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Existence and properties of solutions of the extended play-type hysteresis model

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Abstract

This paper analyses the existence and properties of solutions of the extended play-type model which was proposed in [16] to incorporate hysteresis in unsaturated flow through porous media. The model, when regularised, reduces to a nonlinear degenerate parabolic equation coupled to an ordinary differential equation. This PDE–ODE coupled system, also studied in the fields of bio-film and cellular biology, possesses an interesting mathematical structure which, to our knowledge, remains relatively unexplored. The existence of solutions for the non-degenerate version of the model is shown using the Rothe's method. Furthermore, an L^∞ bound is obtained for the solutions under certain restrictions. Existence of solutions for the degenerate case is then shown assuming that the initial condition is bounded away from the degenerate points. Finally, it is proven that if the solution for the unregularised case exists, then it is contained within physically consistent bounds.

Keywords: Porous flow; Hysteresis; Bio-film models; PDE–ODE coupling

1. Introduction

The Richards equation is commonly used to model the flow of water and air through soil [3,23]. It is obtained by combining the mass balance equation with the Darcy's law [3,23]. In dimensionless form it reads,

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$$\partial_t S + \nabla \cdot [k(S)(\nabla p - \boldsymbol{g})] = 0, \tag{1.1a}$$

where the water saturation S and capillary pressure p are the primary unknowns. The saturation S takes value between 0 and 1, $k : \mathbb{R} \to \mathbb{R}^+$ represents the relative permeability function and $\boldsymbol{g}$ represents the gravitational acceleration assumed to be constant in our case. To close (1.1a), a relation between S and p is generally assumed. In this paper, we use the extended play-type hysteresis model (henceforth called the 'EPH' model) [16] for this purpose which equates S and p as

$$p \in \tfrac{1}{2}(p_c^{(d)}(S) + p_c^{(i)}(S)) - \tfrac{1}{2}(p_c^{(d)}(S) - p_c^{(i)}(S)) \,\mathrm{sign}(\partial_t H(S) + \partial_t p). \tag{1.1b}$$

The functions $p_c^{(i)}$, $p_c^{(d)}$, $H : (0, 1] \to \mathbb{R}$ are determined based on experiments [3,11,23,49] and $\mathrm{sign}(\cdot)$ is the multivalued signum graph given by

$$\mathrm{sign}(\zeta) = \begin{cases} 1 & \text{for } \zeta > 0, \\ [-1, 1] & \text{for } \zeta = 0, \\ -1 & \text{for } \zeta < 0. \end{cases} \tag{1.2}$$

A detailed explanation of the EPH model (1.1b) is given later. On top of being nonlinear, the model (1.1) is also degenerate in the sense that $k(S)$ can vanish and $p_c^{(i)}(S)$, $p_c^{(d)}(S)$ explode as $S \to 0$.

To analyse the model (1.1), we will study another class of problems, i.e.

$$\partial_t u + \nabla \cdot \boldsymbol{F}(u, v) = \nabla \cdot [\mathcal{D}(u, v)\nabla u] + \Psi_1(u, v), \tag{1.3a}$$

$$\partial_t v = \Psi_2(u, v), \tag{1.3b}$$

completed with suitable boundary and initial conditions. The equivalence of (1.3) with the regularised version of (1.1) will be shown in Section 2. System (1.3) has an interesting mathematical structure as it consists of a nonlinear parabolic partial differential equation in a two-way coupling with an ordinary differential equation. Such PDE–ODE coupled systems are used to model the growth of bio-films when the substrate is immobile [17,41], and also to model mitochondrial swelling [18]. However, in our case, the diffusion coefficient D and the advective flux $\boldsymbol{F}$ depend on both the unknowns. This implies that standard techniques, such as the L^1 contraction [36] principle or the Kirchhoff transform [2] cannot directly be applied to this case. Moreover, the maximum principle can be violated for the system (1.3) which gives rise to the overshoot phenomenon in porous flow problems, see [33,48] for detailed discussions on the topic.

Further motivation for our analysis comes from establishing a physically consistent and well-posed model for hysteresis. Hysteresis refers to the phenomenon that if p is measured in an imbibition experiment where $\partial_t S > 0$ at all times then it follows a imbibition pressure curve denoted by $p_c^{(i)}(S)$. On the other hand, if $\partial_t S < 0$ at all times, then a different curve, i.e. drainage curve or $p_c^{(d)}(S)$, is followed. Curves intermediate to $p_c^{(i)}$ and $p_c^{(d)}$ (where $p_c^{(i)}(S) < p < p_c^{(d)}(S)$) are followed if the sign of $\partial_t S$ changes from positive to negative or vice versa. These are referred to as the scanning curves, see Fig. 1. This phenonmena was first documented in 1930 by Haines [20] and since then has been verified by numerous experiments, [34,38,52] being some notable examples. Several different classes of models are available that include hysteresis, Lenhard-Parker

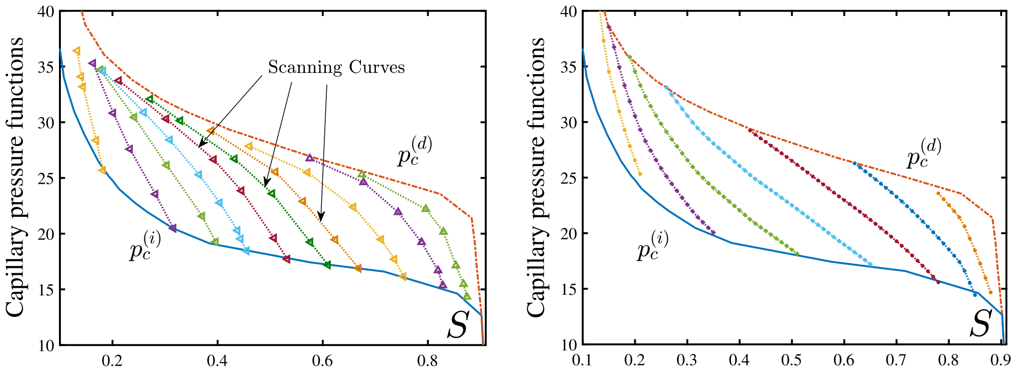

Fig. 1. (left) Plots of $p_c^{(i)}(S)$, $p_c^{(d)}(S)$ and scanning curves from imbibition experiments in [34]. (right) Plot from [16] showing the scanning curves for the extended play-type hysteresis model (1.1b) fitted from the left figure by tuning $H(S)$.

model [37], interfacial area models [21,35], and TCAT (thermodynamically constrained averaging theory) models [29] are examples of such. Review of hysteresis from a mathematical, modelling and physical perspective can be found in [16,29,30,46].

However, due to the complex nature of these models, effects of hysteresis are often ignored in many practical applications and p is simply approximated by some weighted average of $p_c^{(i)}(S)$ and $p_c^{(d)}(S)$ [3,23], e.g.,

$$p = \frac{1}{2}(p_c^{(d)}(S) + p_c^{(i)}(S)). \tag{1.4}$$

Equation (1.1a) along with (1.4) constitutes a nonlinear diffusion problem with diffusivity $-\frac{1}{2}k(S)(p_c^{(i)\prime}(S) + p_c^{(d)\prime}(S))$. The existence of weak solutions of such problems is studied in [1,2] and uniqueness is shown in [14,36] using L^1 contraction principle. However, this simplification introduces significant errors, major examples being in trapping [24], gravity fingering [40], overshoots [33] and redistribution [39]. To account for hysteresis in a comparatively simple way, the play-type hysteresis model was proposed in [5] based on thermodynamic considerations. It reads,

$$p \in \tfrac{1}{2}(p_c^{(d)}(S) + p_c^{(i)}(S)) - \tfrac{1}{2}(p_c^{(d)}(S) - p_c^{(i)}(S))\,\mathrm{sign}(\partial_t S). \tag{1.5}$$

Observe that, this gives $\partial_t S \geq 0$ when $p = p_c^{(i)}(S)$ and $\partial_t S \leq 0$ when $p = p_c^{(d)}(S)$. However, $p_c^{(i)}(S) < p < p_c^{(d)}(S)$ forces that $\partial_t S = 0$ which implies that the scanning curves for this model are vertical lines having constant saturation. Because of this simple structure, it can be considered to be the simplest possible model that addresses hysteresis.

The play-type hysteresis model has been studied extensively analytically due to its local and closed-form structure in contrast to other more complicated models such as the Lenhard-Parker model [37]. If the sign graph in (1.5) is regularised, then together with (1.1a), it constitutes a nonlinear pseudo-parabolic equation for S [12,27,28,45]. Existence results for such pseudo-parabolic equations can be found in [7–9]. The existence of solutions of the regularised play-type hysteresis model with degenerate capillary pressure and permeability functions is shown in [45]. Existence for the two-phase case is discussed in [26]. For the constant relative permeability case, the existence of weak solutions for the unregularised play-type hysteresis model is shown in [43]. In the same paper, an upscaled version of the play-type model is also derived. Uniqueness is shown in [12] for the two-phase regularised case (i.e. with dynamic capillarity). In [44] it is shown that the play-type hysteresis model does not define an L^1-contraction. This leads to unstable planar fronts. Travelling wave solutions are investigated in [4,31–33,48]. For mathematical analysis, in many cases the pseudo-parabolic system is split into an elliptic equation coupled with an ordinary differential equation [8,9,13,27,45].

However, it was pointed out in [16] that due to approximation of the scanning curves by vertical segments, play-type hysteresis model makes certain physically inaccurate predictions. For example, it predicts that in many cases water will not redistribute when two columns having constant but different saturations are joined together. In [33] it is shown that the model predicts infinitely many interior maxima of saturation (called *overshoots* in this context) for high enough injection rates through a long column. However, only finite number of overshoots are observed from experiments [15]. Moreover, it is well documented that numerical methods incorporating the play-type model become unstable if the regularisation parameter is sent to zero [10,42,48,51]. This motivated the extension of the play-type hysteresis model (EPH) given by (1.1b) in [16]. An equivalent expression was derived in [5, Eq. (35)] using thermodynamic arguments. It was used in [16] to cover all cases of horizontal redistribution and in [33] to explain the occurrence of finitely many overshoots, see also [30].

In this paper, we investigate the existence of weak solutions of the regularised EPH model and analyse the properties of its solutions. An alternative form of (1.1) is proposed in Section 2 and mathematical preliminaries are stated. In Section 3, existence of solutions for the model given by (1.3) is proven using Rothe's method. In Section 4, it is shown that a maximum principle holds for the solutions under certain assumptions. This also gives the existence of solutions for the degenerate EPH model. Section 5 is dedicated to investigating the behaviour of the solutions when the regularisation parameter is passed to 0. It is shown that if a limiting solution (S, p) exists then $p \in [p_c^{(i)}(S), p_c^{(d)}(S)]$ almost everywhere.

2. Mathematical formulation

Let $\Omega \subset \mathbb{R}^d$ be a bounded open domain with $\partial\Omega \in C^1$. The $L^2(\Omega)$ inner product in this domain is denoted by $(\cdot,\cdot)$, whereas $\|\cdot\|$ and $\|\cdot\|_p$ denotes the $L^2(\Omega)$ and $L^p(\Omega)$ norms respectively for $1 \le p \le \infty$. For any other space $V(\Omega)$, the norm is denoted by $\|\cdot\|_V$. Further, $W^{k,p}$ denotes the Sobolev space containing functions that have up to k^{th} order derivatives in $L^p(\Omega)$. In particular, $H^k(\Omega) := W^{k,2}(\Omega)$ and $H_0^k := \{u \in H^k : u = 0 \text{ on } \partial\Omega \text{ in a trace sense}\}$. Let $H^{-1}(\Omega)$ denote the dual of H_0^1, and $\langle\cdot,\cdot\rangle$ the duality pairing of $H_0^1(\Omega)$ with $H^{-1}(\Omega)$. The space of functions having up to ℓ^{th} order space derivatives α-Hölder continuous, will be referred to as $C^{\ell,\alpha}$ with $C^\ell := C^{\ell,0}$.

Let $T > 0$ represent a maximum time with $Q := \Omega \times (0, T]$. The Bochner space $L^p(0, T; X(\Omega))$ represents the space of functions $u : Q \to \mathbb{R}$ having norm $\|u\|_{L^p(0,T;X(\Omega))} := \left(\int_0^T \|u(t)\|_X^p dt\right)^{1/p} < \infty$. Finally, we introduce the space

$$\mathcal{W} := \{u \in L^2(0, T; H_0^1(\Omega)) : \partial_t u \in L^2(0, T; H^{-1}(\Omega))\}.$$

Following [47], $\mathcal{W}$ is compactly embedded in $L^2(0, T; L^2(\Omega))$, $\mathcal{W} \hookrightarrow L^2(0, T; L^2(\Omega))$. Moreover, $\mathcal{W}$ is continuously embedded in $C(0, T; L^2(\Omega))$ (space of time continuous functions u with respect to the $L^2(\Omega)$ norm, i.e. $\|u(t_n) - u(t)\| \to 0$ for any $t \in [0, T]$ and $t_n \to t$).

The inequalities that are used repeatedly in our analysis include Cauchy-Schwarz inequality; Poincaré inequality, with C_Ω denoting the constant appearing in it, i.e., $\|u\| \le C_\Omega \|\nabla u\|$ for $u \in H_0^1(\Omega)$; Young's inequality, stating that for $a, b \in \mathbb{R}$ and $\sigma > 0$ one has

$$ab \le \frac{1}{2\sigma}a^2 + \frac{\sigma}{2}b^2, \tag{2.1}$$

and finally the discrete Gronwall's lemma, which states that if $\{y_n\}$, $\{f_n\}$ and $\{g_n\}$ are non-negative sequences and

$$y_n \le f_n + \sum_{0\le k<n} g_k y_k, \qquad \text{for all } n \ge 0, \tag{2.2a}$$

then

$$y_n \le f_n + \sum_{0\le k<n} f_k g_k \exp\left(\sum_{k<j<n} g_j\right), \qquad \text{for all } n \ge 0. \tag{2.2b}$$

Moreover, the identity

$$a(a-b) = \frac{1}{2}[a^2 - b^2 + (a-b)^2] \tag{2.3}$$

will be frequently used. In our notation, $[\cdot]_+$ represents the positive part function, i.e., $[\zeta]_+ := \max\{\zeta, 0\}$. Similarly, $[\zeta]_- := \min\{\zeta, 0\}$. Further, $C > 0$ will denote a generic constant throughout the paper.

2.1. Problem statement and assumptions

For the most part, in this paper, we study the system (1.3). Completed with initial and boundary conditions, it is written in its classical form as

$$(\mathcal{P}s)\begin{cases} \partial_t u + \nabla\cdot \boldsymbol{F}(u,v) = \nabla\cdot[\mathcal{D}(u,v)\nabla u] + \Psi_1(u,v) & \text{in } Q, \quad (2.4)\\ \partial_t v = \Psi_2(u,v) & \text{in } \bar{Q}, \quad (2.5)\\ u(\cdot,0) = u_0(\cdot), \qquad v(\cdot,0) = v_0(\cdot) & \text{in } \Omega, \quad (2.6)\\ u = 0 & \text{on } \partial\Omega\times[0,T]. \quad (2.7)\end{cases}$$

The relation between $(\mathcal{P}s)$ and EPH is established in Section 2.2.

The properties of the functions $\mathcal{D}$, $\Psi_{1/2}$, u_0 and v_0 are as follows:

(A1) $\mathcal{D} \in C^1(\mathbb{R}^2)$; $0 < \mathcal{D}_m \le \mathcal{D}(u,v) \le \mathcal{D}_M < \infty$ for $u, v \in \mathbb{R}$.
(A2) $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^d$ with the j^{th}-component $F_j \in C^1(\mathbb{R}^2)$ satisfying $|\boldsymbol{F}| \le F_M$ for some $F_M > 0$.
(A3) $|\Psi_j(u_1,v_1) - \Psi_j(u_2,v_2)| \le \Psi_u|u_1 - u_2| + \Psi_v|v_1 - v_2|$ for $j \in \{1,2\}$ and constants $\Psi_u, \Psi_v > 0$. Moreover, $\frac{\Psi_j(u,v)-\Psi_j(u,v_1)}{v-v_1}, \frac{\Psi_j(u,v)-\Psi_j(u_1,v)}{u-u_1} \le 0$ for all $u, u_1, v, v_1 \in \mathbb{R}$.
(A4) $u_0 \in L^\infty(\Omega)$ and $v_0 \in H^1(\Omega)\cap L^\infty(\Omega)$, such that $\Psi_2(0, v_0) = 0$ at $\partial\Omega$.

Observe that, (2.5) combined with (A3)–(A4) imply that $v(t)$ restricted to $\partial\Omega$ remains unchanged for all $t > 0$. The weak solution of $(\mathcal{P}s)$ is defined as

Definition 1 *(Weak solution of $(\mathcal{P}s)$).* The pair (u, v) with $u \in \mathcal{W}$ and $v \in H^1(Q)$ is a weak solution of $(\mathcal{P}s)$ if $u(0) = u_0$, $v(0) = v_0$ and it satisfies for all $\phi \in L^2(0, T; H_0^1(\Omega))$ and $\xi \in L^2(Q)$,

$$(\mathcal{P}w) \begin{cases} \int_0^T \langle \partial_t u, \phi\rangle + \int_0^T (\mathcal{D}(u, v)\nabla u, \nabla\phi) = \int_0^T (\boldsymbol{F}(u, v), \nabla\phi) + \int_0^T (\Psi_1(u, v), \phi); & (2.8a) \\ \int_0^T (\partial_t v, \xi) = \int_0^T (\Psi_2(u, v), \xi). & (2.8b) \end{cases}$$

Due to the continuous embedding of $\mathcal{W}$ in $C(0, T; L^2(\Omega))$, $u(0)$ and $v(0)$ are well-defined.

Remark 1 *(Boundary conditions).* For simplicity, a zero Dirichlet condition has been assumed at the boundary for our current analysis. Nevertheless, Definition 1 can be generalised to include Dirichlet, Neumann, and mixed type boundary conditions.

Remark 2 *(Assumptions).* The condition in (A3) that Ψ_j is decreasing with respect to both the variables u and v is not required for proving the existence of the weak solutions. It is used in Section 4 to prove that the solutions are bounded in L^∞. Similarly $u_0, v_0 \in L^\infty(\Omega)$ is only used for proving the L^∞ bound. For proving the existence result in Theorem 3.1, assuming $u_0 \in L^2(\Omega)$ and $v_0 \in H^1(\Omega)$ is sufficient.

2.2. *Relation between the regularised EPH model and $(\mathcal{P}s)$*

Although (1.1) is closer to the expressions of the hysteresis models used in practice, $(\mathcal{P}s)$ is more convenient to analyse mathematically. We show below that the EPH model with the sign$(\cdot)$ graph regularised is a particular case of $(\mathcal{P}s)$. For this purpose, let us assume some properties of the functions $p_c^{(i)}$, $p_c^{(d)}$ and k used in (1.1), (1.4) and (1.5), that are consistent with the data obtained from experiments [23,34,38].

(P1) $p_c^{(i)}, p_c^{(d)} \in C^1((0, 1))$; $p_c^{(i)\prime}(S), p_c^{(d)\prime}(S) < 0$; $p_c^{(d)}(S) > p_c^{(i)}(S)$ for all $S \in (0, 1)$; $p_c^{(i)}(1) = p_c^{(d)}(1)$ and

$$\lim_{S\searrow 0} p_c^{(i)}(S) = \lim_{S\searrow 0} p_c^{(d)}(S) = \lim_{S\nearrow 1} \left[-p_c^{(i)\prime}(S)\right] = \lim_{S\nearrow 1} \left[-p_c^{(d)\prime}(S)\right] = \infty.$$

(P2) $k \in C^1(\mathbb{R})$; $k(S) = k(0) \geq 0$ for $S \leq 0$, $k(S) = k(1)$ for $S \geq 1$ and $k(0) < k(S) < k(1)$ for $0 < S < 1$.

The set of equations (1.1) cannot directly be reduced to the standard weak formulation used for partial differential equations. Thus, we consider an alternative expression to (1.1b) representing the EPH model. Completed with suitable boundary and initial conditions, the model reads

$$
(\text{EPH}) \begin{cases}
\partial_t S = \nabla \cdot [k(S)(\nabla p - \boldsymbol{g})] & \text{in } Q, \quad (2.9)\\
p \in \frac{1}{2}(p_c^{(d)}(S) + p_c^{(i)}(S)) & \\
\quad - \frac{1}{2}(p_c^{(d)}(S) - p_c^{(i)}(S)) \,\text{sign}\,[\partial_t(S + b(p))] & \text{in } \bar{Q}, \quad (2.10)\\
S(\cdot,0) = S_0(\cdot), \quad p(\cdot,0) = p_0(\cdot) & \text{in } \Omega, \quad (2.11)\\
p = 0 & \text{on } \partial\Omega \times [0,T]. \quad (2.12)
\end{cases}
$$

Observe that, for relation (2.10) the scanning curves are given by $S + b(p) = \textit{constant}$, instead of $H(S) + p = \textit{constant}$ as used in [16]. Moreover, if $p = p_c^{(i)}(S)$ in some open subset of Ω, then from (1.2) and (2.10),

$$
\partial_t S + \partial_t b(p_c^{(i)}(S)) \geq 0 \quad \text{or} \quad (1 + b'(p_c^{(i)}(S))\, p_c^{(i)\prime}(S))\, \partial_t S \geq 0.
$$

The directionality imposed by hysteresis then demands that $\partial_t S \geq 0$ in this case. Hence, for consistency $1 + b'(p_c^{(i)}(S))\, p_c^{(i)\prime}(S) > 0$ has to be satisfied. Similar result holds if $p = p_c^{(d)}(S)$ in some open subset of Ω. Combining these observations, we assume that

(P3) $b \in C^1(\mathbb{R})$ with $b(0) = 0$ and

$$
0 < b'(p_c^{(j)}(S)) < -\frac{1}{p_c^{(j)\prime}(S)} \text{ for all } 0 < S < 1 \text{ and } j \in \{i, d\}. \tag{2.13}
$$

Here, (2.13) is the consistency criterion, a counterpart of which was stated in [16, Eq. (2.7)] for $H(S)$.

Remark 3 *(Consistency of the scanning curves).* The inequality (2.13) also guarantees that any scanning curve passing through (S_1, p_1) for arbitrary $S_1 \in (0, 1)$ and $p_1 \in (p_c^{(i)}(S_1), p_c^{(d)}(S_1))$ intersects $p_c^{(i)}$ at some $S_i < 1$ and $p_c^{(d)}$ at some $S_d \in (0, S_i)$, see Fig. 1 (right) for a visual demonstration.

For the initial and boundary conditions we assume:

(P4) $S_0 \in H^1(\Omega)$ and $p_0 \in H_0^1(\Omega)$. Moreover, an $\epsilon > 0$ exists such that $\epsilon \leq S_0 \leq 1 - \epsilon$ and $p_c^{(i)}(S_0) \leq p_0 \leq p_c^{(d)}(S_0)$ a.e. in Ω.

The condition $p_c^{(i)}(S_0) \leq p_0 \leq p_c^{(d)}(S_0)$ comes from the physical constraint that p_0 stays intermediate to $p_c^{(i)}$ and $p_c^{(d)}$ when only hysteretic effects are considered.

Remark 4 *(Degeneracy and physical bounds).* If $\lim_{S \searrow 0} p_c^{(j)}(S) = \infty$ for $j \in \{i, d\}$ or $k(0) = 0$, then $S = 0$ at any interior point in the domain makes the problem degenerate since (2.9) looses its parabolicity. Similarly, the problem becomes degenerate at $S = 1$ since $p_c^{(i)\prime}(1) = p_c^{(d)\prime}(1) =$

$-\infty$. Moreover, S must satisfy the physical bound $0 \le S \le 1$, otherwise $p_c^{(i)}(S)$ and $p_c^{(d)}(S)$ become ill defined. Treating the degeneracy and proving the physical bounds pose extra challenges in considering the problem (EPH) compared to $(\mathcal{P}s)$.

Having stated the properties of the associated functions, we now show the equivalence of the regularised (EPH) model and $(\mathcal{P}s)$. For this purpose, the following transformations are introduced:

$$u := b(p) = \int_0^p b'(\varrho)d\varrho \quad \text{and} \quad v := S + b(p) = S + u. \tag{2.14}$$

We recast (EPH) in terms of u and v. Since $\lim_{S\searrow 0} p_c^{(d)}(S) = \infty$ we get integrating (2.13),

$$b(\infty) - b(p_c^{(d)}(1)) = \int_{p_c^{(d)}(1)}^{\infty} b'(p)\,dp \le -\int_{p_c^{(d)}(1)}^{\infty} \frac{dp}{p_c^{(d)\prime}(p_c^{(d)^{-1}}(p))} \le 1. \tag{2.15}$$

Note that, if $S \to 0$ then $v \to u$. From these observations, we define the constants U_m, V_m, U_M, V_M that will become important later;

$$U_M = V_m = b(\infty), \quad U_m = b(p_c^{(i)}(1)) = b(p_c^{(d)}(1)), \quad V_M = 1 + U_m > V_m. \tag{2.16}$$

The $V_M > V_m$ inequality follows from (2.15). Next, we express $p = p_c^{(i)}(S)$ in terms of a relation between u and v. From (2.14), $p = p_c^{(i)}(S)$ implies $v = v_i(u) := (p_c^{(i)})^{-1}(b^{-1}(u)) + u$. Recalling (P3), $v_i{}'(u) = \frac{1}{b'(p_c^{(i)}(S))p_c^{(i)\prime}(S)} + 1 < 0$ where $S = (p_c^{(i)})^{-1}(b^{-1}(u))$. Hence, the inverse of $v_i(u)$ exists. Let it be denoted by $\rho^{(i)}(u)$. In a similar way $\rho^{(d)}(u)$ is defined. The definitions can alternatively be summarized into

$$\rho^{(i)} := ((p_c^{(i)})^{-1} \circ b^{-1} + 1)^{-1}, \qquad \rho^{(d)} := ((p_c^{(d)})^{-1} \circ b^{-1} + 1)^{-1}. \tag{2.17}$$

From (P1) and (P3), one immediately obtains

(P5) There exists a constant $M_\rho > 0$ such that $-M_\rho < \rho^{(i)\prime}, \rho^{(d)\prime} < 0$ for $v \in [V_m, V_M]$. Further, $\rho^{(d)}(v) > \rho^{(i)}(v)$ for all $v \in (V_m, V_M)$; $\rho^{(i)}(V_m) = \rho^{(d)}(V_m) = U_M$ and $\rho^{(i)}(V_M) = \rho^{(d)}(V_M) = U_m$.

Fig. 2 (right) plots $\rho^{(i)}$ and $\rho^{(d)}$ curves calculated from realistic $p_c^{(i)}$ and $p_c^{(d)}$ curves shown in Fig. 2 (left). By definition, $p = p_c^{(i)}(S)$ iff $u = \rho^{(i)}(v)$, and $p = p_c^{(d)}(S)$ iff $u = \rho^{(d)}(v)$. Furthermore, $\partial_t v = 0$ implies $p_c^{(i)}(S) \le p \le p_c^{(d)}(S)$ which is same as having $\rho^{(i)}(v) \le u \le \rho^{(d)}(v)$. Thus, an equivalent expression to (2.10) is

$$u \in \tfrac{1}{2}(\rho^{(d)}(v) + \rho^{(i)}(v)) - \tfrac{1}{2}(\rho^{(d)}(v) - \rho^{(i)}(v))\,\text{sign}(\partial_t v). \tag{2.18}$$

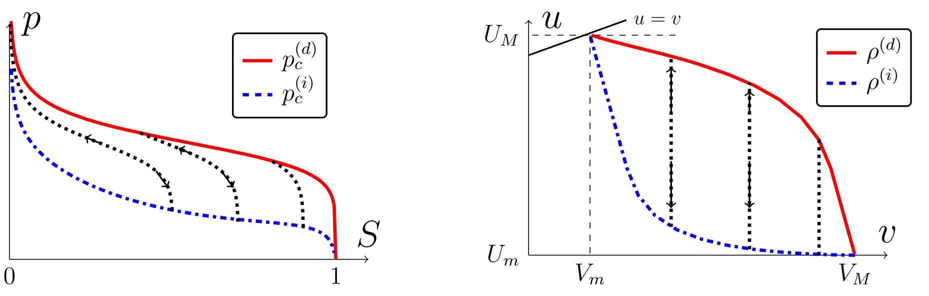

Fig. 2. (left) Realistic $p_c^{(i)}$ and $p_c^{(d)}$ curves in the S–p plane calculated using the van Genuchten model [49]; (right) corresponding $\rho^{(i)}$ and $\rho^{(d)}$ curves in the v–u plain. The $b(p)$ used here is such that $b'(p) = -\frac{1}{2}\max\left(\frac{1}{p_c^{(i)\prime}(p_c^{(i)^{-1}}(p))}, \frac{1}{p_c^{(d)\prime}(p_c^{(d)^{-1}}(p))}\right)$. The black dotted lines correspond to the scanning curves with respect to this choice. In particular, curves for $S + b(p) = 0.5,\ 0.7,\ 0.9$ are shown. The values U_m, U_M, V_m and V_M are marked in the (right) figure.

Since sign$(\cdot)$ is not single-valued, we regularise (2.18) using the expression

$$u \in \tfrac{1}{2}(\rho^{(d)}(v) + \rho^{(i)}(v)) - \tfrac{1}{2}(\rho^{(d)}(v) - \rho^{(i)}(v))\,\mathrm{sign}(\partial_t v) - \tau\partial_t v, \tag{2.19}$$

where $\tau > 0$ is a regularisation parameter. This approach has been used in [13,27,40]. The right most term in (2.19) also has physical significance, since, it gives rise to the dynamic capillarity phenomenon in porous media [12,22,45]. Moreover, expression (2.19) is thermodynamically consistent as it leads to entropy generation as is shown in [5], see specifically equations (28) and (35). Since the function $\frac{1}{2}(\rho^{(d)}(v) - \rho^{(i)}(v))\,\mathrm{sign}(\zeta) + \tau\zeta$ is increasing with respect to ζ, expression (2.19) can be inverted, yielding the relation [5,6,13]

$$\partial_t v = \Phi_\tau(u, v) := \frac{1}{\tau}\begin{cases} \rho^{(d)}(v) - u & \text{when } u > \rho^{(d)}(v), \\ 0 & \text{when } u \in [\rho^{(i)}(v), \rho^{(d)}(v)], \\ \rho^{(i)}(v) - u & \text{when } u < \rho^{(i)}(v). \end{cases} \tag{2.20}$$

Setting in (EPH),

$$\mathcal{D}(u, v) = \frac{k(v-u)}{b'(b^{-1}(u))},\ \boldsymbol{F}(u, v) = \boldsymbol{g}k(v-u),\ \Psi_1(u, v) = \Psi_2(u, v) = \Phi_\tau(u, v), \tag{2.21}$$

we recover $(\mathcal{P}s)$. It is straightforward to verify from (P5) that Ψ_1 and Ψ_2, defined as in (2.21), satisfy Assumption (A3). Similarly, u_0 and v_0 are consistent with (A4) when defined from a (S_0, p_0) pair satisfying (P4). Furthermore, $U_m < u_0 < U_M$ and $V_m < v_0 < V_M$ in Ω. However, to show that $\mathcal{D}(u, v)$ defined in (2.21), satisfies Assumption (A1) ($\mathcal{D} \geq \mathcal{D}_m > 0$) we need to show that $S = v - u$ is bounded away from zero. This is done in Section 4.

Based on our discussion so far, we define the weak solution of the (EPH) model for $\tau > 0$ as

Definition 2 *(Weak solution of (EPH)).* The pair (S, p) with $p \in \mathcal{W}$, $S - S_0 \in \mathcal{W}$ and $S \in [0, 1]$ a.e. in Q is a weak solution of (EPH) for $\tau > 0$ if $p(0) = p_0$, $S(0) = S_0$ and it satisfies for all

$\phi \in L^2(0,T;H_0^1(\Omega))$ and $\xi \in L^2(Q)$,

$$(\mathcal{P}_{\mathrm{EPH}}) \begin{cases} \int_0^T \langle \partial_t S, \phi\rangle = \int_0^T (k(S)[\nabla p - \boldsymbol{g}], \nabla\phi); & (2.22a)\\ \int_0^T (\partial_t (S + b(p)), \xi) = \int_0^T (\Phi_\tau(b(p), S + b(p)), \xi); & (2.22b)\end{cases}$$

where Φ_τ is defined in (2.20).

Observe that, according to Definition 2, S has a trace on $\partial\Omega$ that does not change with time, i.e., it is fixed by S_0.

3. Existence of solutions of $(\mathcal{P}w)$

The main existence result of this section is as follows:

Theorem 3.1. *Assume (A1)–(A4). Then $(\mathcal{P}w)$ has a weak solution (u, v) in the sense of Definition 1. Moreover, $v \in L^\infty(0,T;H^1(\Omega))$ and $\partial_t v \in L^\infty(0,T;L^2(\Omega))$.*

To prove this, we apply Rothe's method [25]. Let the time T be divided into N time steps of width Δt ($T = N\Delta t$) and for any $n \in \{1,....,N\}$ let $\partial_t w$ be approximated by $(w_n - w_{n-1})/\Delta t$ for $w \in \{u, v\}$. Here w_n stands for the value of the variable w at time $t_n := n\Delta t$. The time-discrete solution is defined as

Definition 3 *(Time-discrete solution of $(\mathcal{P}w)$).* For a given $n \in \{1,....,N\}$ and $(u_{n-1}, v_{n-1}) \in (L^2(\Omega))^2$, the time-discrete solution of $(\mathcal{P}w)$ at $t = t_n$ is a pair $(u_n, v_n) \in H_0^1(\Omega) \times L^2(\Omega)$ which satisfies for all $\phi \in H_0^1(\Omega)$ and $\xi \in L^2(\Omega)$,

$$(\mathcal{P}^n_{\Delta t}) \begin{cases} (u_n - u_{n-1}, \phi) + \Delta t\,(\mathcal{D}(u_{n-1}, v_{n-1})\nabla u_n, \nabla\phi) \\ \quad = \Delta t\,(\boldsymbol{F}(u_{n-1}, v_{n-1}), \nabla\phi) + \Delta t\,(\Psi_1(u_n, v_n), \phi); & (3.1a)\\ (v_n, \xi) = (v_{n-1}, \xi) + \Delta t\,(\Psi_2(u_n, v_n), \xi). & (3.1b)\end{cases}$$

For the rest of the section the shorthand $\mathcal{D}_n := \mathcal{D}(u_n, v_n)$, $\boldsymbol{F}_n := \boldsymbol{F}(u_n, v_n)$ and $\Psi_{j,n} := \Psi_j(u_n, v_n)$ will be used extensively for $n \in \{1,\dots,N\}$ and $j \in \{1,2\}$. We show first that the pair (u_n, v_n) exists.

Lemma 3.1. *Assume (A1)–(A3). Then, there exists $\Delta t^* > 0$ such that if $0 < \Delta t < \Delta t^*$, a unique pair (u_n, v_n) solving $(\mathcal{P}^n_{\Delta t})$ in the sense of Definition 3 exists.*

Proof. We define the operator $\mathcal{B} : (L^2(\Omega))^2 \to H_0^1(\Omega) \times L^2(\Omega)$ as follows: $\mathcal{B}(\tilde u, \tilde v) = (u^*, v^*)$, with (u^*, v^*) solving

$$(u^*, \phi) + \Delta t\,(\mathcal{D}_{n-1}\nabla u^* - \boldsymbol{F}_{n-1}, \nabla\phi) = \Delta t\,(\Psi_1(\tilde u, \tilde v), \phi) + (u_{n-1}, \phi), \tag{3.2a}$$

$$(v^*, \xi) = (v_{n-1}, \xi) + \Delta t\,(\Psi_2(\tilde u, \tilde v), \xi), \tag{3.2b}$$

for any $\phi \in H_0^1(\Omega)$ and $\xi \in L^2(\Omega)$. It follows from Lax-Milgram theorem [50, Chapter 2] that $\mathcal{B}$ is well-defined. Next, we apply the Banach fixed point theorem. For two pairs $(\tilde{u}_1, \tilde{v}_1)$ and $(\tilde{u}_2, \tilde{v}_2)$ in $H_0^1(\Omega) \times L^2(\Omega)$, let the outputs of $\mathcal{B}$ be (u_1^*, v_1^*) and (u_2^*, v_2^*). Subtracting the two versions of (3.2a), defining $e_u^* = u_1^* - u_2^*$, substituting $\phi = e_u^*$, and using Young's inequality we get

$$\|e_u^*\|^2 + \Delta t \mathcal{D}_m \|\nabla e_u^*\|^2 \overset{(A1)}{\leq} (e_u^*, e_u^*) + \Delta t\,(\mathcal{D}_{n-1}\nabla e_u^*, \nabla e_u^*) = \Delta t\,(\Psi_1(\tilde{u}_1, \tilde{v}_1) - \Psi_1(\tilde{u}_2, \tilde{v}_2), e_u^*)$$
$$\overset{(2.1)}{\leq} \frac{\Delta t^2}{2}\|\Psi_1(\tilde{u}_1, \tilde{v}_1) - \Psi_1(\tilde{u}_2, \tilde{v}_2)\|^2 + \frac{1}{2}\|e_u^*\|^2$$
$$\overset{(A3)}{\leq} C\Delta t^2[\|\tilde{u}_1 - \tilde{u}_2\|^2 + \|\tilde{v}_1 - \tilde{v}_2\|^2] + \frac{1}{2}\|e_u^*\|^2,$$

for some $C > 0$. This implies,

$$\frac{1}{2}\|e_u^*\|^2 + \Delta t\,\mathcal{D}_m\|\nabla e_u^*\|^2 < C\Delta t^2[\|\tilde{u}_1 - \tilde{u}_2\|^2 + \|\tilde{v}_1 - \tilde{v}_2\|^2]. \tag{3.3}$$

Similarly defining $e_v^* = v_1^* - v_2^*$ we get from (3.2b),

$$\frac{1}{2}\|e_v^*\|^2 < C\Delta t^2[\|\tilde{u}_1 - \tilde{u}_2\|^2 + \|\tilde{v}_1 - \tilde{v}_2\|^2]. \tag{3.4}$$

This clearly shows that $\mathcal{B}$ defines a contraction in $H_0^1(\Omega) \times L^2(\Omega)$ for Δt small enough. More precisely, since $\|u\|_{\Delta t} := \sqrt{\|u\|^2 + 2\Delta t\,\mathcal{D}_m\|\nabla u\|^2}$ is an equivalent norm of $\|u\|_{H_0^1}$, we observe following (3.3)–(3.4) that $\mathcal{B}(u, v)$ is contractive with respect to the norm $\sqrt{\|u\|_{\Delta t}^2 + \|v\|^2}$ for small Δt. Hence, a fixed point (u_n, v_n) of $\mathcal{B}$ exists in $H_0^1(\Omega) \times L^2(\Omega)$, i.e., $\mathcal{B}(u_n, v_n) = (u_n, v_n)$. This proves the lemma. We remark that the condition on Δt is moderate, i.e. (u_n, v_n) exists if $0 < \Delta t \leq C$ where the constant $C > 0$ depends neither on n nor on (u_{n-1}, v_{n-1}). □

From now onward, we assume that Δt is small enough which guarantees the existence of solution pairs to the time discrete problems $(\mathcal{P}_{\Delta t}^n)$. Our goal will be to construct the solution (u, v) from the time-discrete solutions. For this purpose, we introduce the following interpolation functions: for $w \in \{u, v\}$ and $t \in (0, T]$ the piece-wise constant interpolations $\hat{w}_{\Delta t}$, $\check{w}_{\Delta t}$ and the linear interpolation $\bar{w}_{\Delta t}$ are defined in Q so that for $t \in (t_{n-1}, t_n]$ (recall that $t_n = n\Delta t$),

$$\hat{w}_{\Delta t}(t) = w_n,\ \check{w}_{\Delta t}(t) = w_{n-1},\ \bar{w}_{\Delta t}(t) = w_{n-1} + \frac{t - t_{n-1}}{\Delta t}(w_n - w_{n-1}). \tag{3.5}$$

As a first step we show that $\bar{u}_{\Delta t}$ and $(\bar{v}_{\Delta t} - v_0)$ are bounded uniformly in $\mathcal{W}$ and then we would use embedding theorems to construct the weak solution.

Lemma 3.2. *Let (u_n, v_n) be the time-discrete solutions of $(\mathcal{P}_{\Delta t}^n)$ in the sense of Definition 3 for all $n \in \{1, \ldots, N\}$ and let $\hat{u}_{\Delta t}$, $\check{u}_{\Delta t}$, $\bar{u}_{\Delta t}$ and $\hat{v}_{\Delta t}$, $\check{v}_{\Delta t}$, $\bar{v}_{\Delta t}$ be the interpolations defined in (3.5). Then, $\hat{u}_{\Delta t}$, $\check{u}_{\Delta t}$, $\bar{u}_{\Delta t} \in L^2(0, T; H_0^1(\Omega))$ and $\partial_t \bar{u}_{\Delta t} \in L^2(0, T; H^{-1}(\Omega))$ and the corresponding norms are bounded uniformly with respect to Δt. Similarly, the bounds of $\hat{v}_{\Delta t}$, $\check{v}_{\Delta t}$, $\bar{v}_{\Delta t} \in L^\infty(0, T; H^1(\Omega))$ and $\partial_t \bar{v}_{\Delta t} \in L^\infty(0, T; L^2(\Omega))$ are uniform.*

Proof. *(Step 1)* The fact that the functions $\hat{u}_{\Delta t}$, $\check{u}_{\Delta t}$, $\bar{u}_{\Delta t}$ belong to $L^2(0,T;H_0^1(\Omega))$ is direct. We proceed by showing their uniform boundedness. Inserting the test functions $\phi = u_n$ in (3.1a) and $\xi = v_n$ in (3.1b) we get from invoking the identity (2.3) that

$$(\|u_n\|^2 - \|u_{n-1}\|^2 + \|u_n - u_{n-1}\|^2) + \Delta t\, \mathcal{D}_m \|\nabla u_n\|^2 \leq C\Delta t[1 + \|u_n\|^2 + \|v_n\|^2],$$
$$(\|v_n\|^2 - \|v_{n-1}\|^2 + \|v_n - v_{n-1}\|^2) \leq C\Delta t[1 + \|u_n\|^2 + \|v_n\|^2],$$

for some $C > 0$. Here, $(\boldsymbol{F}_{n-1}, \nabla u_n) \leq \frac{1}{2\mathcal{D}_m}\|\boldsymbol{F}_{n-1}\|^2 + \frac{\mathcal{D}_m}{2}\|\nabla u_n\|^2$ and (A2) are used. Combining both inequalities and summing the results up from $n = 1$ to $P \leq N$ yields,

$$\|u_P\|^2 + \|v_P\|^2 + \sum_{k=1}^{P}(\|u_k - u_{k-1}\|^2 + \|v_k - v_{k-1}\|^2) + \mathcal{D}_m \sum_{k=1}^{P}\|\nabla u_k\|^2 \Delta t$$
$$\leq (\|u_0\|^2 + \|v_0\|^2) + 2CP\Delta t + 2C\Delta t \sum_{k=0}^{P}(\|u_k\|^2 + \|v_k\|^2).$$

With $C_0 = \|u_0\|^2 + \|v_0\|^2$ and a generic constant $C > 0$, the discrete Gronwall's lemma (2.2) is applied to obtain:

$$\|u_P\|^2 + \|v_P\|^2 \leq (C_0 + CP\Delta t)\exp\{2CP\Delta t\} \leq (C_0 + CT)\exp\{2CT\}. \tag{3.6}$$

Further, it gives two other important bounds both of which are used later, i.e.

$$\sum_{k=1}^{N}\|\nabla u_k\|^2\Delta t < C_1(T), \quad \sum_{k=1}^{N}(\|u_k - u_{k-1}\|^2 + \|v_k - v_{k-1}\|^2) < C_2(T), \tag{3.7}$$

with $C_{1/2}(T) > 0$ being independent of N. This directly gives the bounds of $\hat{u}_{\Delta t}$, $\check{u}_{\Delta t}$, $\bar{u}_{\Delta t}$ in $L^2(0,T;H_0^1(\Omega))$ since, for example,

$$\int_0^T \|\hat{u}_{\Delta t}\|^2_{H_0^1(\Omega)} dt = \sum_{k=1}^{N}\|\nabla u_k\|^2\Delta t \leq C_1(T),$$

with the rest of the bounds following accordingly.

(Step 2) We need to show that $\hat{v}_{\Delta t}$, $\check{v}_{\Delta t}$, $\bar{v}_{\Delta t} \in L^\infty(0,T;H^1(\Omega))$. So far we have from (3.6) that $\hat{v}_{\Delta t}, \check{v}_{\Delta t}, \bar{v}_{\Delta t} \in L^\infty(0,T;L^2(\Omega))$. Since Definition 3 does not explicitly involve any spatial derivatives of v_n, in order to prove its spatial regularity, we use directional derivatives: $D^h w = (w(\boldsymbol{x} + \hat{\boldsymbol{e}}h) - w(\boldsymbol{x}))/h$ for $\boldsymbol{x} \in \Omega$ and an arbitrary unit vector $\hat{\boldsymbol{e}} \in \mathbb{R}^d$. Choose an open subset $V \subset \Omega$ such that $\text{dist}(V, \partial\Omega) > 2h > 0$. Since $v_n, v_{n-1} \in L^2(\Omega)$, we have from (3.1b) that

$$v_n = v_{n-1} + \Delta t\, \Psi_{2,n} \text{ a.e. in } \Omega, \quad \text{or } D^h v_n = D^h v_{n-1} + \Delta t\, D^h \Psi_{2,n} \text{ a.e. in } V.$$

Then multiplying both sides by $D^h v_n$ and integrating over V we get

$$\int_V (D^h v_n - D^h v_{n-1}) D^h v_n = \Delta t \int_V D^h \Psi_{2,n} D^h v_n \le \Delta t C[\int_V |D^h u_n|^2 + \int_V |D^h v_n|^2],$$

where $C > 0$ does not depend on n or Δt. After summing from $n = 1$ to P, where $P \le N$ is chosen arbitrarily, and using

$$\sum_{k=1}^{P} \int_V |D^h u_k|^2 \Delta t \le \bar{C}_3 \sum_{k=1}^{P} \|\nabla u_k\|^2 \Delta t \overset{(3.7)}{\le} \bar{C}_3 C_1(T) =: C_3$$

for some $\bar{C}_3 > 0$ (see Theorem 3, Chapter 5.8 of [19]), we have (invoking (2.3))

$$\int_V |D^h v_P|^2 - \int_V |D^h v_0|^2 + \sum_{k=1}^{P} \int_V |D^h v_k - D^h v_{k-1}|^2 \le C_3 + 2C \sum_{k=1}^{P} \int_V |D^h v_k|^2 \Delta t.$$

With the application of discrete Gronwall's lemma one obtains that $\int_V |D^h v_n|^2$ is bounded independent of V and h. The smoothness of the boundary $\partial\Omega$ further implies that we can extend V to Ω and hence by applying Theorem 3, Chapter 5.8 of [19] we get that $\|\nabla v_n\|$ is bounded. Consequently, $\hat{v}_{\Delta t}, \check{v}_{\Delta t}, \bar{v}_{\Delta t} \in L^\infty(0, T; H^1(\Omega))$.

(Step 3) Finally, we prove the regularity of time derivatives of $\bar{u}_{\Delta t}$ and $\bar{v}_{\Delta t}$. Observe that, as $\hat{u}_{\Delta t}, \hat{v}_{\Delta t} \in L^\infty(0, T; L^2(\Omega))$, $\Psi_2(\hat{u}_{\Delta t}, \hat{v}_{\Delta t}) \in L^\infty(0, T; L^2(\Omega))$ which gives from (3.1b),

$$\|\partial_t \bar{v}_{\Delta t}\| = \frac{1}{\Delta t}\|v_n - v_{n-1}\| = \|\Psi_{2,n}\| = \|\Psi_2(\hat{u}_{\Delta t}, \hat{v}_{\Delta t})\| \le C_4,$$

for some constant $C_4 > 0$ and $t \in (t_{n-1}, t_n]$. For $\partial_t \bar{u}_{\Delta t}$, from (3.1a) one has

$$\|\partial_t \bar{u}_{\Delta t}\|_{H^{-1}(\Omega)} = \sup_{\|\phi\|_{H_0^1(\Omega)}=1} \langle \partial_t \bar{u}_{\Delta t}, \phi\rangle = \sup_{\|\phi\|_{H_0^1(\Omega)}=1} \left(\frac{u_n - u_{n-1}}{\Delta t}, \phi\right)$$

$$\overset{(A1)}{\le} \mathcal{D}_M \|\nabla u_n\| + \|\boldsymbol{F}_{n-1}\| + \|\phi\|\|\Psi_{1,n}\| \overset{(A2)}{\le} \mathcal{D}_M \|\nabla \hat{u}_{\Delta t}\| + \|F_M\| + C_\Omega \|\Psi_1(\hat{u}_{\Delta t}, \hat{v}_{\Delta t})\|.$$

Here, the Poincaré inequality $\|\phi\| \le C_\Omega \|\nabla\phi\|$ has been used. Since $\hat{u}_{\Delta t} \in L^2(0, T; H_0^1(\Omega))$ and $\Psi_1(\hat{u}_{\Delta t}, \hat{v}_{\Delta t}) \in L^2(Q)$ are uniformly bounded, we have that $\partial_t \bar{u}_{\Delta t} \in L^2(0, T; H^{-1}(\Omega))$. □

Proof of Theorem 3.1. Lemma 3.2 shows that $\bar{u}_{\Delta t}$ and $(\bar{v}_{\Delta t} - v_0)$ are bounded in $\mathcal{W}$ uniformly with respect to Δt. Hence, there exists a sequence of time-steps $\{\Delta t_q\}_{q\in\mathbb{N}}$ with $\lim \Delta t_q = 0$ such that

$$\bar{u}_{\Delta t_q} \rightharpoonup u \text{ and } (\bar{v}_{\Delta t_q} - v_0) \rightharpoonup (v - v_0) \text{ weakly in } \mathcal{W}. \tag{3.8}$$

Due to the compact embedding of $\mathcal{W}$ in $L^2(Q)$, this further implies that

$$\bar{w}_{\Delta t_q} \to w, \text{ strongly in } L^2(Q) \text{ for } w \in \{u, v\}.$$

From (3.7) the following inequalities are obtained,

$$\int_0^T \|\hat{u}_{\Delta t} - \bar{u}_{\Delta t}\|^2 dt = \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \left\| \frac{t_k - t}{\Delta t}(u_k - u_{k-1}) \right\|^2 dt = \frac{\Delta t}{3} \sum_{k=1}^N \|u_k - u_{k-1}\|^2 \leq \frac{C_2}{3}\Delta t,$$

$$\int_0^T \|\check{u}_{\Delta t} - \hat{u}_{\Delta t}\|^2 dt = \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \|(u_k - u_{k-1})\|^2 dt = \Delta t \sum_{k=1}^N \|u_k - u_{k-1}\|^2 \leq C_2 \Delta t.$$

This shows that $\hat{u}_{\Delta t_q} \to u$ in $L^2(Q)$ as $\Delta t_q \to 0$ since $\|u - \hat{u}_{\Delta t_q}\| \leq \|u - \bar{u}_{\Delta t_q}\| + \|\bar{u}_{\Delta t_q} - \hat{u}_{\Delta t_q}\|$. By repeating this argument, one shows that the same holds for $\check{u}_{\Delta t_q}$. Hence,

$$\bar{u}_{\Delta t_q}, \check{u}_{\Delta t_q}, \hat{u}_{\Delta t_q} \to u, \text{ strongly in } L^2(Q). \tag{3.9}$$

In an identical way, for $v_{\Delta t}$ one has

$$\bar{v}_{\Delta t_q}, \check{v}_{\Delta t_q}, \hat{v}_{\Delta t_q} \to v, \text{ strongly in } L^2(Q). \tag{3.10}$$

Finally, due to the bounds of $\partial_t \bar{u}_{\Delta t}$ and $\partial_t \bar{v}_{\Delta t}$ given in Lemma 3.2, there exists a sequence $\Delta t_m \to 0$ such that,

$$\partial_t \bar{w}_{\Delta t_m} \rightharpoonup \partial_t w, \text{ weakly in } L^2(0,T; H^{-1}(\Omega)) \text{ for } w \in \{u, v\}. \tag{3.11}$$

We claim that (u, v) solves $(\mathcal{P}w)$. Let $\Delta t_r \to 0$ be a sequence that satisfies the limits (3.8)–(3.11). From (3.1) we have for $\phi \in L^2(0,T; H_0^1(\Omega))$ and $\xi \in L^2(Q)$,

$$\int_0^T \langle \partial_t \bar{u}_{\Delta t_r}, \phi \rangle + \int_0^T (\mathcal{D}(\check{u}_{\Delta t_r}, \check{v}_{\Delta t_r}) \nabla \hat{u}_{\Delta t_r}, \nabla \phi) = \int_0^T (\Psi_1(\hat{u}_{\Delta t_r}, \hat{v}_{\Delta t_r}), \phi) + \int_0^T (\boldsymbol{F}(\check{u}_{\Delta t_r}, \check{v}_{\Delta t_r}), \nabla \phi),$$

$$\int_0^T \langle \partial_t \bar{v}_{\Delta t_r}, \xi \rangle = \int_0^T (\Psi_2(\hat{u}_{\Delta t_r}, \hat{v}_{\Delta t_r}), \xi).$$

Since $\partial_t \bar{v}_{\Delta t_r} \rightharpoonup \partial_t v \in L^2(\Omega)$, the second equation directly gives (2.8b) in the limit. From (3.11), $\int_0^T \langle \partial_t \bar{u}_{\Delta t_r}, \phi \rangle \to \int_0^T \langle \partial_t u, \phi \rangle$ and (3.9)–(3.10) gives $\int_0^T (\Psi_1(\hat{u}_{\Delta t_r}, \hat{v}_{\Delta t_r}), \phi) \to \int_0^T (\Psi_1(u, v), \phi)$ and $\int_0^T (\boldsymbol{F}(\check{u}_{\Delta t_r}, \check{v}_{\Delta t_r}), \nabla \phi) \to \int_0^T (\boldsymbol{F}(u, v), \nabla \phi)$. Convergence of the second term $(\mathcal{D}(\check{u}_{\Delta t_r}, \check{v}_{\Delta t_r}) \nabla \hat{u}_{\Delta t_r}, \nabla \phi)$ remains to be shown. For this, we first observe that $\mathcal{D}(\check{u}_{\Delta t_r}, \check{v}_{\Delta t_r}) \nabla \hat{u}_{\Delta t_r}$ is bounded in $(L^2(Q))^d$ uniformly with respect to Δt. This means that $\boldsymbol{\zeta} \in L^2(0,T; L^2(\Omega)^d)$ exists such that $\mathcal{D}(\check{u}_{\Delta t_r}, \check{v}_{\Delta t_r}) \nabla \hat{u}_{\Delta t_r} \rightharpoonup \boldsymbol{\zeta}$ weakly. To prove that $\boldsymbol{\zeta} = \mathcal{D}(u, v)\nabla u$, we restrict the test function to $\phi \in C_0^\infty(Q)$. Using the strong convergence of $\mathcal{D}(\check{u}_{\Delta t_r}, \check{v}_{\Delta t_r})$ and the weak convergence of $\nabla \hat{u}_{\Delta t_r}$ one gets with $C_\phi = \|\phi\|_{C^1}$ that

$$\begin{aligned}
&\left| \int_0^T (\mathcal{D}(\check{u}_{\Delta t_r}, \check{v}_{\Delta t_r}) \nabla \hat{u}_{\Delta t_r} - \mathcal{D}(u, v) \nabla u, \nabla \phi) \right| \\
&\leq \left| \int_0^T \big((\mathcal{D}(\check{u}_{\Delta t_r}, \check{v}_{\Delta t_r}) - \mathcal{D}(u, v)) \nabla \hat{u}_{\Delta t_r}, \nabla \phi \big) \right| + \left| \int_0^T (\mathcal{D}(u, v) \nabla (\hat{u}_{\Delta t_r} - u), \nabla \phi) \right| \\
&\leq C_\phi \int_0^T |(\mathcal{D}(\check{u}_{\Delta t_r}, \check{v}_{\Delta t_r}) - \mathcal{D}(u, v), \nabla \hat{u}_{\Delta t_r})| + \left| \int_0^T (\nabla (\hat{u}_{\Delta t_r} - u), \mathcal{D}(u, v) \nabla \phi) \right|
\end{aligned}$$

$$\leq C_2(T)C_\phi \|\mathcal{D}(\check{u}_{\Delta t_r}, \check{v}_{\Delta t_r}) - \mathcal{D}(u,v)\|_{L^2(Q)} + \left| \int_0^T (\nabla(\hat{u}_{\Delta t_r} - u), \mathcal{D}(u,v)\nabla\phi) \right| \to 0,$$

as $\Delta t_r \to 0$. Since the weak limit is unique, we have $\mathcal{D}(\check{u}_{\Delta t_r}, \check{v}_{\Delta t_r})\nabla\hat{u}_{\Delta t_r} \rightharpoonup \boldsymbol{\zeta} = \mathcal{D}(u,v)\nabla u$. This shows that (u,v) is a weak solution of $(\mathcal{P}w)$. □

4. Boundedness and existence of solutions of (EPH)

4.1. L^∞-bounds on u and v

Next, we investigate whether a solution of $(\mathcal{P}w)$ satisfies the maximum principle or not. This is an interesting question primarily due to two reasons. Firstly, the maximum principle is used to prove the existence of solutions of $(\mathcal{P}_{\text{EPH}})$ in the case when $k(0) = 0$. This is discussed in details later. Secondly, for pseudo-parabolic equations arising from the regularisation parameter τ, it is shown in [33,48] that the maximum principle does not hold. Having similar structure to these systems, one might wonder if a maximum principle holds for $(\mathcal{P}w)$. As it turns out, a maximum principle does hold in this case for the variables u and v under certain conditions on the advection term. This is preferred as a property of the EPH since hysteresis alone is known not to cause deviation from the maximum principle [48]. However, it is to be noted that the maximum principle does not necessarily generalise to other advective terms, as is expected in the case of pseudo-parabolic equations.

Proposition 1. *Assume that $\boldsymbol{F}(u,v) = \boldsymbol{F}(u)$. For $v_l := \inf\{v_0\}$ and $v_r := \sup\{v_0\}$ let there exist a pair $\{u_l, u_r\}$ ($u_l < u_r$) such that $u_0(\boldsymbol{x}) \in [u_l, u_r]$ for almost all $\boldsymbol{x} \in \Omega$ and*

$$\Psi_1(u_r, v_l) \leq 0 \leq \Psi_2(u_r, v_l), \quad \Psi_2(u_l, v_r) \leq 0 \leq \Psi_1(u_l, v_r). \tag{4.1}$$

Then the weak solution (u,v) of Definition 1 satisfies $v_l \leq v \leq v_r$ and $u_l \leq u \leq u_r$ a.e.

The rationals behind the assumptions used in the proposition are explained in Section 4.2 in the context of (EPH).

Proof. We only show the proof that $u < u_r$ and $v > v_l$ a.e. and omit the other half of the proof since the arguments are identical. For any $t \in [0,T]$, let $\chi_t : [0,T] \to \{0,1\}$ denote the characteristic function of $[0,t]$, i.e. $\chi_t = 1$ in $[0,t]$ and $\chi_t = 0$ everywhere else. Taking $\phi = [u - u_r]_+ \chi_t \in L^2(0,T; H_0^1(\Omega))$ in (2.8a) and recalling that $\int_0^T (\nabla\zeta, \nabla[\zeta]_+) = \int_0^T \|\nabla[\zeta]_+\|^2$ and $\int_0^T \langle [\zeta]_+, \partial_t\zeta\rangle = \frac{1}{2}\|[\zeta]_+\|^2\Big|_{t=0}^T$ for all $\zeta \in \mathcal{W}$, one has:

$$\frac{1}{2}\|[u(t) - u_r]_+\|^2 + \mathcal{D}_m \int_0^t \|\nabla[u - u_r]_+\|^2 \overset{\text{(A1)}}{\leq} \int_0^t \langle \partial_t u, [u - u_r]_+\rangle + \int_0^t (\mathcal{D}(u,v)\nabla u, \nabla[u - u_r]_+)$$

$$= \int_0^t (\Psi_1(u,v), [u - u_r]_+) + \int_0^t (\boldsymbol{F}(u), \nabla[u - u_r]_+)$$

$$\overset{(4.1)}{\leq} \int_0^t (\Psi_1(u,v) - \Psi_1(u_r, v_l), [u - u_r]_+) + \int_0^t (\boldsymbol{F}(u) - \boldsymbol{F}(u_r), \nabla[u - u_r]_+)$$

$$= \int_0^t (\Psi_1(u,v) - \Psi_1(u_r,v) + \Psi_1(u_r,v) - \Psi_1(u_r,v_l), [u-u_r]_+) + \int_0^t (\partial_u \boldsymbol{F}[u-u_r]_+, \nabla[u-u_r]_+)$$

$$\overset{(A2)}{\leq} \Psi_u \int_0^t \|[u-u_r]_+\|^2 + \int_0^t \left(-\tfrac{\Psi_1(u_r,v)-\Psi_1(u_r,v_l)}{v-v_l}(v_l - v), [u-u_r]_+\right) + C\int_0^t \|[u-u_r]_+\|\|\nabla[u-u_r]_+\|$$

$$\overset{(A3)}{\leq} \Psi_u \int_0^t \|[u-u_r]_+\|^2 + \Psi_v \int_0^t ([v_l - v]_+, [u-u_r]_+) + \frac{C^2}{2\mathcal{D}_m}\int_0^t \|[u-u_r]_+\|^2 + \frac{\mathcal{D}_m}{2}\int_0^t \|\nabla[u-u_r]_+\|^2$$

$$\leq C_m \int_0^t [\|[u-u_r]_+\|^2 + \|[v_l - v]_+\|^2] + \frac{\mathcal{D}_m}{2}\int_0^t \|\nabla[u-u_r]_+\|^2.$$

Here $C,\ C_m > 0$ are constants. The inequality $0 \leq -\frac{\Psi_1(u_r,v)-\Psi_1(u_r,v_l)}{v-v_l} \leq \Psi_v$ follows from (A3). Moreover, $\int_\Omega \boldsymbol{F}(u_r)\cdot\nabla[u-u_r]_+ = \int_{\partial\Omega}[u-u_r]_+\boldsymbol{F}(u_r)\cdot\hat{\boldsymbol{n}} = 0$ is used.

Similarly, using the test function $\xi = [v_l - v]_+\chi_t$ in (2.8b) yields

$$\frac{1}{2}\left\|[v_l - v(t)]_+\right\|^2 = \int_0^t (-\Psi_2(u,v), [v_l - v]_+) \overset{(4.1)}{\leq} \int_0^t (\Psi_2(u_r, v_l) - \Psi_2(u,v), [v_l - v]_+)$$

$$\overset{(A3)}{\leq} C_m \int_0^t [\|[u-u_r]_+\|^2 + \|[v_l - v]_+\|^2].$$

Finally adding them yields for all $t \in [0,T]$ that

$$\left\|[u(t)-u_r]_+\right\|^2 + \left\|[v_l - v(t)]_+\right\|^2 \leq 4C_m \int_0^t [\left\|[u-u_r]_+\right\|^2 + \left\|[v_l - v]_+\right\|^2]. \tag{4.2}$$

Since $\left\|[v_l - v_0]_+\right\| = 0$ and $\left\|[u_0 - u_r]_+\right\|^2 = 0$, we conclude from Gronwall's lemma that $\left\|[u(t)-u_r]_+\right\| = 0$ and $\left\|[v_l - v(t)]_+\right\| = 0$ for all $t > 0$. This proves the proposition. □

4.2. Existence of solutions of (EPH)

The main result of this section is the existence of a solution to $(\mathcal{P}_{\text{EPH}})$. This is obtained first for the case when $k(0) > 0$, and then for $k(0) = 0$ in the absence of convective terms.

Theorem 4.1. *Assume (P1)–(P5).*

(a) If $k(0) > 0$ then a solution of $(\mathcal{P}_{\text{EPH}})$ in the sense of Definition 2 exists.
(b) If $|\boldsymbol{g}| = 0$, then a solution (S,p) of $(\mathcal{P}_{\text{EPH}})$ exists for $k(0) = 0$. Moreover, there exist saturations $0 < S_l < S_r < 1$ such that $S(\boldsymbol{x},t) \in [S_l, S_r]$ and $p(\boldsymbol{x},t) \in [p_c^{(i)}(S_r), p_c^{(d)}(S_l)]$ for almost all $(\boldsymbol{x},t) \in Q$.

To begin with, we observe that (P3) implies $b'(p) \to 0$ for $p \to \pm\infty$ which might cause the model to become degenerate. Therefore, we consider a non-degenerate system that approximates $(\mathcal{P}_{\text{EPH}})$ first. Let $\rho^{(i)}(v)$ and $\rho^{(d)}(v)$, defined in (2.17), be extended to $\mathbb{R}$ such that $\rho^{(i)}(v) = \rho^{(d)}(v) = U_M$ for $v \leq V_m$, and $\rho^{(i)}(v) = \rho^{(d)}(v) = U_m$ for $v \geq V_M$. Since the extended $\rho^{(i)}$ and $\rho^{(d)}$ are bounded, for Φ_τ defined in (2.20), this implies that

$$\|\Phi_\tau(u,v)\|^2 < C[1 + \|u\|^2].$$

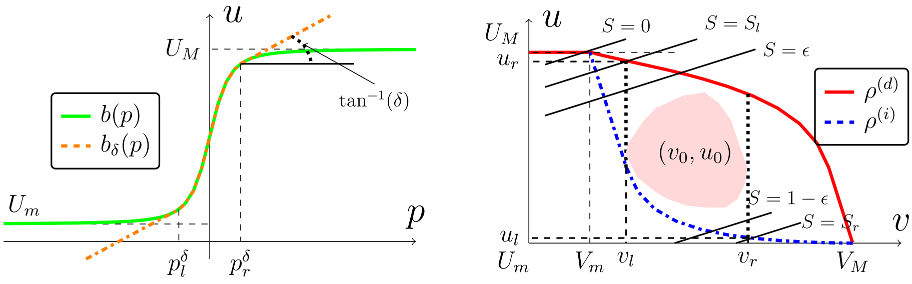

Fig. 3. (left) Schematic for $b(p)$ and $b_\delta(p)$ used in the proof of Theorem 4.1(a). The values p_l^δ, p_r^δ, U_m and U_M are marked. (right) The bounds of u, v and S from Theorem 4.1 (b) shown in the v–u plain. The black parallel lines are representing $v - u = S = constant$. The values v_l, v_r, u_l, u_r, S_l and S_r are shown, as well as the set (v_0, u_0).

For some $\delta > 0$ small enough, let $p_c^{(i)}(1) < p_l^\delta < p_r^\delta < \infty$ be such that $b'(p_l^\delta) = b'(p_r^\delta) = \delta$. Without loss of generality, assume that $b'(p) > \delta$ for $p_l^\delta < p < p_r^\delta$. Define $b_\delta \in C^1(\mathbb{R})$ to be a regularised version of $b(p)$ such that (see Fig. 3 (left))

(Pδ) $b_\delta'(p) \geq \delta > 0$ for all $p \in \mathbb{R}$ with $b_\delta'(p) = \delta$ for $\{p \leq p_l^\delta\} \cup \{p \geq p_r^\delta\}$ and $b_\delta(p) = b(p)$ for $p_l^\delta < p < p_r^\delta$. Clearly, $|b_\delta(p) - b(p)| < C\delta[|p| + 1]$ for some constant $C > 0$.

Consequently, as p_0 satisfies (P4), $p_l^\delta < p_0 < p_r^\delta$ for δ small enough, making $b_\delta(p_0) = b(p_0)$.

We look for solutions $(p_\delta, u_\delta, v_\delta)$, with p_δ, $u_\delta \in \mathcal{W}$ and $v_\delta - S_0 - b(p_0) \in \mathcal{W}$, such that for any $\phi \in L^2(0, T; H_0^1(\Omega))$ and $\xi \in L^2(Q)$ the following is satisfied

$$(\mathcal{P}_\delta) \begin{cases} \int_0^T \langle \partial_t u_\delta, \phi \rangle + \int_0^T (k(v_\delta - u_\delta)[\nabla p_\delta - \boldsymbol{g}], \nabla\phi) = \int_0^T (\Phi_\tau(u_\delta, v_\delta), \phi); & \text{(4.3a)} \\ (\partial_t v_\delta, \xi) = (\Phi_\tau(u_\delta, v_\delta), \xi), \quad u_\delta = b_\delta(p_\delta) \quad \text{in } \bar{Q}; & \text{(4.3b)} \end{cases}$$

with $u_\delta(0) = b(p_0)$ and $v_\delta(0) = S_0 + b(p_0)$.

The existence of such a triplet $(p_\delta, u_\delta, v_\delta)$ follows from Theorem 3.1 by setting $\boldsymbol{F}$, Ψ_1, Ψ_2 as in (2.21) and $\mathcal{D}(u, v) := \frac{k(v-u)}{b_\delta'(b_\delta^{-1}(u))}$. For $k(0) > 0$ it follows directly that all the Assumptions (A1)–(A4) of Theorem 3.1 are satisfied. Hence $(p_\delta, u_\delta, v_\delta)$ exists. We show uniform bounds of p_δ, u_δ and v_δ with respect to δ for this case.

Lemma 4.1. *Assume (P1)–(P5), (Pδ), $k(0) > 0$ and δ small so that $b_\delta(p_0) = b(p_0)$. Let $(p_\delta, u_\delta, v_\delta)$ be a solution of the problem $(\mathcal{P}_\delta)$, i.e. (4.3). Then, p_δ and u_δ are uniformly bounded in $L^2(0, T; H_0^1(\Omega))$ and $\mathcal{W}$ respectively, whereas, $v_\delta - S_0 - b(p_0)$ is uniformly bounded in $\mathcal{W}$.*

Proof. To show this, we first use the test function $\phi = u_\delta$ in (4.3a) which gives

$$\frac{1}{2}[\|u_\delta(T)\|^2 - \|b(p_0)\|^2] + \frac{k(0)}{b_M}\int_0^T \|\nabla u_\delta\|^2$$

$$= \int_0^T \langle \partial_t u_\delta, u_\delta \rangle + \frac{k(0)}{b_M}\int_0^T ({b_\delta}'(p_\delta)\nabla p_\delta, \nabla u_\delta) \leq \int_0^T \langle \partial_t u_\delta, u_\delta \rangle + \int_0^T (k(v_\delta - u_\delta)\nabla p_\delta, \nabla u_\delta)$$

$$\le \int_0^T (k(v_\delta - u_\delta)\,\boldsymbol{g}, \nabla u_\delta) + \int_0^T (\Phi_\tau, u_\delta) \le \frac{b_M}{2k(0)}\|k(1)\,\boldsymbol{g}\|^2 T + \frac{k(0)}{2b_M}\int_0^T \|\nabla u_\delta\|^2 + C\int_0^T [1 + \|u_\delta\|^2].$$

Here $b_M := \max_{\varrho\in\mathbb{R}} b'(\varrho) = \max_{\varrho\in\mathbb{R}} b'_\delta(\varrho)$. Using Gronwall's lemma, we directly get the boundedness of u_δ in $L^2(0,T;H_0^1(\Omega))$. Using the test function $\phi = p_\delta$ in (4.3a) gives

$$\begin{aligned}
&\left\|\int_0^{p_\delta(T)} \varrho b'_\delta(\varrho)d\varrho\right\|_1 - \left\|\int_0^{p_0} \varrho b'_\delta(\varrho)d\varrho\right\|_1 + k(0)\|\nabla p_\delta\|^2 \\
&\le \int_0^T \langle b'(p_\delta)\partial_t p_\delta, p_\delta\rangle + \int_0^T (k(v_\delta - u_\delta)\nabla p_\delta, \nabla p_\delta) = \int_0^T \langle \partial_t u_\delta, p_\delta\rangle + \int_0^T (k(v_\delta - u_\delta)\nabla p_\delta, \nabla p_\delta) \\
&= \int_0^T (k(v_\delta - u_\delta)\,\boldsymbol{g}, \nabla p_\delta) + \int_0^T (\Phi_\tau, p_\delta) \le \int_0^T (k(v_\delta - u_\delta)\,\boldsymbol{g}, \nabla p_\delta) + \int_0^T \|\Phi_\tau\|\|p_\delta\| \\
&\le \frac{\|k(1)\,\boldsymbol{g}\|^2}{k(0)}T + \frac{k(0)}{4}\int_0^T \|\nabla p_\delta\|^2 + \frac{C_\Omega^2}{k(0)}\int_0^T \|\Phi_\tau\|^2 + \frac{k(0)}{4}\int_0^T \|\nabla p_\delta\|^2.
\end{aligned}$$

We have used Poincaré inequality $\|p_\delta\| \le C_\Omega\|\nabla p_\delta\|$, and the fact that $\int_0^p \varrho b'_\delta(\varrho)d\varrho \ge 0$ for all $p \in \mathbb{R}$ here. This gives that p_δ is bounded uniformly in $L^2(0,T;H_0^1(\Omega))$. Finally, by following the steps of Lemma 3.2 (Step 2), one concludes that $v_\delta \in L^\infty(0,T;H^1(\Omega))$ while $\partial_t v_\delta \in L^\infty(0,T;L^2(\Omega))$, the bounds being uniform in both cases. Moreover, following the steps of (Step 3) of Lemma 3.2

$$\|\partial_t u_\delta\|_{H^{-1}} \le k(1)\|\boldsymbol{g}\| + k(1)\|\nabla p_\delta\| + C_\Omega\|\Phi_\tau\|,$$

which shows that $\partial_t u_\delta \in L^2(0,T;H^{-1}(\Omega))$ is bounded uniformly, thus, concluding the proof. □

Proof of Theorem 4.1(a). Define $S_\delta := v_\delta - u_\delta$. From Lemma 4.1 it follows that there exists a sequence $\{\delta_r\}_{r\in\mathbb{N}}$ with $\lim \delta_r = 0$ such that ($\to$ implies strong and $\rightharpoonup$ implies weak convergence)

$$\begin{aligned}
&u_{\delta_r} \to u,\; v_{\delta_r} \to v,\; S_{\delta_r} \to S = v - u \text{ in } L^2(Q);\\
&p_{\delta_r} \rightharpoonup p \text{ in } L^2(0,T;H_0^1(\Omega)),\; \partial_t u_{\delta_r} \rightharpoonup \partial_t u \text{ in } L^2(0,T;H^{-1}(\Omega)), \text{ and } \partial_t v_{\delta_r} \rightharpoonup \partial_t v \text{ in } L^2(Q).
\end{aligned}$$

Hence, following the proof of Theorem 3.1, we conclude that u, v, p and S satisfy

$$\int_0^T \langle \partial_t S, \phi\rangle = \int_0^T (k(S)[\nabla p - \boldsymbol{g}], \nabla\phi) \quad \text{and} \quad \int_0^T (\partial_t v, \xi) = \int_0^T (\Phi_\tau(u,v), \xi),$$

for all $\phi \in L^2(0,T;H_0^1(\Omega))$ and $\xi \in L^2(Q)$. For completeness, we still need to show that $u = b(p)$ and $S \in [0,1]$ a.e. To show the former, observe that $b(p_{\delta_r}) \to u$ in $L^2(Q)$, since,

$$\|u - b(p_{\delta_r})\| \le \|u - u_{\delta_r}\| + \|b_{\delta_r}(p_{\delta_r}) - b(p_{\delta_r})\| \to 0. \tag{4.4}$$

The first term on the right vanishes as $u_\delta \to u$ strongly. For the second, we use (Pδ), giving $\|b_{\delta_r}(p_{\delta_r}) - b(p_{\delta_r})\| \le C\delta_r[1 + \|p_{\delta_r}\|]$ for some $C > 0$, which approaches 0 as $\delta_r \to 0$. Now, from (P3) one gets for $b_M = \max_{\varrho\in\mathbb{R}} b'(\varrho) < \infty$,

$$\|b(p) - b(p_{\delta_r})\|^2 \le b_M(b(p) - b(p_{\delta_r}), p - p_{\delta_r})$$
$$= b_M(b(p) - u, p - p_{\delta_r}) + b_M(u - b(p_{\delta_r}), p - p_{\delta_r}).$$

As $\delta_r \to 0$, the first term in the right vanishes due to the weak convergence of p_{δ_r} and the second term vanishes due to (4.4), thus proving $u = b(p)$.

Finally, $u = b(p) \in [U_m, U_M]$ a.e. implies that $v \in [V_m, V_M]$ a.e. To see this, we use the test function $\xi = [V_m - v]_+$ in (4.3b) to get

$$\tfrac{1}{2}\|[V_m - v]_+(T)\|^2 = \tfrac{1}{2}\int_0^T \partial_t\|[V_m - v]_+\|^2 = (-\Phi_\tau(u, v), [V_m - v]_+)$$
$$= \tfrac{1}{\tau}(u - \rho^{(i)}(v), [V_m - v]_+) \le 0.$$

Here, as $v < V_m$ and $u < U_M$, $\Phi_\tau = \frac{\rho^{(i)}(v)-u}{\tau} \ge \frac{\rho^{(i)}(V_m)-U_M}{\tau} = 0$, see also Fig. 3 (right). This proves that $\|[V_m - v]_+\| = 0$ meaning $v \ge V_m$ a.e. implying that $S = v - u \ge V_m - U_M = 0$. Similarly one shows that $S \le 1$ which concludes our proof. □

Proof of Theorem 4.1(b). Since $k(0) = 0$, we approximate k by $k_\mu : \mathbb{R} \to \mathbb{R}^+$, $\mu > 0$, satisfying $k_\mu(S) = k(\mu) > 0$ for $S < \mu$ and $k_\mu(S) = k(S)$ otherwise. Then the solution (S_μ, p_μ) of ($\mathcal{P}_{\rm EPH}$) exists with k_μ replacing k. Define $u_\mu = b(p_\mu)$ and $v_\mu = S_\mu + b(p_\mu)$, as before. The pair $\{v_l, v_r\}$ is defined as in Proposition 1, i.e. $v_l = \inf\{S_0 + b(p_0)\}$, $v_r = \sup\{S_0 + b(p_0)\}$. Note that, $v_l, v_r \in (V_m, V_M)$ due to (P4), see Fig. 3 (right). Define $u_l, u_r \in (U_m, U_M)$ as

$$u_r = \rho^{(d)}(v_l), \quad u_l = \rho^{(i)}(v_r) \quad \text{giving} \quad \Phi_\tau(v_l, u_r) = \Phi_\tau(v_r, u_l) = 0,$$

see Fig. 3 (right) for clarification. Moreover, since $p_c^{(i)}(S_0) \le p_0 \le p_c^{(d)}(S_0)$, it implies that $u_0 = b(p_0) \in [u_l, u_r]$. Thus, recalling $|\boldsymbol{g}| = 0$, we have from Proposition 1 that $u_l \le u_\mu \le u_r$ and $v_l \le v_\mu \le v_r$ a.e. Consequently, $S_\mu = v_\mu - u_\mu \ge v_l - u_r > 0$. Defining

$$S_l := v_l - u_r \text{ and } S_r := v_r - u_l \text{ one has } 0 < S_l \le S_\mu \le S_r < 1 \text{ a.e. in } Q. \tag{4.5}$$

By taking $\mu < S_l$, one actually gets $k_\mu(S_\mu) \ge k(S_l) > 0$ a.e. in Q. Thus, passing $\mu \to 0$, we obtain the solution. □

5. Behaviour when $\tau \to 0$

In this section we address one other important question: what is the behaviour of (S, p) for the limit $\tau \to 0$ which corresponds to the unregularised EPH model. Ideally $p_c^{(i)}(S) \le p \le p_c^{(d)}(S)$ (or equivalently $\rho^{(i)}(v) \le u \le \rho^{(d)}(v)$) should be satisfied in the pure hysteresis case, i.e., when $\tau \to 0$. One might also wonder whether a limiting solution exists in this case or not. The problem is addressed for the play-type hysteresis model (1.5) in [43, Theorem 3.2] for a reduced case (linear, no advection etc.), however, with stochastic variation of coefficients included. For the nonlinear case, no results are available to our knowledge even for the play-type model. In this section, we take a step towards understanding the limiting case $\tau \to 0$.

Proposition 2. *Let the assumptions of Theorem 4.1 hold. For a given $\tau > 0$, let (S_τ, p_τ) denote a weak solution of $(\mathcal{P}_{\rm EPH})$ in the sense of Definition 2. Further let $u_\tau := b(p_\tau)$ and $v_\tau := S_\tau + b(p_\tau)$. Then, for a constant $C > 0$ independent of τ, one has*

$$\int_0^T \left\| [u_\tau - \rho^{(d)}(v_\tau)]_+ \right\|^2 + \int_0^T \left\| [\rho^{(i)}(v_\tau) - u_\tau]_+ \right\|^2 < C\tau.$$

Proof. Observe that Φ_τ can be rewritten simply as

$$\Phi_\tau(u, v) = -\frac{1}{\tau}[u - \rho^{(d)}(v)]_+ - \frac{1}{\tau}[u - \rho^{(i)}(v)]_-. \tag{5.1}$$

We consider here the case when $|\boldsymbol{g}| = 0$ and $k(0) = 0$, the case $|\boldsymbol{g}| \neq 0$ and $k(0) > 0$ being simpler. From Theorem 4.1(b), S_τ satisfies $0 < S_l \leq S_\tau \leq S_r < 1$ a.e. in Q, where S_l, S_r are independent of τ. For $\mathcal{D}(u, v)$, defined as in (2.21), this implies that there exists $\mathcal{D}_m = k(S_l)/\max_{p\in\mathbb{R}}\{b'(p)\} > 0$ such that (A1) is satisfied. The pair (u_τ, v_τ) satisfies

$$\int_0^T \langle \partial_t u_\tau, \phi\rangle + \int_0^T (\mathcal{D}(u_\tau, v_\tau)\nabla u_\tau, \nabla\phi) = \int_0^T (\Phi_\tau(u_\tau, v_\tau), \phi), \tag{5.2a}$$

$$\int_0^T (\partial_t v_\tau, \xi) = \int_0^T (\Phi_\tau(u_\tau, v_\tau), \xi), \tag{5.2b}$$

for all $\phi \in L^2(0, T; H_0^1(\Omega))$ and $\xi \in L^2(Q)$. Using the test function $\phi = u_\tau$ in (5.2a) yields

$$\frac{1}{2}\left\| u_\tau(T) \right\|^2 + \mathcal{D}_m \int_0^T \left\| \nabla u_\tau \right\|^2 \leq \frac{1}{2}\left\| u_0 \right\|^2 + \int_0^T (\Phi_\tau, u_\tau). \tag{5.3}$$

Using $\xi = \rho^{(d)}(v_\tau)$ in (5.2b) we get

$$\int_0^T \int_\Omega \partial_t \left(\int_{V_m}^{v_\tau} \rho^{(d)}(\varrho)\, d\varrho \right) = \int_0^T (\Phi_\tau, \rho^{(d)}(v_\tau)). \tag{5.4}$$

Subtracting (5.4) from (5.3), one further obtains

$$\frac{1}{2}\left\| u_\tau(T) \right\|^2 + \mathcal{D}_m \int_0^T \left\| \nabla u_\tau \right\|^2 \leq \frac{1}{2}\left\| u_0 \right\|^2 + \left\| \int_{v_0}^{v_\tau(T)} \rho^{(d)}(\varrho)\, d\varrho \right\|_1 + \int_0^T (\Phi_\tau, u_\tau - \rho^{(d)}(v_\tau)).$$

The term $\int_{v_0}^{v_\tau} \rho^{(d)}(\varrho)\, d\varrho$ is bounded in $L^\infty(\Omega)$ since both v_τ (from Proposition 1, $V_m < v_\tau < V_M$ a.e. in Q) and $\rho^{(d)}(\cdot)$ are bounded. The last term is estimated as

$$(\Phi_\tau, u_\tau - \rho^{(d)}(v_\tau)) = -\frac{1}{\tau}\|[u_\tau - \rho^{(d)}(v_\tau)]_+\|^2 - \frac{1}{\tau}([u_\tau - \rho^{(i)}(v_\tau)]_-, u_\tau - \rho^{(d)}(v_\tau))$$
$$\leq -\frac{1}{\tau}\|[u_\tau - \rho^{(d)}(v_\tau)]_+\|^2 - \frac{1}{\tau}([u_\tau - \rho^{(i)}(v_\tau)]_-, u_\tau - \rho^{(i)}(v_\tau))$$
$$\leq -\frac{1}{\tau}\|[u_\tau - \rho^{(d)}(v_\tau)]_+\|^2 - \frac{1}{\tau}\|[u_\tau - \rho^{(i)}(v_\tau)]_-\|^2.$$

Combining everything, we get

$$\frac{1}{2}\left\|u_\tau(T)\right\|^2 + \mathcal{D}_m \int_0^T \left\|\nabla u_\tau\right\|^2 + \frac{1}{\tau}\int_0^T \|[u_\tau - \rho^{(d)}(v_\tau)]_+\|^2 + \frac{1}{\tau}\int_0^T \|[u_\tau - \rho^{(i)}(v_\tau)]_-\|^2$$
$$\leq \frac{1}{2}\left\|u_0\right\|^2 + \left\|\int_{v_0}^{v_\tau(T)} \rho^{(d)}(\varrho)\, d\varrho\right\|_1 . \tag{5.5}$$

This proves the assertion. The proof for the case $k(0) > 0$ is similar. □

This implies that if a limit (u, v) exists of (u_τ, v_τ) as $\tau \to 0$, then $u \in [\rho^{(i)}(v), \rho^{(d)}(v)]$ a.e. in Q implying that the pressure in this limit stays within the bounds of the hysteretic pressure curves. Another immediate consequence of Proposition 2 is the boundedness of $\int_0^T \left\|\nabla u_\tau\right\|^2$ as evident from (5.5). This implies

Corollary 1. *Under assumptions of Proposition 2, there exists $u \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ such that $u_\tau \rightharpoonup u$ in $L^2(0, T; H_0^1(\Omega))$ weakly as $\tau \to 0$.*

6. Conclusion

In this paper, we analysed the extended play-type hysteresis (EPH) model, proposed in [16], for the unsaturated flow case. A modification of the model is proposed that conforms with the standard weak formulation used in analysis. A regularised version of the problem is considered and it is shown to be equivalent to a nonlinear degenerate parabolic equation coupled to an ordinary differential equation. Such PDE–ODE coupled systems are also used in the study of bio-films and in cellular biology.

Using Rothe's method, the existence of weak solutions is first proven for the equivalent system and then for the (EPH) model. Solutions are shown to exist even when the capillary pressure functions blow up and relative permeability vanishes for zero saturation. To cover the latter case, a maximum principle is proven in the absence of an advection term. The consequences of passing the regularisation parameter to zero are explored which indicate that the pressure stays bounded by capillary pressure curves in the limit.

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References

[1] H.W. Alt, S. Luckhaus, Quasilinear elliptic-parabolic differential equations, Math. Z. 183 (3) (1983) 311–341.
[2] H.W. Alt, S. Luckhaus, A. Visintin, On nonstationary flow through porous media, Ann. Mat. Pura Appl. 136 (1) (1984) 303–316.
[3] J. Bear, Hydraulics of Groundwater, McGraw-Hill International Book Co., 1979.
[4] E.E. Behi-Gornostaeva, K. Mitra, B. Schweizer, Traveling wave solutions for the Richards equation with hysteresis, IMA J. Appl. Math. 84 (4) (2019) 797–812.
[5] A.Y. Beliaev, S.M. Hassanizadeh, A theoretical model of hysteresis and dynamic effects in the capillary relation for two-phase flow in porous media, Transp. Porous Media 43 (3) (2001) 487–510.
[6] A.Y. Beliaev, R.J. Schotting, Analysis of a new model for unsaturated flow in porous media including hysteresis and dynamic effects, Comput. Geosci. 5 (4) (2001) 345–368.
[7] M. Bertsch, F. Smarrazzo, A. Tesei, Pseudoparabolic regularization of forward-backward parabolic equations: a logarithmic nonlinearity, Anal. PDE 6 (7) (2013) 1719–1754.
[8] M. Böhm, R.E. Showalter, Diffusion in fissured media, SIAM J. Math. Anal. 16 (3) (1985) 500–509.
[9] M. Böhm, R.E. Showalter, A nonlinear pseudoparabolic diffusion equation, SIAM J. Math. Anal. 16 (5) (1985) 980–999.
[10] M. Brokate, N.D. Botkin, O.A. Pykhteev, Numerical simulation for a two-phase porous medium flow problem with rate independent hysteresis, Physica B, Condens. Matter 407 (9) (2012) 1336–1339.
[11] R.H. Brooks, A.T. Corey, Properties of porous media affecting fluid flow, J. Irrig. Drain. Div. 92 (2) (1966) 61–90.
[12] X. Cao, I.S. Pop, Two-phase porous media flows with dynamic capillary effects and hysteresis: uniqueness of weak solutions, Comput. Math. Appl. 69 (7) (2015) 688–695.
[13] X. Cao, I.S. Pop, Uniqueness of weak solutions for a pseudo-parabolic equation modeling two phase flow in porous media, Appl. Math. Lett. 46 (2015) 25–30.
[14] J. Carrillo, P. Wittbold, Uniqueness of renormalized solutions of degenerate elliptic–parabolic problems, J. Differ. Equ. 156 (1) (1999) 93–121.
[15] D.A. DiCarlo, Experimental measurements of saturation overshoot on infiltration, Water Resour. Res. 40 (4) (2004).
[16] C.J. van Duijn, K. Mitra, Hysteresis and horizontal redistribution in porous media, Transp. Porous Media 122 (2) (2018) 375–399.
[17] H.J. Eberl, E.M. Jalbert, A. Dumitrache, G.M. Wolfaardt, A spatially explicit model of inverse colony formation of cellulolytic biofilms, Biochem. Eng. J. 122 (2017) 141–151.
[18] M.A. Efendiev, M. Otani, H.J. Eberl, Mathematical analysis of a PDE-ODE coupled model of mitochondrial swelling with degenerate calcium ion diffusion, SIAM J. Math. Anal. 52 (1) (2020) 543–569.
[19] L.C. Evans, Partial Differential Equations, Wiley Online Library, 1988.
[20] W.B. Haines, Studies in the physical properties of soil. V. the hysteresis effect in capillary properties, and the modes of moisture distribution associated therewith, J. Agric. Sci. 20 (1) (1930) 97–116.
[21] S.M. Hassanizadeh, W.G. Gray, Mechanics and thermodynamics of multiphase flow in porous media including interphase boundaries, Advances in Water Resources 13 (4) (1990) 169–186.
[22] S.M. Hassanizadeh, W.G. Gray, Thermodynamic basis of capillary pressure in porous media, Water Resour. Res. 29 (10) (1993) 3389–3405.
[23] R. Helmig, Multiphase Flow and Transport Processes in the Subsurface: A Contribution to the Modeling of Hydrosystems, Springer-Verlag, 1997.
[24] V. Joekar-Niasar, F. Doster, R.T. Armstrong, D. Wildenschild, M.A. Celia, Trapping and hysteresis in two-phase flow in porous media: a pore-network study, Water Resour. Res. 49 (7) (2013) 4244–4256.
[25] J. Kačur, Method of Rothe in Evolution Equations, vol. 1192, Springer, 1986.
[26] J. Koch, A. Rätz, B. Schweizer, Two-phase flow equations with a dynamic capillary pressure, Eur. J. Appl. Math. 24 (1) (2013) 49–75.
[27] A. Lamacz, A. Rätz, B. Schweizer, A well-posed hysteresis model for flows in porous media and applications to fingering effects, Adv. Math. Sci. Appl. 21 (2) (2011) 33.
[28] A. Mikelić, A global existence result for the equations describing unsaturated flow in porous media with dynamic capillary pressure, J. Differ. Equ. 248 (6) (2010) 1561–1577.
[29] C.T. Miller, K. Bruning, C.L. Talbot, J.E. McClure, W.G. Gray, Nonhysteretic capillary pressure in two-fluid porous medium systems: definition, evaluation, validation, and dynamics, Water Resour. Res. 55 (8) (2019) 6825–6849.
[30] K. Mitra, Mathematical complexities in porous media flow, PhD thesis, Eindhoven University of Technology & Hasselt University, ISBN 978-90-386-4845-3, 2019.

[31] K. Mitra, T. Köppl, I.S. Pop, C.J. van Duijn, R. Helmig, Fronts in two-phase porous media flow problems: the effects of hysteresis and dynamic capillarity, Stud. Appl. Math. 144 (4) (2020) 449–492.
[32] K. Mitra, A. Rätz, B. Schweizer, Travelling wave solutions for gravity fingering in porous media flows, arXiv preprint, arXiv:2011.10792, 2020.
[33] K. Mitra, C.J. van Duijn, Wetting fronts in unsaturated porous media: the combined case of hysteresis and dynamic capillary pressure, Nonlinear Anal., Real World Appl. 50 (2019) 316–341.
[34] N.R. Morrow, C.C. Harris, Capillary equilibrium in porous materials, Soc. Pet. Eng. 5 (01) (1965) 15–24.
[35] J. Niessner, S.M. Hassanizadeh, A model for two-phase flow in porous media including fluid-fluid interfacial area, Water Resour. Res. 44 (8) (2008).
[36] F. Otto, L^1-contraction and uniqueness for quasilinear elliptic–parabolic equations, J. Differ. Equ. 131 (1) (1996) 20–38.
[37] J.C. Parker, R.J. Lenhard, T. Kuppusamy, A parametric model for constitutive properties governing multiphase flow in porous media, Water Resour. Res. 23 (4) (1987) 618–624.
[38] A. Poulovassilis, Hysteresis of pore water in granular porous bodies, Soil Sci. 109 (1) (1970) 5–12.
[39] P.A.C. Raats, C.J. van Duijn, A note on horizontal redistribution with capillary hysteresis, Water Resour. Res. 31 (1) (1995) 231–232.
[40] A. Rätz, B. Schweizer, Hysteresis models and gravity fingering in porous media, Z. Angew. Math. Mech. 94 (7–8) (2014) 645–654.
[41] Y. Rohanizadegan, S. Sonner, H.J. Eberl, Discrete attachment to a cellulolytic biofilm modeled by an Itô stochastic differential equation, Math. Biosci. Eng. 17 (3) (2020) 2236–2271.
[42] M. Schneider, T. Köppl, R. Helmig, R. Steinle, R. Hilfer, Stable propagation of saturation overshoots for two-phase flow in porous media, Transp. Porous Media 121 (3) (2018) 621–641.
[43] B. Schweizer, Averaging of flows with capillary hysteresis in stochastic porous media, Eur. J. Appl. Math. 18 (3) (2007) 389–415.
[44] B. Schweizer, Instability of gravity wetting fronts for Richards equations with hysteresis, Interfaces Free Bound. 14 (1) (2012) 37–64.
[45] B. Schweizer, The Richards equation with hysteresis and degenerate capillary pressure, J. Differ. Equ. 252 (10) (2012) 5594–5612.
[46] B. Schweizer, Hysteresis in porous media: modelling and analysis, Interfaces Free Bound. 19 (3) (2017) 417–448.
[47] J. Simon, Compact sets in the space $L^p(0, T; B)$, Ann. Mat. Pura Appl. 146 (1) (1986) 65–96.
[48] C.J. van Duijn, K. Mitra, I.S. Pop, Travelling wave solutions for the Richards equation incorporating non-equilibrium effects in the capillarity pressure, Nonlinear Anal., Real World Appl. 41 (Suppl. C) (2018) 232–268.
[49] M.T. van Genuchten, A closed-form equation for predicting the hydraulic conductivity of unsaturated soils, Soil Sci. Soc. Am. J. 44 (5) (1980) 892–898.
[50] E. Zeidler, Applied Functional Analysis: Applications to Mathematical Physics, Applied Mathematical Sciences, vol. 108, Springer, 1995.
[51] H. Zhang, P.A. Zegeling, A numerical study of two-phase flow models with dynamic capillary pressure and hysteresis, Transp. Porous Media 116 (2) (2017) 825–846.
[52] L. Zhuang, C.R. Bezerra Coelho, S.M. Hassanizadeh, M. Th. van Genuchten, Analysis of the hysteretic hydraulic properties of unsaturated soil, Vadose Zone J. 16 (5) (2017) 1–9.