	$\mathfrak{sp}_{p(p+1)} \oplus \mathbb{C}^n$
$p-1/4$	$\mathfrak{so}_{p(2p-1)+4n}$	$\mathfrak{sl}_2^n \oplus \mathbb{C}^n$
$p+1/4$	$\mathfrak{so}_{p(2p+1)+4n}$	$\mathfrak{sl}_2^n \oplus \mathbb{C}^n$

Recall from (4.9) that a strictly positive k is generic if $\prod_{j=1}^{2(n-1)}(jk_1 - 2k_2)$ is nonzero. In particular we already covered all strictly positive non-generic parameters for B_n.

Lemma 6.4. *Theorem 6.2 holds for $\mathbb{H}(\mathbb{C}^n, W(B_n), k)$ when*

$$k_2/k_1 \in (p-1/2, p) \cup (p, p+1/2) \text{ for a } p \in \mathbb{Z}_{>0}.$$

Proof. We only consider $k_2/k_1 \in (p-1/2, p)$, the case $k_2/k_1 \in (p, p+1/2)$ is completely analogous. Define k' by $k'_1 = 1$ and $k'_2 = p - 1/4$.

Notice that all k with $k_2/k_1 \in (p-1/2, p)$ are generic for B_n and for all its parabolic root subsystems. Hence the residual subspaces of $\mathfrak{a}$ for k (Definition 4.6) are canonically in bijection with those for k'. More precisely, every residual subspace has coordinates that are linear functions of k, see Proposition 4.10 and (4.8). If such a linear function gives a residual subspace for k', then it also gives a residual subspace for k, and conversely. From Theorem 4.13 we obtain a canonical bijection between the sets of irreducible discrete series representations of $\mathbb{H}_P(\mathbb{C}^n, W(B_n), k')$ and of $\mathbb{H}_P(\mathbb{C}^n, W(B_n), k)$, say $\delta' \mapsto \delta$, where the image depends continuously on k.

By Lemma 3.34 and (4.6), $\mathrm{Irr}_0(\mathbb{H}(\mathbb{C}^n, W(B_n), k))$ consists of the representations $\pi(P, \delta, 0, \rho)$ with $\rho \in \mathrm{Irr}(\mathbb{C}[\mathfrak{R}_{P,\delta,0}, \natural_{P,\delta,0}])$. Recall from Section 4.1 that the analytic R-group $\mathfrak{R}_{P,\delta,\lambda}$ is defined in terms of the functions $\tilde{c_\alpha}(\lambda)$. Since k is only allowed to vary among generic parameters, the pole order of $\tilde{c_\alpha}|_{\mathfrak{a}^{P*}}$ at $\lambda = 0$ does not depend on k. Hence $\mathfrak{R}_{P,\delta,0}$ does not depend on k either. The intertwining operators $\pi(w, P, \delta, 0)$ ($w \in \mathfrak{R}_{P,\delta,0}$) that span the twisted group algebra can be constructed so that they depend continuously on k. Then $\mathbb{C}[\mathfrak{R}_{P,\delta,0}, \natural_{P,\delta,0}]$ becomes a family of finite dimensional semisimple $\mathbb{C}$-algebras, continuous in k. But such algebras cannot be deformed continuously, so the family is isomorphic to a constant family. That provides a canonical bijection

$$\mathrm{Irr}(\mathbb{C}[\mathfrak{R}_{P,\delta',0}, \natural_{P,\delta',0}]) \to \mathrm{Irr}(\mathbb{C}[\mathfrak{R}_{P,\delta,0}, \natural_{P,\delta,0}]),$$

We plug it into (4.6) and we obtain a bijection

$$\mathrm{Irr}_0\big(\mathbb{H}(\mathbb{C}^n, W(B_n), k')\big) \to \mathrm{Irr}_0\big(\mathbb{H}(\mathbb{C}^n, W(B_n), k)\big), \tag{6.2}$$

where the image depends continuously on k. Finite dimensional representations of the finite group W do not admit continuous deformations, so (6.2) preserves W-types. Knowing that, Theorem 6.2 for $\mathbb{H}(\mathbb{C}^n, W(B_n), k')$, as shown in Lemma 6.3, immediately implies Theorem 6.2 for $\mathbb{H}(\mathbb{C}^n, W(B_n), k)$. $\square$

Now we involve the group Γ that acts on $\mathbb{H}$ via automorphisms of $(\mathfrak{a}^*, \mathfrak{z}^*, R, \Delta)$.

Lemma 6.5. *Let $\mathbb{H}$ be one of the graded Hecke algebras for which we already proved Theorem 6.2. We can choose the total orders on $\mathrm{Irr}_0(\mathbb{H})$ and on $\mathrm{Irr}(W)$ such that, for any*

$\pi \in \mathrm{Irr}_0(\mathbb{H})$ *and any irreducible constituent* π' *of* $\mathrm{Res}_W(\pi)$ *different from* $\zeta_{\mathbb{H}}(\pi)$, $\gamma^*(\pi') > \zeta_{\mathbb{H}}(\pi)$ *for all* $\gamma \in \Gamma$.

Proof. First we assume that R is irreducible and spans $\mathfrak{a}^*$.

We consider a geometric k and we revisit the proof of Lemma 6.3. Replacing $\mathbb{H}$ by an isomorphic algebra, we may assume that k comes from table (5.7). Inspection of the table shows that every automorphism of (R, Δ) can be lifted to an automorphism of (G, T) (Recall that there are no automorphisms of (R, Δ) when R has type B_d, BC_d, G_2 or F_4, apart from the identity.) Hence the function

$$\mathrm{Irr}(W) \to \mathbb{R} : \widetilde{M}_{y,0,0,\rho} \mapsto \dim \mathcal{C}_y^G$$

is Γ-invariant. We replace this function by a function $f_{R,k} : \mathrm{Irr}(W) \to \mathbb{R}$, whose images differ only slightly from $\dim \mathcal{C}_y^G$ and which induces the total order on $\mathrm{Irr}(W)$ claimed in Theorem 6.2.b and exhibited in the proof of Lemma 6.3. Via $\zeta_{\mathbb{H}}$, we also regard $f_{R,k}$ as a function $\mathrm{Irr}_0(\mathbb{H})$. Then Theorem 6.2.c implies that every constituent of $\mathrm{Res}_W(\widetilde{M}_{y,0,r,\rho})$ different from $\widetilde{M}_{y,0,0,\rho}$ is isomorphic to a $\widetilde{M}_{y',0,0,\rho'}$ with

$$f_{R,k}\big(\gamma^*(\widetilde{M}_{y',0,0,\rho'})\big) > f_{R,k}(\widetilde{M}_{y,0,0,\rho}) \qquad \text{for all } \gamma \in \Gamma. \tag{6.3}$$

When $R = G_2$ and k is not geometric, we can use the analysis from [75, §1.4.4] to find a function $f_{R,k} : \mathrm{Irr}(W) \to \mathbb{R}$ with analogous properties. For other non-geometric k, we may assume that $R = B_n$ (recall we imposed that k is geometric for $R = F_4$). Then k is one of the parameter functions considered in Lemma 6.4. In the proof of that lemma we saw that k can be deformed continuously to a geometric parameter function k', while staying generic. That led to a canonical bijection

$$\mathrm{Irr}_0(\mathbb{H}) \to \mathrm{Irr}_0(\mathbb{H}(\mathbb{C}^n, W(B_n), k')), \tag{6.4}$$

which preserves W-types. We define $f_{R,k}$ to be the composition of $f_{R,k'}$ with that bijection, and we transfer it to a function on $\mathrm{Irr}(W)$ via $\zeta_{\mathbb{H}}$. The bijection (6.4) and $\zeta_{\mathbb{H}}$ are Γ-equivariant, if nothing else because (B_n, Δ) does not admit nontrivial automorphisms. Hence all the properties of $f_{R,k'}$ transfer to $f_{R,k}$.

So far we proved the lemma in all cases where R is irreducible and we already had Theorem 6.2, and we made Theorem 6.2.b more explicit by associating the total order to a real-valued function $f_{R,k}$. For a general R we use the decomposition (1.26) of $\mathbb{H}$. It provides a natural bijection

$$\begin{array}{ccc} \mathrm{Irr}_0(\mathbb{H}(\mathfrak{t}_1, W(R_1), k)) \times \cdots \times \mathrm{Irr}_0(\mathbb{H}(\mathfrak{t}_d, W(R_d), k)) & \to & \mathrm{Irr}_0(\mathbb{H}(\mathfrak{t}, W(R), k)) \\ (V_1, \ldots, V_d) & \mapsto & V_1 \otimes \cdots \otimes V_d \otimes \mathbb{C}_0 \end{array}$$

where $\{\mathbb{C}_0\} = \mathrm{Irr}(\mathcal{O}(\mathfrak{z}^*))$. We may assume that all values of the $f_{R_i,k}$ constructed above are algebraically independent and differ from an integer by at most $(8d)^{-1}$ (if not, we can adjust them a bit). Now we define $f_{R,k} : \mathrm{Irr}_0(\mathbb{H}) \to \mathbb{R}$ by

$$f_{R,k}(V_1 \otimes \cdots V_d \otimes \mathbb{C}_0) = \sum\nolimits_{i=1}^d f_{R_i,k}(V_i).$$

and we order $\mathrm{Irr}_0(\mathbb{H})$ accordingly. Via $\zeta_{\mathbb{H}}$ we transfer $f_{R,k}$ and the total order to $\mathrm{Irr}(W)$. Let π' be a constituent of $\mathrm{Res}_W(V_1 \otimes \cdots \otimes V_d \otimes \mathbb{C}_0)$ different from $\zeta_{\mathbb{H}}(V_1 \otimes \cdots \otimes V_d \otimes \mathbb{C}_0)$ and let $\pi'_i \in \mathrm{Irr}(W(R_i))$, $i = 1, \ldots, d$ be its tensor components. At least one of the π'_i is not isomorphic to $\zeta_{\mathbb{H}(\mathfrak{t}_i, W(R_i), k)}(V_i)$. By (6.3) and its analogues for other irreducible R:

$$f_{R,k}(\gamma^*(\pi')) > f_{R,k}(\pi) + 1 - 2d/8d = f_{R,k}(\zeta_{\mathbb{H}}(\pi)) + 3/4 \qquad \text{for all } \gamma \in \Gamma. \quad \square \tag{6.5}$$

Before we continue, we quickly recall how Clifford theory relates the irreducible representations of $\mathbb{H}$ and of $\mathbb{H} \rtimes \Gamma$. For $(\pi, V_\pi) \in \mathrm{Irr}(\mathbb{H})$ we write

$$\Gamma_\pi = \{\gamma \in \Gamma : \gamma^*(\pi) \cong \pi\}.$$

For every $\gamma \in \Gamma_\pi$ we pick a nonzero intertwining operator $I_\gamma : \pi \to \gamma^*(\pi)$. By Schur's lemma I_γ is unique up to scalars, so there exist $\natural_\pi \in \mathbb{C}^\times$ such that

$$I_{\gamma\gamma'} = \natural_\pi(\gamma, \gamma') I_\gamma I_{\gamma'} \qquad \text{for all } \gamma, \gamma' \in \Gamma_\pi. \tag{6.6}$$

Then $\natural_\pi^{\pm 1}$ is a 2-cocycle $\Gamma_\pi \times \Gamma_\pi \to \mathbb{C}^\times$ and the twisted group algebra $\mathbb{C}[\Gamma_\pi, \natural_\pi^{-1}]$ acts on V_π via the I_γ. For every representation (σ, V_σ) of $\mathbb{C}[\Gamma_\pi, \natural_\pi]$, the vector space $V_\pi \otimes_{\mathbb{C}} V_\sigma$ becomes a representation of $\mathbb{H} \rtimes \Gamma_\pi$ by

$$h\gamma \cdot (v_\pi \otimes v_\sigma) = \pi(h) I_\gamma(v_\pi) \otimes \sigma(\gamma) v_\sigma.$$

When σ is irreducible, $V_\pi \otimes V_\sigma$ is also irreducible. Moreover

$$\pi \rtimes \sigma := \mathrm{ind}_{\mathbb{H} \rtimes \Gamma_\pi}^{\mathbb{H} \rtimes \Gamma} (V_\pi \otimes V_\sigma)$$

is an irreducible $\mathbb{H} \rtimes \Gamma$-representation. By [68, Appendix] every irreducible $\mathbb{H} \rtimes \Gamma$-representation is of the form $\pi \rtimes \sigma$, for a pair (π, σ) that is unique up to the Γ-action.

The restriction of $\pi \rtimes \sigma$ to $\mathbb{H}$ has constituents $\gamma^*(\pi)$ for $\gamma \in \Gamma/\Gamma_\pi$, each appearing with multiplicity $\dim V_\sigma$. Since Γ stabilizes $\mathfrak{a}$, $\pi \rtimes \sigma$ has all $\mathcal{O}(\mathfrak{t})$-weights in $\mathfrak{a}$ if and only if that holds for π. As Γ stabilizes Δ, it preserves temperedness of $\mathbb{H}$-representations. Consequently $\pi \rtimes \sigma$ is tempered if and only if π is tempered. In particular $\mathrm{Irr}_0(\mathbb{H} \rtimes \Gamma)$ consists of the representations $\pi \rtimes \sigma$ with $\pi \in \mathrm{Irr}_0(\mathbb{H})$ and $\sigma \in \mathrm{Irr}(\mathbb{C}[\Gamma_\pi, \natural_\pi])$.

Lemma 6.6. *Let $\mathbb{H}$ be one of the graded Hecke algebras for which we already proved Theorem 6.2 and Lemma 6.5. Then Theorem 6.2 holds for $\mathbb{H} \rtimes \Gamma$.*

Proof. The same Clifford theory as above can also be used to relate the irreducible representations of W and of $W \rtimes \Gamma$. Recall that Γ acts on W by automorphisms of the Coxeter system (W, S). In this setting it is known from [3, Proposition 4.3] that the 2-cocycle $\natural_{\pi_W}$ associated to any $\pi_W \in \mathrm{Irr}(W)$ is trivial in $H^2(\Gamma_{\pi_W}, \mathbb{C}^\times)$. Hence we can find I_γ for π_W such that

$$\Gamma_{\pi_W} \to \mathrm{Aut}_{\mathbb{C}}(V_{\pi_W}) : \gamma \mapsto I_\gamma \tag{6.7}$$

is a group homomorphism. Then $\mathrm{Irr}(W \rtimes \Gamma)$ can be parametrized by Γ-orbits of pairs (π_W, σ_W) with $\pi_W \in \mathrm{Irr}(W)$ and $\sigma_W \in \mathrm{Irr}(\Gamma_{\pi_W})$.

We consider any $\pi \in \mathrm{Irr}_0(\mathbb{H})$. By the uniqueness in Theorem 6.2.c, $\zeta_{\mathbb{H}}$ is Γ-equivariant and $\Gamma_\pi = \Gamma_{\zeta_{\mathbb{H}}(\pi)}$. The intertwiners $I_\gamma : \pi \to \gamma^*(\pi)$ also qualify as intertwiners $I_\gamma : \zeta_{\mathbb{H}}(\pi) \to \gamma^*(\zeta_{\mathbb{H}}(\pi))$, because $\zeta_{\mathbb{H}}(\pi)$ is contained in $\mathrm{Res}_W(\pi)$ with multiplicity one. Therefore we can specify a unique $I_\gamma : \pi \to \gamma^*(\pi)$ by the requirement that its restriction to $\zeta_{\mathbb{H}}(\pi)$ equals the I_γ from (6.7). Then

$$\Gamma_\pi \to \mathrm{Aut}_{\mathbb{C}}(V_\pi) : \gamma \mapsto I_\gamma \tag{6.8}$$

is a group homomorphism. Now Clifford theory parametrizes $\mathrm{Irr}_0(\mathbb{H} \rtimes \Gamma)$ via Γ-orbits of pairs (π, σ) with $\pi \in \mathrm{Irr}_0(\mathbb{H})$ and $\sigma \in \mathrm{Irr}(\Gamma_\pi)$. In particular we obtain a bijection (which will be $\zeta_{\mathbb{H} \rtimes \Gamma}$)

$$\mathrm{Irr}_0(\mathbb{H} \rtimes \Gamma) \to \mathrm{Irr}(W \rtimes \Gamma) : \pi \rtimes \sigma \mapsto \zeta_{\mathbb{H}} \rtimes \sigma. \tag{6.9}$$

Let $f_{R,k} : \mathrm{Irr}_0(\mathbb{H}) \to \mathbb{R}$ be as in the proof of Lemma 6.5. We define

$$f_{R,k}(\pi \rtimes \sigma) = \min\{f_{R,k}(\gamma^*(\pi)) : \gamma \in \Gamma\}. \tag{6.10}$$

Notice that the irreducible $W \rtimes \Gamma$-representation $\zeta_{\mathbb{H}} \rtimes \sigma$ appears in $\mathrm{Res}_{W \rtimes \Gamma}(\pi \rtimes \sigma)$. For any other irreducible constituent $\pi'_{W \rtimes \Gamma}$ of $\mathrm{Res}_{W \rtimes \Gamma}(\pi \rtimes \sigma)$, every irreducible W-subrepresentation of $\pi'_{W \rtimes \Gamma}$ is contained in $\mathrm{Res}_W(\gamma^*(\pi))$ for some $\gamma \in \Gamma$. Since $\zeta_{\mathbb{H}}(\pi)$ appears with multiplicity one in $\mathrm{Res}_W(\pi)$, the subspace $\zeta_{\mathbb{H}}(\pi) \rtimes \sigma$ of $\mathrm{Res}_{W \rtimes \Gamma}(\pi \rtimes \sigma)$ exhausts the W-subrepresentations $\mathrm{Res}_W\big(\gamma^*(\zeta_{\mathbb{H}}(\pi))\big)$ in $\mathrm{Res}_{W \rtimes \Gamma}(\pi \rtimes \sigma)$. Hence $\pi'_{W \rtimes \Gamma}$ has W-constituents $\gamma^*(\pi'_W)$ with $\pi'_W \subset \mathrm{Res}_W(\pi)$ but $\pi'_W \not\cong \zeta_{\mathbb{H}}(\pi)$. Lemma 6.5 and (6.5) say that

$$f_{R,k}(\gamma^*(\pi'_W)) > 3/4 + f_{R,k}(\zeta_{\mathbb{H}}(\pi)) \qquad \text{for all } \gamma \in \Gamma. \tag{6.11}$$

Take a total order on $\mathrm{Irr}_0(\mathbb{H} \rtimes \Gamma)$ that refines the partial order defined by $f_{R,k}$. We transfer $f_{R,k}$ and this total order to $\mathrm{Irr}(W \rtimes \Gamma)$ via the bijection (6.9). Then the above verifies parts (a) and (b) of Theorem 6.2 for $\mathbb{H} \rtimes \Gamma$. It follows that there is a unique $\zeta_{\mathbb{H} \rtimes \Gamma}$ that fulfils the requirements, namely (6.9) $\square$

Notice that in Lemma 6.4 we did not allow $k_2/k_1 \in (0, 1/2)$. Other special cases that we skipped in Lemmas 6.3 and 6.4 are:

$$\begin{array}{lcll} \mathbb{H}(\mathbb{C}^n, W(B_n), k) & = & \mathbb{H}(\mathbb{C}^n, W(D_n), k_1) \rtimes \langle s_{e_n} \rangle & \text{when } k_2 = 0, \\ \mathbb{H}(\mathbb{C}^n, W(B_n), k) & = & \mathbb{H}(\mathbb{C}^n, W(A_1)^n, k_2) \rtimes S_n & \text{when } k_1 = 0, \\ \mathbb{H}(\mathbb{C}^4, W(F_4), k) & \cong & \mathbb{H}(\mathbb{C}^4, W(D_4), k_i) \rtimes S_3 & \text{when } k_{3-i} = 0, \\ \mathbb{H}(\mathbb{C}^2, W(G_2), k) & \cong & \mathbb{H}(\mathbb{C}^2, W(A_2), k_i) \rtimes S_2 & \text{when } k_{3-i} = 0. \end{array} \tag{6.12}$$

Let us relate some of these cases.

Lemma 6.7. *Let $k : B_n \to \mathbb{R}_{>0}$ be a parameter function with $k_2/k_1 \in (0, 1/2)$. There exists a canonical bijection*

$$\mathrm{Irr}_0\big(\mathbb{H}(\mathbb{C}^n, W(B_n), k)\big) \to \mathrm{Irr}_0\big(\mathbb{H}(\mathbb{C}^n, W(D_n), k_1) \rtimes \langle s_{e_n} \rangle\big)$$

which preserves $W(B_n)$-types.

Proof. We abbreviate $\mathbb{H} = \mathbb{H}(\mathbb{C}^n, W(B_n), k)$ and $\mathbb{H}' = \mathbb{H}(\mathbb{C}^n, W(D_n), k_1) \rtimes \langle s_{e_n} \rangle$. As explained in the proof of Lemma 6.4, for k within the range of parameters considered in this lemma, $\mathrm{Irr}_0(\mathbb{H})$ is essentially independent of k. By varying k_2 continuously, we can reach the algebra $\mathbb{H}'$, which however may behave differently. By Clifford theory $\mathrm{Irr}_0(\mathbb{H}')$ consists of representations of the following kinds:

(i) $\mathrm{ind}_{\mathbb{H}(\mathbb{C}^n, W(D_n), k_1)}^{\mathbb{H}'}(\pi)$, where $\pi \in \mathrm{Irr}_0(\mathbb{H}(\mathbb{C}^n, W(D_n), k_1))$ is not equivalent with $s_{e_n}^*(\pi)$,

(ii) $V_\pi \otimes V_\sigma$, where $\pi \in \mathrm{Irr}_0(\mathbb{H}(\mathbb{C}^n, W(D_n), k_1))$ is fixed by $s_{e_n}^*$ and $\sigma \in \mathrm{Irr}(\langle s_{e_n} \rangle) = \{\mathrm{triv}, \mathrm{sign}\}$.

Let us investigate what happens when we deform $k_2 = 0$ to a positive but very small real number. Accordingly we replace

$$\pi'_2 = \mathrm{ind}_{\mathbb{H}(\mathbb{C}^n, W(D_n), k_1)}^{\mathbb{H}'}(\pi) \quad \text{by} \quad \pi_2 = \mathrm{ind}_{\mathbb{H}(\mathbb{C}^n, W(D_n), k_1)}^{\mathbb{H}}(\pi).$$

The map

$$wf \mapsto wf \qquad f \in \mathcal{O}(\mathbb{C}^n), w \in W(B_n)$$

is a linear bijection $\mathbb{H}' \to \mathbb{H}$, so $\mathrm{Res}_{W(B_n)} \pi'_2 = \mathrm{Res}_{W(B_n)} \pi_2$.

(i′) We claim that in case (i) π_2 is still irreducible.

As vector spaces $V_{\pi_2} = V_\pi \oplus s_{e_n} V_\pi$. For any nonzero linear subspace V of V_{π_2}, the irreducibility of π'_2 tells us that there exists an $h \in \mathbb{H}'$ such that $\pi'_2(h)V \not\subset V$. For $k_2 > 0$ very small, the corresponding element of $\mathbb{H}$ still satisfies $\pi_2(h)V \not\subset V$. This verifies the claim (i′).

(ii′) In case (ii), we claim that π_2 is reducible.

From Theorem 6.1 we know that

$$|\mathrm{Irr}_0(\mathbb{H})| = |\mathrm{Irr}(W(B_n))| = |\mathrm{Irr}_0(\mathbb{H}')| \tag{6.13}$$

(i) and (ii) provide a way to count the right hand side:

- every $\langle s_{e_n} \rangle$-orbit of length two in $\mathrm{Irr}_0(\mathbb{H}(\mathbb{C}^n, W(D_n), k_1))$ contributes one,
- every $s^*_{e_n}$-fixed element of $\mathrm{Irr}_0(\mathbb{H}(\mathbb{C}^n, W(D_n), k_1))$ contributes two.

The restriction of any element of $\mathrm{Irr}_0(\mathbb{H})$ to $\mathbb{H}(\mathbb{C}^n, W(D_n), k_1)$ has all irreducible constituents in $\mathrm{Irr}_0(\mathbb{H}(\mathbb{C}^n, W(D_n), k_1))$. By Frobenius reciprocity, this implies that it is a constituent of π_2 for some $\pi \in \mathrm{Irr}_0(\mathbb{H}(\mathbb{C}^n, W(D_n), k_1))$.

When $s^*_{e_n}(\pi) \not\cong \pi$, we saw in (i′) that $\{\pi, s^*_{e_n}(\pi)\}$ contributes just one representation to $\mathrm{Irr}_0(\mathbb{H})$. In case $s^*_{e_n}(\pi) \cong \pi$, π_2 can be reducible. It has length at most two, because that is its length as $\mathbb{H}(\mathbb{C}^n, W(D_n), k_1)$-module. When π_2 would contribute only one representation to $\mathrm{Irr}_0(\mathbb{H})$, the sum of the contributions from the cases (i′) and (ii′) would be strictly smaller than the corresponding sum of the contributions from (i) and (ii) to $\mathrm{Irr}(\mathbb{H}')$. However, that would contradict (6.13). We conclude that (ii′) holds.

A π_2 as in (ii′) has length 2, and both its irreducible constituents become isomorphic to π upon restriction to $\mathbb{H}(\mathbb{C}^n, W(D_n), k_1)$. As

$$\mathrm{Res}_{W(B_n)}(\pi_2) = \mathrm{ind}_{W(D_n)}^{W(B_n)} \mathrm{Res}_{W(D_n)}(\pi),$$

the restriction to $W(B_n)$ of the two constituents of π_2 must be $\mathrm{Res}_{W(D_n)}(\pi) \otimes \mathrm{triv}$ and $\mathrm{Res}_{W(D_n)}(\pi) \otimes \mathrm{sign}$. Here $\mathrm{Res}_{W(D_n)}(\pi)$ extends to a representation of $W(B_n)$, while triv and sign are representations of $W(B_n)/W(D_n)$. In combination with case (i′) we see that

$$\mathrm{Res}_{W(B_n)}(\mathrm{Irr}_0(\mathbb{H}')) = \mathrm{Res}_{W(B_n)}(\mathrm{Irr}_0(\mathbb{H})).$$

Hence there is a unique bijection $\mathrm{Irr}_0(\mathbb{H}) \to \mathrm{Irr}_0(\mathbb{H}')$ that preserves W-types. □

Finally, we settle the remaining cases of Theorem 6.2.

Proof. Lemma 6.6 establishes Theorem 6.2 for the algebras in (6.12). Moreover (6.10) gives us a function $f_{R,k}$ that defines a useful partial order on $\mathrm{Irr}_0(\mathbb{H})$ and $\mathrm{Irr}(W)$. With Lemma 6.7 we transfer all that to $\mathbb{H}(\mathbb{C}^n, W(B_n), k)$ with $k_2/k_1 \in (0, 1/2)$. Then we have Theorem 6.2 whenever R is irreducible and spans $\mathfrak{a}^*$. As noted in the proof of Lemma 6.3, that implies Theorem 6.2 for all $\mathbb{H}$ (still with the condition on the parameters for type F_4 components). We finish the proof by applying Lemmas 6.5 and 6.6 another time. □

Suppose that $\mathbb{H} = \mathbb{H}(G, L, \mathcal{L}, \mathbf{r})/(\mathbf{r} - r)$ for a cuspidal local system $\mathcal{L}$ on a nilpotent orbit for L (as in Section 5.2). In terms of Theorem 5.7, $\zeta_{\mathbb{H}}$ from Theorem 6.2 is just the map $\widetilde{M}_{y,0,r,\rho} \mapsto \widetilde{M}_{y,0,0,\rho}$. Here $(y, \rho) \mapsto \widetilde{M}_{y,0,0,\rho}$ is the generalized Springer correspondence from [48], twisted by the sign character of W. So, for a graded Hecke algebra that can be constructed with equivariant homology, Theorem 6.2 recovers a generalized Springer correspondence for W.

Let us be more flexible, and call any nice parametrization of $\mathrm{Irr}(W)$ a generalized Springer correspondence. Then Theorem 6.2 qualifies as such, and we can regard $\zeta_{\mathbb{H}} : \mathrm{Irr}_0(\mathbb{H}) \to \mathrm{Irr}(W)$

as a "generalized Springer correspondence with graded Hecke algebras". This point of view has been pursued in [75], where $\mathrm{Irr}_0(\mathbb{H}(\mathbb{C}^n, W(B_n), k))$ has been parametrized with combinatorial data that mimic the above pairs (y, ρ).

6.2. A generalized Springer correspondence with affine Hecke algebras

With Corollary 3.30 we can translate Theorem 6.2 into a statement about all tempered irreducible representations of $\mathcal{H} = \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})$, in relation with tempered $\mathbb{C}[X \rtimes W]$-representations. We want to generalize that to all irreducible $\mathcal{H}$-representations, at least when $q_s \geq 1$ for all $s \in S_{\mathrm{aff}}$.

Our main tool will be the Langlands classification. We need a version for graded Hecke algebras extended with automorphism groups Γ as in Theorem 6.2. It can be obtained by combining Theorem 3.13 for $\mathbb{H}$ with Clifford theory. Given a Langlands datum (P, τ, λ) for $\mathbb{H}$, let $\Gamma_{P,\tau,\lambda}$ be its stabilizer in Γ, and recall the 2-cocycle $\natural_{P,\tau,\lambda}$ from (6.6). We define a Langlands datum for $\mathbb{H} \rtimes \Gamma$ to be a quadruple (P, τ, λ, ρ), where

- (P, τ, λ) is a Langlands datum for $\mathbb{H}$ (so $\tau \in \mathrm{Irr}(\mathbb{H}_P)$ is tempered and $\lambda \in \mathfrak{a}^{P++} + i\mathfrak{a}^P$),
- $\rho \in \mathrm{Irr}(\mathbb{C}[\Gamma_{P,\tau,\lambda}, \natural_{P,\tau,\lambda}])$.

To such a quadruple we associate the irreducible $\mathbb{H}^P$-representation $((\tau \otimes \lambda) \otimes \rho, V_\tau \otimes_{\mathbb{C}} V_\rho)$ and the $\mathbb{H} \rtimes \Gamma$-representation

$$\pi(P, \tau, \lambda, \rho) = \mathrm{ind}_{\mathbb{H}^P \rtimes \Gamma_{P,\tau,\lambda}}^{\mathbb{H} \rtimes \Gamma} (\tau \otimes \lambda \otimes \rho). \tag{6.14}$$

As a consequence of [79, Corollary 2.2.5] and Section 3.5, we find an extended Langlands classification:

Corollary 6.8. *Let (P, τ, λ, ρ) be a Langlands datum for $\mathbb{H} \rtimes \Gamma$.*

(a) The $\mathbb{H} \rtimes \Gamma$-representation $\pi(P, \tau, \lambda, \rho)$ has a unique irreducible quotient, which we call $L(P, \tau, \lambda, \rho)$.
(b) For every irreducible $\mathbb{H} \rtimes \Gamma$-representation π, there exists a Langlands datum $(P', \tau', \lambda', \rho')$, unique up to the canonical Γ-action, such that $\pi \cong \pi(P', \tau', \lambda', \rho')$.
(c) $L(P, \tau, \lambda, \rho)$ and $\pi(P, \tau, \lambda, \rho)$ are tempered if and only if $P = \Delta$ and $\lambda \in i\mathfrak{a}^\Delta$.

In this context we call $\pi(P, \tau, \lambda, \rho)$ a standard $\mathbb{H} \rtimes \Gamma$-module. When τ has real central character, it follows from Theorem 6.2 that the representation $\mathrm{Res}_{W_P}(\tau) \otimes \lambda$ of $\mathbb{H}(\mathfrak{t}, W_P, 0) = \mathcal{O}(\mathfrak{t}) \rtimes W_P$ has stabilizer $\Gamma_{P,\tau,\lambda}$ in $\{\gamma \in \Gamma : \gamma(P) = P\}$. The $\mathbb{H}^P$-intertwining operators I_γ ($\gamma \in \Gamma_{P,\tau,\lambda}$) from (6.12) are also $\mathcal{O}(\mathfrak{t}) \rtimes W_P$-intertwining operators, so $(\mathrm{Res}_{W_P}(\tau) \otimes \lambda \otimes \rho, V_\tau \otimes_{\mathbb{C}} V_\rho)$ is a well-defined representation of

$$\mathbb{H}(\mathfrak{t}, W_P, 0) \rtimes \Gamma_{P,\tau,\lambda} = \mathcal{O}(\mathfrak{t}) \rtimes W_P \Gamma_{P,\tau,\lambda}.$$

Its parabolic induction is

$$\pi(P, \mathrm{Res}_{W_P}(\tau), \lambda, \rho) = \mathrm{ind}_{\mathcal{O}(\mathfrak{t}) \rtimes W_P \Gamma_{P,\tau,\lambda}}^{\mathcal{O}(\mathfrak{t}) \rtimes W \Gamma} (\mathrm{Res}_{W_P}(\tau) \otimes \lambda \otimes \rho). \tag{6.15}$$

This representation has central character $W\Gamma\lambda$ and the construction mimics that in (6.14), so

$$\pi(P, \mathrm{Res}_{W_P}(\tau), \lambda, \rho) = \pi(P, \tau, \lambda, \rho) \quad \text{as} \quad \mathbb{C}[W\Gamma]\text{-representations.}$$

We note that in general $\pi(P, \mathrm{Res}_{W_P}(\tau), \lambda, \rho)$ differs from $\pi(P, \mathrm{Res}_{W_P \rtimes \Gamma_{P,\tau,\lambda}}(\tau \otimes \lambda \otimes \rho))$, because the latter has central character $0 \in \mathfrak{t}/W\Gamma$. Of course the $\mathbb{H} \rtimes \Gamma$-representation (6.15) may have more than one irreducible quotient, because $\mathrm{Res}_{W_P}(\tau)$ usually is reducible. Recall from the proof of Lemma 6.6 that we can arrange that

$$\Gamma_{P,\tau,\lambda} \to \mathrm{Aut}_{\mathbb{C}}(V_\pi) : \gamma \mapsto I_\gamma$$

is a group homomorphism. Then $\natural_{P,\tau,\lambda} = 1$ and ρ becomes simply an irreducible representation of $\Gamma_{P,\tau,\lambda}$. This construction yields a canonical map

$$\begin{array}{llcl} \mathrm{Res}_{\mathcal{O}(\mathfrak{t}) \rtimes W\Gamma} : & \{\text{standard } \mathbb{H} \rtimes \Gamma\text{-modules}\} & \longrightarrow & \mathrm{Mod}_f(\mathcal{O}(\mathfrak{t}) \rtimes W\Gamma) \\ & \pi(P,\tau,\lambda,\rho) & \mapsto & \pi(P, \mathrm{Res}_{W_P}(\tau), \lambda, \rho). \end{array} \tag{6.16}$$

For $\lambda = 0$, this just the restriction map $\mathrm{Res}_{W \rtimes \Gamma}$, in combination with (6.1). In terms of (4.5) and (4.6), we can express (6.16) as

$$\mathrm{Res}_{\mathcal{O}(\mathfrak{t}) \rtimes W\Gamma}\big(\mathrm{Hom}_{\mathfrak{R}_{P,\delta,\lambda}}\big(\rho, \mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H} \rtimes \Gamma}(\delta \otimes \lambda)\big)\big) = \mathrm{Hom}_{\mathfrak{R}_{P,\delta,\lambda}}\big(\rho, \mathrm{ind}_{\mathcal{O}(\mathfrak{t}) \rtimes W_P}^{\mathcal{O}(\mathfrak{t}) \rtimes W\Gamma}\big(\mathrm{Res}_{W_P}(\delta) \otimes \lambda\big)\big).$$

Here we used [80, Theorem 9.2] to extend the notion of R-groups to $\mathbb{H} \rtimes \Gamma$.

Lemma 6.9. *Let $k : R \to \mathbb{R}$ be a parameter function as in Theorem 6.2, so that in particular Theorem 6.2.b provides a total order $>$ on $\mathrm{Irr}(W_P)$. Let (P,τ,λ,ρ) be a Langlands datum for $\mathbb{H} \rtimes \Gamma$.*

All irreducible constituents of $\pi(P, \mathrm{Res}_{W_P}(\tau), \lambda, \rho)$ different from $\pi(P, \zeta_{\mathbb{H}_P}(\tau), \lambda, \rho)$ are of the form $\pi(P, \tau'_W, \lambda, \rho'_W)$, where $\tau'_W > \zeta_{\mathbb{H}_P}(\tau)$ and $(P, \tau'_W, \lambda, \rho'_W)$ is a Langlands datum for $\mathbb{H}(\mathfrak{t}, W, 0) \rtimes \Gamma = \mathcal{O}(\mathfrak{t}) \rtimes W\Gamma$.

Proof. By Theorem 6.2 every irreducible constituent τ'_W of $\mathrm{Res}_{W_P}(\tau)$ different from $\zeta_{\mathbb{H}_P}(\tau)$ is strictly larger than $\zeta_{\mathbb{H}_P}(\tau)$. Although the $\mathcal{O}(\mathfrak{t}) \rtimes W_P$-representation $\tau'_W \otimes \lambda$ is irreducible, its stabilizer in

$$\Gamma_{P,\lambda} = \{\gamma \in \Gamma : \gamma(P) = P, \gamma(\lambda) = \lambda\}$$

need not be $\Gamma_{P,\tau,\lambda}$. To overcome that, we rather work with $\Gamma_{P,\lambda}$. Put $\tau' = \mathrm{ind}_{\mathbb{H}_P \rtimes \Gamma_{P,\tau,\lambda}}^{\mathbb{H}_P \rtimes \Gamma_{P,\lambda}}(\tau \otimes \rho)$, so that

$$\pi(P,\tau,\lambda,\rho) = \mathrm{ind}_{\mathbb{H}_P \rtimes \Gamma_{P,\lambda}}^{\mathbb{H} \rtimes \Gamma}(\tau' \otimes \lambda).$$

Take $\rho'_W \in \mathrm{Irr}(\Gamma_{P,\tau'_W,\lambda})$ such that

$$\mathrm{ind}_{\mathcal{O}(\mathfrak{t}) \rtimes W_P \Gamma_{P,\tau'_W,\lambda}}^{\mathcal{O}(\mathfrak{t}) \rtimes W_P \Gamma_{P,\lambda}}(\tau'_W \otimes \rho'_W) \tag{6.17}$$

is a subrepresentation of

$$\mathrm{Res}_{W_P \rtimes \Gamma_{P,\lambda}}(\tau') = \mathrm{ind}_{\mathcal{O}(\mathfrak{t}) \times W_P \Gamma_{P,\tau,\lambda}}^{\mathcal{O}(\mathfrak{t}) \rtimes W_P \Gamma_{P,\lambda}}(\mathrm{Res}_{W_P}(\tau) \otimes \rho). \tag{6.18}$$

Then $\pi(P, \tau'_W, \lambda, \rho'_W)$ is a subrepresentation of $\pi(P, \mathrm{Res}_{W_P}(\tau), \lambda, \rho)$, and by Corollary 6.8 it is irreducible. With this construction we can obtain any subrepresentation of (6.18) whose W_P-constituents are not $\Gamma_{P,\lambda}$-associate to $\zeta_{\mathbb{H}_P}(\tau)$.

Since $\zeta_{\mathbb{H}_P}(\tau)$ appears with multiplicity one in $\mathrm{Res}_{W_P}(\tau)$, $\zeta_{\mathbb{H}_P}(\tau) \otimes \lambda \otimes \rho$ exhausts all $\Gamma_{P,\tau,\lambda}$-associates of $\zeta_{\mathbb{H}_P}(\tau)$ in $\mathrm{Res}_{W_P}(\tau) \otimes \lambda \otimes \rho$. Then

$$\mathrm{ind}_{\mathcal{O}(\mathfrak{t}) \times W_P \Gamma_{P,\tau,\lambda}}^{\mathcal{O}(\mathfrak{t}) \rtimes W_P \Gamma_{P,\lambda}}(\zeta_{\mathbb{H}_P}(\tau) \otimes \lambda \otimes \rho) \tag{6.19}$$

exhausts all $\Gamma_{P,\lambda}$-associates of $\zeta_{\mathbb{H}_P}(\tau)$ in

$$\mathrm{ind}_{\mathcal{O}(\mathfrak{t}) \times W_P \Gamma_{P,\tau,\lambda}}^{\mathcal{O}(\mathfrak{t}) \rtimes W_P \Gamma_{P,\lambda}}(\mathrm{Res}_{W_P}(\tau) \otimes \lambda \otimes \rho). \tag{6.20}$$

Consequently (6.19) and the modules (6.17) exhaust the whole of (6.20). This remains the case after inducing everything to $\mathcal{O}(\mathfrak{t}) \rtimes W\Gamma$.

Therefore $\pi(P, \mathrm{Res}_{W_P}(\tau), \lambda, \rho)$ does not have any other subrepresentations besides $\pi(P, \zeta_{\mathbb{H}_P}(\tau), \lambda, \rho)$ and the $\pi(P, \tau'_W, \lambda, \rho'_W)$. $\square$

From (6.16) we will deduce a map whose image consists of irreducible representations of $\mathcal{O}(\mathfrak{t}) \rtimes W\Gamma$. By Corollary 2.1, those are the same as standard $\mathcal{O}(\mathfrak{t}) \rtimes W\Gamma$-modules. We call a central character for $\mathbb{H} \rtimes \Gamma$ real if it lies in $\mathfrak{a}/W\Gamma$.

Proposition 6.10. *Let $\mathbb{H} \rtimes \Gamma$ be as in Theorem 6.2.*

(a) There exists a unique bijection $\zeta_{\mathbb{H}\rtimes\Gamma}$ between:

- *the set of standard $\mathbb{H} \rtimes \Gamma$-modules with real central character,*
- *the set of irreducible $\mathcal{O}(\mathfrak{t}) \rtimes W\Gamma$-representations with real central character,*

such that $\zeta_{\mathbb{H}\rtimes\Gamma}(\pi)$ is always a constituent of $\mathrm{Res}_{\mathcal{O}(\mathfrak{t})\rtimes W\Gamma}(\pi)$.
For suitable total orders on these two sets, the matrix of (the linear extension of) $\zeta_{\mathbb{H}\rtimes\Gamma}$ is the identity, while the matrix of $\mathrm{Res}_{\mathcal{O}(\mathfrak{t})\rtimes W\Gamma}$ is upper triangular and unipotent.

(b) There exists a natural bijection $\zeta'_{\mathbb{H}\rtimes\Gamma}$ from the set of irreducible $\mathbb{H} \rtimes \Gamma$-representations with real central character to the analogous set for $\mathcal{O}(\mathfrak{t}) \rtimes W\Gamma$, such that:

- *for any standard module π, $\zeta_{\mathbb{H}\rtimes\Gamma}(\pi)$ equals $\zeta'_{\mathbb{H}\rtimes\Gamma}$ of the irreducible quotient of π,*
- *for tempered representations it coincides with part (a).*

Proof. (a) By Lemma 6.9, any candidate for such a map must preserve the P and the λ in the Langlands datum of π (from Corollary 6.8). By Lemma 3.3 the condition on the central character of π means that $\lambda \in \mathfrak{a}^{P++}$ and the ingredient τ lies in $\mathrm{Irr}_0(\mathbb{H}_P)$. Hence we can specialize to a map

$$\{(\tau, \rho) : \tau \in \mathrm{Irr}_0(\mathbb{H}_P), \rho \in \mathrm{Irr}(\Gamma_{P,\tau,\lambda})\} \longrightarrow \{(\tau_W, \rho_W) : \tau_W \in \mathrm{Irr}(W_P), \rho_W \in \mathrm{Irr}(\Gamma_{P,\tau_W,\lambda})\}.$$

Here the left hand side parametrizes $\mathrm{Irr}_0(\mathbb{H}_P \rtimes W_{P,\lambda})$ and the right hand side parametrizes $\mathrm{Irr}(W_P \rtimes \Gamma_{P,\lambda})$. For fixed (P, λ) there is a commutative diagram

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{standard } \mathbb{H} \rtimes \Gamma\text{-modules} \\ \text{with real central character} \end{array} \right\} & \xrightarrow{\mathrm{Res}_{\mathcal{O}(\mathfrak{t})\rtimes W\Gamma}} & \mathrm{Mod}_f(\mathcal{O}(\mathfrak{t}) \rtimes W\Gamma) \\ \uparrow \mathrm{ind}_{\mathbb{H}^P \rtimes \Gamma_{P,\lambda}}^{\mathbb{H}\rtimes\Gamma} \otimes \lambda & & \uparrow \mathrm{ind}_{\mathcal{O}(\mathfrak{t})\rtimes W_P\Gamma_{P,\lambda}}^{\mathcal{O}(\mathfrak{t})\rtimes W\Gamma} \otimes \lambda \\ \mathrm{Irr}_0(\mathbb{H}_P \rtimes \Gamma_{P,\lambda}) & \xrightarrow{\mathrm{Res}_{W_P \rtimes \Gamma_{P,\lambda}}} & \mathrm{Mod}_f(\mathcal{O}(\mathfrak{t}_P) \rtimes W_P\Gamma_{P,\lambda}) \end{array} \tag{6.21}$$

where the vertical arrows send $\tau \rtimes \rho$ to $\pi(P, \tau, \lambda, \rho)$ and $\tau_W \rtimes \rho_W$ to $\pi(P, \tau_W, \lambda, \rho_W)$. By Theorem 6.2 for $\mathbb{H}_P \rtimes \Gamma_{P,\lambda}$, the matrix of $\mathrm{Res}_{W_P\rtimes\Gamma_{P,\lambda}}$ (with respect to suitable total orders) is upper triangular and unipotent. Hence there is a unique map $\zeta_{\mathbb{H}\rtimes\Gamma}$ that fulfills the requirements for fixed (P, λ), namely

$$\mathrm{ind}_{\mathbb{H}^P\rtimes\Gamma_{P,\lambda}}^{\mathbb{H}\rtimes\Gamma}(\tau \rtimes \rho \otimes \lambda) \mapsto \mathrm{ind}_{\mathcal{O}(\mathfrak{t})\rtimes W_P\Gamma_{P,\lambda}}^{\mathcal{O}(\mathfrak{t})\rtimes W\Gamma}(\zeta_{\mathbb{H}_P\rtimes\Gamma_{P,\lambda}}(\tau \rtimes \rho) \otimes \lambda). \tag{6.22}$$

This translates to

$$\zeta_{\mathbb{H}\rtimes\Gamma}\pi(P, \tau, \lambda, \rho) = \pi(P, \zeta_{\mathbb{H}_P}(\tau), \lambda, \rho). \tag{6.23}$$

Theorem 6.2 and the commutative diagram (6.21) entail the required properties of the matrices of $\zeta_{\mathbb{H}\rtimes\Gamma}$ and $\mathrm{Res}_{\mathcal{O}(\mathfrak{t})\rtimes W\Gamma}$.

(b) By Corollary 6.8 taking the irreducible quotient of a standard module provides a natural bijection

$$\{\text{standard } \mathbb{H} \rtimes \Gamma\text{-modules}\} \longrightarrow \mathrm{Irr}(\mathbb{H} \rtimes \Gamma). \tag{6.24}$$

Define $\zeta'_{\mathbb{H}\rtimes\Gamma}$ as the composition of the inverse of (6.24) with $\zeta_{\mathbb{H}\rtimes\Gamma}$ from part (a). By Corollary 6.8 every tempered irreducible module is also standard, so for tempered representations the properties of $\zeta_{\mathbb{H}\rtimes\Gamma}$ remain valid for $\zeta'_{\mathbb{H}\rtimes\Gamma}$. □

With Corollary 3.30 we will transfer Proposition 6.10 to affine Hecke algebras with parameters in $\mathbb{R}_{\geq 1}$. We may and will take $\mathcal{H} = \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})$ as in Lemma 3.31. Let $u \in T_{\mathrm{un}}$. By (3.19) the parameter function k_u for $\mathbb{H}(\mathfrak{t}, W(R_u), k_u) \rtimes \Gamma_u$ takes values in $\mathbb{R}_{\geq 0}$. Recall from Lemma 3.32 that Theorems 3.28 and 3.29 are compatible with temperedness and parabolic induction. Let $\pi(P, \tau, \lambda, \rho)$ be a standard module for $\mathbb{H}(\mathfrak{t}, W(R_u), k_u) \rtimes \Gamma_u$, with real central character. Via Theorems 3.28 and 3.29 it corresponds naturally to a standard $\mathcal{H}$-module $\pi(P', \tau', t)$ with $\tau' \in \mathrm{Irr}(\mathcal{H}_{P'})$ tempered and $|t| = \exp(\lambda)$. We define

$$\mathrm{Res}_{\mathcal{O}(T)\rtimes W} : \{\text{standard } \mathcal{H}\text{-modules}\} \longrightarrow \mathrm{Mod}_f(\mathcal{O}(T) \rtimes W)$$

by commutativity of the following diagram (for central characters in $W u \exp(\mathfrak{a})$):

$$\begin{array}{ccc} \{\text{standard } \mathcal{H}\text{-modules}\} & \xrightarrow{\mathrm{Res}_{\mathcal{O}(T)\rtimes W}} & \mathrm{Mod}_f(\mathcal{O}(T) \rtimes W) \\ \uparrow \text{Corollary 3.30} & & \uparrow \text{Corollary 3.30} \\ \left\{ \begin{array}{c} \text{standard } \mathbb{H}(\mathfrak{t}, W(R_u), k_u) \rtimes \Gamma_u- \\ \text{modules with real central character} \end{array} \right\} & \xrightarrow{\mathrm{Res}_{\mathcal{O}(\mathfrak{t})\rtimes W(R_u)\Gamma_u}} & \mathrm{Mod}_f(\mathcal{O}(\mathfrak{t}) \rtimes W(R_u)\Gamma_u) \end{array}$$

The map $\mathrm{Res}_{\mathcal{O}(T)\rtimes W}$ can be made more explicit with (4.3) and Corollary 4.4. In those terms

$$\mathrm{Res}_{\mathcal{O}(T)\rtimes W}\big(\pi(P, \delta, t, \rho)\big) = \mathrm{Hom}_{\mathbb{C}[\mathfrak{R}_{P,\delta,t}, \natural_{P,\delta,t}]}\big(\rho, \pi(P, \mathrm{Res}_{W_P}(\delta), t)\big). \tag{6.25}$$

For tempered standard $\mathcal{H}$-modules (i.e. with $t \in T^P_{\mathrm{un}}$), $\mathrm{Res}_{\mathcal{O}(T)\rtimes W}$ can really be considered as a restriction, see [79, §4.4]. The above map is the natural generalization to all standard $\mathcal{H}$-modules. However, it is not a restriction (along some injective algebra homomorphism) because it can happen that $\pi(P, \delta, t, \rho)$ is reducible but its image (6.25) is irreducible.

We do not know how to extend $\mathrm{Res}_{\mathcal{O}(T)\rtimes W}$ to arbitrary $\mathcal{H}$-representations. The best we can do is to define the W-type of any finite dimensional $\mathcal{H}$-representation, in the following way:

- By decomposing it as in (3.3), we may assume that all its $\mathcal{O}(T)$-weights lie in a single W-orbit, say in $W u \exp(\mathfrak{a})$ with $u \in T_{\mathrm{un}}$.
- Apply Theorems 3.28 and 3.29 to produce a representation of $\mathbb{H}(\mathfrak{t}, W(R_u), k_u) \rtimes \Gamma_u$.
- Restrict to $\mathbb{C}[W(R_u) \rtimes \Gamma_u]$ and then induce to $\mathbb{C}[W]$.

Of course this mimics the earlier W-type maps for graded Hecke algebras. In [79, §4.1–4.2] it was shown that the W-type of an $\mathcal{H}$-representation can also be obtained via a continuous deformation of $\mathbf{q}$ to 1.

We are ready to transfer Proposition 6.10 to affine Hecke algebras:

Theorem 6.11. *Let $\mathcal{H} = \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q})$ be an affine Hecke algebra with parameter functions $\lambda, \lambda^* : R \to \mathbb{R}_{\geq 0}$. Suppose that the restrictions $\lambda_i = \lambda_i^*$ to any type F_4 component R_i of R satisfy: either λ_i is geometric or $\lambda_i(\alpha) = 0$ for an $\alpha \in R_i$.*

(a) There exists a unique bijection

$$\zeta_{\mathcal{H}} : \{\textit{standard } \mathcal{H}\textit{-modules}\} \longrightarrow \mathrm{Irr}(\mathcal{O}(T) \rtimes W)$$

such that $\zeta_{\mathcal{H}}(\pi)$ *is always a constituent of* $\mathrm{Res}_{\mathcal{O}(T)\rtimes W}(\pi)$.
There exists a total order on $\mathrm{Irr}(\mathcal{O}(T)\rtimes W)$ *such that, if we transfer it via* $\zeta_{\mathcal{H}}$*, the matrix of the* $\mathbb{Z}$*-linear map*

$$\mathrm{Res}_{\mathcal{O}(T)\rtimes W} : \mathbb{Z}\{\textit{standard } \mathcal{H}\textit{-modules}\} \longrightarrow \mathbb{Z}\,\mathrm{Irr}(\mathcal{O}(T)\rtimes W)$$

becomes upper triangular and unipotent.

(b) There exists a natural bijection

$$\zeta'_{\mathcal{H}} : \mathrm{Irr}(\mathcal{H}) \longrightarrow \mathrm{Irr}(X\rtimes W)$$

such that:

- *for a standard module* π, $\zeta_{\mathcal{H}}(\pi)$ *equals* $\zeta'_{\mathcal{H}}$ *of the irreducible quotient of* π,
- *for irreducible tempered* $\mathcal{H}$*-representations it coincides with part (a).*

The allowed parameter functions for a type F_4 component of R are

$$(\lambda(\alpha),\lambda(\beta)) \in \{(k,k),(2k,k),(k,2k),(4k,k),(k,0),(0,k),(0,0)\},$$

where $k\in\mathbb{R}_{>0}, \alpha\in F_4$ is a short root and $\beta\in F_4$ is a long root. Like we mentioned after Theorem 6.2, we believe that Theorem 6.11 is valid for all parameter functions $\lambda,\lambda^* : R\to\mathbb{R}_{\geq 0}$.

We point out that the uniqueness/naturality is essential in Theorem 6.11. Without that condition, it would be much easier to derive it from [79, §2.3].

Proof. By Lemma 3.31 and the asserted naturality of $\zeta_{\mathcal{H}}$ and $\zeta'_{\mathcal{H}}$, we may assume that $\lambda(\alpha)\geq\lambda^*(\alpha)$ for all $\alpha\in R$. For any $u\in T_{\mathrm{un}}$, the parameter function k_u from (3.19) takes values in $\mathbb{R}_{\geq 0}$.

For any irreducible component R_i of R not of type F_4, we claim that R_i does not possess a root subsystem isomorphic to F_4. This can be seen with a case-by-case consideration of irreducible root systems. To get a root subsystem of type F_4, R_i needs to possess roots of different lengths, so it has type B_n, C_n, F_4 or G_2. The rank of G_2 is too low, so $R\not\supseteq G_2$. In type B_n (resp. C_n) the short (resp. long) roots form a subsystem of type $(A_1)^n$, and that does not contain D_4. As both the long and the short roots in F_4 form a root system of type D_4, we can conclude that $B_n\not\supseteq R_i\not\supseteq C_n$.

Now we apply Corollary 3.30 and reduce the theorem to the graded Hecke algebras $\mathbb{H}(\mathrm{t},W(R_u),k_u)\rtimes\Gamma_u$, for all $u\in T_{\mathrm{un}}$. By the above, the parameter function k_u fulfils the requirements of Theorem 6.2. Finally, we apply Proposition 6.10. □

We note that Theorem 6.11 provides a natural bijection between the irreducible or standard modules of two affine Hecke algebras with the same root datum but different parameters $q_s\geq 1$.

When $\mathcal{H}$ arises from a cuspidal local system $\mathcal{L}$ on a nilpotent orbit for a Levi subgroup L of G, Theorem 6.11 is related to Theorem 5.10. We claim that in this case

$$\begin{array}{rcl} \mathrm{Res}_{\mathcal{O}(T)\rtimes W}(\overline{E}_{s,u,\rho}) &=& \overline{E}_{s,u,\rho}, \\ \zeta_{\mathcal{H}(G,L,\mathcal{L})}(\overline{E}_{s,u,\rho}) = \zeta'_{\mathcal{H}(G,L,\mathcal{L})}(\overline{M}_{s,u,\rho}) &=& \overline{M}_{s,u,\rho}, \end{array} \tag{6.26}$$

where the terms on the right are representations of $\mathcal{O}(T)\rtimes W$, the version of $\mathcal{H}(G,L,\mathcal{L})$ with $\mathbf{q}=1$. The reasons for the compatibility between these two ways to go from $\mathcal{H}$-modules to $\mathcal{O}(T)\rtimes W$-modules are:

- the constructions that led to $\zeta_{\mathcal{H}}$ are analogous to those behind Theorem 5.10,

- for graded Hecke algebras from cuspidal local systems we imposed such compatibility in the proof of Lemma 6.3.

Theorem 5.10 with $\mathbf{q} = 1$ can be considered as a generalized Springer correspondence for the (extended) affine Weyl group $X \rtimes \Gamma = W_{\mathrm{aff}} \rtimes \Omega$, with geometric data (s, u, ρ). Consequently Theorem 6.11 can also be regarded as a generalized Springer correspondence of sorts, where the geometric data have been replaced by standard or irreducible modules of an affine Hecke algebra with (nearly arbitrary) parameters $q_s \in \mathbb{R}_{\geq 1}$.

6.3. Consequences for type B_n/C_n Hecke algebras

We illustrate the power of Theorem 6.2 with two results that rely on techniques developed in Section 6.1.

Lemma 6.12. *Let $\mathbb{H}$ be as in Theorem 6.2 and let $\tilde{\xi} \in \tilde{\Xi}^+$. Then the 2-cocycle $\natural_{\tilde{\xi}}$ of $\mathfrak{R}_{\tilde{\xi}}$ is trivial and* $\mathrm{End}_{\mathbb{H}}(\pi(\tilde{\xi})) \cong \mathbb{C}[\mathfrak{R}_{\tilde{\xi}}]$.

Proof. Write $\tilde{\xi} = (P, \delta, \lambda)$, so $\delta \in \mathrm{Irr}(\mathbb{H}^P), \lambda \in \mathfrak{t}^P$ and $\mathfrak{R}_{\tilde{\xi}} \subset \mathrm{Stab}_W(P)$. Then $\mathfrak{R}_{\tilde{\xi}}$ also stabilizes $(P, \delta, z\lambda)$ for any $z \in \mathbb{C}$. The operators $\pi(w, P, \delta, z\lambda)$ with $w \in \mathfrak{R}_{\tilde{\xi}}$ and $z \in \mathbb{C}$ come from (3.15), they are rational in z and by Theorem 3.23 they are regular for $z \in \mathbb{R}_{\geq 0}$. Thus the family of projective $\mathfrak{R}_{\tilde{\xi}}$-representations $\pi(P, \delta, z\lambda)$ depends continuously on $z \in \mathbb{R}_{\geq 0}$. Since a finite group has only finitely many isomorphism classes of projective representations of a given dimension, it follows that our family is constant (up to isomorphism). Hence it suffices to prove the lemma in the case $\lambda = 0$.

We apply the proof of Lemma 6.6 to the algebra $\mathbb{H}^P \rtimes \mathrm{Stab}_W(P)$. From (6.8) we get a group homomorphism

$$\mathfrak{R}_{\tilde{\xi}} \to \mathrm{Aut}_{\mathbb{C}}(V_\delta) : w \mapsto I_w$$

such that $\mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H}}(I_w)$ is a scalar multiple of $\pi(w, P, \delta, 0)$. This shows that $\natural_{\tilde{\xi}}$ is trivial. Now the second claim follows from (4.4). □

Recall the notion of genericity for parameter functions $k : B_n \to \mathbb{R}$ from (4.9).

Theorem 6.13. *Let k be a generic parameter function for $\mathbb{H}(\mathbb{C}^n, W(B_n), k)$.*

(a) All R-groups $\mathfrak{R}_\xi$ with $\xi \in \tilde{\Xi}^+$ are trivial.
(b) The map (4.6) provides a bijection $\tilde{\Xi}^+ \to \mathrm{Irr}(\mathbb{H}(\mathbb{C}^n, W(B_n), k))$.

Proof. Note that every parabolic subalgebra of $\mathbb{H}(\mathbb{C}^n, W(B_n), k)$ is the tensor product of an algebra of the same kind (but of smaller rank) and some graded Hecke algebras of type A (for which we saw in Example 4.3 that all R-groups are trivial).
(a) For discrete series representations, this follows from [66, Proposition 3.9 and subsequent remark]. That also implies part (a) for the discrete series of parabolic subalgebras of $\mathbb{H}(\mathbb{C}^n, W(B_n), k)$. By [80, Theorem 9.1]

$$\mathrm{End}_{\mathbb{H}}(\pi(P, \delta, \lambda)) \subset \mathrm{End}_{\mathbb{H}}(\pi(P, \delta, 0)).$$

Together with (4.4) this implies that $\mathfrak{R}_{(P,\delta,\lambda)}$ is trivial whenever $\mathfrak{R}_{(P,\delta,0)}$ is trivial. Therefore it suffices to consider the case $\lambda = 0$.

For every fixed P, Theorem 6.2 and [66, Proposition 3.9] tell us that the irreducible discrete representations of $\mathbb{H}^P = \mathbb{H}(\mathbb{C}^n, W_P, k)$ naturally give rise to a basis of the space of elliptic representations W_P. By the latter we mean the representation ring of W_P modulo the span of the representations induced from proper parabolic subgroups. The dimension of this space equals the number of conjugacy classes of W_P that are elliptic, that is, do not intersect any proper parabolic subgroup of W_P [70, §2]. The union over P of these sets of elliptic conjugacy classes, altogether up to $W(B_n)$-equivalence, is precisely the set of conjugacy classes of $W(B_n)$. That is also the dimension of the representation ring of $W(B_n)$. It follows that the set

$$\{\mathrm{ind}_{W_P}^{W}(\zeta_{\mathbb{H}^P}(\delta)) : (P, \delta, 0) \in \tilde{\Xi}\}/W(B_n)$$

is a basis of the representation ring of $W(B_n)$. In particular it has $|\mathrm{Irr}(W(B_n))|$ elements. By Theorem 6.2.c the bijection $\zeta_{\mathbb{H}^P}$ is equivariant for $\mathrm{Stab}_W(P)/W_P$. Hence the number of $W(B_n)$-orbits of elements $(P, \delta, 0) \in \tilde{\Xi}$ is also $|\mathrm{Irr}(W(B_n))|$.

Suppose that $\mathfrak{R}_{(P,\delta,0)}$ is nontrivial for some $(P, \delta, 0)$. By Lemma 6.12 there is more than one ρ in $\mathrm{Irr}(\mathbb{C}[\mathfrak{R}_{(P,\delta,0)}, \natural_{(P,\delta,0)}]) = \mathrm{Irr}(\mathfrak{R}_{(P,\delta,0)})$. Therefore the forgetful map

$$\big\{(P, \delta, 0, \rho) : (P, \delta, 0) \in \tilde{\Xi}^+, \rho \in \mathrm{Irr}(\mathfrak{R}_{(P,\delta,0)})\big\}/\mathcal{W}_{\tilde{\Xi}} \to \tilde{\Xi}^+/\mathcal{W}_{\tilde{\Xi}}$$

is not injective. With (4.6) we see that $\mathrm{Irr}_0(\mathbb{H})$ has more elements than $\mathrm{Irr}(W)$. That contradicts Theorem 6.2.

(b) This follows directly from (a) and (4.6). □

We want to use Corollary 3.30 to obtain a version of Theorem 6.13 for affine Hecke algebras. Let T be the "diagonal" maximal torus of SO_{2n+1} and consider the root datum

$$\mathcal{R}(B_n) := \mathcal{R}(SO_{2n+1}, T)$$

Let $\mathcal{H}(\mathcal{R}(B_n), \lambda, \lambda^*, \mathbf{q})$ be an affine Hecke algebra as in Lemma 3.31. As in Example 4.11 there are three independent parameters

$$q_1 = \mathbf{q}^{\lambda(e_i \pm e_j)}, q_2 = \mathbf{q}^{\lambda(e_i)}, q_0 = \mathbf{q}^{\lambda^*(e_i)}$$

while Lemma 3.31 ensures that $q_1, q_2, q_0 \in \mathbb{R}_{\geq 1}$ and $q_2 \geq q_0$. For $u \in T_{\mathrm{un}}$, the parameter function k_u from (3.19) satisfies

$$\begin{aligned} k_u(e_i \pm e_j) &= \lambda(e_i \pm e_j)\log(\mathbf{q}) =: k_1, \\ k_u(e_i) &= (\lambda(e_i) \pm \lambda^*(e_i))\log(\mathbf{q})/2 =: k_2^{\pm}. \end{aligned} \tag{6.27}$$

Motivated by (4.9) and (6.27), we call q and (λ, λ^*) generic if

$$q_1 \neq 1 \text{ and } q_2 q_0^{\pm 1} \neq q_1^j \text{ for any } j \in \mathbb{Z} \text{ with } |j| \leq 2(n-1).$$

Proposition 6.14. *Assume that (λ, λ^*) is generic and as in Lemma 3.31. Then all R-groups for $\mathcal{H}(\mathcal{R}(B_n), \lambda, \lambda^*, \mathbf{q})$ are trivial and Corollary 4.4.d provides a bijection*

$$\Xi^+ \to \mathrm{Irr}\big(\mathcal{H}(\mathcal{R}(B_n), \lambda, \lambda^*, \mathbf{q})\big).$$

Proof. Consider any $u \in T_{\mathrm{un}} \cong (S^1)^n$. Replacing it by a $W(B_n)$-conjugate, we may assume that it is of the form

$$u = (1)^{m_+}(-1)^{m_-}(\lambda_1)^{m_1} \cdots (\lambda_d)^{m_d},$$

where $\lambda_i \in S^1 \setminus \{1, -1\}$ and $\lambda_i \neq \lambda_j^{\pm 1}$ for any i, j. The notation $(\lambda)^m$ means that m consecutive coordinates of u are equal to λ. Then

$$R_u \cong B_{m_+} \times B_{m_-} \times A_{m_1 - 1} \times \cdots \times A_{m_d - 1}$$

and $W(B_n)_u = W(R_u)$, $\Gamma_u = 1$. Hence the algebra $\mathcal{H}(\mathcal{R}_t, \lambda, \lambda^*, \mathbf{q})$ appearing in Theorem 3.28 is isomorphic to

$$\mathcal{H}(\mathcal{R}(B_{m_+}), \lambda, \lambda^*, \mathbf{q}) \otimes \mathcal{H}(\mathcal{R}(B_{m_-}), \lambda, \lambda^*, \mathbf{q}) \otimes \mathcal{H}_{m_1}(q_1) \otimes \cdots \otimes \mathcal{H}_{m_d}(q_1).$$

Next we apply Theorem 3.29 and we end up with the graded Hecke algebra

$$\begin{aligned} &\mathbb{H}(\mathbb{C}^{m_+}, W(B_{m_+}), k_1, k_2^+) \otimes \mathbb{H}(\mathbb{C}^{m_-}, W(B_{m_-}), k_1, k_2^-) \otimes \\ &\quad \mathbb{H}(\mathbb{C}^{m_1}, S_{m_1}, k_1) \otimes \cdots \otimes \mathbb{H}(\mathbb{C}^{m_d}, S_{m_d}, k_1). \end{aligned} \tag{6.28}$$

The genericity of (λ, λ^*) implies that the parameters for every tensor factor of (6.28) are generic and Lemma 3.31 ensures that they are positive. Corollary 3.30 says that the relevant part of the module category of $\mathcal{H}(\mathcal{R}(B_n), \lambda, \lambda^*, \mathbf{q})$ is equivalent with the relevant part of the module category of (6.28). We know from Theorem 6.13 and Example 4.3 that all R-groups of (6.28) are trivial. With a view on the construction of R-groups in Section 4.1, we conclude that the R-groups for $\mathcal{H}(\mathcal{R}(B_n), \lambda, \lambda^*, \mathbf{q})$ are also trivial.

We conclude with an application of Corollary 4.4. □

We point out that Theorem 6.13.b and Proposition 6.14 give an effective classification of the irreducible representations of the involved Hecke algebras, because all discrete series representations of parabolic subalgebras have been classified entirely in terms of residual points, see Theorem 4.13 and the subsequent discussion.

It is interesting to compare this with [42, Theorem D]. Kato provides a geometric construction and classification of the irreducible representations of $\mathcal{H}(\mathcal{R}(B_n), \lambda, \lambda^*, \mathbf{q})$ for all generic parameters (λ, λ^*). We stress that this is a much wider range of q-parameters than in Proposition 6.14, most of them are complex. The indexing set for $\mathrm{Irr}(\mathcal{H}(\mathcal{R}(B_n), \lambda, \lambda^*, \mathbf{q}))$ in [42] is a generalization of Kazhdan–Lusztig parameters, only with an exotic version of the nilpotent cone. Unfortunately the results of [42] do not extend to non-generic parameters.

Acknowledgements

We thank George Lusztig for some useful clarifications of his work and the referee for his helpful suggestions.

References

[1] A.-M. Aubert, Local Langlands and Springer correspondences, in: A.-M. Aubert, M. Mishra, A. Roche, S. Spallone (Eds.), Representations of Reductive p-adic Groups, in: Progress in Mathematics, vol. 328, Birkhäuser, 2019, pp. 1–37.

[2] A.-M. Aubert, P.F. Baum, R.J. Plymen, M. Solleveld, Conjectures about p-adic groups and their noncommutative geometry, Contemp. Math. 691 (2017) 15–51.

[3] A.-M. Aubert, P.F. Baum, R.J. Plymen, M. Solleveld, The principal series of p-adic groups with disconnected centre, Proc. Lond. Math. Soc. 114 (5) (2017) 798–854.

[4] A.-M. Aubert, A. Moussaoui, M. Solleveld, Generalizations of the Springer correspondence and cuspidal Langlands parameters, Manuscripta Math. 157 (2018) 121–192.

[5] A.-M. Aubert, A. Moussaoui, M. Solleveld, Graded Hecke algebras for disconnected reductive groups, in: W. Müller, S.W. Shin, N. Templier (Eds.), Geometric Aspects of the Trace Formula, Simons Symposia, Springer, 2018, pp. 23–84.

[6] A.-M. Aubert, A. Moussaoui, M. Solleveld, Affine Hecke algebras for Langlands parameters, 2019, arXiv:1701.03593.

[7] D. Barbasch, A. Moy, A unitarity criterion for p-adic groups, Invent. Math. 98 (1989) 19–37.

[8] D. Barbasch, A. Moy, Reduction to real infinitesimal character in affine Hecke algebras, J. Amer. Math. Soc. 6 (3) (1993) 611–635.

[9] J. Bernstein, A. Zelevinsky, Representations of the group $GL(n, F)$ where F is a local non-archimedean field, Uspekhi Mat. Nauk 31 (3) (1976) 5–70.
[10] N. Bourbaki, Groupes et Algèbres de Lie. Chapitres IV, V et VI, in: Éléments de Mathḿatique, vol. XXXIV, Hermann, 1968.
[11] C.J. Bushnell, P.C. Kutzko, The Admissible Dual of GL(N) Via Compact Open Subgroups, in: Annals of Mathematics Studies, vol. 129, Princeton University Press, 1993.
[12] R.W. Carter, Finite Groups of Lie Type. Conjugacy Classes and Complex Characters, in: Pure and Applied Mathematics, John Wiley & Sons, New York, 1985.
[13] W. Casselman, Introduction to the theory of admissible representations of p-adic reductive groups, 1995, preprint.
[14] I. Cherednik, A unification of Knizhnik-Zamolodchikov and Dunkl operators via affine Hecke algebras, Invent. Math. 106 (1991) 411–431.
[15] I. Cherednik, Double Affine Hecke Algebras, in: London Mathematical Society Lecture Note Series, vol. 319, Cambridge University Press, 2005.
[16] N. Chriss, V. Ginzburg, Representation Theory and Complex Geometry, Birkhäuser, 1997.
[17] D. Ciubotaru, On unitary unipotent representations of p-adic groups and affine Hecke algebras with unequal parameters, Represent. Theory 12 (2008) 453–498.
[18] D. Ciubotaru, M. Kato, S. Kato, On characters and formal degrees of discrete series of affine Hecke algebras of classical types, Invent. Math. 187 (2012) 589–635.
[19] D. Ciubotaru, P.E. Trapa, Functors for unitary representations of classical real groups and affine Hecke algebras, Adv. Math. 227 (4) (2011) 1585–1611.
[20] D. Ciubotaru, P.E. Trapa, Duality between $GL(n\mathbb{R})GL(n\mathbb{Q}_p)$, and the degenerate affine Hecke algebra for $\mathfrak{gl}(n)$, Amer. J. Math. 134 (1) (2012) 141–170.
[21] C.W. Curtis, I. Reiner, Representation Theory of Finite Groups and Associative Algebras, in: Pure and Applied Mathematics, vol. 11, John Wiley & Sons, 1962.
[22] P. Delorme, E.M. Opdam, The Schwartz algebra of an affine Hecke algebra, J. Reine Angew. Math. 625 (2008) 59–114.
[23] P. Delorme, E.M. Opdam, Analytic R-groups of affine Hecke algebras, J. Reine Angew. Math. 658 (2011) 133–172.
[24] V.G. Drinfeld, Degenerate affine Hecke algebras and Yangians, Funktsional. Anal. i Prilozhen. 20 (1) (1986) 69–70.
[25] S. Evens, The Langlands classification for graded Hecke algebras, Proc. Amer. Math. Soc. 124 (4) (1996) 1285–1290.
[26] M. Geck, G. Pfeiffer, Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras, in: London Mathematical Society Monographs New Series, vol. 21, Oxford University Press, 2000.
[27] D. Goldberg, A. Roche, Hecke algebras and SL_n-types, Proc. Lond. Math. Soc. 90 (1) (2005) 87–131.
[28] A. Gyoja, Modular representation theory over a ring of higher dimension with applications to Hecke algebras, J. Algebra 174 (2) (1995) 553–572.
[29] A. Gyoja, K. Uno, On the semisimplicity of Hecke algebras, J. Math. Soc. Japan 41 (1) (1989) 75–79.
[30] G.J. Heckman, E.M. Opdam, Yang's system of particles and Hecke algebras, Ann. of Math. 145 (1) (1997) 139–173.
[31] F. Hivert, A. Schilling, N.M. Thiéry, The biHecke monoid of a finite Coxeter group and its representations, Algebra Number Theory 7 (3) (2013) 595–671.
[32] F. Hivert, N.M. Thiéry, The Hecke group algebra of a Coxeter group and its representation theory, J. Algebra 321 (8) (2009) 2230–2258.
[33] R.B. Howlett, G. Lehrer, Induced cuspidal representations and generalised Hecke rings, Invent. Math. 58 (1980) 37–64.
[34] R.B. Howlett, G. Lehrer, Representations of generic algebras and finite groups of Lie type, Trans. Amer. Math. Soc. 280 (2) (1983) 753–779.
[35] J.E. Humphreys, Reflection Groups and Coxeter Groups, in: Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, 1990.
[36] N. Iwahori, On the structure of a Hecke ring of a Chevalley group over a finite field, J. Fac. Sci. Univ. Tokyo Sect. I 10 (1964) 215–236.
[37] N. Iwahori, Generalized Tits system (Bruhat decompostition) on p-adic semisimple groups, in: Algebraic Groups and Discontinuous Subgroups, in: Proc. Sympos. Pure Math., vol. 9, American Mathematical Society, 1966, pp. 71–83.

[38] N. Iwahori, H. Matsumoto, On some Bruhat decomposition and the structure of the Hecke rings of the p-adic Chevalley groups, Inst. Hautes Études Sci. Publ. Math. 25 (1965) 5–48.

[39] V.R.F. Jones, Hecke algebra representations of braid groups and link polynomials, Ann. of Math. 126 (1987) 335–388.

[40] S.-I. Kato, Irreducibility of principal series representations for Hecke algebras of affine type, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 28 (3) (1981) 929–943.

[41] S.-I. Kato, A realization of irreducible representations of affine Weyl groups, Indag. Math. 45 (2) (1983) 193–201.

[42] Syu Kato, An exotic Deligne–Langlands correspondence for symplectic groups, Duke Math. J. 148 (2) (2009) 305–371.

[43] D. Kazhdan, G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math. 53 (2) (1979) 165–184.

[44] D. Kazhdan, G. Lusztig, Proof of the Deligne–Langlands conjecture for Hecke algebras, Invent. Math. 87 (1987) 153–215.

[45] A. Kirillov, Lectures on affine Hecke algebras and Macdonald's conjectures, Bull. Amer. Math. Soc. 34 (1997) 251–292.

[46] A.W. Knapp, Representation Theory of Semisimple Groups. An Overview Based on Examples, in: Princeton Mathematical Series, vol. 36, Princeton University Press, 1986.

[47] C. Kriloff, A. Ram, Representations of graded Hecke algebras, Represent. Theory 6 (2002) 31–69.

[48] G. Lusztig, Intersection cohomology complexes on a reductive group, Invent. Math. 75 (2) (1984) 205–272.

[49] G. Lusztig, Equivariant K-theory and representations of Hecke algebras, Proc. Amer. Math. Soc. 94 (2) (1985) 337–342.

[50] G. Lusztig, Cells in affine Weyl groups III, J. Fac. Sci. Univ. Tokyo 34 (2) (1987) 223–243.

[51] G. Lusztig, Cuspidal local systems and graded Hecke algebras, Publ. Math. Inst. Hautes Études Sci. 67 (1988) 145–202.

[52] G. Lusztig, Affine Hecke algebras and their graded version, J. Amer. Math. Soc. 2 (3) (1989) 599–635.

[53] G. Lusztig, Classification of unipotent representations of simple p-adic groups, Int. Math. Res. Not. 11 (1995) 517–589.

[54] G. Lusztig, Cuspidal local systems and graded Hecke algebras. II, in: Representations of Groups, in: Canadian Mathematical Society Conference Proceedings, vol. 16, 1995, pp. 216–275.

[55] G. Lusztig, Cuspidal local systems and graded Hecke algebras. III, Represent. Theory 6 (2002) 202—242.

[56] G. Lusztig, Hecke Algebras with Unequal Parameters, in: CRM Monograph Series, vol. 18, American Mathematical Society, 2003.

[57] I.G. Macdonald, Affine Hecke Algebras and Orthogonal Polynomials, in: Cambridge Tracts in Mathematics, vol. 157, Cambridge University Press, 2003.

[58] H. Matsumoto, Générateurs et relations des groupes de Weyl généralisés, C. R. Math. Acad. Sci. Paris 258 (1964) 3419–3422.

[59] H. Matsumoto, Analyse Harmonique dans les Systèmes de Tits Bornologiques de Type Affine, in: Lecture Notes in Mathematics, vol. 590, Springer-Verlag, 1977.

[60] M. Miyauchi, S. Stevens, Semisimple types for p-adic classical groups, Math. Ann. 358 (2014) 257–288.

[61] L. Morris, Tamely ramified intertwining algebras, Invent. Math. 114 (1) (1993) 1–54.

[62] E.M. Opdam, Harmonic analysis for certain representations of graded Hecke algebras, Acta Math. 175 (1995) 75–121.

[63] E.M. Opdam, On the spectral decomposition of affine Hecke algebras, J. Inst. Math. Jussieu 3 (4) (2004) 531–648.

[64] E.M. Opdam, Spectral transfer morphisms for unipotent affine Hecke algebras, J. Inst. Math. Jussieu 3 (4) (2004) 531–648.

[65] E.M. Opdam, Spectral correspondences for affine Hecke algebras, Adv. Math. 286 (2016) 912–957.

[66] E.M. Opdam, M. Solleveld, Homological algebra for affine Hecke algebras, Adv. Math. 220 (2009) 1549–1601.

[67] E.M. Opdam, M. Solleveld, Discrete series characters for affine Hecke algebras and their formal dimensions, Acta Math. 205 (2010) 105–187.

[68] A. Ram, J. Rammage, Affine Hecke algebras, cyclotomic Hecke algebras and Clifford theory, in: A Tribute to C.S. Seshadri (Chennai 2002), in: Trends in Mathematics, Birkhäuser, 2003, pp. 428–466.

[69] M. Reeder, Nonstandard intertwining operators and the structure of unramified principal series representations, Forum Math. 9 (1997) 457–516.

[70] M. Reeder, Euler–Poincaré pairings and elliptic representations of Weyl groups and p-adic groups, Compos. Math. 129 (2001) 149–181.
[71] M. Reeder, Isogenies of Hecke algebras and a Langlands correspondence for ramified principal series representations, Represent. Theory 6 (2002) 101–126.
[72] D. Renard, Représentations des Groupes Réductifs p-adiques, in: Cours Spécialisés, vol. 17, Société Mathématique de France, 2010.
[73] A. Roche, Types and Hecke algebras for principal series representations of split reductive p-adic groups, Ann. Sci. Éc. Norm. Supér. 31 (3) (1998) 361–413.
[74] A.J. Silberger, Introduction to harmonic analysis on reductive p-adic groups, in: Mathematical Notes, vol. 23, Princeton University Press, 1979.
[75] K. Slooten, A Combinatorial Generalization of the Springer Correspondence for Classical Type (Ph.D. thesis), Universiteit van Amsterdam, 2003.
[76] K. Slooten, Generalized Springer correspondence and Green functions for type B/C graded Hecke algebras, Adv. Math. 203 (1) (2006) 34–108.
[77] M. Solleveld, Periodic Cyclic Homology of Affine Hecke Algebras (Ph.D. thesis), Universiteit van Amsterdam, 2007.
[78] M. Solleveld, Homology of graded Hecke algebras, J. Algebra 323 (2010) 1622–1648.
[79] M. Solleveld, On the classification of irreducible representations of affine Hecke algebras with unequal parameters, Represent. Theory 16 (2012) 1–87.
[80] M. Solleveld, Parabolically induced representations of graded Hecke algebras, Algebr. Represent. Theory 15 (2) (2012) 233–271.
[81] M. Solleveld, Hochschild homology of affine Hecke algebras, J. Algebra 384 (2013) 1–35.
[82] M. Solleveld, Topological K-theory of affine Hecke algebras, Ann. K-Theory 3 (3) (2018) 395–460.
[83] M. Solleveld, On completions of Hecke algebras, in: A.-M. Aubert, M. Mishra, A. Roche, S. Spallone (Eds.), Representations of Reductive p-adic Groups, in: Progress in Mathematics, vol. 328, Birkhäuser, 2019, pp. 207–262.
[84] M. Solleveld, Endomorphism algebras and Hecke algebras for reductive p-adic groups, 2020, arXiv:2005.07899.
[85] T.A. Springer, Linear Algebraic Groups, second ed., in: Progress in Mathematics, vol. 9, Birkhäuser, 1998.
[86] J.-L. Waldspurger, La formule de Plancherel pour les groupes p-adiques (d'après Harish-Chandra), J. Inst. Math. Jussieu 2 (2) (2003) 235–333.
[87] R. Yan, Isomorphisms between affine Hecke algebras of type $\widetilde{A_2}$, J. Algebra 324 (5) (2010) 984–999.
[88] A.V. Zelevinsky, Induced representations of reductive p-adic groups II. On irreducible representations of GL(n), Ann. Sci. Éc. Norm. Supér. (4) 13 (2) (1980) 165–210.