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Positive scalar curvature and an equivariant Callias-type index theorem for proper actions

Hao Guo, Peter Hochs and Varghese Mathai

For a proper action by a locally compact group G on a manifold M with a G-equivariant Spin-structure, we obtain obstructions to the existence of complete G-invariant Riemannian metrics with uniformly positive scalar curvature. We focus on the case where M/G is noncompact. The obstructions follow from a Callias-type index theorem, and relate to positive scalar curvature near hypersurfaces in M. We also deduce some other applications of this index theorem. If G is a connected Lie group, then the obstructions to positive scalar curvature vanish under a mild assumption on the action. In that case, we generalise a construction by Lawson and Yau to obtain complete G-invariant Riemannian metrics with uniformly positive scalar curvature, under an equivariant bounded geometry assumption.

1. Introduction

Let G be a locally compact group, acting properly on a manifold M. Suppose that M has a G-equivariant Spin-structure. The results in this paper are about the following question.

Question 1.1. *When does M admit a complete, G-invariant Riemannian metric with uniformly positive scalar curvature?*

We are mainly interested in the case where M/G is noncompact.

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The literature on the nonequivariant case of Question 1.1 is too vast to summarise, but important work where M is noncompact was done in [Gromov and Lawson 1983]. A more refined perspective on the nonequivariant case is to consider a manifold X, and let M be its universal cover, and G its fundamental group. This allows one to construct obstructions to metrics of positive scalar curvature in terms of G-equivariant index theory on M, refining index-theoretic obstructions on X. If X is compact, this is the origin of the even more refined Rosenberg index [1986a; 1983; 1986b], in KO-theory of the real maximal group C^*-algebra of G.

More generally, if G is a discrete group not necessarily acting freely on M, then $X := M/G$ is an orbifold, whence Question 1.1 becomes the question of whether X admits an orbifold metric of positive scalar curvature.

We consider the case where G is not necessarily discrete, and does not necessarily act freely. Results on this case of Question 1.1 in the case where G is an almost-connected Lie group and M/G is compact were obtained in [Guo et al. 2019b; Piazza and Posthuma 2019]. In this paper, we focus on the case where M/G is noncompact.

Results on positive scalar curvature. We first obtain obstructions to G-invariant Riemannian metrics with positive scalar curvature, both in the K-theory of the maximal or reduced group C^*-algebra of G, and in terms of numerical topological invariants generalising the $\hat{A}$-genus. If G is a connected Lie group, then these obstructions vanish under a mild assumption, as shown in [Hochs and Mathai 2016]. In that case, we construct G-invariant Riemannian metrics with uniformly positive scalar curvature, under an equivariant bounded geometry assumption.

Our most general obstruction result is the following.

Theorem 1.2 (Theorem 2.1). *Let $H \subset M$ be a G-invariant, cocompact hypersurface with trivial normal bundle that partitions M into two open sets. If M admits a complete, G-invariant Riemannian metric with nonnegative scalar curvature, and positive scalar curvature in a neighbourhood of H, then*

$$\mathrm{index}_G(D^H) = 0 \quad \textit{in } K_*(C^*(G)),$$

for a Spin-*Dirac operator D^H on H.*

In the case where M is the universal cover of a manifold X and G is its fundamental group, this becomes Theorem A in [Cecchini 2020].

Theorem 1.2 implies topological obstructions to G-invariant Riemannian metrics with positive scalar curvature. Let $g \in G$, and let $H^g \subset H$ be the fixed-point set of g. Let $\mathcal{N} \to H^g$ be its normal bundle in H, and let $R^{\mathcal{N}}$ be the curvature of the Levi-Civita connection restricted to $\mathcal{N}$. The *g-localised $\hat{A}$-genus* of H is

$$\hat{A}_g(H) := \int_{H^g} c^g \frac{\hat{A}(H^g)}{\det(1 - g e^{-R^{\mathcal{N}}/2\pi i})^{1/2}},$$

for a cutoff function c^g on X^g. If G acts freely, then $\hat{A}_g(H) = 0$ if $g \neq e$, and $\hat{A}_e(H)$ is the $\hat{A}$-genus of H/G. In general, for example in the orbifold case, $\hat{A}_g(H)$ may be nonzero for different g.

Corollary 1.3 (Corollary 2.4). *Consider the setting of Theorem 1.2. Suppose that either*

- *G is any locally compact group and $g = e$*;
- *G is discrete and finitely generated and g is any element*; *or*
- *G is a connected semisimple Lie group and g is a semisimple element.*

Then $\hat{A}_g(H)$ is independent of the choice of G-invariant Riemannian metric, *and $\hat{A}_g(H) = 0$.*

Theorem 2 in [Hochs and Mathai 2016] is a generalisation of the vanishing result of Atiyah and Hirzebruch [1970] to the noncompact case. It states that, if G is connected, and not every stabiliser of its action on H is maximal compact, then $\mathrm{index}_G(D^H) = 0$. This implies that $\hat{A}_g(H) = 0$ as well. In view of Theorem 1.2 and Corollary 1.3, this makes it a natural question whether M admits a G-invariant Riemannian metric with positive scalar curvature if G is a connected Lie group. The answer, given in the present paper, turns out to be *yes* under a certain equivariant bounded geometry assumption.

Suppose that G is a connected Lie group, and let $K < G$ be a maximal compact subgroup. Abels' slice theorem [1974] implies that there is a diffeomorphism $M \cong G \times_K N$, for a K-invariant submanifold $N \subset M$. Consider the infinitesimal action map

$$\varphi : N \times \mathfrak{k} \to TN$$

mapping $(y, X) \in N \times \mathfrak{k}$ to $(d/dt)|_{t=0} \exp(tX)y$. The action by K on N is said to have *no shrinking orbits* with respect to a K-invariant Riemannian metric on N if the pointwise operator norm of φ as a map from $\mathfrak{k}$ to a tangent space is uniformly positive outside a neighbourhood of the fixed-point set N^K. We say that N has *K-bounded geometry* if it has bounded geometry and no shrinking orbits.

Theorem 1.4. *Suppose that K is nonabelian*, *and that K acts effectively on N with compact fixed-point set. If there exists a K-invariant Riemannian metric on N for which N has K-bounded geometry*, *then the manifold $G \times_K N$ admits a G-invariant metric with uniformly positive scalar curvature.*

In the compact case, the Atiyah–Hirzebruch vanishing theorem [1970] implies that the obstruction $\hat{A}(N)$ to Riemannian metrics of positive scalar curvature vanishes if K acts nontrivially on N. Lawson and Yau [1974] constructed such metrics, under mild conditions. In a similar way, Theorem 1.4 complements the vanishing result in [Hochs and Mathai 2016].

A Callias-type index theorem. Two effective sources of index-theoretic obstructions to metrics of positive scalar curvature on noncompact manifolds are coarse index theory and Callias-type index theory. For some results involving coarse index theory, see for example [Schick and Zadeh 2018] and the literature on the coarse Novikov conjecture, in particular [Fu et al. 2020] for the equivariant setting we are interested in here. We will use Callias-type index theory.

Not assuming that M is Spin for now, and letting G be any locally compact group, we consider a G-equivariant Dirac-type operator D on a G-equivariant vector bundle $S \to M$. A *Callias-type operator* is of the form $D + \Phi$, for a G-equivariant endomorphism Φ of S such that $D + \Phi$ is uniformly positive outside a cocompact set. Then this operator has an index $\operatorname{index}_G(D + \Phi) \in K_*(C^*(G))$, constructed in [Guo 2021]. (See Theorem 4.2 in [Guo et al. 2019a] for a realisation of this index in terms of coarse geometry.)

Theorem 1.5 (G-Callias-type index theorem). *We have*

$$\operatorname{index}_G(D + \Phi) = \operatorname{index}_G(D_N),$$

for a Dirac operator D_N on a G-invariant, cocompact hypersurface $N \subset M$.

See Theorem 3.10. Versions of this result where G is trivial were proved in [Anghel 1993; Bott and Seeley 1978; Bunke 1995; Callias 1978; Kucerovsky 2001]. Versions for operators on bundles of modules over operator algebras were proved in [Braverman and Cecchini 2018; Cecchini 2020]. Parts of our proof of Theorem 1.5 are based on strategy similar to that in the proof of the index theorem in [Cecchini 2020].

We deduce Theorem 2.1 from Theorem 1.5. This approach is an equivariant generalisation of the obstructions to metrics of positive scalar obtained in [Anghel 1993; Bunke 1995; Cecchini 2020].

If $g \in G$, then under conditions, there is a subalgebra $A(G) \subset C^*(G)$ such that $K_*(A(G)) = K_*(C^*(G))$, and there is a well-defined trace τ_g on $A(G)$ given by

$$\tau_g(f) = \int_{G/Z} f(hgh^{-1})\, d(hZ),$$

where Z is the centraliser of g. In various settings, including the three cases in Corollary 1.3, there are index formulas for the number $\tau_g(\operatorname{index}_G(D))$; see [Hochs and Wang 2018; Wang 2014; Wang and Wang 2016]. These index formulas imply that, in the setting of Theorem 1.2,

$$\tau_g(\operatorname{index}_G(D^H)) = \hat{A}_g(H).$$

Hence Theorem 1.2 implies Corollary 1.3.

Theorem 1.4 is proved via a generalisation of Lawson and Yau's arguments [1974], together with a result from [Guo et al. 2019a] that allows one to induce metrics of positive scalar curvature from N to $M = G \times_K N$.

Apart from using Theorem 1.5 to prove Theorem 2.1, we obtain some further applications, on the image (Corollary 7.1) and cobordism invariance (Corollary 7.2) of the analytic assembly map; on the Callias-type index of Spin^c-Dirac operators (Corollary 7.3); on induction of Callias-type indices from compact groups to non-compact groups (Corollary 7.7); and on the Spin^c-version [Paradan and Vergne 2017] of the *quantisation commutes with reduction* problem [Guillemin and Sternberg 1982; Meinrenken 1998; Paradan 2001; Tian and Zhang 1998] for Spin^c-Callias type operators (Corollary 7.8).

Outline. We state our obstruction and existence results in Section 2: Theorem 2.1, Corollary 2.4 and Theorem 2.11. In Section 3, we state the equivariant Callias-type index theorem, Theorem 3.10, which is proved in Sections 4 and 5. We then deduce Theorem 2.1 and Corollary 2.4 in Section 6A. Theorem 2.11 is proved in Section 6B. In Section 7, we obtain some further applications of the Callias-type index theorem, Corollaries 7.1–7.8.

Notation and conventions. All manifolds, vector bundles, group actions and other maps between manifolds are implicitly assumed to be smooth. If a Hilbert space H is mentioned, the inner product on that space will be denoted by $(-\,,-)_H$, and the corresponding norm by $\|\cdot\|_H$. Spaces of continuous sections are denoted by Γ; spaces of smooth sections by Γ^∞. Subscripts c denote compact supports.

If G is a group, and H is a subgroup acting on a set S, then we write $G \times_H S$ for the quotient of $G \times S$ by the H-action given by

$$h \cdot (g, s) = (gh^{-1}, hs),$$

for $h \in G$, $g \in G$ and $s \in S$. If S is a manifold, G is Lie group and H is a closed subgroup, then this action is proper and free, and $G \times_H S$ is a manifold.

A continuous group action, and also the space acted on, is said to be *cocompact* if the quotient space is compact.

2. Results on positive scalar curvature

Except in Sections 2B and 6B, which concern the existence results, G will be a locally compact group, acting properly on a manifold M. We do not assume that M/G is compact and are in fact interested mainly in the case where it is not. The group G may have infinitely many connected components, and may for example be an infinite discrete group.

2A. *Obstructions.* For a proper, cocompact action by G on a manifold N, a G-equivariant elliptic differential operator D on N has an equivariant index

$$\mathrm{index}_G(D) \in K_*(C^*(G))$$

defined by the analytic assembly map [Baum et al. 1994]. Here $C^*(G)$ is the maximal or reduced group C^*-algebra of G, and the index takes values in its even K-theory if D is odd with respect to a grading, and in odd K-theory otherwise.

Our most general obstruction result is the following.

Theorem 2.1. *Let M be a complete Riemannian* Spin*-manifold, on which a locally compact group G acts properly and isometrically. Let $H \subset M$ be a G-invariant, cocompact hypersurface with trivial normal bundle such that $M \setminus H = X \cup Y$ for disjoint open subsets $X, Y \subset M$. If the scalar curvature on M is nonnegative, and positive in a neighbourhood of H, then the* Spin*-Dirac operator D^H on H, acting on sections of the restriction of the spinor bundle on M to H, satisfies*

$$\mathrm{index}_G(D^H) = 0 \quad \textit{in } K_*(C^*(G)).$$

We will deduce this result from an equivariant index theorem for Callias-type operators, Theorem 3.10, which may be of independent interest and has some other applications as well.

Remark 2.2. If M is the universal cover of a manifold X, and G is the fundamental group of X acting on M in the usual way, then Theorem 2.1 reduces to Theorem A in [Cecchini 2020], by the Miščenko–Fomenko realisation [1979] of the analytic assembly map in that case.

Theorem 2.1 implies a set of topological obstructions to G-invariant positive scalar curvature metrics on M. Let X be any Riemannian manifold on which G acts properly, isometrically and cocompactly. Let $g \in G$, and let $X^g \subset X$ be its fixed-point set. (Properness of the action implies that $X^g = \varnothing$ if g is not contained in a compact subset of G.) Let $\mathcal{N} \to X^g$ be the normal bundle to X^g in X. The connected components of X^g are submanifolds of X of possibly different dimensions, so the rank of $\mathcal{N}$ may jump between these components. In what follows, we implicitly apply all constructions to the connected components of X^g and add the results together.

By a *cutoff function* we will mean a smooth function $c : M \to [0, 1]$ such that $\mathrm{supp}(c)$ has compact intersection with each G-orbit, and for each $x \in M$ we have

$$\int_G c(g^{-1}x)\, dg = 1.$$

We will also use cutoff functions, defined analogously, for other group actions.

Let $R^{\mathcal{N}}$ be the curvature of the Levi-Civita connection restricted to $\mathcal{N}$. Let $\hat{A}(X^g)$ be the $\hat{A}$-class of X^g. Let $Z_G(g) < G$ be the centraliser of g. Let c^g be a cutoff function for the action by $Z_G(g)$ on X^g.

Definition 2.3. The *g-localised $\hat{A}$-genus of X* is

$$\hat{A}_g(X) := \int_{X^g} c^g \frac{\hat{A}(X^g)}{\det(1 - g e^{-R^{\mathcal{N}}/2\pi i})^{1/2}}.$$

If $g = e$, then

$$\hat{A}_e(X) = \int_X c^e \hat{A}(X)$$

is the L^2-$\hat{A}$ genus of X used in [Wang 2014]. If G is discrete and acts properly and freely on X, then $\hat{A}_e(X) = \hat{A}(X/G)$.

Corollary 2.4. *Suppose that either*

- *G is any locally compact group and $g = e$;*
- *G is discrete and finitely generated and g is any element*; *or*
- *G is a connected semisimple Lie group and g is a semisimple element.*

Let M be a manifold on which G acts properly and that admits a G-equivariant Spin-*structure. Let $H \subset M$ be a G-invariant*, *cocompact hypersurface such that $M \setminus H = X \cup Y$ for disjoint open subsets $X, Y \subset M$. The localised $\hat{A}$-genus $\hat{A}_g(H)$ is independent of the choice of a Riemannian metric. If M admits a complete*, *G-invariant Riemannian metric whose scalar curvature is nonnegative*, *and positive in a neighbourhood of H*, *then $\hat{A}_g(H) = 0$.*

Theorem 2 in [Hochs and Mathai 2016] is a generalisation of the vanishing theorem of Atiyah and Hirzebruch [1970] to actions by noncompact groups. It states that if G is a connected Lie group, and not all stabilisers of the action by G on H are maximal compact, then $\mathrm{index}_G(D^H) = 0$. So, in this setting, the obstructions in Theorem 2.1 and Corollary 2.4 vanish. This makes it a natural question whether Riemannian metrics as in Theorem 2.1 exist if G is a connected Lie group. A partial affirmative answer to that question is given in Section 2B.

This also means that the natural place to look for examples and applications where Theorem 2.1 and Corollary 2.4 yield nontrivial obstructions is the setting where G has infinitely many connected components. (The vanishing result generalises directly to the case where G has finitely many connected components.) As noted in Remark 2.2, Theorem 2.1 implies Theorem A in [Cecchini 2020], in the case where G is the fundamental group of M/G and M is its universal cover. More generally, if G is discrete, then M/G is an orbifold. Then Theorem 2.1 and Corollary 2.4 lead to obstructions to orbifold metrics on M/G with nonnegative scalar curvature, and positive scalar curvature near the suborbifold N/G. If G acts freely, then $\hat{A}_g(H)$ is zero if $g \neq e$ (and $\hat{A}_e(H) = \hat{A}(H/G)$), but in the orbifold case the localised $\hat{A}$-genera for nontrivial elements g are additional obstructions to positive scalar curvature.

2B. ***Existence for connected Lie groups.*** In this subsection, we suppose that G is a connected Lie group. As pointed out at the end of the previous subsection, for such groups the obstructions to G-invariant Riemannian metrics with positive scalar curvature in Theorem 2.1 and Corollary 2.4 vanish under a mild assumption on the action, so it is natural to investigate existence of such metrics.

If G is connected, Abels' global slice theorem [1974] implies we have a diffeomorphism $M \cong G \times_K N$, for a K-invariant submanifold $N \subset M$. Our existence result, Theorem 2.11, supposes that such a slice N has *K-bounded geometry*, a notion introduced in Definition 2.10.

Suppose a compact, connected Lie group K acts isometrically on a complete Riemannian manifold (N, g_N). Let b denote a bi-invariant Riemannian metric on K. For each $y \in N$ we have a linear map

$$\varphi_y : \mathfrak{k} \to T_y N \tag{2.5}$$

defined by $\varphi_y(X) := (d/dt)|_{t=0} \exp(tX)y$ for $X \in \mathfrak{k}$. Define a pointwise norm function

$$\|\varphi\| : N \to \mathbb{R}, \quad y \mapsto \|\varphi_y\|, \tag{2.6}$$

where $\|\cdot\|$ denotes the linear operator norm with respect to b and g_N.

Definition 2.7. We say that the action of K on N has *no shrinking orbits* if, for any neighbourhood U of the fixed-point set N^K, there exists a constant $C_U > 0$ such that for all $y \in N \setminus U$ we have

$$\|\varphi(y)\| \geq C_U,$$

where the norm function is taken with respect to the Riemannian metric g.

We remark that the condition of the action having no shrinking orbits is independent of b.

Example 2.8. Suppose that $N = \mathbb{R}^2$, on which $K = \mathrm{SO}(2)$ acts in the natural way. Let $\psi \in C^\infty(\mathbb{R}^2)$ be positive and rotation-invariant, and consider the Riemannian metric on $\mathbb{R}^2$ equal to ψ^2 times the Euclidean metric. Then for all $y \in \mathbb{R}^2$ and $X \in \mathbb{R} \cong \mathfrak{so}(2)$,

$$\varphi_y(X) = X \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} y.$$

So $\|\varphi\|(y) = \psi(y)\|y\|$, where $\|y\|$ is the Euclidean norm of y. Hence the action has no shrinking orbits if and only if the function $y \mapsto \psi(y)\|y\|$ has a positive lower bound outside a neighbourhood of $(\mathbb{R}^2)^{\mathrm{SO}(2)} = \{0\}$.

Now let us define the notion of K-bounded geometry, which is a strengthening of the standard notion of bounded geometry.

Definition 2.9. A Riemannian manifold has *bounded geometry* if

- its injectivity radius is positive;
- for each $l \geq 0$ there exists $C_l > 0$ such that $\|\nabla^l R\|_\infty \leq C_l$, where R is the Riemann curvature tensor.

Definition 2.10. The action of K on N is said to have *K-bounded geometry* if it has no shrinking orbits and N has bounded geometry.

Our main existence result, proved in Section 6B, is the following.

Theorem 2.11. *Let G be a connected Lie group, and $K < G$ a maximal compact subgroup with nonabelian identity component. Let N be a manifold admitting an effective action by K with compact fixed-point set. If there exists a Riemannian metric on N such that the K-action has K-bounded geometry, then the manifold $G \times_K N$ admits a G-invariant metric with uniformly positive scalar curvature.*

This result may be viewed as a strengthening of the vanishing of the obstructions to G-invariant metrics of positive scalar curvature in Theorem 2.1 and Corollary 2.4 in the case of connected Lie groups, by the result in [Hochs and Mathai 2016], in the same way that the construction in [Lawson and Yau 1974] of metrics of positive scalar curvature strengthens the vanishing of the $\hat{A}$-genus as in the vanishing theorem of [Atiyah and Hirzebruch 1970] in the compact case. (See the diagram on page 233 of [Lawson and Yau 1974].)

Remark 2.12. By Abels' slice theorem [1974], every manifold with a proper action by a connected Lie group G is of the form $G \times_K N$ in Theorem 2.11. The condition that N^K is compact is equivalent to the condition that the points in $G \times_K N$ whose stabilisers in G are maximal compact form a cocompact set.

We note that the proof of Theorem 2.11 makes essential use of the assumption that the subgroup K has nonabelian identity component in constructing a positive scalar curvature metric on $G \times_K N$ (see the proof of Proposition 6.2), as does the proof of Lawson and Yau's original result [1974].

In the case of S^1 acting freely on a closed manifold M, Bérard-Bergery [1983] shows that M has an S^1-invariant metric of positive scalar curvature if and only if M/S^1 has a metric of positive scalar curvature. This gives many examples of manifolds with vanishing index invariants but no invariant metrics of scalar curvature. On the other hand, Wiemeler [2016, Theorem 2.4] shows that in the opposite extreme when the S^1-fixed-point set has low codimension, M also admits an S^1-invariant metric of positive scalar curvature.

3. An index theorem

We will deduce Theorem 2.1, and hence Corollary 2.4, from an equivariant index theorem for Callias-type operators, Theorem 3.10. This is based on equivariant index theory for such operators with respect to proper actions, developed in [Guo 2021]. The proof of the index theorem involves several arguments analogous to those in [Cecchini 2020].

3A. ***The G-Callias-type index.*** From now on, M will be a complete Riemannian manifold on which G acts properly and isometrically. Let $S \to M$ denote a $\mathbb{Z}/2$-graded, G-equivariant Clifford module over M, and D an odd-graded Dirac operator on $\Gamma^\infty(S)$, associated to a G-invariant Clifford connection on S via the Clifford action by TM on S.

Let Φ be an odd, G-equivariant, fibrewise Hermitian vector bundle endomorphism of S.

Definition 3.1. The endomorphism Φ is *admissible* for D if

- the operator $D\Phi + \Phi D$ on $\Gamma^\infty(S)$ is a vector bundle endomorphism; and
- there are a cocompact subset $Z \subset M$ and a constant $C > 0$ such that we have the pointwise estimate
$$\Phi^2 \geq \|D\Phi + \Phi D\| + C \tag{3.2}$$
on $M \setminus Z$.

In this setting the operator $D + \Phi$ is called a *G-Callias-type operator*.

In the rest of the paper, we will use the following notation. Let μ denote the modular function on G. Let $C^*(G)$ be either the reduced or maximal group C^*-algebra of G. Let G act on sections of the bundle S by

$$(gs)(x) := g(s(g^{-1}x)),$$

for s a section, $g \in G$ and $x \in M$.

Equip the space $\Gamma_c(S)$ with a right $C_c(G)$-action defined by

$$(s_1 \cdot b)(x) := \int_G (gs)(x) \cdot b(g^{-1})\mu(g)^{-\frac{1}{2}}\, dg \tag{3.3}$$

and a $C_c(G)$-valued inner product defined by

$$(s_1, s_2)_{C^*(G)}(g) := \mu(g)^{-\frac{1}{2}}(s_1, gs_2)_{L^2(S)}, \tag{3.4}$$

for $s_1, s_2 \in \Gamma_c^\infty(S)$, $b \in C_c(G)$, $g \in G$, and $x \in M$. Let $\mathcal{E}$ be the Hilbert $C^*(G)$-module completion of $\Gamma_c^\infty(S)$ with respect to this structure.

The definition of the equivariant index of G-Callias-type operators is based on the following result, Theorem 4.19 in [Guo 2021].

Theorem 3.5. *There is a continuous G-invariant cocompactly supported function f on M such that*

$$F := (D+\Phi)((D+\Phi)^2 + f)^{-\frac{1}{2}} \tag{3.6}$$

is a well-defined, adjointable operator on $\mathcal{E}$ and $(\mathcal{E}, F)$ is a Kasparov $(\mathbb{C}, C^*(G))$-cycle. Its class in $KK(\mathbb{C}, C^*(G))$ is independent of the function f chosen.

For details about the definition of the operator F, we refer to Definition 4.11 in [Guo 2021].

Definition 3.7. The *G-index* of the G-Callias-type operator $D+\Phi$ is the class

$$\text{index}_G(D+\Phi) := [\mathcal{E}, F] \in K_0(C^*(G)) = KK(\mathbb{C}, C^*(G)),$$

as in Theorem 3.5.

3B. ***Hypersurfaces and the index theorem.*** In the setting of the previous subsection, we now suppose that $S = S_0 \oplus S_0$ for an ungraded, G-equivariant Clifford module S_0 over M, where the first copy of S_0 is the even part of S, and the second copy is the odd part. Suppose that

$$D = \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix} \tag{3.8}$$

for a Dirac operator D_0 on S_0 and that

$$\Phi = \begin{pmatrix} 0 & i\Phi_0 \\ -i\Phi_0 & 0 \end{pmatrix} \tag{3.9}$$

for a Hermitian endomorphism Φ_0 of S_0. (The conditions on $D\Phi + \Phi D$ then become conditions on $[D_0, \Phi_0]$.)

Let Z be as in Definition 3.1. Let $M_- \subset M$ be a G-invariant, cocompact subset containing Z in its interior such that $N := \partial M_-$ is a smooth submanifold of M. Let M_+ be the closure of the complement of M_-, so that $N = M_- \cap M_+$ and $M = M_- \cup M_+$. In this and similar settings, we write

$$M = M_- \cup_N M_+.$$

By (3.2), the restriction of Φ_0 to N is fibrewise invertible. Let $S_+^N \to N$ and $S_-^N \to N$ be its positive and negative eigenbundles. (These are vector bundles, even though eigenbundles for single eigenvalues may not be.) Clifford multiplication by the unit normal vector field $\hat{n}$ to N pointing into M_+, times $-i$, defines G-invariant gradings on $S_\pm^N$.

Let ∇^{S_0} be the Clifford connection on S_0 used to define D_0. By restriction and projection, it defines connections $\nabla^{S_\pm^N}$ on $S_\pm^N$. The Clifford action by $TM|_N$ on $S_0|_N$ preserves $S_\pm^N$ by the first condition in Definition 3.1; see also Remark 1.2

in [Anghel 1993]. Therefore the connections $\nabla^{S^N_\pm}$ define Dirac operators $D^{S^N_\pm}$ on $\Gamma^\infty(S^N_\pm)$. These operators are odd-graded. Because N is cocompact, $D^{S^N_+}$ has an equivariant index

$$\mathrm{index}_G(D^{S^N_+}) \in K_0(C^*(G))$$

defined by the analytic assembly map [Baum et al. 1994].

Theorem 3.10 (G-Callias-type index theorem). *We have*

$$\mathrm{index}_G(D+\Phi) = \mathrm{index}_G(D^{S^N_+}) \quad \textit{in } K_0(C^*(G)). \tag{3.11}$$

Versions of this result where G is trivial were proved in [Anghel 1993; Bott and Seeley 1978; Bunke 1995; Callias 1978; Kucerovsky 2001]. Versions for operators on bundles of modules over operator algebras are proved in [Braverman and Cecchini 2018; Cecchini 2020].

There are various index theorems for the image of the right-hand side of (3.11) under traces [Hochs and Wang 2018; Wang 2014; Wang and Wang 2016] or pairings with higher cyclic cocycles [Hochs et al. 2020; Pflaum et al. 2015; Piazza and Posthuma 2019]. Via these results, Theorem 3.10 yields topological expressions for the corresponding images of the left-hand side of (3.11). The results of [Hochs and Wang 2018; Wang 2014; Wang and Wang 2016] will be used to deduce Corollary 2.4 from Theorem 2.1.

4. Properties of the G-Callias-type index

To prove Theorem 3.10, we will use the properties of the index of Definition 3.7 that we describe below.

4A. ***Sobolev modules.*** First we recall the definition of Sobolev Hilbert $C^*(G)$-modules from [Guo 2021]. Let M, G, S and D be as in Section 3A.

Definition 4.1. For each nonnegative integer j, define $\Gamma_c^{\infty,j}(S)$ to be the pre-Hilbert $C_c(G)$-module whose underlying vector space is $\Gamma_c^\infty(S)$, equipped with the right $C_c(G)$-action defined by (3.3) and $C_c(G)$-valued inner product defined by

$$\langle e_1, e_2\rangle_{\mathcal{E}^j} = \sum_{k=0}^{j} (D^k e_1, D^k e_2)_{C^*(G)},$$

where $e_1, e_2 \in \Gamma_c^\infty(S)$ and $(-,-)_{C^*(G)}$ is as in (3.4). Here we set D^0 equal to the identity operator. Denote by $\mathcal{E}^j(S) = \mathcal{E}^j$ the vector space completion of $\Gamma_c^{\infty,j}(S)$ with respect to the norm induced by $\langle -,-\rangle_{\mathcal{E}^j}$, and extend naturally the $C_c^\infty(G)$-action to a $C^*(G)$-action, and $\langle -,-\rangle_{\mathcal{E}^j}$ to a $C^*(G)$-valued inner product on $\mathcal{E}^j$, to give it the structure of a Hilbert $C^*(G)$-module. We call $\mathcal{E}^j$ the *j-th G-Sobolev module with respect to D.*

The module $\mathcal{E}$ defined above Theorem 3.5 equals $\mathcal{E}^0$. The following version of the Rellich lemma holds for Sobolev modules (Theorem 3.12 in [Guo 2021]).

Theorem 4.2. *Let f be a continuous G-invariant cocompactly supported function on M. Then multiplication by f defines an element of $\mathcal{K}(\mathcal{E}^s, \mathcal{E}^t)$ whenever $s > t$.*

We will state and prove a homotopy invariance result, Proposition 4.11, for the index in Definition 3.7, which will be of use later. A hypothesis in this result is that a certain vector bundle endomorphism defines adjointable operators on the Sobolev modules $\mathcal{E}^0$ and $\mathcal{E}^1$. In order to check this condition in some geometric situations relevant to us, we will need Propositions 4.3 and 4.4 below.

Proposition 4.3. *A smooth, G-invariant, uniformly bounded bundle endomorphism of S defines an element of $\mathcal{L}(\mathcal{E}^0)$.*

Proof. Let Ψ be a smooth, G-invariant, uniformly bounded bundle endomorphism of S. Since Ψ is uniformly bounded, it defines a bounded operator on $L^2(S)$. Let $\|\Psi\|$ denote its operator norm, let c be a cutoff function on M, and let Ψ^* be the pointwise adjoint of Ψ. Since the operator $\Psi_1 := \Psi^*\Psi - \|\Psi\|^2$ is positive on $L^2(S)$, it has a positive square root Q that one observes is G-invariant. For a fixed $e \in \Gamma_c^\infty(S)$, the function

$$g \mapsto (c\Psi_1(ge), ge)_{L^2(S)} = (\sqrt{c}Q(ge), \sqrt{c}Q(ge))_{L^2(S)}$$

has compact support in G, by properness of the G-action. Thus the map $G \to L^2(S)$ defined by $g \mapsto \sqrt{c}Q(ge)$ has compact support in G.

It follows that for any unitary representation π of G on a Hilbert space H and $h \in H$,

$$v := \int_G \mu(g)^{-\frac{1}{2}} \sqrt{c}Q(ge) \otimes \pi(g)h \, dg$$

is a well-defined vector in $L^2(S) \otimes H$. By computations similar to those in the proof of Proposition 5.4 in [Guo 2021], one sees that $\|v\|_{L^2(S)\otimes H}$ equals

$$\int_G \int_G \langle gc\Psi_1 g^{-1}e, e\rangle_{\mathcal{E}^0}(g') \, dg \cdot (\pi(g')h, h)_H \, dg'.$$

Thus, for any unitary representation π of G,

$$\pi\biggl(\int_G \langle gc\Psi_1 g^{-1}e, e\rangle_{\mathcal{E}^0} \, dg \biggr) = \pi(\langle \Psi_1 e, e\rangle_{\mathcal{E}^0})$$

is a positive operator, where we let $f \in C_c(G)$ act on H by

$$\pi(f)h := \int_G f(g)\pi(g)h \, dg.$$

It follows that the element

$$\langle \Psi_1 e, e\rangle_{\mathcal{E}^0} = \bigl\langle (\Psi^*\Psi - \|\Psi\|^2)e, e\bigr\rangle_{\mathcal{E}^0} = \langle \Psi e, \Psi e\rangle_{\mathcal{E}^0} - \|\Psi\|^2 \langle e, e\rangle_{\mathcal{E}^0}$$

is positive in $C^*(G)$. Hence Ψ extends to an operator on all of $\mathcal{E}^0$. Similarly, Ψ^* defines an operator on all of $\mathcal{E}^0$ that one checks is the adjoint of Ψ. $\square$

Proposition 4.4. *Suppose that there are a G-invariant, cocompact subset $K \subset M$ and a G-invariant, cocompact hypersurface $N \subset M$ such that there are a G-equivariant isometry $M \setminus K \cong N \times (0, \infty)$ and a G-equivariant vector bundle isomorphism $S|_{M\setminus K} \cong S|_N \times (0, \infty)$. Let Ψ be a G-equivariant vector bundle endomorphism of S. Suppose that, on $M \setminus K$, Ψ and D are constant in the factor $(0, \infty)$ of $M \setminus K \cong N \times (0, \infty)$. Then Ψ defines an element of $\mathcal{L}(\mathcal{E}^1)$.*

The proof uses the next lemma. To state it, let $H^1(S)$ be the completion of $\Gamma_c^\infty(S)$ with respect to the inner product

$$(-\,,-)_{H^1(S)} = (-\,,-)_{L^2(S)} + (D-\,,D-)_{L^2(S)}.$$

Lemma 4.5. *Let M and S be as in Proposition 4.4. Let Θ be a bounded, positive operator on $H^1(S)$ such that*

- *Θ preserves the subspace $\Gamma_c^\infty(S)$;*
- *for any $e \in \Gamma_c^\infty(S)$, the function $a : G \to \mathbb{R}$ given by $g \mapsto (\Theta(ge), ge)_{H^1(S)}$ has compact support in G.*

Then

$$\int_G \langle g\Theta g^{-1}e, e\rangle_{\mathcal{E}^1}\, dg$$

is a positive element of $C^*(G)$.

Proof. Let Q be the positive square root of Θ in $\mathcal{B}(H^1(S))$. Since a has compact support, and $(\Theta(ge), ge)_{H^1(S)} = (Q(ge), Q(ge))_{H^1(S)}$, the map $G \to H^1(S)$ defined by $g \mapsto Q(ge)$ has compact support in G. As in the proof of Proposition 4.3, one finds that for any unitary representation π of G on a Hilbert space H and $h \in H$,

$$v := \int_G \mu(g)^{-\frac{1}{2}} Q(ge) \otimes \pi(g)h\, dg$$

is a well-defined vector in $H^1(S) \otimes H$ and that

$$\int_G \int_G \langle gQ^2 g^{-1}e, e\rangle_{\mathcal{E}^1}(g')\, dg \cdot (\pi(g')h, h)_H\, dg' = \|v\|_{H^1(S)\otimes H} \geq 0.$$

Similarly to the proof of Proposition 4.3, we deduce that $\int_G \langle g\Theta g^{-1}e, e\rangle_{\mathcal{E}^1}\, dg$ is a positive element of $C^*(G)$. $\square$

Proof of Proposition 4.4. Because of the forms of M and S, there is a canonical (up to equivalence) first Sobolev norm $\|\cdot\|_1$ on sections of S that is G-invariant, and invariant under the relevant class of translations in the factor $(0, \infty)$ of $N \times (0, \infty)$. Because Ψ is an order-zero differential operator constant on the factor $(0, \infty)$, it defines a bounded operator with respect to $\|\cdot\|_1$. Due to the form of D, the norm on $H^1(S)$ is equivalent to $\|\cdot\|_1$, and so Ψ defines a bounded operator on $H^1(S)$.

Let $\pi_N : M \setminus K \cong N \times (0, \infty) \to N$ be the natural projection. Let c be a cutoff function on M such that

$$c|_{M\setminus K} = \pi_N^* c_N$$

for a cutoff function c_N on N. Let Ψ^* and c^* denote the respective adjoints of Ψ and c in $\mathcal{B}(H^1(S))$. Then the operator

$$\Psi_1 := \tfrac{1}{2}(c\Psi^*\Psi + \Psi^*\Psi c^*)$$

is bounded and self-adjoint on $H^1(S)$ with norm at most $\|\Psi\|^2\|c\|$, where the norms are taken in $\mathcal{B}(H^1(S))$.

Let c' be a smooth, nonnegative function on M that is identically 1 on the support of c and such that

$$c'|_{M\setminus K} = \pi_N^* c'_N$$

for a compactly supported function c'_N on N. Consider the endomorphism $\Psi_2 := (c')^*c'\|\Psi\|^2\|c\| - \Psi_1$ of S. For the same reasons as for Ψ, it defines a bounded operator on $H^1(S)$. Fix $e \in \Gamma_c^\infty(S)$. Because Ψ is a positive bounded operator on $H^1(S)$, we may apply Lemma 4.5 with $\Theta = \Psi_2$ to conclude that

$$\int_G \langle (g\Psi_2 g^{-1})e, e\rangle_{\mathcal{E}^1}\, dg = \int_G \big\langle \big(g((c')^*c')g^{-1}\|\Psi\|^2\|c\|\big)e, e\big\rangle_{\mathcal{E}^1}\, dg - \langle \Psi^*\Psi e, e\rangle_{\mathcal{E}^1} \tag{4.6}$$

is positive in $C^*(G)$. Define $b : M \times G \to \mathbb{R}$ by

$$b(x, g) := b_x(g) := c'(g^{-1}x).$$

By construction of c', the quantities

$$C_1(x) := \int_G c'(g^{-1}x)^2\, dg \quad \text{and} \quad C_2(x) := \|dc'\|_\infty^2 \cdot \operatorname{vol}(\operatorname{supp}_G(b_x))$$

are bounded as functions of $x \in M$ (see also Remark 4.7 below). A direct calculation shows that

$$\begin{aligned}
\Big\| \int_G \big\langle \big(g((c')^*c')g^{-1}\|\Psi\|^2\|c\|\big)e, e\big\rangle_{\mathcal{E}^1}\, dg \Big\|_{C^*(G)} \\
&= \|\Psi\|^2\|c\| \Big\| \int_G \langle c'g^{-1}e, c'g^{-1}e\rangle_{\mathcal{E}^1}\, dg \Big\|_{C^*(G)} \\
&\leq \|\Psi\|^2\|c\| \big(\|C_1\|_\infty \|e\|_{\mathcal{E}^1}^2 + \|C_2\|_\infty \|e\|_{\mathcal{E}^0}^2\big) \\
&\leq C_3\|\Psi\|^2\|e\|_{\mathcal{E}^1}^2
\end{aligned}$$

for some constant C_3. Together with the positivity of (4.6), this implies that

$$\|\Psi e\|^2_{\mathcal{E}^1} = \|\langle e, \Psi^*\Psi e\rangle_{\mathcal{E}^1}\|_{C^*(G)} \leq C_3\|\Psi\|^2\|e\|^2_{\mathcal{E}^1},$$

so that Ψ extends to an operator on all of $\mathcal{E}^1$. Similarly, the $H^1(S)$-adjoint Ψ^* defines an operator on $\mathcal{E}^1$ that one checks is the adjoint of Ψ. □

Remark 4.7. As can be seen from the proof, the conclusion of Proposition 4.4 holds more broadly for any M on which the functions C_1 and C_2 on M are bounded.

4B. ***Vanishing.*** Two cases where the index of Definition 3.7 vanishes are straightforward to prove, but we state them here because they will be used in various places.

Lemma 4.8. *If (3.2) holds on all of M, then* $\mathrm{index}_G(D+\Phi) = 0$.

Proof. In this setting, the operator F in (3.6) is invertible. This implies that the KK-cycle $(\mathcal{E}, F)$ is operator-homotopic to the degenerate cycle $(\mathcal{E}, F(F^*F)^{-\frac{1}{2}})$. □

Lemma 4.9. *If M/G is compact, and D and Φ are of the forms (3.8) and (3.9), then* $\mathrm{index}_G(D) = 0$.

Proof. In this setting, Φ is bounded, and the cycle $(\mathcal{E}, F)$ is operator-homotopic to $(\mathcal{E}, D/\sqrt{D^2+1})$.

In general, let A be a C^*-algebra, let $\mathcal{E}_0$ be a Hilbert A-module, and set $\mathcal{E} := \mathcal{E}_0 \oplus \mathcal{E}_0$. Suppose there is an adjointable operator F on $\mathcal{E}$ such that $(\mathcal{E}, F)$ is a Kasparov $(\mathbb{C}, A)$-cycle and F is of the form

$$F = \begin{pmatrix} 0 & F_0 \\ F_0 & 0 \end{pmatrix}, \tag{4.10}$$

for a (necessarily self-adjoint) $F_0 \in \mathcal{L}(\mathcal{E}_0)$. Then $[\mathcal{E}, F] = 0 \in K_0(A)$. Because $D/\sqrt{D^2+1}$ is of the form (4.10), the claim follows. □

4C. ***Homotopy invariance.*** The index of Definition 3.7 has a homotopy invariance property analogous to Proposition 4.1 in [Cecchini 2020]. This homotopy invariance applies in a more general setting than Callias-type operators.

Let P be an odd, G-equivariant Dirac-type operator on a $\mathbb{Z}/2$-graded Clifford module $\mathcal{S}$, and let Ψ be an odd, smooth G-equivariant, uniformly bounded Hermitian vector bundle endomorphism of $\mathcal{S}$. Fix $t_0 < t_1 \in \mathbb{R}$. For $t \in [t_0, t_1]$, consider the operator $P_t := P + t\Psi$.

Proposition 4.11 (homotopy invariance). *Suppose that*:

(1) *For $j = 0, 1$, the endomorphism Ψ defines an adjointable operator on the Sobolev module $\mathcal{E}^j$ of Definition 4.1.*

(2) *There is a nonnegative, G-invariant, cocompactly supported function $f \in C^\infty(M)$ such that for all $t \in [t_0, t_1]$ the operator $P_t^2 + f : \mathcal{E}^2 \to \mathcal{E}^0$ is invertible, with inverse in $\mathcal{L}(\mathcal{E}^0, \mathcal{E}^2)$.*

Then $\operatorname{index}_G(P_t) \in K_0(C^*(G))$ *is independent of $t \in [t_0, t_1]$.*

Proof. The proof proceeds identically to the proof of Proposition 4.1 in [Cecchini 2020], with the exceptions that Theorem 4.2 should be substituted for Lemma 4.2 in [Cecchini 2020], and Lemmas 4.4 and 4.6(a) in [Guo 2021] should be substituted for Lemmas 1.4 and 1.5 in [Bunke 1995], respectively. □

Corollary 4.12. *If $D + \Phi$ is a G-Callias-type operator on $S \to M$, and Ψ is a G-equivariant, odd vector bundle endomorphism of S that equals zero outside a cocompact set, then* $\operatorname{index}_G(D + \Phi) = \operatorname{index}_G(D + \Phi + \Psi)$.

Proof. We set $P = D + \Phi$ and apply Proposition 4.11. Since Ψ is cocompactly supported, the first condition in Proposition 4.11 holds by Proposition 3.5 in [Guo 2021]. For the same reason, $\Phi + t\Psi$ is a Callias-type potential for all $t \in \mathbb{R}$, so by Theorem 5.6 in [Guo 2021], the second condition in Proposition 4.11 holds for $t \in [0, 1]$, where a priori the function f may depend on t. But since Ψ is zero outside a cocompact set, we can choose f independent of t. The claim then follows from Proposition 4.11. □

Remark 4.13. In Proposition 4.11, it is not assumed that Ψ is a Callias-type potential in the sense of Definition 3.1. We will use Proposition 4.11 in this greater generality in the proof of Lemma 5.5.

Remark 4.14. Corollary 4.12 can be used to give an alternative proof of Lemma 4.8: this corollary implies that in the setting of that lemma,

$$\operatorname{index}_G(D + \Phi) = \operatorname{index}_G(D - \Phi) = \operatorname{index}_G((D + \Phi)^*) = -\operatorname{index}_G(D + \Phi).$$

See also Corollary 4.9 in [Cecchini 2020].

4D. ***A relative index theorem.*** We will use an analogue of Bunke's relative index theorem [1995, Theorem 1.2]. For $j = 1, 2$, let M_j, S_j, D_j and Φ_j, respectively, be as M, S, D and Φ were before. Suppose that there are a G-invariant hypersurface $N_j \subset M_j$ and a G-invariant tubular neighbourhood U_j of N_j and that there is a G-equivariant isometry $\psi : U_1 \to U_2$ such that

- $\psi(N_1) = N_2$;
- $\psi^*(S_2|_{U_2}) \cong S_1|_{U_1}$;
- $\psi^*(\nabla^{S_2}|_{U_2}) = \nabla^{S_1}|_{U_1}$, where ∇^{S_j} is the Clifford connection used to define D_j;
- $\Phi_1|_{U_1}$ corresponds to $\Phi_2|_{U_2}$ via ψ.

Suppose that $M_j = X_j \cup_{N_j} Y_j$ for closed, G-invariant subsets $X_j, Y_j \subset M_j$. We identify N_1 and N_2 via ψ and write N for this manifold when we do not want to distinguish between N_1 and N_2. Using the map φ, form

$$M_3 := X_1 \cup_N Y_2 \quad \text{and} \quad M_4 := X_2 \cup_N Y_1.$$

For $j = 3, 4$, let S_j, D_j and Φ_j be obtained from the corresponding data on M_1 and M_2 by cutting and gluing along $U_1 \cong U_2$ via ψ.

Theorem 4.15. *In the above situation*,

$$\mathrm{index}_G(D_1 + \Phi_1) + \mathrm{index}_G(D_2 + \Phi_2) = \mathrm{index}_G(D_3 + \Phi_3) + \mathrm{index}_G(D_4 + \Phi_4).$$

Proof. This proof is an adaptation of the proof of Theorem 1.14 in [Bunke 1995], with some results from [Guo 2021] added. For $j = 1, 2, 3, 4$, let $\mathcal{E}_j$ and F_j be as $\mathcal{E}$ and F above and in Theorem 3.5, for the data indexed by j. Using superscripts op to denote opposite gradings, we write $\mathcal{E} := \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \mathcal{E}_3^{\mathrm{op}} \oplus \mathcal{E}_4^{\mathrm{op}}$ and $F := F_1 \oplus F_2 \oplus F_3 \oplus F_4$. We will show that

$$[\mathcal{E}, F] = 0 \quad \text{in } KK_0(\mathbb{C}, C^*(G)), \tag{4.16}$$

which is equivalent to the theorem.

For $j = 1, 2$, let $\chi_{X_j}, \chi_{Y_j} \in C^\infty(M_j)$ be real-valued functions such that

- $\mathrm{supp}(\chi_{X_j}) \subset X_j \cup U_j$ and $\mathrm{supp}(\chi_{Y_j}) \subset Y_j \cup U_j$;
- $\psi^*(\chi_{X_2}|_{U_2}) = \chi_{X_1}|_{U_1}$ and $\psi^*(\chi_{Y_2}|_{U_2}) = \chi_{Y_1}|_{U_1}$;
- $\chi_{X_j}^2 + \chi_{Y_j}^2 = 1$.

We view pointwise multiplication by these functions as operators

$$\begin{aligned} \chi_{X_1} &: \mathcal{E}_1 \to \mathcal{E}_3, & \chi_{X_2} &: \mathcal{E}_1 \to \mathcal{E}_4, \\ \chi_{Y_1} &: \mathcal{E}_2 \to \mathcal{E}_3, & \chi_{Y_2} &: \mathcal{E}_2 \to \mathcal{E}_4. \end{aligned} \tag{4.17}$$

The adjoints of these operators map in the opposite directions, and are also given by pointwise multiplication by the respective functions. Using these multiplication operators, and the grading operator γ, we form the operator

$$X := \gamma \begin{pmatrix} 0 & 0 & -\chi_{X_1}^* & \chi_{X_2}^* \\ 0 & 0 & -\chi_{Y_1}^* & \chi_{Y_2}^* \\ \chi_{X_1} & \chi_{Y_1} & 0 & 0 \\ \chi_{X_2} & -\chi_{Y_2} & 0 & 0 \end{pmatrix}$$

on $\mathcal{E}$. Then X is an odd, self-adjoint, adjointable operator on $\mathcal{E}$ such that $X^2 = 1$. As such, it generates a Clifford algebra Cl.

We claim $XF + FX$ is a compact operator. This is based on the Rellich lemma for Hilbert $C^*(G)$-modules, Theorem 4.2. Let χ be one of the functions χ_{X_j} or χ_{Y_j}, viewed as an operator from $\mathcal{E}_k$ to $\mathcal{E}_l$ as in (4.17). Let $f_j \in C^\infty(M)$ be as in Theorem 3.5, for the operator $D_j + \Phi_j$. For $j = 1, 2, 3, 4$ and $\lambda \in \mathbb{R}$, the operator $(D_j^2 + f_j^2 + \lambda^2)$ on $\mathcal{E}_j$ is invertible by Lemma 4.6 in [Guo 2021], and we denote its inverse by $R_j(\lambda)$.

Then, as in the proof of Theorem 1.14 in [Bunke 1995], and using Proposition 4.12 in [Guo 2021], we find that the operator

$$\chi^* \circ F_l - F_k \circ \chi^* : \mathcal{E}_l \to \mathcal{E}_k \tag{4.18}$$

equals

$$\frac{2}{\pi} \int_{\mathbb{R}} \big(-\operatorname{grad}(\chi) R_l(\lambda) + D_k^2 R_k(\lambda) \operatorname{grad}(\chi) R_l(\lambda) \\ + D_k R_k(\lambda) \operatorname{grad}(f_k) R_k(\lambda) \operatorname{grad}(\chi) R_l(\lambda) \\ + D_k R_k(\lambda) \operatorname{grad}(\chi) D_l R_l(\lambda) \big)\, d\lambda. \tag{4.19}$$

Theorem 4.2, together with Lemma 4.6(a) in [Guo 2021], implies that for all cocompactly supported continuous functions φ on M_j, the compositions $\varphi D_j^n R_j(\lambda)$, $D_j^n \varphi R_j(\lambda)$ and $D_j^n R_j(\lambda)\varphi$ are compact operators on $\mathcal{E}_j$ if $n = 0$, 1, and adjointable operators if $n = 1$. So all the terms in the integrand in (4.19) are compact operators. By Lemmas 4.6 and 4.8 in [Guo 2021], the norm of the integrand in (4.19) is bounded by $a(b + \lambda^2)^{-1}$ for constants $a, b > 0$. So the integral converges in the operator norm on $\mathcal{L}(\mathcal{E})$, and we conclude that (4.18) is a compact operator on $\mathcal{E}$. This implies that $XF + FX$ is a compact operator.

Because X generates Cl and X anticommutes with F modulo compacts, the pair $(\mathcal{E}, F)$ is a Kasparov (Cl, $C^*(G)$)-cycle. Its class in $KK(\mathrm{Cl}, C^*(G))$ is mapped to the left-hand side of (4.16) by the pullback along the inclusion map $\mathbb{C} \hookrightarrow \mathrm{Cl}$. That map is zero by Lemma 1.15 in [Bunke 1995], so (4.16) follows. □

Theorem 4.15 implies the following version of Proposition 5.9 in [Cecchini 2020].

Corollary 4.20. *In the setting of Theorem 4.15, suppose that for $j = 1, 2$, the set X_j is cocompact and contains a set Z_j for Φ_j as in Definition 3.1. Then*

$$\operatorname{index}_G(D_1 + \Phi_1) = \operatorname{index}_G(D_2 + \Phi_2).$$

Proof. This fact can be deduced from Theorem 4.15 in exactly the same way Proposition 5.9 in [Cecchini 2020] is deduced from Theorem 5.7 there. References to Corollaries 3.4 and 4.9 and Theorem 5.7 in that paper should be replaced by references to Lemmas 4.8 and 4.9 and Theorem 4.15, respectively, here. □

The crucial assumption in Corollary 4.20 is that all data near N_1 can be identified with the corresponding data near N_2.

5. Proof of the G-Callias-type index theorem

The first and most important step in the proof of Theorem 3.10 is Proposition 5.1, which states that $\mathrm{index}_G(D+\Phi)$ equals the index of a G-Callias-type operator on the manifold $N \times \mathbb{R}$, which we will call the cylinder on N. See Figure 1. Such an approach is taken in proofs of various other index theorems for Callias-type operators; see for example [Anghel 1993; Braverman and Cecchini 2018; Bunke 1995; Cecchini 2020].

In this section, we consider the setting of Section 3B. In particular, D and Φ are assumed to be of the forms (3.8) and (3.9).

5A. ***An index on the cylinder.*** Let $S_{\pm}^{N\times\mathbb{R}} \to N\times\mathbb{R}$ be the pullbacks of the bundles of $S_{\pm}^{N} \to N$ defined in Section 3B along the projection $N\times\mathbb{R} \to N$. They are Clifford modules, with Clifford actions

$$\hat{c}(v,t) = c(v + t\hat{n})$$

for $v \in TN$ and $t \in \mathbb{R}$, where c is the Clifford action by TM on S (which preserves S_+^N, as pointed out in Section 3B), and $\hat{n}$ is the normal vector field to N in the direction of M_+. Let $D_0^{S_+^{N\times R}}$ be the Dirac operator on $\Gamma^\infty(S_+^{N\times\mathbb{R}})$ defined by this Clifford action, and the pullback to $N\times\mathbb{R}$ of the restriction to N of the Clifford connection $\nabla^{S_+^N}$ used to define $D_0^{S_+^N}$.

Let $\chi \in C^\infty(\mathbb{R})$ be an odd function such that $\chi(t) = t$ for all $t \geq 2$. We also denote its pullback to $N\times\mathbb{R}$ by χ. Then pointwise multiplication by χ is an admissible endomorphism for $D_0^{S_+^{N\times\mathbb{R}}}$. Whenever a Dirac operator with a subscript 0 is given, we will remove that subscript to denote the corresponding Dirac operator on two copies of the Clifford module in question, as in (3.8). In the current setting, this gives us the Dirac operator

$$D^{S_+^{N\times\mathbb{R}}} = \begin{pmatrix} 0 & D_0^{S_+^{N\times\mathbb{R}}} \\ D_0^{S_+^{N\times\mathbb{R}}} & 0 \end{pmatrix}$$

on $\Gamma^\infty(S_+^{N\times\mathbb{R}} \oplus S_+^{N\times\mathbb{R}})$. We also consider the admissible endomorphism

$$\chi^{N\times\mathbb{R}} = \begin{pmatrix} 0 & i\chi \\ -i\chi & 0 \end{pmatrix}$$

of $S_+^{N\times\mathbb{R}} \oplus S_+^{N\times\mathbb{R}}$.

Proposition 5.1. *We have*

$$\mathrm{index}_G(D+\Phi) = \mathrm{index}_G(D^{S_+^{N\times\mathbb{R}}} + \chi^{N\times\mathbb{R}}). \tag{5.2}$$

The proof of Proposition 5.1 that we give below is an analogue of the proof of Theorem 5.4 in [Cecchini 2020]. We give this proof in Sections 5B and 5C,

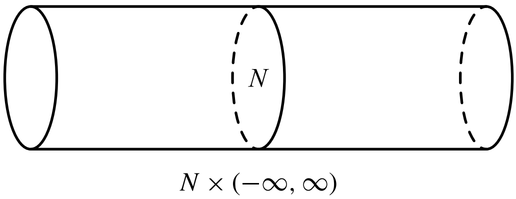

Figure 1. The cylinder $N \times \mathbb{R}$.

referring to [Cecchini 2020] for details in some places, and using results from [Guo 2021] and from Section 4.

5B. ***Attaching a half-cylinder.*** Let $S_0^{N\times\mathbb{R}} \to N \times \mathbb{R}$ be the pullback of $S_0|_N \to N$. We choose U small enough that $S_0|_U \cong S_0^{N\times\mathbb{R}}|_U$.

Because the sets X_j are cocompact in Corollary 4.20, we initially compare the left-hand side of (5.2) to an index on a manifold where only M_+ is replaced by a half-cylinder $N \times [1, \infty)$. To be more precise, $\operatorname{index}_G(D + \Phi)$ is invariant under changes in the Riemannian metric on cocompact sets because the Kasparov $(\mathbb{C}, C^*(G))$-cycles corresponding to two G-invariant Riemannian metrics differing only a cocompact set are homotopic by convexity of the space of G-invariant Riemannian metrics. We choose a metric such that there is a neighbourhood U of N that admits an isometry onto $N \times \left(\frac{1}{4}, \frac{7}{4}\right)$ mapping N onto $N \times \{1\}$ and $U \cap M_-$ onto $N \times \left(\frac{1}{4}, 1\right]$ (see Figure 2).

By Corollary 4.12, the index of $D + \Phi$ does not change if we change Φ_0 in a cocompact set. So we may assume that Φ_0 is constant in the direction normal to N inside U; i.e., for all $n \in N$ and $t \in \left(\frac{1}{4}, \frac{7}{4}\right)$, $\Phi_0(n, t) = \Phi_0^N(n)$, for an endomorphism Φ_0^N of $S_0|_N$. We further choose U such that a set Z as in Definition 3.1 is contained in $M_- \setminus U$.

Let $\nabla^{S_0^{N\times\mathbb{R}}}$ be the pullback of $\nabla^{S_0}|_N$ to a connection on $S_0^{N\times\mathbb{R}}$. We choose the Clifford connection ∇^{S_0} to define D_0 so that on U, it equals the restriction of $\nabla^{S_0^{N\times\mathbb{R}}}$ to $N \times \left(\frac{1}{4}, \frac{7}{4}\right)$.

For this structure near N, we can form the Riemannian manifold (see Figure 3) $M_C := M_- \cup_N (N \times [1, \infty))$, and define the Clifford module $S_0^C \to M_C$ so that it

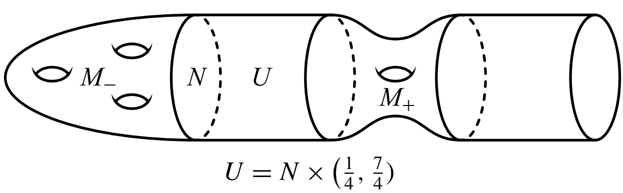

Figure 2. The manifold M.

Figure 3. The manifold M_C.

equals S_0 on M_- and $S_0^{N\times\mathbb{R}}$ on $N \times \left(\frac{1}{4}, \infty\right)$. (Note that $N \times \left(\frac{1}{4}, 1\right] \subset M_-$, so that $N \times \left(\frac{1}{4}, \infty\right) \subset M_C$.) Let $\nabla^{S_0^C}$ denote the Clifford connection on S_0^C corresponding to ∇^{S_0} on M_- and to $\nabla^{S_0^{N\times\mathbb{R}}}$ on $N \times \left(\frac{1}{4}, \infty\right)$. Let D_0^C be the resulting Dirac operator.

We define an endomorphism Φ_0^C of S_0^C that is equal to Φ_0 on M_- and to the pullback of Φ_0^N on $N \times \left(\frac{1}{4}, \infty\right)$. Recall that by removing the subscript 0 from D_0^C we refer to the construction (3.8). Similarly, when we remove the subscript 0 from Φ_0^C, we will be referring to the endomorphism Φ^C defined by Φ_0^C as in (3.9). Then Corollary 4.20 immediately implies that

$$\mathrm{index}_G(D + \Phi) = \mathrm{index}_G(D^C + \Phi^C). \tag{5.3}$$

The connection $\nabla^{S_0^{N\times\mathbb{R}}}$, and therefore the corresponding Dirac operator, does not preserve the decomposition $S_0^{N\times\mathbb{R}} = S_+^{N\times\mathbb{R}} \oplus S_-^{N\times\mathbb{R}}$. With respect to this decomposition, that Dirac operator has the form

$$\begin{pmatrix} D_0^{S_+^{N\times\mathbb{R}}} & A \\ B & D_0^{S_-^{N\times\mathbb{R}}} \end{pmatrix} \tag{5.4}$$

for vector bundle homomorphisms $A : S_-^{N\times\mathbb{R}} \to S_+^{N\times\mathbb{R}}$ and $B : S_+^{N\times\mathbb{R}} \to S_-^{N\times\mathbb{R}}$. (See Section 5.16 in [Braverman and Cecchini 2018], or use the fact that the difference of two connections is an endomorphism-valued one-form.) The Dirac operator D_0^C equals this operator on $N \times \left(\frac{1}{4}, \infty\right)$. Let $\nabla^{S_\pm^{N\times\mathbb{R}}}$ be the pullback of $\nabla^{S_\pm^N}$ to a connection on $S_\pm^{N\times\mathbb{R}}$. Consider a Clifford connection $\nabla^{\tilde{S}_0^C}$ on S_0^C that is equal to the direct sum of $\nabla^{S_+^{N\times\mathbb{R}}}$ and $\nabla^{S_-^{N\times\mathbb{R}}}$ on $N \times \left(\frac{1}{2}, \infty\right)$ and to ∇^{S_0} on $M_- \setminus U$. Then the corresponding Dirac operator $\tilde{D}_0^C$ is equal to

$$\begin{pmatrix} D_0^{S_+^{N\times\mathbb{R}}} & 0 \\ 0 & D_0^{S_-^{N\times\mathbb{R}}} \end{pmatrix}$$

on $N \times \left(\frac{1}{2}, \infty\right)$ and to D_0 on $M_- \setminus U$.

Lemma 5.5. *There exists a $\lambda \geq 1$ such that $\lambda\Phi^C$ is admissible for $\tilde{D}_0^C$. For such λ,*

$$\mathrm{index}_G(D^C + \Phi^C) = \mathrm{index}_G(\tilde{D}^C + \lambda\Phi^C). \tag{5.6}$$

Proof. Existence of λ with the desired property can be established as in the proof of Lemma 5.13 in [Cecchini 2020]. The equality (5.6) can be proved via a linear homotopy; again see the proof of Lemma 5.13 in [Cecchini 2020] for details, where references to Proposition 4.1 there should be replaced by references to Proposition 4.11 here, and one uses Propositions 4.3 and 4.4 to check that the first condition in Proposition 4.11 is satisfied.

To be explicit, an application of Proposition 4.11 shows that

$$\mathrm{index}_G(D^C+\lambda\Phi^C)=\mathrm{index}_G(\tilde{D}^C+\lambda\Phi^C).$$

This follows from Proposition 4.11 by setting $P=D^C+\lambda\Phi^C$, $\Psi=\left(\begin{smallmatrix}0 & A\\ B & 0\end{smallmatrix}\right)$ with A and B as in (5.4), $t_0=-1$ and $t_1=0$. Note that although Ψ is not a Callias-type potential here, Proposition 4.11 still applies, since Ψ defines an element of $\mathcal{L}(\mathcal{E}^0)$ by Proposition 4.3 and an element of $\mathcal{L}(\mathcal{E}^1)$ by Proposition 4.4. (The conditions of Proposition 4.4 hold because of the forms of D^C, Φ^C and Ψ.) A more straightforward application of Proposition 4.11 yields $\mathrm{index}_G(D^C+\Phi^C)=\mathrm{index}_G(D^C+\lambda\Phi^C)$. □

Let $\chi\in C^\infty(\mathbb{R})$ be an odd function such that

$$\chi(t)=\begin{cases}0 & \text{if } -\frac{1}{4}\le t\le \frac{1}{4},\\ 1 & \text{if } \frac{3}{4}\le t\le \frac{3}{2},\\ t & \text{if } t\ge 2.\end{cases}$$

(The property that χ is unbounded in a way that makes it proper is only used in the proof of Lemma 5.12; see Lemma 5.11.) Such functions form a subset of the set of functions χ in Section 5A, but Proposition 5.1 for general χ follows from the case for this class of functions, because the index on the right-hand side of (5.2) does not change if we modify χ in a cocompact set. (And, at any rate, to prove Theorem 3.10, we only need Proposition 5.1 to hold for one such function χ.)

Let γ^N be the grading operator on $S_0|_N$ that equals ± 1 on $S^N_\pm$. Let $\gamma^{N\times\mathbb{R}}$ be its pullback to $S_0^{N\times\mathbb{R}}$. Let Φ_0^χ be the endomorphism of S_0^C equal to $\chi\gamma^{N\times\mathbb{R}}$ on $N\times\left(\frac{1}{4},\infty\right)$ and equal to zero on the rest of M_C.

Lemma 5.7. $$\mathrm{index}_G(\tilde{D}^C+\lambda\Phi^C)=\mathrm{index}_G(\tilde{D}^C+\Phi^\chi).$$

Proof. This can be proved via a linear homotopy between $\lambda\Phi^C$ and Φ^χ. The details are precisely as in the proof of Lemma 5.15 in [Cecchini 2020], with references to Propositions 4.1 and 5.9 in [Cecchini 2020] replaced by references to Proposition 4.11 (combined with Propositions 4.3 and 4.4) and Corollary 4.20, respectively, in the present paper. □

5C. *Proof of Proposition 5.1.* Let M_-^- be the manifold M_- with reversed orientation. Form the manifold

$$M_C^- := (N \times (-\infty, 1]) \cup_N M_-^-.$$

See Figure 4. (The notation is motivated by the fact that M_C with reversed orientation is naturally equal to $(N \times (-\infty, -1]) \cup_N M_-^-$, which can be identified with M_C^- via a shift over a distance 2.) We identify the subset $U \cap M_- \cong N \times \left(\frac{1}{4}, 1\right]$ with $N \times \left[1, \frac{7}{4}\right)$, so that $N \times \left(-\infty, \frac{7}{4}\right) \subset M_C^-$.

Let $S_0^- \to M_-^-$ be equal to the vector bundle $S_0|_{M_-}$, but with the opposite Clifford action (where $v \in TM_-^-$ acts as $c(-v)$). Let $S_0^{C,-} \to M_C^-$ be the Clifford module that is equal to $S_0^{N\times\mathbb{R}}$ on $N \times \left(-\infty, \frac{5}{4}\right]$ and to S_0^- on M_-^-. From the Clifford connections $\nabla^{S_\pm^{N\times\mathbb{R}}}$ on $S_\pm^{N\times\mathbb{R}}$ and a Clifford connection $\nabla^{S_0^-}$ on S_0^-, construct a Clifford connection $\nabla^{S_0^{C,-}}$ on $S_0^{C,-}$ by

$$\nabla^{S_0^{C,-}} := \begin{cases} \nabla^{S_+^{N\times\mathbb{R}}} \oplus \nabla^{S_-^{N\times\mathbb{R}}} & \text{on } N \times \left(-\infty, \frac{5}{4}\right), \\ \nabla^{S_0^-} & \text{on } M_-^-. \end{cases}$$

Using this connection, we obtain the Dirac operator $D_0^{C,-}$ on $\Gamma^\infty(S_0^{C,-})$. Then $D_0^{C,-}$ equals $\tilde{D}_0^C$ on $N \times \left(\frac{1}{2}, \frac{5}{4}\right)$.

The function χ equals 1 on $\left(\frac{3}{4}, \frac{5}{4}\right)$. Thus, on this interval, both Φ_0^χ and $\left(\begin{smallmatrix}\chi & 0\\ 0 & -1\end{smallmatrix}\right)$ are equal to $\gamma^{N\times\mathbb{R}}$. (Here we use 2×2 matrix notation with respect to the decomposition $S_0^{N\times\mathbb{R}} = S_+^{N\times\mathbb{R}} \oplus S_-^{N\times\mathbb{R}}$.) So we can define the endomorphism $\Phi_0^{C,-}$ of $S_0^{C,-}$ by setting it equal to Φ_0^χ on $\left(N \times \left(\frac{3}{4}, 1\right]\right) \cup_N M_-^-$ and equal to

$$\begin{pmatrix} \chi & 0 \\ 0 & -1 \end{pmatrix} \tag{5.8}$$

on $N \times \left(-\infty, \frac{5}{4}\right)$, where $S_0^{C,-} = S_0^{N\times\mathbb{R}} = S_+^{N\times\mathbb{R}} \oplus S_-^{N\times\mathbb{R}}$.

Lemma 5.9. $$\operatorname{index}_G(D^{C,-} + \Phi^{C,-}) = 0.$$

Proof. Since $\chi = -1$ on $\left(-\frac{3}{2}, -\frac{3}{4}\right)$, we can define the endomorphism $\tilde{\Phi}_0^{C,-}$ of $S_0^{C,-}$ by setting it equal to $\Phi_0^{C,-}$ on $N \times \left(-\infty, -\frac{3}{4}\right)$ (where it equals (5.8)) and equal to -1 on $(N \times (-1, 1]) \cup_N M_-^-$. For that endomorphism, the estimate (3.2) holds on all of M. Therefore, by Lemma 4.8,

$$\operatorname{index}_G(D^{C,-} + \tilde{\Phi}^{C,-}) = 0.$$

The claim now follows from Corollary 4.12. □

Proof of Proposition 5.1. Consider the cylinder $N \times \mathbb{R}$ as in Figure 1. The data $(M_C, S_0^C, \Phi_0^\chi)$ and $(M_C^-, S_0^{C,-}, \Phi_0^{C,-})$ coincide in a neighbourhood of $N \times \{1\}$. By cutting along $N \times \{1\}$ and gluing, we obtain the corresponding data

$$(N \times \mathbb{R}, S_0^{N\times\mathbb{R}}, \Phi_0^{N\times\mathbb{R}}) \quad \text{and} \quad (M_- \cup_N M_-^-, S_0^{M_-\cup_N M_-^-}, \Phi_0^{M_-\cup_N M_-^-}).$$

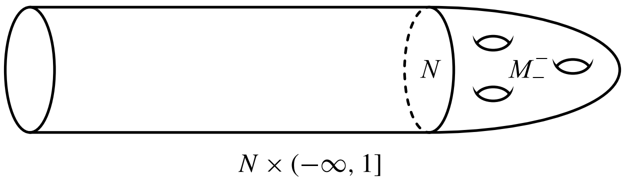

Figure 4. The manifold M_C^-.

See Figure 5. To be explicit,

$$\Phi_0^{N\times\mathbb{R}} = \begin{pmatrix} \chi & 0 \\ 0 & -1 \end{pmatrix} \tag{5.10}$$

on $S_+^{N\times\mathbb{R}} \oplus S_-^{N\times\mathbb{R}}$.

Theorem 4.15 implies that

$$\begin{aligned}&\operatorname{index}_G(\tilde{D}^C + \Phi^\chi) + \operatorname{index}_G(D^{C,-} + \Phi^{C,-}) \\ &\qquad = \operatorname{index}_G(D^{N\times\mathbb{R}} + \Phi^{N\times\mathbb{R}}) + \operatorname{index}_G(D^{M_-\cup_N M_-^-} + \Phi^{M_-\cup_N M_-^-}).\end{aligned}$$

By Lemmas 4.9 and 5.9, this implies that

$$\operatorname{index}_G(\tilde{D}^C + \Phi^\chi) = \operatorname{index}_G(D^{N\times\mathbb{R}} + \Phi^{N\times\mathbb{R}}).$$

The connection $\tilde{\nabla}^{S_0^{N\times\mathbb{R}}}$ on $S_0^{N\times\mathbb{R}}$ that is obtained from cutting and gluing the connections $\nabla^{S_0^C}$ and $\nabla^{S_0^{C,-}}$ is the direct sum connection $\nabla^{S_+^{N\times\mathbb{R}}} \oplus \nabla^{S_+^{N\times\mathbb{R}}}$. Therefore the corresponding Dirac operator $D_0^{S_0^{N\times\mathbb{R}}}$ equals

$$D_0^{S_0^{N\times\mathbb{R}}} = \begin{pmatrix} D_0^{S_+^{N\times\mathbb{R}}} & 0 \\ 0 & D_0^{S_+^{N\times\mathbb{R}}} \end{pmatrix}.$$

By the explicit form (5.10) of $\Phi_0^{N\times\mathbb{R}}$, the operator $D_0^{S_0^{N\times\mathbb{R}}} \pm i\Phi_0^{N\times\mathbb{R}}$ on $\Gamma^\infty(S_0^{N\times\mathbb{R}})$ is the direct sum of the operators $D_0^{S_+^{N\times\mathbb{R}}} \pm i\chi$ on $\Gamma^\infty(S_+^{N\times\mathbb{R}})$ and $D_0^{S_-^{N\times\mathbb{R}}} \mp i$ on $\Gamma^\infty(S_-^{N\times\mathbb{R}})$. So

$$\operatorname{index}_G(D^{N\times\mathbb{R}} + \Phi^{N\times\mathbb{R}}) = \operatorname{index}_G(D^{S_+^{N\times\mathbb{R}}} + \chi^{N\times\mathbb{R}}) + \operatorname{index}_G\left(D^{S_-^{N\times\mathbb{R}}} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right).$$

Lemma 4.8 then implies that the second term on the right-hand side is zero, whence

$$\operatorname{index}_G(\tilde{D}^C + \Phi^\chi) = \operatorname{index}_G(D^{S_+^{N\times\mathbb{R}}} + \chi^{N\times\mathbb{R}}).$$

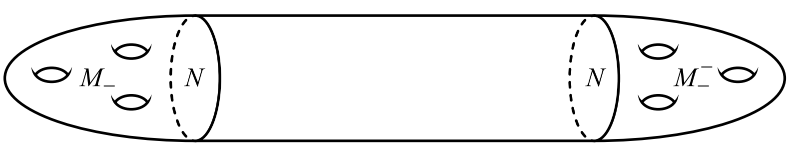

Figure 5. The manifold $M_- \cup_N M_-^-$.

The claim now follows from (5.3) in conjunction with Lemmas 5.5 and 5.7. As pointed out before Lemma 5.7, the case for the class of functions χ we have used implies the case of the more general functions χ allowed in Proposition 5.1. □

5D. *Proof of Theorem 3.10.* Let $\mathcal{E}_{N\times\mathbb{R}}$ be the Hilbert $C^*(G)$-module constructed from $\Gamma_c(S_+^{N\times\mathbb{R}})$ as in Section 3A. We write $D_\chi := D^{S_+^{N\times\mathbb{R}}} + \chi^{N\times\mathbb{R}}$ for brevity.

Lemma 5.11. *For all $a > 0$ the operator $(D_\chi^2 + a)^{-1}$ on $\mathcal{E}_{N\times\mathbb{R}}$ is compact.*

Proof. This is (a special case of) an analogue of Theorem 2.40 in [Ebert 2016]. The proof proceeds in the same way, with the difference that the operator D_χ^2 can only be bounded below by a function h that is G*-proper*, in the sense that the inverse image of a compact set is cocompact instead of compact. One chooses the bump functions g_n in the proof of Theorem 2.40 in [Ebert 2016] to be G-invariant and cocompactly supported. Theorem 4.2, along with Lemma 4.6(a) in [Guo 2021], implies that $g_n(D_\chi^2 + a)^{-1}$ is a compact operator on $\mathcal{E}_{N\times\mathbb{R}}$. And as in the proof of Theorem 2.40 in [Ebert 2016], one shows that $g_n(D_\chi^2 + a)^{-1}$ converges to $(D_\chi^2 + a)^{-1}$ in the operator norm on $\mathcal{L}(\mathcal{E}_{N\times\mathbb{R}})$. □

We will use an analogue of Theorem 6.6 in [Cecchini 2020].

Lemma 5.12. *The operator*

$$D_\chi(D_\chi^2 + f)^{-\frac{1}{2}} - D_\chi(D_\chi^2 + 1)^{-\frac{1}{2}} \tag{5.13}$$

lies in $\mathcal{K}(\mathcal{E}_{N\times\mathbb{R}})$.

Proof. By Proposition 4.12 in [Guo 2021], and as in (6.6) in [Cecchini 2020], the operator (5.13) equals

$$\frac{2}{\pi}\int_0^\infty D_\chi(D_\chi^2 + f + \lambda^2)^{-1}(f-1)(D_\chi^2 + 1 + \lambda^2)^{-1}\, d\lambda. \tag{5.14}$$

The operator $(D_\chi^2 + 1 + \lambda^2)^{-1}$ is compact by Lemma 5.11. Lemmas 4.6 and 4.8 in [Guo 2021] imply that the integrand in (5.14) is bounded by $a(b+\lambda^2)^{-1}$ for certain $a, b > 0$, so the integral converges in the operator norm. It therefore defines a compact operator. □

Theorem 3.10 follows from Proposition 5.1 and the following fact.

Proposition 5.15. *In the setting of Proposition 5.1,*

$$\mathrm{index}_G(D_\chi) = \mathrm{index}_G(D^{S_+^N}).$$

Proof. Suppose that $\mathcal{E}_N$ is the Hilbert $C^*(G)$-module constructed from $\Gamma_c(S_+^N)$ as in Section 3A. Then $\mathcal{E}_{N\times\mathbb{R}} = \mathcal{E}_N \otimes L^2(\mathbb{R})$. So Lemma 5.12 implies that $\mathrm{index}_G(D_\chi)$ is represented by the unbounded Kasparov cycle

$$(\mathcal{E}_N \otimes L^2(\mathbb{R}) \otimes \mathbb{C}^2, D_\chi). \tag{5.16}$$

The Callias-type operator D_χ on

$$\Gamma^\infty(S_+^{N\times\mathbb{R}} \oplus S_+^{N\times\mathbb{R}}) = \Gamma^\infty(S_+^N) \otimes C^\infty(\mathbb{R}) \otimes \mathbb{C}^2$$

is equal to

$$\begin{pmatrix} 0 & D^{S_+^N} \\ D^{S_+^N} & 0 \end{pmatrix} \otimes 1_{C^\infty(\mathbb{R})} + \gamma_{S_+^N} \otimes \begin{pmatrix} 0 & i\frac{d}{dt} \\ i\frac{d}{dt} & 0 \end{pmatrix} + 1_{\Gamma_c^\infty(S_+^N)} \otimes \begin{pmatrix} 0 & -i\chi \\ i\chi & 0 \end{pmatrix}, \tag{5.17}$$

where $\gamma_{S_+^N}$ is the grading operator on S_+^N, equal to $-i$ times Clifford multiplication by the unit normal vector field $\hat{n}$ on N pointing into M_+ (so that $\gamma_{S_+^N} \otimes i(d/dt) = c(\hat{n}) \otimes (d/dt)$). Let $\mathcal{E}_N^\pm$ denote the even- and odd-graded parts of $\mathcal{E}_N$, and let $D_\pm^{S_+^N}$ denote the restriction of $D^{S_+^N}$ to even- and odd-graded sections of S_+^N, respectively. With respect to the decomposition

$$\begin{aligned} &\mathcal{E}_N \otimes L^2(\mathbb{R}) \otimes \mathbb{C}^2 \\ &\quad = (\mathcal{E}_N^+ \otimes L^2(\mathbb{R})) \oplus (\mathcal{E}_N^- \otimes L^2(\mathbb{R})) \oplus (\mathcal{E}_N^+ \otimes L^2(\mathbb{R})) \oplus (\mathcal{E}_N^- \otimes L^2(\mathbb{R})), \end{aligned} \tag{5.18}$$

the operator (5.17) equals

$$D_\chi = \begin{pmatrix} 0 & 0 & 1_{\mathcal{E}_N^+} \otimes \big(i\frac{d}{dt} - i\chi\big) & D_-^{S_+^N} \otimes 1_{L^2(\mathbb{R})} \\ 0 & 0 & D_+^{S_+^N} \otimes 1_{L^2(\mathbb{R})} & 1_{\mathcal{E}_N^-} \otimes \big(-i\frac{d}{dt} - i\chi\big) \\ 1_{\mathcal{E}_N^+} \otimes \big(i\frac{d}{dt} + i\chi\big) & D_-^{S_+^N} \otimes 1_{L^2(\mathbb{R})} & 0 & 0 \\ D_+^{S_+^N} \otimes 1_{L^2(\mathbb{R})} & 1_{\mathcal{E}_N^-} \otimes \big(-i\frac{d}{dt} + i\chi\big) & 0 & 0 \end{pmatrix}. \tag{5.19}$$

The kernel of $i(d/dt) \pm i\chi$ in $C^\infty(\mathbb{R})$ is one-dimensional and spanned by the function

$$f_\pm(t) = e^{\mp \int_0^t \chi(u)\,du}.$$

By the properties of χ, $f_+ \in L^2(\mathbb{R})$, while $f_- \notin L^2(\mathbb{R})$. It follows that $i(d/dt) - i\chi$ is invertible on the appropriate domain, whereas $i(d/dt) + i\chi$ is zero on $\mathbb{C}f_+$ and invertible on $f_+^\perp$.

Consider the submodules

$$\begin{aligned} \mathcal{E}_1 &:= (\mathcal{E}_N^+ \otimes \mathbb{C}f_+) \oplus 0 \oplus 0 \oplus (\mathcal{E}_N^- \otimes \mathbb{C}f_+), \\ \mathcal{E}_2 &:= \mathcal{E}_1^\perp = (\mathcal{E}_N^+ \otimes f_+^\perp) \oplus (\mathcal{E}_N^- \otimes L^2(\mathbb{R})) \oplus (\mathcal{E}_N^+ \otimes L^2(\mathbb{R})) \oplus (\mathcal{E}_N^- \otimes f_+^\perp) \end{aligned}$$

of (5.18). These are preserved by the operator D_χ. (For $\mathcal{E}_1$, this is immediate from (5.19); for $\mathcal{E}_2$, this follows from the fact that D_χ is symmetric and preserves $\mathcal{E}_1$.) We find that the cycle (5.16) decomposes as

$$\left(\mathcal{E}_1, \begin{pmatrix} 0 & 0 & 0 & D_-^{S_+^N} \otimes 1_{\mathbb{C}f_+} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ D_+^{S_+^N} \otimes 1_{\mathbb{C}f_+} & 0 & 0 & 0 \end{pmatrix} \right) \oplus (\mathcal{E}_2, D_\chi|_{\mathcal{E}_2}). \tag{5.20}$$

The operator $D_\chi|_{\mathcal{E}_2}$ is essentially self-adjoint on the initial domain of compactly supported smooth sections by Proposition 5.5 in [Guo 2021], and its square has a positive lower bound in the Hilbert $C^*(G)$-module sense. Thus its self-adjoint closure is invertible, so that the second term in (5.20) is homotopic to a degenerate cycle. The first term represents $\mathrm{index}_G(D^{S_+^N})$. □

Remark 5.21. A similar argument in the case where G is trivial is hinted at below Lemma 4.1 in [Kucerovsky 2001].

6. Proofs of results on positive scalar curvature

6A. ***Obstruction results.*** We now deduce Theorem 2.1 from Theorem 3.10, and Corollary 2.4 from Theorem 2.1 and index theorems in [Hochs and Wang 2018; Wang and Wang 2016; Wang 2014].

Proof of Theorem 2.1. This proof is an adaptation of the proof of Theorem 2.1 in [Anghel 1993].

First suppose that M is odd-dimensional. Let κ denote scalar curvature. Let $K \subset \overline{X}$ be a cocompact subset of M such that $H \subset K$, $\kappa > 0$ on K, and the distance from $X \setminus K$ to Y is positive. Let $\chi \in C^\infty(M)^G$ be a function such that $\chi(x) = 1$ for all $x \in Y$ and $\chi(x) = -1$ for all $x \in X \setminus K$. Consider the operator D as in (3.8), where D_0 is the Spin-Dirac operator on M, and the admissible endomorphism Φ as in (3.9), where Φ_0 is pointwise multiplication by χ. Then the set M_- in Section 3B can be chosen to be cocompact as required and so that $N = N^- \cup H$, where $f|_{N^-} = -1$. In this setting,

$$S_+^N = S_0|_H, \tag{6.1}$$

where $S_0 \to M$ is the spinor bundle. (This is consistent with Corollary 7.3.)

For any $\lambda \in \mathbb{R}$, Lichnerowicz's formula implies that

$$(D_0 \pm i\lambda\chi)^2 = D_0^2 \pm i\lambda c(d\chi) + \lambda^2\chi^2 \geq \tfrac{1}{4}\kappa - \lambda\|d\chi\| + \lambda^2\chi^2.$$

On $M \setminus K$, the function on the right-hand side equals $\frac{1}{4}\kappa + \lambda^2 \geq \lambda^2$. Since κ is G-invariant, and positive on the cocompact set K, it has a positive lower bound on that set. Further, $d\chi$ is G-invariant and cocompactly supported, hence bounded. So we can choose $\lambda > 0$ small enough that $\frac{1}{4}\kappa - \lambda\|d\chi\| > 0$ on K. It follows that $(D_0 \pm i\lambda\chi)^2$ has a positive lower bound. This implies that $D + \Phi$ is invertible, so $\mathrm{index}_G(D + \Phi) = 0$. The claim then follows by Theorem 3.10 and (6.1).

If M is even-dimensional, then the claim follows by applying the result in the odd-dimensional case to the manifold $M \times S^1$. □

Proof of Corollary 2.4. This follows from Theorem 2.1 and the following index formulas for traces defined by orbital integrals applied to $\mathrm{index}_G(D^H)$:

- Theorem 6.10 in [Wang 2014] if G is any locally compact group and $g = e$.
- Theorem 6.1 in [Wang and Wang 2016] if G is discrete and finitely generated and g is any element.
- Proposition 4.11 in [Hochs and Wang 2018] if G is a connected semisimple Lie group and g is a semisimple element.

These results imply that, in the setting of Corollary 2.4,

$$\tau_g(\mathrm{index}_G(D^H)) = \hat{A}_g(H)$$

for a trace τ_g. As $\mathrm{index}_G(D^H)$ is independent of the choice of Riemannian metric, so is $\hat{A}_g(H)$. Finally, vanishing of $\mathrm{index}_G(D^H)$ implies vanishing of $\hat{A}_g(H)$ for all g as above. □

6B. ***Existence result.*** In the remainder of this section, we prove Theorem 2.11 by generalising a construction in [Lawson and Yau 1974]. We first prove in this subsection an extension of Theorem 3.8 in that work, namely, Proposition 6.2.

To prepare, let us recall the steps in the construction of Lawson and Yau's positive scalar curvature metrics, which we denote by $\tilde{g}_t$, on a compact manifold N.

Let K be a compact Lie group acting on N. Consider the principal K-bundle defined by the map

$$p : K \times N \to N, \quad (k, y) \mapsto k^{-1}y.$$

Take a K-invariant Riemannian metric g_N on N. Let b be a bi-invariant Riemannian metric on K. Let $\hat{g}$ denote the lift of g_N to the orthogonal complement of $\ker(Tp)$ in $T(K \times N)$ with respect to the product metric $b \oplus g_N$ on $K \times N$.

For each $t > 0$, let $\hat{b}_{t^2}$ be the lift of the metric $t^2 b$ on K to $TK \times N \subset T(K \times N)$. Then

$$g_t := \hat{g} \oplus \hat{b}_{t^2}$$

is a Riemannian metric on the total space $K \times N$. One can check that, for each t, g_t is invariant under the left K-action on $K \times N$ defined by

$$l \cdot (k, y) = (kl^{-1}, y).$$

Thus g_t descends, via the projection onto the second factor,

$$\pi : K \times N \to N, \quad (k, y) \mapsto y,$$

to a K-invariant metric $\tilde{g}_t$ on N. Further, one sees that π is a Riemannian submersion with respect to the metrics g_t and $\tilde{g}_t$ on the total space and base respectively.

The next proposition shows that, under the conditions stated, for all sufficiently small t, $\tilde{g}_t$ has positive scalar curvature outside a neighbourhood of the fixed-point set. It is an adaptation of the proof of Theorem 3.8 in [Lawson and Yau 1974] to the more general setting when N is noncompact but has K-bounded geometry; see Definition 2.10.

Proposition 6.2. *Let N be a manifold with an action by a nonabelian, compact, connected Lie group K. Fix a bi-invariant metric on K. If g_N is a K-invariant Riemannian metric on N with K-bounded geometry, then for any neighbourhood U of the fixed-point set N^K, there exists $t_U > 0$ such that for all $t \leq t_U$, the metric $\tilde{g}_t$ constructed above has uniform positive scalar curvature on $N \setminus U$.*

Proof. We will follow the steps in the proof of Theorem 3.8 in [Lawson and Yau 1974] and show where the assumptions of bounded geometry and no shrinking orbits (Definition 2.7) are needed to obtain the conclusion.

For each $y \in N \setminus N^K$, there is an orthogonal splitting $\mathfrak{k} = \mathfrak{k}_y \oplus \mathfrak{p}_y$, where $\mathfrak{k}_y$ is the Lie subalgebra of the isotropy subgroup K_y of y. The map φ_y from (2.5) restricts to an injection on $\mathfrak{p}_y$. Denote the orthogonal complement of $\varphi_y(\mathfrak{p}_y)$ in T_yN by V_y. Then

$$T_{(e,y)}(K \times N) \cong \mathfrak{k}_y \oplus \mathfrak{p}_y \oplus \varphi_y(\mathfrak{p}_y) \oplus V_y.$$

For each $y \in N \setminus N^K$, choose an orthonormal basis $\{e_1(y), \dots, e_{l_y}(y)\}$ of $\mathfrak{p}_y$ with respect to b such that for all $j, k = 1, \dots, l_y$,

$$g\big(\varphi_y(e_j(y)), \varphi_y(e_k(y))\big) = \sigma_j^2(y)\delta_{jk}$$

for some continuous, positive functions σ_j. For each $j = 1, \dots, l_y$, define a function $\lambda_j : N \setminus N^K \to (0, \infty)$ by

$$\lambda_j(y) := \sigma_j(y)(1 + \sigma_j(y)^2).$$

By the calculations in the proofs of Propositions 3.6 and 3.7 in [Lawson and Yau 1974] and the assumption that g_N has bounded geometry, for any neighbourhood U of N^K, the scalar curvature of $\tilde{g}_t$ at any $y \in N \setminus U$ is bounded below by

$$\sum_{j,k=1}^{l_y} \frac{1}{t^2} \frac{\lambda_j(y)^2 \lambda_k(y)^2}{(t^2 + \lambda_j(y)^2)(t^2 + \lambda_k(y)^2)} \big\| [e_j(y), e_k(y)] \big\|_b^2 + O(1) \tag{6.3}$$

as $t \to 0$, where the $O(1)$ term is independent of y. Since the K-action has no shrinking orbits with respect to g_N, there exists $c_U > 0$ such that $\lambda_j(y) > c_U$ for each $y \in N \setminus U$ and $j = 1, \dots, l_y$. In particular, for $t \leq c_U$, the expression (6.3) is bounded below by

$$\sum_{j,k=1}^{l_y} \frac{1}{4t^2} \big\| [e_j(y), e_k(y)] \big\|_b^2 + O(1). \tag{6.4}$$

Now, without loss of generality, we may assume that $K = \mathrm{SU}(2)$ or $K = \mathrm{SO}(3)$, as any compact, connected, nonabelian Lie group has such a subgroup. Since K

has no subgroups of codimension 1, we have $l_y = \dim \mathfrak{p}_y \geq 2$ at each $y \in N \setminus N^K$. For all j and k, $\|[e_j(y), e_k(y)]\|_b^2$ is 4 times the sectional curvature of the plane spanned by $e_j(y)$ and $e_k(y)$ with respect to the metric b, and this is constant in y, and positive for $K = \mathrm{SU}(2)$ or $K = \mathrm{SO}(3)$. Thus for any neighbourhood U of N^K, there exists $t_U > 0$ such that for all $t \leq t_U$, the expression (6.4), and hence also (6.3), is uniformly positive outside U. It follows that for all such t, the scalar curvature of $\tilde{g}_t$ is uniformly positive outside U. □

We now deduce Theorem 2.11 from the following noncompact generalisation of the main result in [Lawson and Yau 1974].

Theorem 6.5. *Let N be a manifold that admits an effective action by a compact, connected, nonabelian Lie group K such that the fixed-point set N^K is compact. If there exists a K-invariant Riemannian metric on N such that the K-action has K-bounded geometry, then N admits a K-invariant metric with uniformly positive scalar curvature.*

Proof. Since N^K is compact and the action is effective, by Section 4 of [Lawson and Yau 1974] there exists $t_0 > 0$, a K-invariant neighbourhood U of K with compact closure, and a K-invariant Riemannian metric g' on N such that each metric $\tilde{g}'_t$, constructed from g' as in Section 6B, has positive scalar curvature on U for $0 < t < t_0$.

Fix a bi-invariant metric b on K. Let g'' be a K-invariant metric on N for which the K-action has K-bounded geometry. Let $\{f_1, f_2\}$ be a smooth, K-invariant partition of unity on N such that $f_1 \equiv 1$ on U and $f_1 \equiv 0$ on $N \setminus U'$, where U' is a relatively compact neighbourhood of N^K containing the closure of U. Then

$$g_N := f_1 g' + f_2 g''$$

is a K-invariant Riemannian metric on N. Applying the prescription in Section 6B to g_N, we obtain a family $\{\tilde{g}_t\}_{t>0}$ of K-invariant metrics on N. We claim that for sufficiently small t, $\tilde{g}_t$ has uniformly positive scalar curvature on N.

To see this, let $\|\varphi\|$ be the norm function (2.6) associated to the metric g_N. Since g'' and g_N coincide on $N \setminus U'$, and g'' has K-bounded geometry, there exists $C_{U'} > 0$ such that $\|\varphi\|(y) \geq C_{U'}$ for all $y \in N \setminus U'$. One sees that g_N has K-bounded geometry, and so by Proposition 6.2, there exists $t_1 = (t_1)_U > 0$ such that for all $t \leq t_1$, $\tilde{g}_t$ has uniformly positive scalar curvature on $N \setminus U$. It follows that for all $t \leq \min\{t_0, t_1\}$, $\tilde{g}_t$ has uniformly positive scalar curvature on N. □

Proof of Theorem 2.11. In the setting of Theorem 2.11, Theorem 6.5 implies that N admits a K-invariant metric with uniformly positive scalar curvature. By Theorem 4.6 in [Guo et al. 2019a] (see also Theorem 58 in [Guo et al. 2019b]), this metric induces a G-invariant Riemannian metric on $G \times_K N$ of uniformly positive scalar curvature. □

7. Further applications of the Callias-type index theorem

We used Theorem 3.10 to prove Theorem 2.1 in Section 6A. We give some other applications of Theorem 3.10 here.

7A. ***The image of the assembly map.*** If M/G is noncompact, and G is not known to satisfy Baum–Connes surjectivity, then it is a priori unclear if $\mathrm{index}_G(D+\Phi)$ lies in the image of the Baum–Connes assembly map [Baum et al. 1994]; see the question raised on page 3 of [Guo 2021]. Theorem 3.10 implies that this is in fact the case for G-Callias-type operators as defined above:

Corollary 7.1. *The Callias-index of $D+\Phi$ lies in the image of the Baum–Connes assembly map.*

7B. ***Cobordism invariance of the assembly map.*** Theorem 3.10 leads to a perspective on cobordism invariance of the analytic assembly map.

Corollary 7.2. *Let X be an odd-dimensional Riemannian manifold with boundary N, on which G acts properly and isometrically, preserving N, such that M/G is compact. Suppose that a neighbourhood U of N is G-equivariantly isometric to $N\times[0,\varepsilon)$, for some $\varepsilon>0$. Let $S_0^X\to X$ be a G-equivariant Clifford module, and consider the Clifford module $S_0^X|_N\to N$, graded by i times Clifford multiplication by the inward-pointing unit normal. Suppose that $S_0^X|_U\cong S_0^X|_N\times[0,\varepsilon)$. Let D^N be the Dirac operator on $S_0^X|_N$ associated to a G-invariant Clifford connection ∇^N for the Clifford action by TN on $S_0^X|_N$. Then* $\mathrm{index}_G(D^N)=0$.

Proof. Form the manifold M by attaching the cylinder $N\times(0,\infty)$ to X along U. Extend S_0^X and the Clifford action to M in the natural way. We write S_0 for the extension of S_0^X to M. The connection ∇^N pulls back to a Clifford connection on $S_0|_{N\times(0,\infty)}$. Because the G-invariant Clifford connections form an affine space, we can extend this pulled-back connection to all of M using a G-invariant Clifford connection on $X\setminus N$ and a partition of unity. Let D_0 be the associated Dirac operator.

Let Φ_0 be the identity endomorphism on S_0. Then Φ as in (3.9) is admissible for D as in (3.8), and (3.2) holds on all of M. Then $\mathrm{index}_G(D+\Phi)=0$ by Lemma 4.8. In this case, $S_+^N=S_0^X|_N$, so by Theorem 3.10, $\mathrm{index}_G(D^N)=0$. □

7C. **Spin^c*-Dirac operators.***

Corollary 7.3. *Consider the setting of Theorem 3.10, and assume that M is odd-dimensional and D_0 is a* Spin^c*-Dirac operator. Then there is a union N^+ of connected components of N and a* Spin^c*-structure on N^+ with spinor bundle $S_0|_{N^+}$ such that*

$$\mathrm{index}_G(D+\Phi)=\mathrm{index}_G(D^{S_0|_{N^+}})\quad \text{in } K_0(C^*(G)),\tag{7.4}$$

where $D^{S_0|_{N^+}}$ is a Spin^c*-Dirac operator on the spinor bundle $S_0|_{N^+}\to N^+$. If N is connected, then* $\mathrm{index}_G(D+\Phi)=0$.

Proof. Since S_0 is an irreducible Clifford module, and $S_+^N \subset S_0|_N$ is invariant under the Clifford action of $TM|_N$, over each connected component X of N the bundle $S_+^N|_X$ is either zero or $S_0|_X$. Since M is odd-dimensional, $S_0|_X$ is the spinor bundle of the Spin^c-structure of X that it inherits from M. So (7.4) follows.

If N is connected, then we either have $N^+ = \varnothing$, in which case $\text{index}_G(D+\Phi) = 0$ because S_+^N is the zero bundle, or $N^+ = N$, in which case (7.4) and Lemma 4.8 imply that

$$\text{index}_G(D+\Phi) = \text{index}_G\left(D + \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\right) = 0. \qquad \square$$

There is a converse to Corollary 7.3 in the following sense. Let $N^+ \subset N$ be any union of connected components; there is a finite number of such subsets of N since N/G is compact. We can define an admissible endomorphism Φ_0 such that (7.4) holds, by taking Φ_0 to be multiplication by a G-invariant function on M that equals 1 on N^+ and -1 on $N \setminus N^+$, and is identically 1 or -1 outside a cocompact set. Thus given any hypersurface N bounding a cocompact set, and any set N^+ of connected components of N, we have an index

$$\text{index}_G^{N^+}(D) := \text{index}_G(D+\Phi), \tag{7.5}$$

with Φ and Φ_0 related as in (3.9), independent of the choice of Φ_0, with the property that Φ_0 is positive-definite on N^+ and negative-definite on $N \setminus N^+$.

Versions of the index (7.5) are sometimes used in applications of Callias-type index theorems to obstructions to positive scalar curvature for Spin-manifolds; see [Anghel 1993; Cecchini 2020] and the proof of Theorem 2.1 in Section 6A.

7D. ***Induction.*** Suppose that G is an almost connected, reductive Lie group, and let $K < G$ be maximal compact. In [Guo et al. 2019b; Hochs 2009; Hochs and Wang 2018] some results were proved relating G-equivariant indices to K-equivariant indices via Dirac induction. Such results allow one to deduce results in equivariant index theory for actions by noncompact groups from corresponding results for compact groups. This was applied to obtain results in geometric quantisation [Hochs 2009; 2015; Hochs and Mathai 2017; Mathai and Zhang 2010] and geometry of group actions [Guo et al. 2019b; Hochs and Mathai 2016]. Corollary 7.7 below is a version of this idea for the index of Definition 3.7.

To state this corollary, we consider the setting of Section 3B. Using Abels' slice theorem, we write $M = G \times_K Y$ for a K-invariant submanifold $Y \subset M$ and $S_0 = G \times_K S_0|_Y$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. Then $TM \cong G \times_K (TY \oplus \mathfrak{p})$. We assume that the G-invariant Riemannian metric on M is induced by a K-invariant Riemannian metric on Y and an $\text{Ad}(K)$-invariant inner product on $\mathfrak{p}$ via this identification.

We assume for simplicity that the adjoint representation $\mathrm{Ad}: K \to \mathrm{SO}(\mathfrak{p})$ lifts to the double cover $\mathrm{Spin}(\mathfrak{p})$ of $\mathrm{SO}(\mathfrak{p})$. (This is true for a double cover of G.) Then the standard Spin representation $S_{\mathfrak{p}}$ of $\mathrm{Spin}(\mathfrak{p})$ may be viewed as a representation of K. We assume that $S_0|_Y = S_0^Y \otimes S_{\mathfrak{p}}$ for a Clifford module $S_0^Y \to Y$. The Clifford action by $TM|_Y$ on $S_0|_Y$ equals $c_Y \otimes 1_{S_{\mathfrak{p}}} + 1_{S_0^Y} \otimes c_{\mathfrak{p}}$ for a Clifford action c_Y by TY on S_0^Y and the Clifford action $c_{\mathfrak{p}}$ of $\mathfrak{p}$ on $S_{\mathfrak{p}}$. We choose the G-invariant connection ∇^{S_0} so that $\nabla^{S_0}|_Y = \nabla^{S_0^Y} \otimes 1_{S_{\mathfrak{p}}} + 1_{S_0^Y} \otimes \nabla^{S_{\mathfrak{p}}}$ for Clifford connections $\nabla^{S_0^Y}$ on S_0^Y and $\nabla^{S_{\mathfrak{p}}}$ on $Y \times S_{\mathfrak{p}}$

Since Φ_0 is G-equivariant, it is determined by its restriction to Y, which is a K-equivariant endomorphism of $S_0|_Y$. We assume that $\Phi_0|_Y = \Phi_0^Y \otimes 1_{\mathfrak{p}}$, for a K-equivariant endomorphism Φ_0^Y of S_0^Y. (What follows remains true if $\Phi_0 = \Phi_0^Y \otimes 1_{\mathfrak{p}} + 1 \otimes \Phi_0^{\mathfrak{p}}$ for an $\mathrm{Ad}(K)$-invariant endomorphism of $S_{\mathfrak{p}}$, but this requires a small extra argument that we omit here.)

Consider the Dirac operator $D_0^Y := c_Y \circ \nabla^{S_0^Y}$ on $\Gamma^\infty(S_0^Y)$. Form D^Y from D_0^Y as in (3.8) and Φ^Y from Φ_0^Y as in (3.9). Let $R(K)$ be the representation ring of K and

$$\mathrm{D\text{-}Ind}_K^G : R(K) \to K_*(C^*(G)) \tag{7.6}$$

be the Dirac induction map [Baum et al. 1994].

Corollary 7.7. *The operator $D^Y + \Phi^Y$ is a K-equivariant Callias-type operator, and*

$$\mathrm{index}_G(D + \Phi) = \mathrm{D\text{-}Ind}_K^G(\mathrm{index}_K(D^Y + \Phi^Y)).$$

Proof. Theorem 3.10 implies that $\mathrm{index}_G(D + \Phi) = \mathrm{index}_G(D^{S_+^N})$. Write $Y^N := Y \cap N$, so that $N = G \times_K Y^N$. Then Y^N is a compact manifold. Define $D^{S_+^{Y^N}}$ analogously to D^Y. The induction result for cocompact actions, Theorem 4.5 in [Hochs 2009], Theorem 5.3 in [Hochs and Wang 2018] or Theorem 46 in [Guo et al. 2019b], implies that

$$\mathrm{index}_G(D^{S_+^N}) = \mathrm{D\text{-}Ind}_K^G(\mathrm{index}_K(D^{S_+^{Y^N}})).$$

Another application of Theorem 3.10, now with G replaced by K, or Theorem 1.5 in [Anghel 1993] with a compact group action added, shows that $\mathrm{index}_K(D^{S_+^{Y^N}}) = \mathrm{index}_K(D^Y + \Phi^Y)$. □

7E. *Callias quantisation commutes with reduction.* Theorem 3.11 in [Guo et al. 2021] is a *quantisation commutes with reduction* result for the equivariant index of Spin^c-Callias-type operators. This result applies to reduction at the trivial representation of G; i.e., to an index defined in terms of G-invariant sections of S. Using Theorem 3.10, we can generalise this result to reduction at more general representations, or more precisely, at arbitrary generators of $K_0(C_r^*(G))$. Furthermore, this result is "exact" rather than asymptotic as Theorem 3.11 in [Guo et al. 2021], in the sense that one does not need to consider high powers of a line bundle.

In the setting of Section 3B, we now assume that M is odd-dimensional, and that S_0 is the spinor bundle for a G-equivariant Spin^c-structure. Let D_0 be the Spin^c-Dirac operator on $\Gamma(S_0)$, defined by the Clifford connection corresponding to a connection ∇^L on the determinant line bundle L.

The Spin^c-*moment map* associated to ∇^L is the map $\mu : M \to \mathfrak{g}^*$ such that for all $X \in \mathfrak{g}$,

$$2\pi i \langle \mu, X\rangle = \mathcal{L}_X - \nabla^L_{X^M} \quad \text{in } \operatorname{End}(L) = C^\infty(M, \mathbb{C}),$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to X, and X^M is the vector field induced by X. The *reduced space* at an element $\xi \in \mathfrak{g}^*$ is defined as $M_\xi := \mu^{-1}(\xi)/G_\xi$, where G_ξ is the stabiliser of ξ. This reduced space is noncompact in general, and may not be smooth. But the reduced space $N_\xi := (\mu^{-1}(\xi) \cap N)/G_\xi$ is compact. It is not always a smooth manifold, but if it is, and $\xi \in \mathfrak{k}^*$, then we have an identification $N_\xi \cong Y^N_\xi := (\mu^{-1}(\xi) \cap Y \cap N)/K_\xi$, with Y and Y^N as in Section 7D, including Spin^c-structures. See Propositions 3.13 and 3.14 in [Hochs and Mathai 2017]. Let $N^+ \subset N$ be as in Corollary 7.3. Then we similarly have $N^+_\xi \cong Y^{N^+}_\xi := (\mu^{-1}(\xi) \cap Y \cap N^+)/K_\xi$ in the smooth case, including Spin^c-structures.

There is a nontrivial way to define a Spin^c-*quantisation* $Q^{\mathrm{Spin}^c}(Y^{N^+}_\xi) \in \mathbb{Z}$, even when $Y^{N^+}_\xi$ is not smooth, described in detail in Section 5.1 of [Paradan and Vergne 2017]. Motivated by the identification $N^+_\xi \cong Y^{N^+}_\xi$ in the smooth case, we define $Q^{\mathrm{Spin}^c}(N^+_\xi) := Q^{\mathrm{Spin}^c}(Y^{N^+}_\xi)$ for $\xi \in \mathfrak{k}^*$.

Let $T < K$ be a maximal torus, and fix a positive root system for (K, T). Let $V \in \hat{K}$ have highest weight $\lambda \in i\mathfrak{t}^* \hookrightarrow \mathfrak{k}^* \hookrightarrow \mathfrak{g}^*$. (The first inclusion is defined by $\mathfrak{t}^* \cong (\mathfrak{k}^*)^{\mathrm{Ad}^*(T)}$, the second by the Cartan decomposition.) Following [Paradan and Vergne 2017; 2018], we call an element $\xi \in \mathfrak{k}^*$ an *ancestor* of V if the coadjoint orbit $\mathrm{Ad}^*(K)\xi$ is admissible in the sense of [Paradan and Vergne 2018], and its K-equivariant Spin^c-quantisation is V. There exists a finite set $A(V)$ of ancestors representing all different such coadjoint orbits.

Let $C^*_r(G)$ be the reduced group C^*-algebra of G and D-Ind^G_K the Dirac induction map (7.6). By the Connes–Kasparov conjecture, proved in [Chabert et al. 2003; Lafforgue 2002; Wassermann 1987], the abelian group $K_*(C^*_r(G))$ is free, with generators $\text{D-Ind}^G_K[V]$, where V runs over $\hat{K}$.

Recall the definition of the Callias index of Spin^c-Dirac operators (7.5).

Corollary 7.8 (Callias quantisation commutes with reduction). *We have*

$$\operatorname{index}^{N^+}_G(D) = \bigoplus_{V \in \hat{K}} \sum_{\xi \in A(V)} Q^{\mathrm{Spin}^c}(N^+_\xi)\, \text{D-Ind}^G_K[V] \quad \textit{in } K_*(C^*_r(G)). \tag{7.9}$$

Proof. By Corollary 7.3, $\operatorname{index}_G^{N^+}(D) = \operatorname{index}_G(D^{S_0|_{N^+}})$, where now $D^{S_0|_{N^+}}$ is a Spin^c-Dirac operator on N^+. Theorem 4.6 in [Hochs and Mathai 2017] implies that $\operatorname{index}_G(D^{S_0|_{N^+}})$ equals

$$\bigoplus_{V\in\hat{K}} \sum_{\xi\in A(V)} Q^{\mathrm{Spin}^c}(N^+_\xi)\,\text{D-Ind}_K^G[V]. \qquad \square$$

Remark 7.10. In cases where M_ξ is smooth and N_ξ is a hypersurface in M_ξ, which is a transversality condition between N and $\mu^{-1}(\xi)$, one can use Theorem 1.5 in [Anghel 1993] (more precisely, its special case for Spin^c-Dirac operators, which is the nonequivariant case of Corollary 7.3) to express the Spin^c-quantisation $Q^{\mathrm{Spin}^c}(N^+_\xi)$ as the index of a Callias-type operator on M_ξ.

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HAO GUO: haoguo@math.tamu.edu
Department of Mathematics, Texas A&M University, College Station, TX, United States

PETER HOCHS: p.hochs@math.ru.nl
Institute for Mathematics, Astrophysics and Particle Physics, Radboud University, Nijmegen, Netherlands

VARGHESE MATHAI: mathai.varghese@adelaide.edu.au
School of Mathematical Sciences, University of Adelaide, Adelaide SA, Australia

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