

1 Tuple Interpretations for Higher-Order Complexity

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6 Abstract

7 We present a style of algebra interpretations for many-sorted and higher-order term rewriting based
8 on interpretations to *tuples*; intuitively, a term is mapped to a sequence of values identifying for
9 instance its evaluation cost, size and perhaps other values. This could give a more finegrained notion
10 of the complexity of a term or TRS than notions such as runtime or derivational complexity.

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15 1 Introduction

16 In the study of complexity of term rewriting systems, it is common to consider termination
17 techniques: if a TRS can be proved terminating by a certain technique, this typically implies
18 a specific bound on the number of steps that may be needed to reduce a term in that TRS
19 to normal form (see, e.g., [2, 4, 5, 7]). Some approaches (e.g., [5, 7]) consider *interpretations*
20 of terms s . Interpretations impose a natural bound on reduction length for given terms.

21 By their nature, interpretations to natural numbers do not tend to give *tight* bounds.
22 Consider for example the term rewriting system implementing addition, which is given
23 by the rules $\text{add}(x, 0) \rightarrow x$ and $\text{add}(x, \text{s}(y)) \rightarrow \text{s}(\text{add}(x, y))$. An interpretation would need
24 to be monotonic, and have $\llbracket \ell \rrbracket > \llbracket r \rrbracket$ for both rules. This leads to for instance: $\llbracket 0 \rrbracket = 0$,
25 $\llbracket \text{s}(x) \rrbracket = \llbracket x \rrbracket + 1$ and $\llbracket \text{add}(x, y) \rrbracket = \llbracket x \rrbracket + 2 \cdot \llbracket y \rrbracket + 1$. With these choices, we indeed have:

$$\begin{aligned} \llbracket \text{add}(x, 0) \rrbracket &= \llbracket x \rrbracket + 1 > \llbracket x \rrbracket = \llbracket x \rrbracket \\ \llbracket \text{add}(x, \text{s}(y)) \rrbracket &= \llbracket x \rrbracket + 2 \cdot \llbracket y \rrbracket + 2 > \llbracket x \rrbracket + 2 \cdot \llbracket y \rrbracket + 1 = \llbracket \text{s}(\text{add}(x, y)) \rrbracket \end{aligned}$$

27 But $\llbracket \text{add}(\text{s}^n(0), \text{s}^m(0)) \rrbracket = n + 2m + 1$, even though only $n + m + 1$ steps can be done before
28 reaching normal form. This is because the interpretation captures not only the reduction
29 cost, but also the size of the normal form. This is not problematic for the example above,
30 because the result is still *linear* runtime complexity. However, for *exponential* bounds, the
31 consequences are more severe: consider $\mathcal{O}(2^n)$ versus $\mathcal{O}(2^{3n}) = \mathcal{O}(8^n)$. And particularly
32 when considering higher-order term rewriting, exponential bounds are often very relevant.

33 The situation could be improved by splitting interpretations into separate *cost* and *size*
34 components, as was done for conditional rewriting in [6]. For instance, in the example above we
35 could take $\llbracket \text{add}(x, y) \rrbracket_{\text{size}} = \llbracket x \rrbracket_{\text{size}} + \llbracket y \rrbracket_{\text{size}}$ and $\llbracket \text{add}(x, y) \rrbracket_{\text{cost}} = \llbracket x \rrbracket_{\text{cost}} + \llbracket y \rrbracket_{\text{cost}} + \llbracket y \rrbracket_{\text{size}} + 1$.
36 More generally, we could interpret terms to tuples of arbitrary size. This essentially generalises
37 matrix interpretations [7] as well, by mapping terms to a vector but not imposing restrictions
38 on the shape of interpretation functions. This could be particularly useful for many-sorted and
39 higher-order term rewriting systems, where the choice of tuple length may be type-dependent.

40 The present short paper explores the ideas above. It documents work in progress with
41 the aim to help establish a more fine-grained notion of complexity for term rewriting—which
42 captures both time, space and perhaps other properties such as the shape of normal forms.
43 The technique we develop may also be useful for resource analysis of higher-order programs.

2 Preliminaries: many-sorted and higher-order term rewriting

We assume familiarity with first-order term rewriting. In *many-sorted* rewriting, all function symbols have a sequence of input sorts, and an output sort; and terms must be well-typed.

► **Example 1.** The TRS \mathcal{R}_+ , for arithmetic and lists, has six function symbols: $0 :: \text{nat}$, $\text{nil} :: \text{list}$, $s :: \text{nat} \Rightarrow \text{nat}$, $\text{add} :: \text{nat} \times \text{nat} \Rightarrow \text{nat}$, $\text{mult} :: \text{nat} \times \text{nat} \Rightarrow \text{nat}$, $\text{dList} :: \text{list} \rightarrow \text{list}$, and $\text{cons} :: \text{nat} \times \text{natlist} \rightarrow \text{natlist}$. It is given by rules of sort nat and list , as follows:

$$\begin{array}{ll}
\text{add}(x, 0) \rightarrow x & \text{d}(0) \rightarrow 0 \\
\text{add}(x, s(y)) \rightarrow s(\text{add}(x, y)) & \text{d}(s(x)) \rightarrow s(\text{d}(x)) \\
\text{mult}(x, 0) \rightarrow 0 & \text{dList}(\text{nil}) \rightarrow \text{nil} \\
\text{mult}(x, s(y)) \rightarrow \text{add}(x, \text{mult}(x, y)) & \text{dList}(\text{cons}(x, q)) \rightarrow \text{cons}(\text{d}(x), \text{dList}(q))
\end{array}$$

For *higher-order* rewriting, we use a formalism where function symbols take a sequence of *simple types* as input (i.e., generated from a set \mathcal{B} of sorts and a right-associative binary type constructor \Rightarrow) and a sort as output; term formation allows for function application ($f(s_1, \dots, s_m) : \iota$ if $f : \sigma_1 \times \dots \times \sigma_m \Rightarrow \iota$ is a symbol and each $s_i : \sigma_i$), as well as application (i.e., if $s : \sigma \Rightarrow \tau$ and $t : \sigma$ then $s t : \tau$) and λ -abstraction as in the simply-typed λ -calculus. The β -reduction rule $(\lambda x.s) t \rightarrow s[x := t]$ is always included in the reduction relation $\rightarrow_{\mathcal{R}}$.

► **Example 2.** Let $\mathcal{R}_{\text{fold}}$ be the higher-order TRS with symbols $\text{nil} :: \text{list}$, $\text{cons} :: \text{nat} \times \text{list} \Rightarrow \text{list}$, $\text{map} :: (\text{nat} \Rightarrow \text{nat}) \times \text{list} \Rightarrow \text{list}$ and $\text{foldl} :: (\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}) \times \text{nat} \times \text{list} \Rightarrow \text{nat}$ and rules:

$$\begin{array}{ll}
\text{foldl}(f, z, \text{nil}) \rightarrow z & \text{map}(f, \text{nil}) \rightarrow \text{nil} \\
\text{foldl}(f, z, \text{cons}(x, q)) \rightarrow \text{foldl}(f, (f z x), q) & \text{map}(f, \text{cons}(x, q)) \rightarrow \text{cons}(f x, \text{map}(f, q))
\end{array}$$

3 First-Order type-based interpretation

It is common in the rewriting literature to use termination proofs to assess the difficulty of rewriting a term to a normal form [2, 5]. For example, in [5], Hofbauer gives a first upper-bound for the derivational complexity of first-order TRS's with polynomial interpretation as termination proofs. This technique has been extended to other termination proofs as well [2]. Polynomial interpretations are a form of *algebra interpretations*:

► **Definition 3** (adapted from [8]). An algebra \mathcal{A} for many-sorted first-order terms consists of a mapping from each sort $\iota \in \mathcal{B}$ to a well-founded set $(A_\iota, >_\iota, \geq_\iota)$ together with an interpretation function \mathcal{J} which assigns to each $f :: \iota_1 \times \dots \times \iota_m \Rightarrow \kappa \in \mathcal{F}$ a monotonic function $\mathcal{J}_f \in A_{\iota_1} \rightarrow \dots \rightarrow A_{\iota_m} \rightarrow A_\kappa$ (monotonic: $\mathcal{J}_f(\dots, x, \dots) > \mathcal{J}_f(\dots, y, \dots)$ if $x > y$).

If α is a mapping from variables of sort ι to A_ι , term interpretation is defined recursively with $\llbracket x \rrbracket_\alpha^{\mathcal{J}} = \alpha(x)$ and $\llbracket f(s_1, \dots, s_m) \rrbracket_\alpha^{\mathcal{J}} = \mathcal{J}_f(\llbracket s_1 \rrbracket_\alpha^{\mathcal{J}}, \dots, \llbracket s_m \rrbracket_\alpha^{\mathcal{J}})$. We usually omit α and \mathcal{J} and just write $\llbracket s \rrbracket$. Termination follows if $\llbracket \ell \rrbracket_\alpha^{\mathcal{J}} > \llbracket r \rrbracket_\alpha^{\mathcal{J}}$ for all α and a fixed \mathcal{J} .

If each $A_\iota = \mathbb{N}$, then $\llbracket s \rrbracket$ gives a worst-case boundary on the number of rewriting steps starting from s (as observed in the introduction); this can be used to bound the number of steps starting from an arbitrary term of size n , depending on the shape of the interpretation.

As an alternative, we consider interpretations with $A_\iota = \mathbb{N}^{K_\iota}$. We let $(n_1, \dots, n_{K_\iota}) \geq (n'_1, \dots, n'_{K_\iota})$ if each $n_i \geq n'_i$, and $(n_1, \dots, n_{K_\iota}) > (n'_1, \dots, n'_{K_\iota})$ if $n_1 > n'_1$ and each $n_i \geq n'_i$. For example, we let $A_{\text{nat}} = \mathbb{N}^2$ and $A_{\text{list}} = \mathbb{N}^3$. Intuitively, the first component in both cases indicates “cost”: the number of steps needed to reduce a term to normal form. The second component of A_{nat} represents the size of the natural number, and the second and third component of A_{list} represent the list length and maximum element size respectively.

88 ► **Example 4.** Consider the signature of Example 1. We set its interpretation as follows
 89 below, where s_c is syntactic sugar for $\llbracket s \rrbracket_1$ (the cost component of s), s_s and s_l are $\llbracket s \rrbracket_2$ (the
 90 size or length component) and s_m is $\llbracket s \rrbracket_3$ (the component for maximum element size).

$$\begin{aligned}
 \llbracket 0 \rrbracket &= \langle 0, 1 \rangle & \llbracket \text{nil} \rrbracket &= \langle 0, 0, 0 \rangle \\
 \llbracket s(x) \rrbracket &= \langle x_c, x_s + 1 \rangle & \llbracket \text{cons}(x, q) \rrbracket &= \langle x_c + q_c, q_l + 1, \max(x_s, q_m) \rangle \\
 91 \quad \llbracket d(x) \rrbracket &= \langle 1 + x_c + x_s, 2 \cdot x_s \rangle & \llbracket \text{dList}(q) \rrbracket &= \langle 1 + q_c + q_l \cdot (2 + q_m), q_l, 2 \cdot q_m \rangle \\
 \llbracket \text{add}(x, y) \rrbracket &= \langle 1 + x_c + y_c + y_s, x_s + y_s \rangle \\
 \llbracket \text{mult}(x, y) \rrbracket &= \langle 1 + x_c + y_c + x_s \cdot (2 + x_c + x_s \cdot y_s), x_s \cdot y_s \rangle
 \end{aligned}$$

92 We can easily check that $\llbracket \ell \rrbracket > \llbracket r \rrbracket$ for all rewrite rules $\ell \rightarrow r$; that is, there is a strict decrease
 93 in the “cost” component and a weak decrease (with \geq) in the others. For example:

$$\begin{aligned}
 &\llbracket \text{dList}(\text{cons}(x, q)) \rrbracket \\
 &= \langle 1 + (x_c + q_c) + (q_l + 1) \cdot (2 + \max(x_s, q_m)), q_l + 1, 2 \cdot \max(x_s, q_m) \rangle \\
 94 \quad &= \langle 3 + x_c + q_c + \max(x_s, q_m) + q_l \cdot (2 + \max(x_s, q_m)), q_l + 1, 2 \cdot \max(x_s, q_m) \rangle \\
 &> \langle 2 + x_c + q_c + x_s + q_l \cdot (2 + q_m), q_l + 1, \max(2 \cdot x_s, 2 \cdot q_m) \rangle \\
 &= \langle (1 + x_c + x_s) + (1 + q_c + q_l \cdot (2 + q_m)), q_l + 1, \max(2 \cdot x_s, 2 \cdot q_m) \rangle \\
 &= \llbracket \text{cons}(d(x), \text{dList}(q)) \rrbracket
 \end{aligned}$$

95 Note that our interpretation method has some similarities with matrix interpretations [3], as
 96 each term is associated to an n -tuple. However, the interpretation function is not restricted
 97 to linear multivariate polynomials, allowing interpretations such as those for `cons` and `mult`.
 98 Tuple interpretations give information on more than just the cost of evaluating a term.

99 ► **Example 5** (Bounds for arithmetic). We have $\llbracket \text{dList}(\text{cons}(s^3(0), \text{cons}(s^1(0), \text{nil}))) \rrbracket = \langle 11, 2, 6 \rangle$.
 100 Given the way the interpretation was constructed, this implies that an evaluation to normal
 101 form takes at most 11 steps, and the normal form has length at most 2 and a greatest element
 102 at most $s^6(0)$. The cost component is not tight: it only takes 8 steps to evaluate the term
 103 (11 is the maximum number of steps to evaluate `dList`(q) for *any* constructor-list q of length
 104 2 and with greatest element $s^3(0)$). The other two values are tight.

105 4 Higher-order type-based interpretations

106 In first-order term rewriting, the difficulty of a TRS is often measured as *runtime* or
 107 *derivational* complexity: both measures are parametrised by the size of an initial term. This
 108 is not a good measure for terms with immediate subterms of higher-order type: the behaviour
 109 of such subterms on given arguments should be considered, as the next example shows.

110 ► **Example 6.** Consider $\mathcal{R}_+ \cup \mathcal{R}_{\text{foldl}}$. The evaluation cost of a term `foldl`(F, t, q) depends almost
 111 completely on the behaviour of the functional subterm F , and not only on its evaluation cost.
 112 If F is $\lambda x. \lambda y. d(x)$ —so a *size-increasing* term—evaluating `foldl`(F, t, q) takes exponentially
 113 many steps, even though `d` runs in linear steps and F is executed only $|q|$ times. Thus,
 114 higher-order rewriting in particular is a natural place to separate cost and size.

115 Algebra interpretations for higher-order rewriting were defined in [8]. Essentially, the
 116 interpretations of Definition 3 are extended by letting $A_{\sigma \Rightarrow \tau}$ be the set of *weakly monotonic*
 117 *functions* from A_σ to A_τ (that is, $f(\dots, x, \dots) \geq_\tau f(\dots, y, \dots)$ if $x \geq_\sigma y$), with $>_{\sigma \Rightarrow \tau}$
 118 and $\geq_{\sigma \Rightarrow \tau}$ being pointwise comparisons. While the author of [8] and followup work used
 119 \mathbb{N} for A_ι (with $\iota \in \mathcal{B}$), the method needs no modification when tuple interpretations are
 120 used instead. For elements of $A_{\iota \Rightarrow \sigma}$, we moreover limit interest to functions f such that
 121 always $f(x_1, x_2, \dots, x_n)_i = f(x'_1, x_2, \dots, x_n)_i$ for $i > 1$; that is, the size, length and “greatest
 122 element” components do not depend on the cost component (but may depend on each other).

4 Tuple Interpretations for Higher-Order Complexity

123 ► **Example 7.** Let $A_{\text{nat}} = \mathbb{N}^2$ and $A_{\text{list}} = \mathbb{N}^3$ as before, and assume `cons` and `nil` are interpreted
124 as in Example 4. We can use the following interpretation for `map`:

$$125 \quad \llbracket \text{map}(f, q) \rrbracket = \langle 1 + q_c + 2 \cdot q_l + (q_l + 1) \cdot \llbracket f \rrbracket(q_c, q_m)_1, \quad q_l, \quad \llbracket f \rrbracket(q_c, q_m)_2 \rangle$$

126 This expresses that the list length is retained (as the length component is just q_l), the greatest
127 element of the result `map` is bounded by the value of f on the greatest element of q , and
128 the evaluation cost is mostly expressed by a linear number of f steps. For $\llbracket \text{map}(\lambda x. d(x), q) \rrbracket$
129 we obtain $\langle 1 + q_c + 2 \cdot q_l + (q_l + 1) \cdot (1 + q_c + q_m), q_l, 2 \cdot q_m \rangle$. This is slightly larger than
130 $\llbracket \text{dList}(q) \rrbracket$ (which evaluates to the same term), but has a similar order of magnitude.

131 For `foldl`, we can use an interpretation like the one below, where $Q_{g,h,a,m} : \mathbb{N}^2 \rightarrow \mathbb{N}^2$ is
132 defined as follows: $Q_{g,h,a,m}(c, s) = \langle c + a + g(c, s, a, m), h(s, m) \rangle$; the superscript denotes
133 repeated function application (e.g., $Q^3(x) = Q(Q(Q(x)))$) and $+$ indicates placewise addition.

$$134 \quad \llbracket \text{foldl}(f, z, q) \rrbracket = \langle 1 + q_l + q_c, 0 \rangle + \llbracket f \rrbracket(\langle 0, 0 \rangle) + Q_{g,h,q_c,q_m}^{q_l}(\llbracket z \rrbracket)$$

135 Where $g := \lambda xc, xs, yc, ys. \llbracket f \rrbracket(\langle xc, xs \rangle, \langle yc, ys \rangle)_1$ and $h := \lambda xs, ys. \llbracket f \rrbracket(\langle 0, xs \rangle, \langle 0, ys \rangle)_2$ (re-
136 spectively, the cost and size parts of $\llbracket f \rrbracket$). This is much harder to read, but can still be used
137 to gain an idea of the complexity for specific (groups of) instantiations of f .

5 Discussion

139 This paper aims to start a line of research for termination and complexity analysis of
140 higher-order term rewriting. We abandon the classical notions of derivational and runtime
141 complexity that are often used for this task, since these do not naturally match the behaviour
142 of higher-order terms. We separate cost and size (and other structural properties) in our
143 analysis, which is a similar idea (but very different angle) to analysis using *sized types* [1].

144 In the future, we plan to further develop the method, and find interpretations to other
145 classic higher-order functions that often occur as part of larger systems. We aim to investigate
146 properties of the technique, and hope to find connections both in the broader area of
147 computational complexity and in the analysis of term rewriting. We are also interested in
148 automating the construction of interpretations, and in applications in functional programming.

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