

COARSE GEOMETRY AND CALLIAS QUANTISATION

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ABSTRACT. Consider a proper, isometric action by a unimodular, locally compact group G on a complete Riemannian manifold M . For equivariant elliptic operators that are invertible outside a cocompact subset of M , we show that a localised index in the K -theory of the maximal group C^* -algebra of G is well-defined. The approach is based on the use of maximal versions of equivariant localised Roe algebras, and many of the technical arguments in this paper are used to handle the ways in which they differ from their reduced versions.

By using the maximal group C^* -algebra instead of its reduced counterpart, we can apply the trace given by integration over G to recover an index defined earlier by the last two authors, and developed further by Braverman, in terms of sections invariant under the group action. This leads to refinements of index-theoretic obstructions to Riemannian metrics of positive scalar curvature on noncompact manifolds, and also on orbifolds and other singular quotients of proper group actions. As a motivating application in another direction, we prove a version of Guillemin and Sternberg's quantisation commutes with reduction principle for equivariant indices of Spin^c Callias-type operators.

CONTENTS

1. Introduction	1
2. Equivariant localised maximal Roe algebras	5
3. Results	9
4. Regularity and localisation	15
5. Averaging maps	26
6. The invariant index	33
7. Quantisation commutes with reduction	36
Acknowledgments	39
References	40

1. INTRODUCTION

Background. Let M be a complete Riemannian manifold, and let D be an elliptic differential operator on a vector bundle $E \rightarrow M$. The *coarse index* [44] of D lies in $K_*(C^*(M))$, the K -theory group of the *Roe algebra* $C^*(M)$ of M . This Roe algebra is the closure in the operator norm of the algebra of locally compact,

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bounded operators on $L^2(E)$ that enlarge supports of sections by a finite amount. If M is compact, then $C^*(M)$ is the algebra of compact operators, and the coarse index of D is its Fredholm index. A strength of the coarse index is that it applies very generally, without any assumptions on compactness of M , or on the behaviour of D at infinity. Coarse index theory has a range of applications, for example to Riemannian metrics of positive scalar curvature [46], and to the Novikov conjecture [50, 51]. A central role here is played by the coarse Baum–Connes conjecture [43].

The general applicability of the coarse index can come at the cost of computability. For that reason, it is worth looking for special cases, or variations, where a version of the coarse index is more explicit or computable. One useful approach is Roe’s *localised coarse index* [45]. If D^2 is positive outside a subset $Z \subset M$ in a suitable sense, then Roe constructed a *localised coarse index*

$$\text{index}^Z(D) \in K_*(C^*(Z)).$$

The special case where Z is compact is already of interest: then D is Fredholm (by Theorem 2.1 in [2]), and its localised coarse index generalises the Gromov–Lawson index [17], the Atiyah–Patodi–Singer index on compact manifolds with boundary [5], and the index of Callias-type Dirac operators [3, 11, 32] $D = \tilde{D} + \Phi$, where \tilde{D} is a Dirac operator, and Φ is a vector bundle endomorphism making D invertible at infinity.

The localised coarse index was generalised to an equivariant version in [20], for a proper, isometric action by a unimodular locally compact group G on M , preserving all structure including D . Then, if Z/G is compact, one obtains a *localised equivariant index*

$$(1.1) \quad \text{index}_{G,\text{red}}^{\text{loc}}(D) \in K_*(C_{\text{red}}^*(G)),$$

where $C_{\text{red}}^*(G)$ is the reduced group C^* -algebra of G . The fact that this index lies in $K_*(C_{\text{red}}^*(G))$ is useful, because that K -theory group is independent of M , and it is a very well-studied object that is central to many problems in geometry, topology and group theory. In particular, it is large enough to contain relevant group-theoretic information. And importantly, there is a range of traces and higher cyclic cocycles on subalgebras of $C_{\text{red}}^*(G)$ that allows one to obtain a number from the index (1.1), for which one can then find a topological expression. Examples of such expressions are the equivariant Atiyah–Patodi–Singer index theorems in [14, 30, 49], in the case of manifolds with boundary.

Results. This paper is about the construction and application of a *maximal* localised equivariant coarse index, taking values in the K -theory of the maximal group C^* -algebra $C_{\text{max}}^*(G)$

$$(1.2) \quad \text{index}_G^{\text{loc}}(D) \in K_*(C_{\text{max}}^*(G)).$$

The first result in this paper is that this index is well-defined: see Theorem 3.1 and Proposition 3.3.

The index (1.2) has several advantages over (1.1). From a general point of view, the natural map from $C_{\text{max}}^*(G)$ to $C_{\text{red}}^*(G)$ maps the index in $K_*(C_{\text{max}}^*(G))$ to the one in $K_*(C_{\text{red}}^*(G))$, so the former is a more refined invariant. On a more practical level, the integration map $I: L^1(G) \rightarrow \mathbb{C}$ extends to a trace on $C_{\text{max}}^*(G)$ (not on $C_{\text{red}}^*(G)$); this can also be viewed as an algebra homomorphism $I: C_{\text{max}}^*(G) \rightarrow \mathbb{C}$.

That means the induced map $I_*: K_0(C_{\max}^*(G)) \rightarrow K_0(\mathbb{C})$ on K -theory can be applied to the index (1.2), to yield the integer

$$(1.3) \quad I_*(\text{index}_G^{\text{loc}}(D)) \in K_0(\mathbb{C}) = \mathbb{Z}.$$

Morally, applying the integration trace I should correspond to taking the G -invariant part of the equivariant index. The second result in this paper, Theorem 3.9, is that this is indeed the case in a precise sense:

$$(1.4) \quad I_*(\text{index}_G^{\text{loc}}(D)) = \text{index}(D)^G,$$

where the right hand side is the Fredholm index of D restricted to G -invariant sections that are square integrable transversally to orbits in a certain sense. The latter index was defined in [25], and developed further by Braverman [9].

In the example where D_X is an elliptic operator on a possibly noncompact manifold X , invertible outside a compact set, and D is its lift to the universal cover of X , (1.4) implies that I_* maps the $\pi_1(X)$ -equivariant, localised, maximal index of D to the Fredholm index of D_X . This means that the index (1.2) refines the Gromov–Lawson index, the index of Callias-type operators, as well as the Atiyah–Patodi–Singer index. One application of this fact is that it leads to refinements of obstructions to Riemannian metrics of positive scalar curvature defined through the Gromov–Lawson and Callias indices on Spin manifolds. This is analogous to the way in which the image of D under the analytic assembly map [6] for the maximal group C^* -algebra of $\pi_1(X)$ refines the index of D_X in the case where X is compact. The robustness of (1.4) allows one to generalise this to orbifolds, and more generally to metrics invariant under a proper group action. This is in contrast to a version for the reduced group C^* -algebra in the compact case, where I is replaced by the von Neumann trace, and the analogue of (1.4) only holds because the actions is free and the group is discrete. Furthermore, an analogue of Atiyah’s L^2 -index theorem used in the reduced version is not available yet for noncompact manifolds (but see Theorem 2.20 in [10] for an analogue). Another approach to K -theoretic obstructions to positive scalar curvature, in terms of Callias operators, was developed in [13]. Explicit applications to positive scalar curvature will be explored in future work.

A completely different application that motivates the development of the index (1.2) and Theorem 3.9. is a version of Guillemin and Sternberg’s *quantisation commutes with reduction* principle [18] for Callias-type Spin^c-Dirac operators. That principle was initially stated and proved for compact Kähler and symplectic manifolds [38, 39, 41, 47]. This principle was extended in various directions, including results for proper actions by possibly noncompact groups, with possibly noncompact orbit spaces, see [25] for the symplectic case and [26] for Spin^c-manifolds. The index, or quantisation, used in those papers, was defined just in terms of sections invariant under the group action. Furthermore, the index was only well-defined after a suitable order zero term was added to the operator in question. The first of these issues was partially remedied in [29], where the quantisation commutes with reduction principle was proved for an index with values in the completed representation ring of a maximal compact subgroup of G .

Since the work of Paradan and Vergne [42], the quantisation commutes with reduction principle is known to be a general property of equivariant indices of Spin^c-Dirac operators in general, and not just of geometric quantisation in the narrow sense. For a Callias-type operator $D = \tilde{D} + \Phi$, where \tilde{D} is a Spin^c-Dirac

operator, the third result in this paper, Theorem 3.11, states that the quantisation commutes with reduction principle holds, in the sense that

$$(1.5) \quad I_*(\text{index}_G^{\text{loc}}(\tilde{D} + \Phi)) = \text{index}(D_0),$$

where D_0 is a Dirac operator on a *reduced space* M_0 , a Spin^c -analogue of a reduced space in symplectic geometry, for high enough powers of the determinant line bundle of the Spin^c -structure. In this setting, the use of the maximal localised coarse index allows us to prove such a result in the setting of noncompact groups and orbit spaces, for a truly equivariant index in $K_0(C_{\max}^*(G))$, which is defined without the need of an added term.

The equality (1.5) already appears to be new in the case where G is compact. Then $\tilde{D} + \Phi$ is Fredholm, and has an equivariant index in the usual sense. In this case, a version of the shifting trick in symplectic geometry applies to yield information about the multiplicities in that index of all irreducible representations of G . Such a result would apply for example to an equivariant version of Callias' treatment [12] of fermions in the field of magnetic $\text{SU}(2)$ -monopoles.

Techniques used. The key ingredient in the construction of the index (1.2) is the notion of a *maximal localised equivariant Roe algebra* for arbitrary unimodular, locally compact groups. This involves the notion of an *admissible module*, which was defined in [52] for discrete groups, and in [20] in general. In the non-equivariant, non-localised case, the natural maximal norm for such algebras was shown to be well-defined in [16]. In the equivariant, localised case, this is less clear, and getting around this is a step in the construction of the algebras we need.

The construction of the index (1.2) is very different from the construction of the reduced version (1.1) in [20]. Instead of viewing D as an unbounded operator on $L^2(E)$, we view it as an unbounded operator on a maximal localised equivariant Roe algebra A , viewed as a Hilbert C^* -module over itself. The reason for this is that the localisation results in [45] that make the definition of the localised coarse index possible do not directly carry over to the norm on the maximal Roe algebra. Indeed, it is not even clear if the operators involved lie in the unlocalised maximal Roe algebra, let alone if they localise in a suitable way.

We prove versions of Roe's localisation results for D as an operator on A , thus allowing us to define (1.2). To do this we prove that the functional calculus for such operators on A is well-defined. This was done in [21] for the uniform maximal Roe algebra; in our setting it works for usual maximal Roe algebras due to localisation at a cocompact set.

To prove the equality (1.4), we use various *averaging maps*, which map G -equivariant operators on M to operators on M/G . Comparing such maps for operators on $L^2(E)$ and on A to the integration trace I then leads to a proof of (1.4).

Using (1.4), we see that the left hand side of (1.5) equals a more concrete index in terms of G -invariant sections. For the latter index, we obtain localisation estimates that allow us to show that this index equals the right hand side of (1.5). These localisation estimates build on those in [25, 26, 37, 47], but a fundamental difference is that we now need the key deformation term to go to zero at infinity, rather than grow towards infinity.

Outline of this paper. We start by defining equivariant localised maximal Roe algebras in Section 2. That allows us to state the three results in the paper mentioned above, in Section 3. Well-definedness of the index (1.2) is proved in Section 4. To prepare for the proof of (1.4), we construct several averaging maps in section 5. In Section 6, we use these maps to prove (1.4). We conclude this paper by using (1.4) and some localisation estimates to prove (1.5) in Section 7.

2. EQUIVARIANT LOCALISED MAXIMAL ROE ALGEBRAS

Throughout this paper, G will be a unimodular, locally compact group, with a Haar measure dg .

We assume that G admits a left-invariant distance function d_G for which there are $a, b > 0$ such that for all $r > 0$, any ball in G of radius r has volume at most ae^{br} . This is the case, for example, if the connected component $G_0 < G$ is a Lie group, and G/G_0 is finitely generated. See also [22] for general second-countable groups. This volume growth condition is used in the proof of Lemma 4.4, which is a step in the proof of Theorem 3.1, which in turn is the basis of the functional calculus of operators on Hilbert C^* -modules that we use.

2.1. Equivariant $C_0(X)$ -modules. Let (X, d) be a metric space in which all closed balls are compact. Suppose that G acts properly and isometrically on X .

A G -equivariant $C_0(X)$ -module is a Hilbert space \mathcal{H}_X equipped with a unitary representation π of G , and a $*$ -homomorphism $\rho: C_0(X) \rightarrow \mathcal{B}(\mathcal{H}_X)$, such that for all $g \in G$ and $\varphi \in C_0(X)$,

$$\pi(g)\rho(\varphi)\pi(g)^{-1} = \rho(g \cdot \varphi).$$

Here $(g \cdot \varphi)(x) = \varphi(g^{-1}x)$, for all $x \in X$. We will omit the representations π and ρ from the notation, and for example write $\varphi \cdot \xi := \rho(\varphi)\xi$, for $\varphi \in C_0(X)$ and $\xi \in \mathcal{H}_X$.

Fix a G -equivariant $C_0(X)$ -module \mathcal{H}_X . Let $\mathcal{B}(\mathcal{H}_X)^G$ be the algebra of G -equivariant bounded operators on \mathcal{H}_X . An operator $T \in \mathcal{B}(\mathcal{H}_X)$ is said to be *locally compact* if for all $\varphi \in C_0(X)$, the operators φT and $T\varphi$ are compact. And T has *finite propagation* if there is an $r > 0$ such that for all $\varphi, \psi \in C_0(X)$ whose supports are at least a distance r apart,

$$\varphi T \psi = 0.$$

In that case, the infimum of such numbers r is the *propagation* of T . The G -equivariant reduced Roe algebra of X with respect to \mathcal{H}_X is the closure in the operator norm of the algebra of locally compact operators in $\mathcal{B}(\mathcal{H}_X)^G$ with finite propagation. In this paper, we will use an algebra that differs from the equivariant reduced Roe algebra in two ways: we consider a *localised* version, and complete it in a *maximal* norm.

A relevant example of a G -equivariant $C_0(X)$ -module is the space $L^2(E)$ of square integrable sections of a G -equivariant, Hermitian vector bundle $E \rightarrow X$, with respect to a G -invariant measure dx on X . The algebra $C_0(X)$ acts on $L^2(E)$ by pointwise multiplication, and G acts in the usual way. Consider the vector bundle $\text{Hom}(E) := E \boxtimes E^* \rightarrow X \times X$. Let $C_{\text{ker}}^*(X; L^2(E))^G$ be the algebra of locally compact operators $T \in \mathcal{B}(L^2(E))^G$ with finite propagation, for which there is a bounded, measurable¹ section κ of $\text{Hom}(E)$ such that for all $s \in L^2(E)$ and

¹One can also work with continuous sections; the main reason we use measurable sections is that the map $\kappa \mapsto \tilde{\kappa}$ in Lemma 4.3 does not preserve continuity.

$x \in X$,

$$(Ts)(x) = \int_X \kappa(x, x')s(x') dx'.$$

We will identify such operators with their kernels κ .

2.2. Admissible modules and the maximal Roe algebra. In Definition 2.2 in [52], the notion of an *admissible* Γ -equivariant $C_0(X)$ -module was introduced, for discrete groups Γ . In Definition 2.4 in [20], this was extended to general unimodular, locally compact groups G , in the case where X/G is compact. The main difference between the discrete and general group case is the role played by local slices in the sense of Palais [40] in the non-discrete case.

Suppose that X/G is compact. A G -equivariant $C_0(X)$ -module \mathcal{H}_X is defined to be admissible if there is a G -equivariant, unitary isomorphism

$$\mathcal{H}_X \cong L^2(G) \otimes \mathcal{H},$$

for a Hilbert space \mathcal{H} , such that locally compact operators on \mathcal{H}_X are mapped to locally compact operators on $L^2(G) \otimes \mathcal{H}$, and operators with finite propagation are mapped to operators with finite propagation, in both cases with respect to the pointwise action by $C_0(G)$.

The point of using admissible modules is that the resulting equivariant Roe algebras encode the relevant group-theoretic information. It is clear that such information may be lost in the example where X is a point, acted on trivially by a compact group, and one uses the non-admissible module \mathbb{C} .

By Theorem 2.7 in [20], an example of an admissible module is $L^2(E) \otimes L^2(G)$. Here $E \rightarrow X$ is as at the end of the previous subsection, $C_0(X)$ acts pointwise on the factor $L^2(E)$, and G acts diagonally, with respect to the left regular representation of G in $L^2(G)$. By definition of admissibility, we have an isomorphism

$$(2.1) \quad L^2(E) \otimes L^2(G) \cong L^2(G) \otimes \mathcal{H}$$

with the properties above. Let $C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G$ be the algebra of G -equivariant, locally compact operators on $L^2(E) \otimes L^2(G)$ with finite propagation, given by bounded, measurable kernels

$$\kappa_G: G \times G \rightarrow \mathcal{K}(\mathcal{H})$$

via the isomorphism (2.1). Explicitly, for such a κ_G , the corresponding operator T is defined by

$$(T(f \otimes \xi))(g) = \int_G f(g')\kappa(g, g')\xi dg',$$

for $f \in L^2(G)$, $\xi \in \mathcal{H}$ and $g \in G$. If X/G is compact, then Theorem 2.11 in [20] states that $C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G$ is isomorphic to a dense subalgebra of $C^*(G) \otimes \mathcal{K}(\mathcal{H})$, where $C^*(G)$ is either the reduced or maximal group C^* -algebra of G . (To be precise, $C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G$ is isomorphic to the convolution algebra of compactly supported, bounded, measurable functions on G with values in the algebra of compact operators on \mathcal{H} .) This implies that the *maximal norm* of an element $\kappa \in C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G$,

$$(2.2) \quad \|\kappa\|_{\max} := \sup_{\eta} \|\eta(\kappa)\|_{\mathcal{B}(\mathcal{H}_{\eta})},$$

where the supremum is over all $*$ -representations

$$\eta: C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G \rightarrow \mathcal{B}(\mathcal{H}_{\eta}),$$

is finite. Then

$$(2.3) \quad \|\kappa\|_{\max} = \|\kappa_G\|_{C_{\max}^*(G) \otimes \mathcal{B}(\mathcal{H})},$$

so the completion of $C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G$ in the maximal norm equals

$$(2.4) \quad C_{\max}^*(X; L^2(E) \otimes L^2(G))^G \cong C_{\max}^*(G) \otimes \mathcal{K}(\mathcal{H}).$$

Since this algebra is independent of the admissible module used, we will denote it by

$$C_{\max}^*(X)^G := C_{\max}^*(X; L^2(E) \otimes L^2(G))^G.$$

Remark 2.1. In the case where G is trivial, and X is not assumed to be compact but is only assumed to have bounded geometry, finiteness of the maximal norm (2.2) was proved by Gong, Wang and Yu, see Lemma 3.4 in [16]. See also Lemma 1.10 in [48]. This generalises directly to free cocompact actions by discrete groups, see Lemma 3.16 in [16]. For the case of non-cocompact actions, see [21].

2.3. The map $\oplus 0$ and the maximal norm for non-admissible modules. We will use a completion of $C_{\ker}^*(X; L^2(E))^G$ in a version of the maximal norm. It is unclear a priori if an analogue of the supremum (2.2) is finite, however. We therefore define the norm we use via an embedding of $C_{\ker}^*(X; L^2(E))^G$ into $C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G$, which has a well-defined maximal norm if X/G is compact, as we saw at the end of the previous subsection.

Let $C_{\text{alg}}^*(X; \mathcal{H}_X)^G$ be the algebra of bounded, G -equivariant, locally compact operators on an equivariant $C_0(X)$ -module \mathcal{H}_X , with finite propagation. In Section 3.2 in [20], a map

$$(2.5) \quad \oplus 0: C_{\text{alg}}^*(X; L^2(E))^G \rightarrow C_{\text{alg}}^*(X; L^2(E) \otimes L^2(G))^G$$

is defined as follows. Let $\chi \in C(X)$ be a function whose support has compact intersections with all G -orbits, and has the property that for all $x \in X$,

$$(2.6) \quad \int_G \chi(gx)^2 dx = 1.$$

(The integrand is compactly supported by properness of the action.) We can and will choose such a function that is bounded above by 1. Such a function will be called a *cutoff function*. The map $j: L^2(E) \rightarrow L^2(E) \otimes L^2(G)$, given by

$$(2.7) \quad (j(s))(x, g) = \chi(g^{-1}x)s(x)$$

for $s \in L^2(E)$, $x \in X$ and $g \in G$, is an isometric, G -equivariant embedding. Let $p: L^2(E) \otimes L^2(G) \rightarrow j(L^2(E))$ be the orthogonal projection. The map

$$(2.8) \quad \oplus 0: \mathcal{B}(L^2(E)) \rightarrow \mathcal{B}(L^2(E) \otimes L^2(G))$$

that maps $T \in \mathcal{B}(L^2(E))$ to $jTj^{-1}p$ is an injective $*$ -homomorphism, and preserves equivariance, local compactness, and finite propagation. Hence it restricts to an injective $*$ -homomorphism (2.5). (The notation $\oplus 0$ reflects the fact that $T \oplus 0$ equals jTj^{-1} on the image of j , and zero on its orthogonal complement.)

Lemma 2.2. *The map (2.5) maps $C_{\ker}^*(X; L^2(E))^G$ into $C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G$.*

Proof. The key point is that (2.5) maps kernels with finite propagation in M to kernels with finite propagation in G ; that follows from the explicit expression for this map in Lemma 4.3. (See also (11) in [20].) \square

If X/G is compact, then for $\kappa \in C_{\ker}^*(X; L^2(E))^G$, we define its *maximal norm* as

$$(2.9) \quad \|\kappa\|_{\max} := \|\kappa \oplus 0\|_{\max}.$$

For different choices of j used in the definition of the map $\oplus 0$, the corresponding operators $\kappa \oplus 0$ are conjugate via isometries. This implies that the norm $\|\cdot\|_{\max}$ does not depend on the choice of j . We denote the completion of $C_{\ker}^*(X; L^2(E))^G$ in this norm by $C_{\max}^*(X; L^2(E))^G$.

Remark 2.3. On any $*$ -algebra, one can define a maximal norm analogous to (2.2), if this supremum is finite for all elements of the algebra. The norm (2.9) is *not* this maximal norm. The reason why we use the norm (2.9) instead of the generally defined maximal norm is that for the algebra $C_{\ker}^*(X; L^2(E))^G$, it does not seem obvious a priori if the supremum in (2.2) is finite. For free actions by discrete groups, this is shown in Lemmas 3.4 and 4.13 in [16]. Furthermore, the equality (2.4), as well as the key ingredient for our use of functional calculus on Hilbert C^* -modules, Theorem 3.1, are true for the norm (2.9).

Remark 2.4. In the case of reduced Roe algebras, defined with respect to the operator norm for a $C_0(X)$ -module, the algebra $C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G$ is dense in $C_{\text{alg}}^*(X; L^2(E) \otimes L^2(G))^G$. See Proposition 5.11 in [20]. In that case, kernels and operators can be used more or less interchangeably, but this is less clear for the maximal completions we use here.

2.4. Localised maximal Roe algebras. Let $Z \subset X$ be a G -invariant subset. Let \mathcal{H}_X be a G -equivariant $C_0(X)$ -module. An operator $T \in \mathcal{B}(\mathcal{H}_X)^G$ is *supported near Z* if there is an $r > 0$ such that for all φ whose support is at least a distance r away from Z , the operators φT and $T\varphi$ are zero. Let $C_{\ker}^*(X; Z, \mathcal{H}_X)^G$ be the algebra of elements of $C_{\ker}^*(X; \mathcal{H}_X)^G$ supported near Z .

For $r \geq 0$ and any subset $Y \subset X$, we write

$$\text{Pen}(Y, r) := \{x \in X; d(x, Y) \leq r\}.$$

Then we have a natural isomorphism

$$(2.10) \quad C_{\ker}^*(X; Z, \mathcal{H}_X)^G = \varinjlim_r C_{\ker}^*(\text{Pen}(Z, r); \mathcal{H}_X)^G.$$

Now suppose that Z/G is compact. The algebra $C_{\ker}^*(X; Z, \mathcal{H}_X)^G$ is then independent of Z , as long as Z/G is compact. For this reason, we write

$$C_{\ker}^*(X; \mathcal{H}_X)_{\text{loc}}^G := C_{\ker}^*(X; Z, \mathcal{H}_X)^G.$$

For every $r > 0$, we have the norm $\|\cdot\|_{\max}$ on $C_{\ker}^*(\text{Pen}(Z, r); L^2(E))^G$. Let $\|\cdot\|_{\max}$ be the resulting norm on $C_{\ker}^*(X, L^2(E))_{\text{loc}}^G$ via (2.10).

Definition 2.5. The *localised, G -maximal equivariant Roe algebra of X for $L^2(E)$* , denoted by $C_{\max}^*(X; L^2(E))_{\text{loc}}^G$, is the completion of $C_{\ker}^*(X, L^2(E))_{\text{loc}}^G$ in the norm $\|\cdot\|_{\max}$.

The *localised, G -maximal equivariant Roe algebra of X* , denoted by $C_{\max}^*(X)_{\text{loc}}^G$, is the completion of $C_{\ker}^*(X; L^2(E) \otimes L^2(G))_{\text{loc}}^G$ in the norm $\|\cdot\|_{\max}$.

By construction, $C_{\max}^*(X; L^2(E))_{\text{loc}}^G$ is isometrically embedded into $C_{\max}^*(X)_{\text{loc}}^G$. By (2.4) and (2.10),

$$C_{\max}^*(X)_{\text{loc}}^G \cong C_{\max}^*(G) \otimes \mathcal{K}(\mathcal{H}).$$

3. RESULTS

Our first result is the fact that a maximal version of the localised equivariant index of [20] is well-defined, see Theorem 3.1 and Proposition 3.3 and Definition 3.4. We will show that that index is an equivariant refinement of the index defined in terms of invariant sections in [9, 25, 37], see Theorem 3.9. The quantisation commutes with reduction results for proper, non-cocompact actions in [25, 26] only involved sections invariant under a group action. In Theorem 3.11, we generalise this to the equivariant index of Definition 3.4, in the case of Callias-type Spin^c -Dirac operators.

3.1. The localised maximal equivariant index. From now on, we suppose that $X = M$, a complete Riemannian manifold, and that d is the Riemannian distance corresponding to a G -invariant Riemannian metric. We suppose that $E \rightarrow M$ is a smooth, G -equivariant, Hermitian vector bundle and D a symmetric, first order, elliptic, G -equivariant differential operator on sections of E . Suppose that D has finite propagation speed, i.e. if σ_D is its principal symbol, then

$$\sup\{\|\sigma_D(\xi)\|; \xi \in T^*M, \|\xi\| = 1\} < \infty.$$

Then D is essentially self adjoint as an unbounded operator on $L^2(E)$, see Proposition 10.2.11 in [24].

Let $Z \subset M$ be a closed, cocompact G -invariant subset. Let $C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$ be the algebra of smooth kernels in $C_{\ker}^*(M; L^2(E))_{\text{loc}}^G$. Then D acts on $\kappa \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$ by

$$(D\kappa)(m, m') := (D \otimes 1_{E_m^*}(\kappa(-, m')))(m).$$

Here we used the fact that for every $m' \in M$, $\kappa(-, m')$ is a smooth section of $E \otimes E_{m'}^*$.

For A a C^* -algebra and \mathcal{M} a Hilbert A -module, we write $\mathcal{L}_A(\mathcal{M})$ and $\mathcal{K}_A(\mathcal{M})$ for the C^* -algebras of bounded adjointable operators and compact operators on \mathcal{M} , respectively. We can view A as a right Hilbert C^* -module over itself, with A -valued inner product

$$(3.1) \quad \langle a, b \rangle := a^*b,$$

for $a, b \in A$. Then $\mathcal{K}_A(A) \cong A$, with the isomorphism being given by identifying the operator

$$\theta_{a,b} : c \mapsto a\langle b, c \rangle$$

with left multiplication by ab^* . We also have that $\mathcal{L}_A(A)$ is the multiplier algebra of $\mathcal{K}_A(A)$.

To simplify notation, we will from now on use A to denote the G -maximal, localised equivariant Roe algebra $C_{\max}^*(M; L^2(E))_{\text{loc}}^G$. Then A is a Hilbert module over itself. We will use functional calculus for self-adjoint, regular operators on the Hilbert A -module A . (For a uniform version of the maximal Roe algebra, this was developed in [21].) This functional calculus applies to D because of the following result.

Theorem 3.1. *The unbounded operator D on the Hilbert A -module A is essentially self-adjoint and regular.*

This theorem is proved in Subsection 4.6. Because of Theorem 3.1, we can apply the following general result (see [33], [23] Theorem 3.1 and [15] Theorem 1.19) about functional calculus on Hilbert C^* -modules to the self-adjoint closure of D .

Theorem 3.2. *Let B be a C^* -algebra and \mathcal{M} a Hilbert B -module. Let $C(\mathbb{R})$ be the $*$ -algebra of complex-valued continuous functions on \mathbb{R} . For any regular, essentially self-adjoint operator T on \mathcal{M} , there is a $*$ -preserving linear map*

$$\pi_T : C(\mathbb{R}) \rightarrow \mathcal{R}_B(\mathcal{M}),$$

with values in the set $\mathcal{R}_B(\mathcal{M})$ of regular operators on \mathcal{M} , such that:

- (i) π_T restricts to a $*$ -homomorphism $\pi_T : C_b(\mathbb{R}) \rightarrow \mathcal{L}_B(\mathcal{M})$;
- (ii) If $|f(t)| \leq |g(t)|$ for all $t \in \mathbb{R}$, then $\text{dom}(\pi_T(g)) \subseteq \text{dom}(\pi_T(f))$;
- (iii) If $(f_n)_{n \in \mathbb{N}}$ is a sequence in $C(\mathbb{R})$ for which there exists $F \in C(\mathbb{R})$ such that $|f_n(t)| \leq |F(t)|$ for all $t \in \mathbb{R}$, and if f_n converge to a limit function $f \in C(\mathbb{R})$ uniformly on compact subsets of \mathbb{R} , then $\pi_T(f_n)x \mapsto \pi_T(f)x$ for each $x \in \text{dom}(\pi_T(f))$;
- (iv) $\pi_T(\text{Id}) = T$;
- (v) $\|\pi_T(f)\|_{\mathcal{L}_B(\mathcal{M})} \leq \sup_{\lambda \in \text{spec}_{\mathcal{M}}(T)} |f(\lambda)|$;

In the context of this theorem, we write $f(T) := \pi_T(f)$.

Suppose that there are a G -equivariant, Hermitian vector bundle $F \rightarrow M$, a differential operator $P : \Gamma^\infty(E) \rightarrow \Gamma^\infty(F)$, a G -equivariant vector bundle endomorphism R of E , and a constant $c > 0$ such that

$$(3.2) \quad D^2 = P^*P + R,$$

and $R \geq c^2$, fibrewise outside Z . (The use of c^2 instead of c is a convention here, which implies that $D \geq c$ outside Z in an appropriate sense.)

In this setting, and when G is trivial but without assuming Z to be compact, Roe [45] developed localised index theory with values in the K -theory of a reduced completion of $C_{\text{alg}}^*(M; Z, L^2(E))$. We will use an equivariant version of this index theory for the maximal completion, in terms of admissible modules. The reason for using the maximal completion is that we then obtain an index in the K -theory of $C_{\text{max}}^*(G)$, to which we can apply an integration map to recover the G -invariant index from [25] as a special case, see Theorem 3.9. The construction of the localised index is based on the following analogue of Lemma 2.3 and Theorem 2.4 in [45].

Proposition 3.3. *If $f \in C_c(\mathbb{R})$ is supported in $[-c, c]$, then*

$$f(D) \in A = \mathcal{K}_A(A) \subset \mathcal{L}_A(A).$$

This proposition is proved in Subsection 4.7

Let $b : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous, increasing, odd function, such that $b(x) = \pm 1$ for all $x \in \mathbb{R}$ with $|x| \geq c$. Then $b^2 - 1$ has the property of the function f in Proposition 3.3. So, in particular, $b(D) \in \mathcal{L}_A(A)$ is invertible modulo $\mathcal{K}_A(A)$, and hence has an index in

$$K_*(\mathcal{K}_A(A)) = K_*(A).$$

This index lies in even K -theory if D is odd with respect to a G -invariant grading on E , and in odd K -theory otherwise. See for example Definition 3.2 in [20] for details.

Explicitly, consider the case where D is odd with respect to a G -invariant grading

$$E = E_+ \oplus E_-.$$

Let $C_{\ker}^{\infty}(X; L^2(E))_{\text{loc}}^G$ be the algebra of kernels in $C_{\ker}^{\infty}(X; L^2(E))^G$ supported near Z . Let $b(D)_+$ be the restriction of $b(D)$ to kernels in $C_{\ker}^{\infty}(X; L^2(E))_{\text{loc}}^G$ that are sections of $E_+ \otimes E^*$. Then $b(D)_+$ is invertible modulo $\mathcal{K}_A(A)$, and its inverse is the restriction of $b(D)$ to $E_- \otimes E^*$. Hence this operator defines a class $[b(D)_+] \in K_1(\mathcal{L}_A(A)/\mathcal{K}_A(A))$, and the index of $b(D)$ is defined as

$$(3.3) \quad \partial[b(D)_+] \in K_0(A),$$

where $\partial: K_1(\mathcal{L}_A(A)/\mathcal{K}_A(A)) \rightarrow K_0(\mathcal{K}_A(A))$ is the boundary map in the six-term exact sequence corresponding to the ideal $\mathcal{K}_A(A) \subset \mathcal{L}_A(A)$. For ungraded operators, one uses the projection $\frac{1}{2}(b(D) + 1)$ in $\mathcal{L}_A(A)/\mathcal{K}_A(A)$ and applies the boundary map to its class in even K -theory to obtain the index of $b(D)$ in $K_1(A)$.

Definition 3.4. The *localised, maximal, equivariant index* of D is the image of the index of $b(D)$ in $K_*(A)$ described above under the map

$$\oplus 0: K_*(A) \rightarrow K_*(C_{\max}^*(G)).$$

It is denoted by $\text{index}_G^{\text{loc}}(D)$.

Remark 3.5. One could consider (3.3) (and its analogue in $K_1(A)$ in the non-graded case) as a localised index of D , defined in terms of the non-admissible $C_0(M)$ -module $L^2(E)$. Two advantages of the index in Definition 3.4 over (3.3) are that it takes values in a K -theory group independent of X or E , and that the application of the map $\oplus 0$ on K -theory means that the index of Definition 3.4 captures group-theoretic information that is not encoded in (3.3). This is clear in the example where G is compact, M is a point, and D is the zero operator on $E = V \in \hat{K}$, as discussed in Example 3.8 in [20]. This illustrates why it is useful to use the admissible $C_0(M)$ -module $L^2(E) \otimes L^2(G)$.

Another approach to constructing the index of Definition 3.4 would be to use an extension map $\oplus 1$, extending operators by the identity operator on the orthogonal complement to $j(L^2(E))$, before applying boundary maps. See Definition 3.6 and Lemma 3.7 in [20].

Example 3.6. If D is a Dirac-type operator associated to a Clifford connection ∇ on E , then

$$D^2 = \nabla^* \nabla + R,$$

for a vector bundle endomorphism R of E . (If D is a Spin-Dirac operator, then R is scalar multiplication by a quarter of scalar curvature, by Lichnerowicz' formula.) If $R \geq c^2$ outside Z , then the condition on D holds, with $F = E \otimes T^*M$ and $P = \nabla$. This is the situation considered in [45], for G trivial.

Example 3.7. Let \tilde{D} be a G -equivariant Dirac operator on E , and let Φ be a G -equivariant vector bundle endomorphism of E . Suppose that $\{\tilde{D}, \Phi\} := \tilde{D}\Phi + \Phi\tilde{D}$ is a vector bundle endomorphism of E , and that

$$(3.4) \quad \{\tilde{D}, \Phi\} + \Phi^2 \geq c^2$$

fibrewise outside Z . Then $D := \tilde{D} + \Phi$ satisfies the conditions on D as above, with $F = E$, $P = \tilde{D}$ and $R = \{\tilde{D}, \Phi\} + \Phi^2$.

This type of operator is a *Callias-type operator*. Indices of Callias-type operators equivariant under proper actions were studied in [19, 20].

The main application of the maximal localised index in this paper, Theorem 3.11, is about the maximal localised index of Callias-type operators.

3.2. The invariant index. Integrating L^1 -functions over G extends to a trace on $C_{\max}^*(G)$. We will see in Theorem 3.9 that applying this trace to the localised index of D recovers an index defined in terms of G -invariant sections in [25]. This fact will be used in the proof of Theorem 3.11. It can also be used to obtain refined index theoretic information on non-compact manifolds; see Remark 3.10.

Let $\chi \in C^\infty(M)$ a function with the property (2.6). Consider the space $\Gamma_{\text{tc}}(E)^G$ of transversally compactly supported sections of E , defined as the space of continuous, G -invariant sections of E whose supports have compact images in M/G under the quotient map. The Hilbert space $L_T^2(E)^G$ of G -invariant, transversally L^2 -sections of E is the completion of $\Gamma_{\text{tc}}(E)^G$ in the inner product

$$(s_1, s_2)_{L_T^2(E)^G} := (\chi s_1, \chi s_2)_{L^2(E)}.$$

The space $L_T^2(E)^G$ is independent of the choice of χ ; see Lemma 4.4 in [25].

Suppose that D is odd with respect to a G -invariant grading $E = E_+ \oplus E_-$. In Proposition 4.7 in [25], it is shown that D defines a Fredholm operator \bar{D} from a suitable Sobolev space inside $L_T^2(E)^G$ into $L_T^2(E)^G$. In Proposition 4.8 in the same paper, it is deduced that the space

$$\ker(D) \cap L_T^2(E)^G$$

is finite-dimensional, and that the index of \bar{D} equals

$$(3.5) \quad \dim(\ker(D) \cap L_T^2(E_+)^G) - \dim(\ker(D) \cap L_T^2(E_-)^G).$$

Definition 3.8. The G -invariant index of D , denoted by $\text{index}(D)^G$, is the number (3.5).

In [9], Braverman further develops the theory of this index, when applied to Dirac operators with an added zero-order term that is relevant to geometric quantisation, and in particular proves that it is invariant under a suitable notion of cobordism.

The map from $L^1(G)$ to \mathbb{C} given by integrating functions over G extends continuously to a $*$ -homomorphism, or a trace

$$I: C_{\max}^*(G) \rightarrow \mathbb{C}.$$

The integer

$$I_*(\text{index}_G^{\text{loc}}(D)) \in K_0(\mathbb{C}) = \mathbb{Z}$$

plays the role of the G -invariant part of the localised index of D , and this will be made precise in Theorem 3.9 below.

If M/G is compact, then all smooth sections of E are transversally L^2 . Then the G -invariant index of D equals

$$\dim(\ker(D_+)^G) - \dim(\ker(D_-)^G),$$

where D_\pm is the restriction of D to sections of E_\pm . This index was developed and applied by Mathai and Zhang in [37], with an appendix by Bunke. In Theorem 2.7 and Proposition D.3 in that paper, it is shown that the index can be recovered from the equivariant index of D in $K_0(C_{\max}^*(G))$, defined via the analytic assembly map, if one applies the integration trace I . We will show that this generalises to the index in Definition 3.4 in the non-cocompact case.

Theorem 3.9. *We have*

$$(3.6) \quad I_*(\text{index}_G^{\text{loc}}(D)) = \text{index}(D)^G \in \mathbb{Z}.$$

This theorem is proved in Section 6.

Remark 3.10. Theorem 3.9 allows us to construct a more refined invariant of operators that are invertible at infinity in the non-equivariant case than their Fredholm index. Suppose that M is the universal cover of a manifold \bar{M} and that $G = \Gamma$ is the fundamental group of \bar{M} , acting on M in the natural way. Let \bar{D} be an elliptic operator on \bar{M} that is invertible at infinity in the appropriate sense, so that it lifts to a Γ -equivariant operator on M satisfying the conditions of Theorem 3.9. That theorem then implies that

$$(3.7) \quad I_*(\text{index}_\Gamma^{\text{loc}}(D)) = \text{index}(\bar{D}).$$

In this sense, $\text{index}_\Gamma^{\text{loc}}(D)$ refines the Fredholm index of \bar{D} , much like the image of D under the analytic assembly map for the maximal group C^* -algebra refines the Fredholm index of \bar{D} if \bar{M} is compact. This can for example be used to obtain stronger obstructions to Riemannian metrics of positive scalar curvature than classical obstructions from Callias index theory [3] or the Gromov–Lawson index [17].

In the compact case, one can use the assembly map for the reduced group C^* -algebra here, and use the von Neumann trace to recover the index of \bar{D} by Atiyah’s L^2 -index theorem [4]. This is only possible because the action is free and the group is discrete. In the noncompact case, an analogue of this for the reduced localised index is not yet available. Moreover, Theorem 3.9 does not rely on discreteness of the group acting or freeness of the action, which leads to refined obstructions to positive scalar curvature on possibly noncompact orbifolds (refining Kawasaki’s orbifold index, in the compact case) and more generally to metrics of positive scalar curvature invariant under a proper group action (refining the invariant index of Definition 3.8).

Concrete applications to positive scalar curvature will be explored in future work.

Proposition 2.4 in [27] shows that the index defined in [28] is another refinement of the invariant index. That index applies to Dirac operators with certain deformation terms added that are relevant to geometric quantisation. It takes values in the completion of the representation ring of a maximal compact subgroup of the group acting.

3.3. Callias quantisation commutes with reduction. In [25, 26], the *quantisation commutes with reduction* principle of Guillemin and Sternberg [18, 38, 39, 41, 47] and its Spin^c -version [42] is generalised to proper actions by possibly noncompact groups, with possibly noncompact orbit spaces, for suitably high powers of the prequantum or determinant line bundle in question. These results in [25, 26] are stated in terms of the invariant index of Definition 3.8. The result in [25] in the symplectic setting generalises the result in [37] from compact to noncompact orbit spaces. This is generalised to the Spin^c -setting in [26].

These were the first results on a version of the quantisation commutes with reduction principle where both the group and orbit space were allowed to be noncompact, but two drawbacks were that the invariant index used

- (1) was only well-defined after a deformation term (Clifford multiplication by the Kirwan vector field) was added to the relevant Dirac operator;
- (2) only involved G -invariant sections, and therefore provided no information about the parts of the kernel of D on which G acts nontrivially.

The second point was partially addressed in Theorem 2.13 in [27], a quantisation commutes with reduction result for non-cocompact actions, where quantisation takes values in the completion of the representation ring of a maximal compact subgroup.

We are now able to remedy both points, in the case of Callias-type Spin^c -Dirac operators.

Let $D = D_\Phi$ be a Callias-type operator as in Example 3.7. Suppose that E has a G -invariant $\mathbb{Z}/2$ grading, and that \tilde{D} and Φ , and hence D , are odd for this grading. Suppose that $E = \mathcal{S}$ is the spinor bundle of a G -equivariant Spin^c -structure on M , and let $L \rightarrow M$ be its determinant line bundle. (The assumption that E is $\mathbb{Z}/2$ -graded now means that M is even-dimensional.) Suppose that \tilde{D} is a Spin^c -Dirac operator on \mathcal{S} . The Clifford connection on \mathcal{S} used to define \tilde{D} can be constructed locally from a G -invariant, Hermitian connection ∇^L on L and the connection on the spinor bundle for a local Spin -structure; see e.g. Proposition D.11 in [35]. This also induces a Clifford connection on the spinor bundle $\mathcal{S} \otimes L^p$, for any $p \in \mathbb{Z}_{\geq 0}$. Let \tilde{D}^{L^p} be the corresponding Spin^c -Dirac operator on $\mathcal{S} \otimes L^p$. Set

$$D_p := \tilde{D}^{L^p} + \Phi \otimes 1_{L^p}.$$

We have

$$(3.8) \quad \{\tilde{D}^{L^p}, \Phi \otimes 1_{L^p}\} = \{\tilde{D}, \Phi\} \otimes 1_{L^p},$$

where $\{-, -\}$ denotes the anticommutator. In what follows, we will omit ‘ $\otimes 1_{L^p}$ ’ from the notation. By (3.4) and (3.8), we have

$$(3.9) \quad \{\tilde{D}^{L^p}, \Phi\} + \Phi^2 \geq c^2$$

outside Z , for all p . Hence D_p has an index

$$(3.10) \quad \text{index}_G^{\text{loc}}(D_p) \in K_0(C_{\max}^*(G)).$$

The quantisation commutes with reduction principle in general is an equality between the invariant part of the equivariant index of a Spin^c -Dirac operator and the index of a Dirac operator localised at the level set of a *moment map*. The invariant part of the index will now be represented by the image of (3.10) under the integration trace I .

The Spin^c -*moment map* associated to ∇^L is the map

$$\mu: M \rightarrow \mathfrak{g}^*$$

such that for all $X \in \mathfrak{g}$,

$$2\pi i \langle \mu, X \rangle = \mathcal{L}_X - \nabla_X^L, \quad \in \text{End}(L) = C^\infty(M, \mathbb{C}),$$

where \mathcal{L}_X denotes the Lie derivative with respect to X , and X^M is the vector field induced by X . Our sign convention is that for all $X \in \mathfrak{g}$ and $m \in M$,

$$(3.11) \quad X^M(m) = \left. \frac{d}{dt} \right|_{t=0} \exp(-tX) \cdot m.$$

Suppose that 0 is a regular value of μ , and that G acts freely on $\mu^{-1}(0)$. Suppose that the *reduced space*

$$M_0 := \mu^{-1}(0)/G$$

is compact. In Lemma 3.3 in [26], a condition is given for M_0 to inherit a Spin^c -structure from M , with determinant line bundle L_0^p , with $L_0 = (L|_{\mu^{-1}(0)})/G \rightarrow M_0$. This is true for example if G is semisimple, see Proposition 3.5 and Example 3.6

in [26]. It is also true in the symplectic setting, where the Spin^c -structure on M is associated to a G -invariant almost complex structure compatible with a G -invariant symplectic form, together with a G -equivariant, Hermitian line bundle on M . From now on, we assume such a Spin^c -structure on M_0 exists. Let $D_{M_0}^{L^p}$ be a Spin^c -Dirac operator on M_0 for this Spin^c -structure.

Theorem 3.11 (Callias Spin^c -quantisation commutes with reduction). *There is a $p_0 \in \mathbb{Z}_{\geq 0}$ such that for all $p \geq p_0$,*

$$I_*(\text{index}_G(D_p)) = \text{index}(D_{M_0}^{L^p}) = \int_{M_0} \hat{A}(M_0) e^{\frac{p}{2}c(L_0)}.$$

This theorem is proved in Section 7.

Remark 3.12. In the case of Callias-type operators $D = \tilde{D} + \Phi$, as in Example 3.7, an index in $K_0(C^*(G))$ was constructed directly in [19]. Here $C^*(G)$ can be either the reduced or maximal group C^* -algebra. Let us denote this index by

$$\text{index}_G^C(\tilde{D} + \Phi) \in K_0(C^*(G)).$$

Theorem 4.2 in [20] states that, for the reduced group C^* -algebra and Roe algebra, this index of Callias-type operators is a special case of the localised index:

$$\text{index}_G^C(\tilde{D} + \Phi) = \text{index}_G^{\text{loc}}(\tilde{D} + \Phi) \in K_0(C_{\text{red}}^*(G)).$$

Via analogous arguments, one can show that this equality still holds for the maximal group C^* -algebra and Roe algebra. Then Theorem 3.11 implies that, under the conditions in that theorem,

$$I_*(\text{index}_G^C(\tilde{D}_p + \Phi)) = \text{index}(D_{M_0}^{L^p}).$$

Remark 3.13. As far as we are aware, Theorem 3.11 was not known in the case where G is compact, so that D is Fredholm in the classical sense. In that case, by the standard shifting trick (see for example Corollary 1.2 in [38], and [42]), Theorem 3.11 implies expressions for the multiplicities of all irreducible representations of G in the equivariant index of D_p . One can handle cases where a reduced space $\mu^{-1}(\text{Ad}^*(G)\xi)/G$ is not smooth by

- (1) using orbifold line bundles and indices if ξ is a regular value of μ in an appropriate sense;
- (2) using reduced spaces at nearby regular values if ξ is a singular value of μ , as in [39, 41, 42]. An alternative approach is developed in [36].

4. REGULARITY AND LOCALISATION

In this section, we prove Theorem 3.1 and Proposition 3.3, which imply that the localised maximal index of Definition 3.4 is well-defined.

4.1. Unbounded operators on maximal operator modules. Our first goal is to make sense of D as an unbounded, regular, essentially self-adjoint operator on certain maximal operator modules that we now introduce.

Let M_1 and M_2 be Riemannian manifolds equipped with proper, isometric G -actions. Let E_1 and E_2 be Hermitian G -vector bundles over M_1 and M_2 respectively. Consider the vector bundle $\text{Hom}(E_2, E_1) := E_1 \boxtimes E_2^* \rightarrow M_1 \times M_2$.

Definition 4.1. Denote by

$$\mathcal{H}_{\ker}^\infty(E_1, E_2)_{\text{loc}}^G \subset \Gamma^\infty(\text{Hom}(E_2, E_1))$$

the \mathbb{C} -vector space of smooth G -invariant, cocompactly supported, finite propagation kernels. Here we say that a kernel κ has cocompact support if there exists cocompact subsets U_1 and U_2 of M_1 and M_2 respectively such that $\text{supp}(\kappa) \subseteq U_1 \times U_2$.

There is no natural product or $*$ -operation on $\mathcal{H}_{\ker}^\infty(E_1, E_2)_{\text{loc}}^G$. However, it admits a natural action of $C_{\ker}^\infty(M_2; L^2(E_2))_{\text{loc}}^G$ from the right, given by composition of kernels. Further, it has a $C_{\ker}^\infty(M_2; L^2(E_2))_{\text{loc}}^G$ -valued inner product given by

$$\langle \kappa, \kappa' \rangle := \kappa^* \kappa',$$

for $\kappa, \kappa' \in \mathcal{H}_{\ker}^\infty(E_1, E_2)_{\text{loc}}^G$, defined through the usual adjoint and multiplication of kernels. This makes $\mathcal{H}_{\ker}^\infty(E_1, E_2)_{\text{loc}}^G$ a pre-Hilbert $C_{\ker}^\infty(M_2; L^2(E_2))_{\text{loc}}^G$ -module.

Now taking the simultaneous completions of

$$\mathcal{H}_{\ker}^\infty(E_1, E_2)_{\text{loc}}^G \quad \text{and} \quad C_{\ker}^\infty(M_2; L^2(E_2))_{\text{loc}}^G$$

(see p. 5 of [34]) gives a Hilbert $C_{\max}^*(M_2; L^2(E))_{\text{loc}}^G$ -module that we denote by $\mathcal{H}_{\max}(E_1, E_2)_{\text{loc}}^G$.

In the case $M_1 = M_2 = M$ and $E_1 = E_2 = E$, equipped with the same G -action, then $\mathcal{H}_{\ker}^\infty(E, E)_{\text{loc}}^G = C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$. In this case $\mathcal{H}_{\max}(E, E)_{\text{loc}}^G$ is the G -maximal equivariant localised Roe algebra $C_{\max}^*(M; L^2(E))_{\text{loc}}^G$ of Definition 2.5.

To introduce the idea of an unbounded operator on these operator modules, let us first consider $C_{\max}^*(M; L^2(E))_{\text{loc}}^G$ as a right Hilbert module over itself, as in Subsection 3.1. We will consider D as an unbounded, densely defined operator on this Hilbert module. Note that D acts naturally on smooth Schwartz kernels via differentiation on the first coordinate, so it defines a map

$$D : C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G \rightarrow C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G.$$

More generally, we may consider the situation when M_1 and M_2 are proper G -manifolds, and E_1, E_2 are Hermitian G -vector bundles over M_1 and M_2 respectively. Let D be a symmetric, G -equivariant differential operator acting on a E_1 . Then D defines an unbounded, symmetric operator

$$D : \mathcal{H}_{\max}(E_1, E_2)_{\text{loc}}^G \rightarrow \mathcal{H}_{\max}(E_1, E_2)_{\text{loc}}^G$$

with initial domain $\mathcal{H}_{\ker}^\infty(E_1, E_2)_{\text{loc}}^G$.

4.2. Kernels on $M \times M$ and on $G \times G$. Via the isomorphism (2.1), operators on $L^2(E)$ with smooth kernels define kernels on $G \times G$, in a way that we make explicit in Lemma 4.3. Using that lemma, we obtain estimates for certain kernels on $G \times G$ in Subsection 4.3, which are then used to prove the case of Theorem 3.1 for cocompact actions, Proposition 4.9, in Subsection 4.5.

In order to make estimates in the norm on $C_{\max}^*(M; L^2(E))_{\text{loc}}^G$, we need to work with the maximal norm on the Roe algebra of the corresponding admissible module, namely $L^2(E) \otimes L^2(G)$. Thus let j and p be the inclusion and projection maps defined in Subsection 2.3, and consider the map $\oplus 0$ as in (2.8). Let

$$(4.1) \quad \Psi : L^2(G) \otimes \mathcal{H} \rightarrow L^2(E) \otimes L^2(G)$$

be the G -equivariant unitary isomorphism as in (2.1), as defined in (21) in [20] and reviewed below.

We assume for the rest of this subsection that $M = G \times_K N$ for a single slice $N \subset M$. We then have $\mathcal{H} = L^2(K \backslash G) \otimes L^2(E|_N)$, and the isomorphism Ψ is pullback along a G -equivariant, measure preserving bijection

$$\psi: M \times G \rightarrow G \times K \backslash G \times N.$$

We will use the explicit form of this bijection from Lemma 5.2 in [20]: for a fixed Borel section $\phi: K \backslash G \rightarrow G$, and for $g, h \in G$ and $y \in N$,

$$\psi(gy, n) = (h\phi(Kg^{-1}h)^{-1}, Kg^{-1}h, \phi(Kg^{-1}h)h^{-1}gy).$$

We will use an expression for the inverse of this map.

Lemma 4.2. *There is a measurable map $\eta: G \times K \backslash G \times N \rightarrow K$ such that for all $x_1, x_2 \in G$ and $n \in N$,*

$$\psi^{-1}(x_1, Kx_2, n) = (x_1n, x_1\eta(x_1, Kx_2, n)x_2).$$

Proof. Let $x_1, x_2, g, h \in G$ and $n, y \in N$, and suppose that

$$\psi(gy, h) = (x_1, Kx_2, n).$$

Then the elements

$$\begin{aligned} k &:= \phi(Kg^{-1}h)h^{-1}g \\ k' &:= x_2h^{-1}g \end{aligned}$$

lie in K . They depend measurably on g, h and x_2 , and hence on x_1, Kx_2 and n because ψ and its inverse are measurable. One then finds that $gy = x_1n$ and $h = x_1kk'^{-1}x_2$, so the claim follows. \square

For any $T \in \mathcal{B}(L^2(E))$, the operator \tilde{T} on $L^2(G) \otimes \mathcal{H}$ corresponding to $T \oplus 0$ can be written as

$$(4.2) \quad \tilde{T} := \Psi^{-1}(T \oplus 0)\Psi = \Psi^{-1} \circ j \circ T \circ j^{-1} \circ p \circ \Psi.$$

We use the same notation for Schwartz kernels of such operators.

Lemma 4.3. *Let η be as in Lemma 4.2. Then for all $\kappa \in \Gamma(\text{Hom}(E))^G$ defining a bounded operator on $L^2(E)$, and all $g, g', h, h' \in G$ and $y, y' \in N$,*

$$\tilde{\kappa}(g, Kh, y; g', Kh', y') = \chi(h^{-1}\eta(g, Kh, y)y)\chi(\phi(Kh')^{-1}y')\kappa(gy, g'y').$$

Proof. By (4.15) in [30] (or a direct check), we have for all $\zeta \in L^2(E) \otimes L^2(G)$ and $m \in M$,

$$(j^{-1} \circ p)(\zeta)(m) = \int_G \chi(g^{-1}m)\zeta(m, g) dg.$$

Using this equality, the definition $\Psi = \psi^*$, the definition (2.7) of j and the definition (4.2) of the tilde operation, we find that for all $\varphi \in L^2(G) \otimes L^2(K \backslash G) \otimes L^2(E|_N)$ and all $g, h \in G$ and $y \in N$,

$$(4.3) \quad (\tilde{\kappa}\varphi)(\psi(gy, h)) = \chi(h^{-1}gy) \int_G \int_G \int_N \chi(g''^{-1}g'y')\kappa(gy, g'y')\varphi(\psi(g'y', g'')) dy' dg' dg''.$$

Substituting

$$\begin{aligned} g_1 &= g'^{-1}g'' \quad \text{for } g', \\ g_2 &= g''\phi(Kg'^{-1}g'')^{-1} \quad \text{for } g'', \text{ and} \\ y_1 &= \phi(Kg'^{-1}g'')g''^{-1}g'y' \quad \text{for } y', \end{aligned}$$

and using unimodularity of G , we compute that the right hand side of (4.3) equals

$$\chi(h^{-1}gy) \int_G \int_G \int_N \chi(\phi(Kg_1)^{-1}y_1)\kappa(gy, g_2y_1)\varphi(g_2, Kg_1, y_1) dy_1 dg_1 dg_2.$$

So for all $g, h, g_1, g_2 \in G$ and $y, y_1 \in N$,

$$\tilde{\kappa}(\psi(gy, h); g_2, Kg_1, y_1) = \chi(h^{-1}gy)\chi(\phi(Kg_1)^{-1}y_1)\kappa(gy, g_2y_1).$$

The claim now follows from Lemma 4.2. \square

4.3. Estimates for kernels on $G \times G$. We will use Lemma 4.3 to obtain estimates (Lemmas 4.4 and 4.5) for certain kernels $\tilde{\kappa}$ that are used in the proof of Proposition 4.9. In Lemma 4.3, it was assumed that $M = G \times_K N$ for a single slice N . That is true for almost connected G [1], but for general cocompact M , one needs a finite set of slices [40]. We state Lemmas 4.4 and 4.5 in that more general setting. In their proofs, we consider the case of a single slice first, and discuss how that implies the more general case.

Suppose that M/G is compact. Then by Palais' slice theorem [40], there are finitely many compact subgroups $K_j < G$ and relatively compact submanifolds $N_j \subset M$ such that

- (1) for each j , the open set $U_j := G \cdot N_j$ is G -equivariantly diffeomorphic to $G \times_{K_j} N_j$,
- (2) the union of the closures of the sets U_j is all of M , and
- (3) the closures of any two of the sets U_j intersect in a set of measure zero.

Then the Hilbert space \mathcal{H} in (4.1) may be taken to be

$$\mathcal{H} = \bigoplus_j L^2(K_j \backslash G) \otimes L^2(E|_{N_j}),$$

and the map Ψ in (4.1) is defined in terms of the decomposition $L^2(E) = \bigoplus_j L^2(E|_{U_j})$, and maps

$$\Psi_j: L^2(G) \otimes L^2(E|_{N_j}) \rightarrow L^2(E|_{U_j}) \otimes L^2(G)$$

defined as in the case of a single slice.

As before, a G -equivariant bounded operator defines operators $T \oplus 0$ on $L^2(E) \otimes L^2(G)$ and $\tilde{T} = \Psi^{-1}(T \oplus 0)\Psi$ on $L^2(G) \otimes \mathcal{H}$. If such an operator T is defined by a smooth (or measurable) kernel $\kappa \in \Gamma(\text{Hom}(E))^G$, then \tilde{T} is defined by a measurable kernel $\tilde{\kappa}$. For all $g, g' \in G$, the operator $\tilde{\kappa}(g, g')$ on \mathcal{H} is given by a kernel

$$\tilde{\kappa}(g, g') \in \bigoplus_{j,k} \Gamma(\text{Hom}(K_j \backslash G \times E|_{N_j}, K_k \backslash G \times E|_{N_k})).$$

Its component $\tilde{\kappa}_{j,k}(g, g')$ in $\Gamma(\text{Hom}(K_j \backslash G \times E|_{N_j}, K_k \backslash G \times E|_{N_k}))$ is a section of the vector bundle

$$\text{Hom}(K_j \backslash G \times E|_{N_j}, K_k \backslash G \times E|_{N_k}) \rightarrow (K_j \backslash G \times N_j) \times (K_k \backslash G \times N_k).$$

Whenever the following expressions converge, we set

$$\|\tilde{\kappa}_{j,k}(g, g')\|_{2,\infty} := \sup_{(K_k h', y') \in K_k \backslash G \times N_k} \left(\int_{K_j \backslash G \times N_j} \|\tilde{\kappa}_{j,k}(g, g')(K_j h, y; K_k h', y')\|^2 d(K_j h) dy \right)^{1/2};$$

$$\|\tilde{\kappa}\|_{L^1(G), 2,\infty} := \sum_{j,k} \int_G \|\tilde{\kappa}_{j,k}(e, g)\|_{2,\infty} dg.$$

The norm in the integrand in the top line is the operator norm on $\text{Hom}(E_{y'}, E_y)$.

Lemma 4.4. *Suppose that $\kappa_0 \in \Gamma^\infty(\text{Hom}(E))^G$ has finite propagation and defines a bounded operator on $L^2(E)$. For $\mu > 0$, let κ be the kernel of the composition $(D + i\mu)^{-1} \circ \kappa_0$ of bounded operators on $L^2(E)$. Then for μ large enough, the norm $\|\tilde{\kappa}\|_{L^1(G), 2,\infty}$ is well-defined and finite.*

Let $\sigma: M \times M \rightarrow \text{End}(E)$ be bounded, and G -invariant in the sense that for all $g \in G$ and $m, m' \in M$,

$$\sigma(gm, gm') = g\sigma(m, m')g^{-1}.$$

Then for all $\kappa \in \Gamma(\text{Hom}(E))^G$, we define the section $\sigma\kappa \in \Gamma(\text{Hom}(E))^G$ by

$$(4.4) \quad (\sigma\kappa)(m, m') = \sigma(m, m')_m \circ \kappa(m, m'),$$

for $m, m' \in M$. Here $\sigma(m, m')_m \in \text{End}(E_m)$ is the value of $\sigma(m, m') \in \text{End}(E)$ at m . We will identify the element $\tilde{\sigma\kappa} \in \mathcal{B}(L^2(G) \otimes \mathcal{H})$ with the map $G \rightarrow \mathcal{B}(\mathcal{H})$ defined by $g \mapsto \tilde{\sigma\kappa}(e, g)$.

Lemma 4.5. *Suppose that $\kappa \in \Gamma(\text{Hom}(E))^G$ is such that $\|\tilde{\kappa}\|_{L^1(G), 2,\infty}$ is finite, and suppose that $\sigma: M \times M \rightarrow \text{End}(E)$ is bounded and G -invariant. Then $\tilde{\sigma\kappa} \in L^1(G) \otimes \mathcal{B}(\mathcal{H})$, and*

$$\|\tilde{\sigma\kappa}\|_{L^1(G) \otimes \mathcal{B}(\mathcal{H})} \leq \|\sigma\|_\infty \|\tilde{\kappa}\|_{L^1(G), 2,\infty}.$$

4.4. Proofs of Lemmas 4.4 and 4.5. We first prove Lemma 4.5. This is based on the following two lemmas.

Lemma 4.6. *Let $\tau \in \Gamma(\text{Hom}(K_j \backslash G \times E|_{N_j}, K_k \backslash G \times E|_{N_k}))$, and suppose that $\|\tau\|_{2,\infty}$ is finite. Then τ defines a bounded operator from $L^2(K_j \backslash G) \otimes L^2(E_{N_j})$ to $L^2(K_k \backslash G) \otimes L^2(E_{N_k})$, and its operator norm is at most equal to $\|\tau\|_{2,\infty}$.*

Proof. This can be checked directly, and is in fact a special case of a more general statement about operators defined by measurable kernels. \square

Lemma 4.7. *Suppose that $\sigma: M \times M \rightarrow \text{End}(E)$ is bounded and G -invariant, and let $\kappa \in \Gamma(\text{Hom}(E))^G$. Let $g, g' \in G$, and suppose that $\|\tilde{\kappa}(g, g')\|_{2,\infty}$ is finite. Then*

$$\|\tilde{\sigma\kappa}(g, g')\|_{2,\infty} \leq \|\sigma\|_\infty \|\tilde{\kappa}(g, g')\|_{2,\infty}.$$

Proof. This follows directly from Lemma 4.3 and the fact that $\|\chi\|_\infty \leq 1$. \square

Lemma 4.5 follows from Lemmas 4.6 and 4.7.

The proof of Lemma 4.4 starts with an off-diagonal estimate for the kernel of $(D + i\mu)^{-1}$.

Lemma 4.8. *Let $\mu > 0$, and let κ_μ be the Schwartz kernel of $(D + i\mu)^{-1}$. There exists a constant $C_\mu > 0$ such that for all $m, m' \in M$ satisfying $d(m, m') \geq 1$,*

$$\|\kappa_\mu(m, m')\| \leq C_\mu e^{-\frac{\mu}{2}d(m, m')},$$

where the norm is taken fibrewise in $E \boxtimes E^*$.

Proof. This is Lemma 3.3 in [21]. We remark that the proof works because M and E have bounded geometry. \square

Proof of Lemma 4.4. Let κ be as in the lemma. Lemma 4.8 and the fact that κ_0 has finite propagation imply that for all $\mu > 0$, there is an $A_\mu > 0$ such that for all $m, m' \in M$,

$$(4.5) \quad \|\kappa(m, m')\| \leq A_\mu e^{-\mu d(m, m')/2}.$$

First suppose that $M = G \times_K N$ for a single slice N . Because M/G is compact, the slice N and the support of χ are compact. So by properness of the action by G on M , the function

$$\chi_N: g \mapsto \int_N \chi(g^{-1}y) dy$$

on G has compact support. Thus $\phi^{-1}(\text{supp}(\chi_N)) \subset K \backslash G$ has finite volume.

Using Lemma 4.3, the estimate (4.5), and the fact that $\|\chi\|_\infty \leq 1$, we find that

$$(4.6) \quad \|\tilde{\kappa}\|_{L^1(G), 2, \infty} \leq A_\mu \text{vol}(\phi^{-1}(\text{supp}(\chi_N))) \int_G \sup_{y' \in N} \left(\int_N e^{-\mu d(y, gy')} dy \right)^{1/2} dg.$$

For fixed $y' \in N$, the map $g \mapsto gy'$ is a quasi-isometry from G to M . Together with compactness of N , this implies that there are $a, b > 0$ such that for all $g \in G$ and all $y' \in N$,

$$\int_N e^{-\mu d(y, gy')} dy \leq a e^{-b\mu d_G(e, g)}.$$

By the volume growth condition at the start of Section 2, we can choose μ large enough so that the right hand side of (4.6) converges. \square

4.5. The cocompact case of Theorem 3.1.

Proposition 4.9. *Let M be a cocompact G -manifold. Then D is a regular and essentially self-adjoint operator on the Hilbert $C_{\max}^*(M; L^2(E))^G$ -module $C_{\max}^*(M; L^2(E))^G$.*

Proof. Proposition 4.1 in [31] (which is based on Lemmas 9.7 and 9.8 in [34]) states that D is both essentially self-adjoint and regular if there is a $\mu > 0$ such that $D + i\mu$ and $D - i\mu$ have dense ranges. We will find a $\mu > 0$ such that $D + i\mu$ has dense range; the argument for $D - i\mu$ is similar.

Let $\kappa_0 \in C_{\ker}^\infty(M, L^2(E))^G$, and let $\mu > 0$. Consider the composition

$$\kappa := (D + i\mu)^{-1} \kappa_0$$

of bounded operators on $L^2(E)$. It is unclear a priori if κ lies in the initial domain of $D + i\mu$ as an unbounded operator on the Hilbert C^* -module $C_{\max}^*(M; L^2(E))^G$.

To remedy this, we consider a family $\{f_\varepsilon\}_{\varepsilon \in (0,1]}$ of smooth functions on $M \times M$, invariant under the diagonal action by G , such that for all $\varepsilon \in (0, 1]$,

- (1) $f_\varepsilon = 1$ on $\text{Pen}(\text{supp}(\kappa_0), 1/\varepsilon)$;
- (2) $f_\varepsilon = 0$ outside $\text{Pen}(\text{supp}(\kappa_0), 3/\varepsilon)$;
- (3) $\|df_\varepsilon\|_\infty \leq \varepsilon$.

Because M is complete, the sets $\text{Pen}(\text{supp}(\kappa_0), 3/\varepsilon)$ are cocompact (with respect to the diagonal action by G). So the functions f_ε are cocompactly supported.

We write $d_1 f_\varepsilon \in \Gamma^\infty(M \times M, T^*M \times M)$ for the derivative of such a function f_ε in the first factor. Then for all $\varepsilon \in (0, 1]$, $f_\varepsilon \kappa$ lies in the domain of the Hilbert C^* -module operator $D + i\mu$, and

$$(D + i\mu)(f_\varepsilon \kappa) = f_\varepsilon(D + i\mu)\kappa + \sigma_D(d_1 f_\varepsilon)\kappa = \kappa_0 + \sigma_D(d_1 f_\varepsilon)\kappa.$$

Here the composition $\sigma_D(d_1 f_\varepsilon)\kappa$ is defined as in (4.4).

By (2.3) and (2.9), we find that

$$\begin{aligned} \|(D + i\mu)(f_\varepsilon \kappa) - \kappa_0\|_{\max} &= \|\sigma_D(d_1 f_\varepsilon)\kappa\|_{\max} \\ &= \|\widetilde{\sigma_D(d_1 f_\varepsilon)\kappa}\|_{C^*_{\max}(G) \otimes \mathcal{B}(\mathcal{H})} \\ &\leq \|\widetilde{\sigma_D(d_1 f_\varepsilon)\kappa}\|_{L^1(G) \otimes \mathcal{B}(\mathcal{H})}. \end{aligned}$$

Lemmas 4.4 and 4.5 imply that, for μ large enough (independently of κ_0), the right hand side is at most equal to

$$(4.7) \quad \|\sigma_D(d_1 f_\varepsilon)\|_\infty \|\tilde{\kappa}\|_{L^1(G), 2, \infty}.$$

Because D has finite propagation speed, there is a $C > 0$ such that $\|\sigma_D(\xi)\| \leq C\|\xi\|$ for all $\xi \in T^*M$. So (4.7) is at most equal to

$$C\|d_1 f_\varepsilon\|_\infty \|\tilde{\kappa}\|_{L^1(G), 2, \infty} \leq C\varepsilon \|\tilde{\kappa}\|_{L^1(G), 2, \infty}.$$

We conclude that, for μ large enough, any element $\kappa_0 \in C^\infty_{\ker}(M, L^2(E))^G$ can be approximated arbitrarily closely by an element in the image of $D + i\mu$. Since $C^\infty_{\ker}(M, L^2(E))^G$ is dense in $C^*_{\max}(M, L^2(E))^G$, it follows that $D + i\mu$ has dense range. By the first paragraph of this proof, the operator D is regular and essentially self-adjoint. \square

Proposition 4.9 generalises to kernels on the product of two different manifolds.

Proposition 4.10. *Let M_1 and M_2 be proper G -manifolds with and E_1, E_2 G -vector bundles over M_1 and M_2 respectively. Suppose that M_1 is cocompact, and let D be a symmetric, elliptic, first-order differential operator with finite propagation speed acting on sections of E_1 . Then the operator*

$$D : \mathcal{H}_{\max}(E_1, E_2)_{\text{loc}}^G \rightarrow \mathcal{H}_{\max}(E_1, E_2)_{\text{loc}}^G$$

*is regular and essentially self-adjoint in the sense of Hilbert $C^*_{\max}(M_2, L^2(E_2))_{\text{loc}}^G$ -modules.*

The proof of Proposition 4.9 can be applied with minimal changes to prove Proposition 4.10. We have only given the details for $M_1 = M_2$ for notational simplicity. The key point is that any kernel with finite propagation has cocompact support. Note that for this, it is only necessary that either M_1 or M_2 be cocompact.

4.6. Proof of Theorem 3.1. We will prove Theorem 3.1 using Proposition 4.9 for the cocompact case.

The following lemma will be used in a few places.

Lemma 4.11. *Let S be a G -equivariant vector bundle homomorphism of E , whose fibrewise norm is bounded. Then $S \in \mathcal{L}_A(A)$, and $\|S\|_{\mathcal{L}_A(A)} \leq \|S\|_\infty$.*

Proof. The endomorphism $\|S\|_\infty 1_E - S$ of E is fibrewise nonnegative. Let $T := (\|S\|_\infty 1_E - S)^{1/2}$ be its fibrewise square root. Then for all $\kappa \in C_{\ker}^*(M; L^2(E))_{\text{loc}}^G$,

$$\langle \kappa, (\|S\|_\infty - S)\kappa \rangle = \langle T\kappa, T\kappa \rangle \geq 0.$$

So $S \leq \|S\|_\infty$ in $\mathcal{L}_A(A)$. \square

Proof of Theorem 3.1. As pointed out at the start of the proof of Proposition 4.9, it is enough to prove that the operators $D \pm i$ have ranges that are dense in $C_{\max}^*(M; L^2(E))_{\text{loc}}^G$. We prove this for $D + i$, with the case of $D - i$ being similar.

Analogously to the existence of the functions f_ε in the proof of Proposition 4.9, completeness of M implies that there exists a family $\{a_\varepsilon\}_{\varepsilon \in (0,1]}$ of G -invariant, cocompactly supported smooth functions taking values in $[0, 1]$, such that:

- (1) $\text{supp}(a_{\varepsilon_1}) \subseteq \text{supp}(a_{\varepsilon_2})$ whenever $\varepsilon_2 \leq \varepsilon_1$;
- (2) $a_{\varepsilon_1}^{-1}(1) \subseteq a_{\varepsilon_2}^{-1}(1)$ whenever $\varepsilon_2 \leq \varepsilon_1$;
- (3) $\bigcup_\varepsilon \{a_\varepsilon^{-1}(1)\} = M$;
- (4) $\sup_{m \in M} \|da_\varepsilon(m)\| \leq \varepsilon$.

Now let $\kappa \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$. Since κ has cocompact support, for small enough ε , we have $\text{supp}(\kappa) \subseteq a_\varepsilon^{-1}(1) \times a_\varepsilon^{-1}(1)$. Let U_ε be a G -invariant, relatively cocompact neighbourhood of $\text{supp}(a_\varepsilon)$. Denote the double of its closure \overline{U}_ε by $\overline{U}_\varepsilon^+$, noting that there exists a G -invariant collar neighbourhood of $\partial\overline{U}_\varepsilon$ inside $\overline{U}_\varepsilon^+$. By restricting the various geometric structures on M to U_ε and extending to $\overline{U}_\varepsilon^+$, we obtain a Dirac operator D_ε acting on a bundle E_ε over the double.

As in Subsection 4.1, D_ε defines an operator on $C_{\ker}^\infty(\overline{U}_\varepsilon^+, L^2(E_\varepsilon))^G$ that extends to an unbounded operator on the maximal completion $C_{\max}^*(\overline{U}_\varepsilon^+, L^2(E_\varepsilon))^G$, whose norm we will denote by $\|\cdot\|_{\varepsilon, \max}$. By Proposition 4.9 and cocompactness of $\overline{U}_\varepsilon^+$, D_ε is regular as an unbounded Hilbert C^* -module operator. So there exists a sequence $\{e_{\varepsilon, j}\}_{j \in \mathbb{N}}$ in $C_{\ker}^\infty(\overline{U}_\varepsilon^+, L^2(E_\varepsilon))^G$ such that

$$(4.8) \quad (D_\varepsilon + i)e_{\varepsilon, j} \rightarrow \kappa.$$

in $\|\cdot\|_{\varepsilon, \max}$. Since the action of the operator $(D_\varepsilon + i)^{-1}$ on κ preserves support in the second coordinate, we may assume that

$$\text{pr}_2(\text{supp}(e_{\varepsilon, j})) \subseteq \text{pr}_2(\text{supp}(\kappa)),$$

where $\text{pr}_2 : \overline{U}_\varepsilon^+ \times \overline{U}_\varepsilon^+ \rightarrow \overline{U}_\varepsilon^+$ is the projection onto the second factor, so that $a_\varepsilon e_{\varepsilon, j}$ lies in the domain of D . For each j ,

$$(4.9) \quad \begin{aligned} (D + i)(a_\varepsilon e_{\varepsilon, j}) - \kappa &= (D_\varepsilon + i)(a_\varepsilon e_{\varepsilon, j}) - \kappa \\ &= a_\varepsilon((D_\varepsilon + i)e_{\varepsilon, j} - \kappa) + \sigma_D(da_\varepsilon)e_{\varepsilon, j}, \end{aligned}$$

acting again on the first coordinate.

Because D has finite propagation speed, there is a $C > 0$ such that $\|\sigma_D(\xi)\| \leq C\|\xi\|$ for all $\xi \in T^*M$. By Lemma 4.11, this implies that $\|\sigma_D(da_\varepsilon)\|_{\mathcal{L}_A(A)} \leq C\|da_\varepsilon\|_\infty$. It follows that

$$\|\sigma_D(da_\varepsilon)\kappa\|_{\max} \leq C\|da_\varepsilon\|_\infty \|\kappa\|_{\max}.$$

Similarly,

$$\|a_\varepsilon\kappa\|_{\max} \leq \|\kappa\|_{\max}.$$

Thus $\sigma_D(da_\varepsilon)$ and a_ε define bounded multipliers of $C_{\max}^*(M; L^2(E))_{\text{loc}}^G$ with norms bounded above by their supremum norms. Combining this with (4.9) gives

$$\begin{aligned} \|(D+i)(a_\varepsilon e_{\varepsilon,j}) - \kappa\|_{\max} &= \|a_\varepsilon((D_\varepsilon+i)e_{\varepsilon,j} - \kappa) + \sigma_D(da_\varepsilon)e_{\varepsilon,j}\|_{\max} \\ &\leq \|(D_\varepsilon+i)e_{\varepsilon,j} - \kappa\|_{\max} + C\|da_\varepsilon\|_\infty \|e_{\varepsilon,j}\|_{\max}. \end{aligned}$$

The algebra $C_{\max}^*(\overline{U}_\varepsilon; L^2(E|_{\overline{U}_\varepsilon}))^G$ is a subalgebra of the admissible Roe algebra $C_{\max}^*(\overline{U}_\varepsilon; L^2(E|_{\overline{U}_\varepsilon}) \otimes L^2(G))^G$, which is itself a common subalgebra of both $C_{\max}^*(M)_{\text{loc}}^G$ and $C_{\max}^*(\overline{U}_\varepsilon^+; L^2(E_\varepsilon) \otimes L^2(G))^G$. This implies that for any kernel $\kappa' \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$ supported on $U_\varepsilon \times U_\varepsilon$,

$$(4.10) \quad \|\kappa'\|_{\max} = \|\kappa'\|_{\varepsilon, \max}$$

as both sides are equal to the norm of the image of κ' in the algebra

$$C_{\max}^*(\overline{U}_\varepsilon; L^2(E|_{\overline{U}_\varepsilon}) \otimes L^2(G))^G.$$

Also note that (4.8) implies that there exists j_0 (dependent on ε) such that for all $j \geq j_0$,

$$(4.11) \quad \|e_{\varepsilon,j}\|_{\varepsilon, \max} \leq 2\|\kappa\|_{\varepsilon, \max}.$$

By (4.10) and (4.11), we have for all $j \geq j_0$,

$$\begin{aligned} \|(D+i)(a_\varepsilon e_{\varepsilon,j}) - \kappa\|_{\max} &\leq \|(D_\varepsilon+i)e_{\varepsilon,j} - \kappa\|_{\varepsilon, \max} + C\|da_\varepsilon\|_\infty \|e_{\varepsilon,j}\|_{\varepsilon, \max} \\ &\leq \|(D_\varepsilon+i)e_{\varepsilon,j} - \kappa\|_{\varepsilon, \max} + 2C\varepsilon\|\kappa\|_{\max}. \end{aligned}$$

Now if any $\varepsilon' > 0$ is given, choose $\varepsilon \in (0, 1]$ so that $2C\varepsilon\|\kappa\|_{\max} < \varepsilon'/2$. For this ε , choose j so that $j \geq j_0$, and $\|(D_\varepsilon+i)e_{\varepsilon,j} - \kappa\|_{\varepsilon, \max} < \varepsilon'/2$. Then $\|(D+i)(a_\varepsilon e_{\varepsilon,j}) - \kappa\|_{\max} < \varepsilon'$. So any element κ of the dense subspace

$$C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G \subset C_{\max}^*(M; L^2(E))_{\text{loc}}^G$$

can be approximated arbitrarily closely by elements in the image of $D+i$. \square

A straightforward adaptation of the above proof, together with Proposition 4.10, gives the following result in the more general situation involving two different bundles and manifolds.

Theorem 4.12. *Let M_1 and M_2 be proper, isometric G -manifolds with and E_1, E_2 Hermitian G -vector bundles over M_1 and M_2 respectively. Suppose that M_1 is cocompact, and let D be a symmetric, G -equivariant, first order differential operator acting on sections of E_1 , with finite propagation speed. Then the unbounded operator*

$$D : \mathcal{H}_{\max}(E_1, E_2)_{\text{loc}}^G \rightarrow \mathcal{H}_{\max}(E_1, E_2)_{\text{loc}}^G$$

is regular and essentially self-adjoint.

4.7. Generalised Fredholmness. We will prove Proposition 3.3 by adapting the method in [45] to the Hilbert A -module A . (Recall from Subsection 3.1 that $A = C_{\max}^*(M; L^2(E))_{\text{loc}}^G$.)

We begin by establishing a useful property of the wave operator group associated to an essentially self-adjoint regular operator, following Proposition 3.4 of [23].

Lemma 4.13. *Let D be an essentially self-adjoint regular operator on A . Then the wave operator group $\{e^{itD}\}_{t \in \mathbb{R}}$ satisfies the wave equation: for $\kappa \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$,*

$$\frac{d}{dt} e^{itD} \kappa = i D e^{itD} \kappa.$$

Moreover, each operator e^{itD} has propagation at most $|t|$, in the sense that it does not increase the propagation of κ by more than $|t|$.

Proof. The function $s \mapsto e^{its}$ is in $C_b(\mathbb{R})$. Thus for each $t \in \mathbb{R}$, the operator e^{itD} is bounded adjointable and unitary. For each $t \in \mathbb{R}$, the difference quotient

$$\frac{e^{i(t+h)s} - e^{its}}{h} \rightarrow i s e^{its}$$

as $h \rightarrow 0$, uniformly in s in compact sets. Furthermore, this difference quotient is bounded uniformly by $|1 + s|$. The wave equation property then follows from the third point in Theorem 3.2. The finite propagation property can be proved exactly as in Proposition 3.4 in [23]. \square

Corollary 4.14. *Let K be a cocompact subset of M_1 . Let $r > 0$. Let D_1 and D_2 be essentially self-adjoint differential operators on M_1 that are equal on $\text{Pen}(K, r)$. Then for a kernel $\kappa \in \mathcal{H}_{\ker}^\infty(E_1, E_2)_{\text{loc}}^G$ supported on $K \times M_2$,*

$$e^{itD_1} \kappa = e^{itD_2} \kappa$$

if $-r \leq t \leq r$.

Now suppose that D is as in Subsection 3.1, and in particular that it satisfies (3.2). We will use Corollary 4.14 to establish a norm estimate for $f(D)$ in $\mathcal{L}_A(A)$ when f has compactly supported Fourier transform.

Lemma 4.15. *The operator D on the Hilbert module $C_{\max}^*(M; L^2(E))_{\text{loc}}^G$ satisfies*

$$D^2 \geq c^2$$

with respect to the Hilbert module inner product.

Proof. For any $\kappa \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$,

$$(4.12) \quad \langle D^2 \kappa, \kappa \rangle = \langle (P^* P + R) \kappa, \kappa \rangle = \langle P \kappa, P \kappa \rangle + \langle R \kappa, \kappa \rangle.$$

As in the proof of Lemma 4.11, we find that $\langle R \kappa, \kappa \rangle \geq c^2 \langle \kappa, \kappa \rangle$. So the right hand side of (4.12) is at least equal to $c^2 \langle \kappa, \kappa \rangle$. \square

Lemma 4.15 is the place where we use form (3.2) of D^2 , rather than a slightly milder positivity condition on D^2 outside Z . We continue to use c to denote the constant below (3.2), which was also used in Lemma 4.15.

Lemma 4.16. *Let $r > 0$. Suppose $f \in \mathcal{S}(\mathbb{R})$ is a function with Fourier transform \hat{f} supported in $[-r, r]$. Let φ be a smooth, bounded, G -invariant function with support*

disjoint from $\text{Pen}(Z, 2r)$. Then

$$\|f(D)\varphi\|_{\mathcal{L}_A(A)} \leq \|\varphi\|_\infty \cdot \sup\{|f(\lambda)|; |\lambda| \geq c\}.$$

The same estimate applies to $\varphi f(D)$.

Proof. For $n = 1, 2$, let $U_n = \{m \in M; d(m, Z) > nr\}$. Let \overline{U}_1^+ denote the double of \overline{U}_1 . By extending the various geometric structures on U_1 to \overline{U}_1^+ , we may extend the Dirac operator $D|_{U_1}$ to an operator \tilde{D} on \overline{U}_1^+ acting on the extension $\tilde{E} \rightarrow \overline{U}_1^+$ of $E|_{\overline{U}_1}$. Then \tilde{D} is an unbounded symmetric operator on $\mathcal{H}_{\max}(\overline{U}_1^+, M)_{\text{loc}}^G$ with initial domain $\mathcal{H}_{\ker}^\infty(\overline{U}_1^+, M)_{\text{loc}}^G$. By Lemma 4.15, we have $\tilde{D}^2 \geq c^2$. Since \overline{U}_1^+ is complete, Theorem 4.12 implies that \tilde{D} is essentially self-adjoint and regular.

Now for any $\kappa \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$ with support contained in $U_2 \times M$, Corollary 4.14 implies that

$$e^{itD}\kappa = e^{it\tilde{D}}\kappa$$

for all $-r \leq t \leq r$. Together with the equality

$$f(D) = \frac{1}{\pi} \int_{-r}^r \hat{f}(t)e^{itD} dt,$$

this implies that $f(D)\varphi = f(\tilde{D})\varphi$. The bound $\tilde{D}^2 \geq c^2$ implies, by the fifth point of Theorem 3.2, that

$$\|f(\tilde{D})\|_{\mathcal{L}_A(A)} \leq \sup\{|f(\lambda)|; |\lambda| \geq c\}.$$

Together with the fact that φ defines an element of $\mathcal{L}_A(A)$ with norm at most $\|\varphi\|_\infty$ (a special case of Lemma 4.11), this gives

$$\|f(D)\varphi\|_{\mathcal{L}_A(A)} \leq \|\varphi\|_\infty \cdot \sup\{|f(\lambda)|; |\lambda| \geq c\}.$$

□

With these preparations, we can now finish the proof of Proposition 3.3. Again, we follow the idea of Roe [45].

Proof of Proposition 3.3. Let $f \in C_c(\mathbb{R})$ be supported in $[-c, c]$, and let $\varepsilon > 0$. There exists a smooth function g with compactly supported Fourier transform such that

$$\sup\{|g(\lambda) - f(\lambda)|; \lambda \in \mathbb{R}\} < \varepsilon.$$

This implies that $|g(\lambda)| < \varepsilon$ when $|\lambda| > c$. Suppose that $\text{supp}(\hat{g}) \subseteq [-r, r]$ for some $r > 0$. Let $\psi : M \rightarrow [0, 1]$ be a smooth G -invariant function such that

$$\psi(m) = \begin{cases} 1 & \text{for } m \in \text{Pen}(Z, 2r) \\ 0 & \text{for } m \in M \setminus \text{Pen}(Z, 3r). \end{cases}$$

We can write

$$f(D) = \psi g(D)\psi + (1 - \psi)g(D)\psi + g(D)(1 - \psi) + (f(D) - g(D)).$$

Now the first term on the right hand side is a G -invariant cocompactly supported smooth kernel. The second and third terms each have maximal norm bounded by ε by Lemma 4.16, while the maximal norm of the last term is bounded by ε by the fifth point in Theorem 3.2. Thus, for any $\varepsilon > 0$, $f(D)$ lies within 3ε of a G -invariant cocompactly supported smooth kernel. Thus $f(D)$ is in the completion $C_{\max}^*(M; L^2(E))_{\text{loc}}^G$. □

5. AVERAGING MAPS

In Subsections 5.1–5.3, we return to the general setting of Subsection 2.1, of a metric space (X, d) rather than a Riemannian manifold.

The main tools in the proof of Theorem 3.9 are several *averaging maps*, which map G -equivariant operators on X to operators on X/G . In this section, we introduce these maps, and prove their properties that we need. We then use these maps in Section 6 to prove Theorem 3.9.

5.1. Averaging kernels. Consider the setting of Subsection 2.1. Consider the action by $G \times G$ on $\Gamma(\text{Hom}(E))$ given by

$$(5.1) \quad ((g, g') \cdot \kappa)(x, x') := g\kappa(g^{-1}x, g'x')g',$$

for $g, g' \in G$, $x, x' \in X$ and $\kappa \in \Gamma(\text{Hom}(E))$. Let $\Gamma(\text{Hom}(E))^{G \times G} \subset \Gamma(\text{Hom}(E))$ be the subspace of sections invariant under this action.

Let d_G be the metric on X/G induced by d :

$$d_G(Gx, Gx') := \inf_{g \in G} d(gx, x'),$$

for $x, x' \in X$. Consider the measure $d(Gx)$ on X/G such that for all $\varphi \in C_c(X)$,

$$(5.2) \quad \int_X \varphi(x) dx = \int_{X/G} \int_G \varphi(gx) dg d(Gx).$$

(See for example [7, Chapter VII, Section 2.2, Proposition 4b].)

Consider the Hilbert space $L_T^2(E)^G$ defined in Subsection 3.2. We view it as a $C_0(X/G)$ -module by pointwise multiplication after pullback along the quotient map. Let $C_{\ker}^*(X/G; L_T^2(E)^G)$ be the subalgebra of $\mathcal{B}(L_T^2(E)^G)$ of locally compact operators T with finite propagation, given by a continuous kernel $\kappa \in \Gamma(\text{Hom}(E))^{G \times G}$ via

$$(Ts)(x) = \int_{X/G} \kappa(x, x')s(x') d(Gx'),$$

for $x \in X$ and $s \in L_T^2(E)^G$. The integral is independent of the Borel section $X/G \rightarrow X$ used implicitly, by G -invariance of s and $G \times G$ -invariance of κ . We will identify operators in $C_{\ker}^*(X/G; L_T^2(E)^G)$ with their kernels.

For $\kappa \in C_{\ker}^*(X; L^2(E))^G$ and $x, x' \in X$, set

$$\text{Av}(\kappa)(x, x') := \int_G g\kappa(g^{-1}x, x') dg.$$

The integrand is bounded, measurable and compactly supported, because κ has finite propagation and the action is proper.

Lemma 5.1. *For every*

$$\kappa \in C_{\ker}^*(X; L^2(E))^G,$$

$\text{Av}(\kappa)$ is an element of $C_{\ker}^(X/G; L_T^2(E)^G)$. This defines a surjective $*$ -homomorphism*

$$\text{Av}: C_{\ker}^*(X; L^2(E))^G \rightarrow C_{\ker}^*(X/G; L_T^2(E)^G).$$

Proof. Let $\kappa \in C_{\ker}^*(X; L^2(E))^G$. It follows from a computation involving G -equivariance of κ and left and right invariance of dg that $\text{Av}(\kappa)$ is $G \times G$ -invariant. Furthermore, the propagation of $\text{Av}(\kappa)$ in X/G is at most equal to the propagation of κ in X . So indeed $\text{Av}(\kappa) \in C_{\ker}^*(X/G; L_T^2(E)^G)$.

Using the fact that the Haar measure dg is invariant under inversion, one computes directly that for all $\kappa \in C_{\ker}^*(X; L^2(E))^G$,

$$\text{Av}(\kappa^*) = \text{Av}(\kappa)^*.$$

Let $\kappa, \kappa' \in C_{\ker}^*(X; L^2(E))^G$ be given. Then a computation involving (5.2) shows that

$$\text{Av}(\kappa\kappa') = \text{Av}(\kappa) \text{Av}(\kappa').$$

For surjectivity, let χ be a cutoff function as in (2.6). Choose this function such that, in addition to its other properties,

$$(5.3) \quad d(x, x') \leq d_G(Gx, Gx') + 1$$

for all $x, x' \in \text{supp}(\chi)$. In other words, the support of χ mainly extends transversally to G -orbits. (See Lemma 5.2 below.)

Suppose that $\kappa_G \in C_{\ker}^*(X/G; L_T^2(E)^G)$, and let r be its propagation in X/G . Define $\kappa \in \Gamma(\text{Hom}(E))$ by

$$\kappa(x, x') = \int_G \chi(gx)^2 \chi(gx')^2 dg \cdot \kappa_G(x, x'),$$

for $x, x' \in X$.

We first claim that κ has finite propagation. Indeed, let $x, x' \in X$ be such that $d(x, x') > r + 1$. If $d_G(Gx, Gx') > r$, then $\kappa_G(x, x') = 0$, so $\kappa(x, x') = 0$. So suppose that $d_G(Gx, Gx') \leq r$. If $g \in G$, and $\chi(gx)^2 \chi(gx')^2$ is nonzero, then (5.3) implies that

$$d(x, x') = d(gx, gx') \leq r + 1,$$

a contradiction. So $\chi(gx)^2 \chi(gx')^2 = 0$ for all $g \in G$, and hence $\kappa(x, x') = 0$.

Right invariance of dg implies that for all $x, x' \in X$ and $g \in G$,

$$g^{-1} \kappa(gx, gx') g = \kappa(x, x'),$$

i.e. κ is invariant under the restriction of the action (5.1) to the diagonal. This implies that the operator on $L^2(E)$ defined by κ is G -equivariant.

The property (2.6) of χ and left invariance of dg imply that

$$\text{Av}(\kappa) = \kappa_G,$$

so Av is surjective. □

Lemma 5.2. *There is a cutoff function χ , satisfying (2.6), such that for all $x, x' \in \text{supp}(\chi)$,*

$$d(x, x') \leq d_G(Gx, Gx') + 1.$$

Proof. Let $Y \subset X$ be a subset intersecting all G -orbits, such that for all $y, y' \in Y$,

$$d(y, y') \leq d_G(Gy, Gy') + 1/2.$$

Let U be the open set of all points in X closer than $1/8$ to Y . Then the intersection of U with any G -orbit is open in that orbit. For all $u, u' \in U$, choose $y, y' \in Y$ such that $d(u, y) < 1/8$ and $d(u', y') < 1/8$. Then

$$d(u, u') < d(y, y') + 1/4 \leq d_G(Gy, Gy') + 3/4 < d_G(Gu, Gu') + 1.$$

Let $\tilde{\chi}$ be any nonnegative continuous function on X such that the interior of its support is U . Then the function χ , given by

$$\chi(x) := \frac{\tilde{\chi}(x)}{\left(\int_G \tilde{\chi}(gx)^2 dg\right)^{1/2}}$$

for $x \in X$, has the desired properties. \square

5.2. Averaging operators on Hilbert spaces. We will use an extension of the homomorphism Av to more general bounded operators on $L^2(E)$, not necessarily given by integrable kernels.

We choose a partition of unity $\{\psi_j\}_{j=1}^\infty$ on X , which restricts to a compactly supported partition of unity on every orbit, such that there is an $r > 0$ such that for all j , the set of k for which $d(\text{supp } \psi_j, \text{supp } \psi_k) \leq r$ is finite.

Let $\mathcal{B}_{\text{fp}}(L^2(E))^G$ be the algebra of G -equivariant, bounded operators on $L^2(E)$ with finite propagation. Given $T \in \mathcal{B}_{\text{fp}}(L^2(E))^G$ and $s \in L^2_T(E)^G$, note that $\psi_j s \in L^2(E)$. Hence $T(\psi_j s)$ is well-defined, and for $x \in X$, we set

$$(5.4) \quad (\text{Av}_{L^2}(T)s)(x) := \sum_j (T(\psi_j s))(x).$$

Lemma 5.3. *The sum (5.4) converges for all $x \in X$, and the result is independent of the choice of the partition of unity $\{\psi_j\}$.*

Proof. Fix $x \in X$. Let $f \in C_c(X)$ be a function such that $f(x) = 1$. Let r be greater than both the diameter of $\text{supp } f$ and the propagation of T , and such that

$$d(\text{supp } \psi_j, \text{supp } \psi_k) \leq 2r$$

for only finitely many k , for any fixed j . Then there are only finitely many j for which

$$d(\text{supp } f, \text{supp } \psi_j) \leq r.$$

Hence $fT\psi_j$ is nonzero for only finitely many j , and we see that

$$(\text{Av}_{L^2}(T)s)(x) = (f \text{Av}(T)s)(x) = \sum_j (fT(\psi_j s))(x)$$

is a finite sum, and hence converges.

Let $\{\psi'_j\}$ be another partition of unity on X , with the same properties as $\{\psi_j\}$. We will write

$$\text{Av}_{L^2}^{\{\psi_j\}}(T) \quad \text{and} \quad \text{Av}_{L^2}^{\{\psi'_j\}}(T)$$

for the operators defined by (5.4) using these two partitions of unity. As above, let $J \subset \mathbb{N}$ be a finite set such that $fT\psi_j$ and $fT\psi'_j$ are zero if $j \notin J$. Then

$$fT \sum_{j \in J} \psi_j = fT = fT \sum_{j \in J} \psi'_j.$$

Since the finite sum over J commutes with T , we conclude that

$$\begin{aligned} (\text{Av}_{L^2}^{\{\psi_j\}}(T)s)(x) &= (f \text{Av}_{L^2}^{\{\psi_j\}}(T)s)(x) = \sum_{j \in J} (fT(\psi_j s))(x) \\ &= \sum_{j \in J} (fT(\psi'_j s))(x) = (\text{Av}_{L^2}^{\{\psi'_j\}}(T)s)(x). \end{aligned}$$

\square

Lemma 5.4. *The construction (5.4) defines a *-homomorphism*

$$\text{Av}_{L^2}: \mathcal{B}_{\text{fp}}(L^2(E))^G \rightarrow \mathcal{B}(L_T^2(E))^G.$$

Proof. Let $T \in \mathcal{B}_{\text{fp}}(L^2(E))^G$ and $s \in L_T^2(E)^G$. We first claim that $\text{Av}_{L^2}(T)s \in L_T^2(E)^G$. Indeed, the properties of the functions ψ_j , and finite propagation of T imply that $\chi \text{Av}(T)s \in L^2(E)$. And for all $g \in G$ and $x \in X$, one checks, using G -equivariance of T and G -invariance of s , that

$$(g \cdot (\text{Av}(T)s))(x) = \sum_j T((g \cdot \psi_j)s)(x).$$

Since $\{g \cdot \psi_j\}_{j=1}^\infty$ is a partition of unity with the same properties as $\{\psi_j\}_{j=1}^\infty$, the second part of Lemma 5.3 implies that the right hand side equals $(\text{Av}_{L^2}(T)s)(x)$.

Boundedness of the operator $\text{Av}_{L^2}(T)$ on $L_T^2(E)^G$ follows from boundedness and finite propagation of T , via the fact that the sum $\chi \sum_j T\psi_j$ is finite.

If T' is an other operator in $\mathcal{B}_{\text{fp}}(L^2(E))^G$, then for all $x \in X$,

$$\begin{aligned} (\text{Av}_{L^2}(TT')s)(x) &= \sum_j (TT'\psi_j s)(x) \\ &= \sum_{j,k} (T\psi_k T'\psi_j s)(x) = (\text{Av}_{L^2}(T) \text{Av}_{L^2}(T')s)(x). \end{aligned}$$

Here we used the fact that the sum over k is finite for each j , since T' has finite propagation. One checks directly that Av_{L^2} preserves *-operations. \square

Lemma 5.5. *For an operator $T \in C_{\text{ker}}^*(X, L^2(E))^G$, with kernel κ , the operator $\text{Av}_{L^2}(T)$ is an element of $C_{\text{ker}}^*(X/G; L_T^2(E))^G$, and its kernel is $\text{Av}(\kappa)$.*

Proof. This follows from a direct computation involving (5.2). \square

If s_1 and s_2 are sections of E such that their pointwise inner product $(s_1, s_2)_E$ is in $L^1(X)$, we will say that the inner product $(s_1, s_2)_{L^2(E)}$ converges, and define it as the integral of $(s_1, s_2)_E$ over X .

Lemma 5.6. *For all $T \in \mathcal{B}_{\text{fp}}(L^2(E))^G$, $s \in \Gamma_{\text{tc}}(E)^G$ and $\sigma \in \Gamma_c(E)$,*

$$|(\text{Av}_{L^2}(T)s, \sigma)_{L^2(E)}| \leq \|T\| |(s, \sigma)_{L^2(E)}|.$$

In particular, this inner product converges.

Proof. Let $J \subset \mathbb{N}$ be a finite subset (depending on T , s and σ) such that $\sum_{j \in J} \psi_j = 1$ on $\text{supp}(\sigma)$, and for all $j \in \mathbb{N} \setminus J$, $(T\psi_j s, \sigma)_E = 0$. Then

$$\begin{aligned} |(\text{Av}(T)s, \sigma)_{L^2(E)}| &= |(T \sum_{j \in J} \psi_j s, \sigma)_{L^2(E)}| \\ &\leq \|T\| |(\sum_{j \in J} \psi_j s, \sigma)_{L^2(E)}| = \|T\| |(s, \sigma)_{L^2(E)}|. \end{aligned}$$

\square

5.3. Relation to the integration trace. In this subsection, we suppose that X/G is compact. Consider the map $\oplus 0$ in (2.5).

Lemma 5.7. *For all $\kappa \in C_{\ker}^*(X; L^2(E))^G$,*

$$\| \text{Av}(\kappa) \|_{\mathcal{B}(L_T^2(E)^G)} = \| (I \otimes 1)(\kappa \oplus 0) \|_{\mathcal{B}(\mathcal{H})}.$$

Lemma 5.8. *If X/G is compact, then Av has a unique extension to a $*$ -homomorphism*

$$\text{Av}: C_{\max}^*(X; L^2(E))^G \rightarrow \mathcal{K}(L_T^2(E)^G).$$

Proof. If X/G is compact, then $C_{\ker}^*(X/G; L_T^2(E)^G) \subset \mathcal{K}(L_T^2(E)^G)$. So the claim is that Av is continuous with respect to the norm $\| \cdot \|_{\max}$ and the operator norm on $\mathcal{K}(L_T^2(E)^G)$. And Lemma 5.7 implies that for all $\kappa \in C_{\ker}^*(M; L^2(E))^G$,

$$\| \text{Av}(\kappa) \|_{\mathcal{B}(L_T^2(E)^G)} \leq \| \kappa \oplus 0 \|_{\max} = \| \kappa \|_{\max},$$

because $I \otimes 1$ is a $*$ -representation of $C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G \subset L^1(G) \otimes \mathcal{K}(\mathcal{H})$. \square

Proposition 5.9. *The following diagram commutes:*

$$\begin{array}{ccc} K_0(C_{\max}^*(X; L^2(E))^G) & \xrightarrow{\oplus 0} & K_0(C_{\max}^*(X)^G) \\ \text{Av} \downarrow & & \cong \downarrow \\ K_0(\mathcal{K}) = \mathbb{Z} & \xleftarrow{I_*} & K_0(C_{\max}^*(G)). \end{array}$$

As in Section 4.4 of [30], consider the map

$$\widetilde{\text{TR}}: C_{\ker}^*(X; L^2(E))^G \rightarrow L^1(G) \otimes \mathcal{K}(L^2(E))$$

such that for $\kappa \in C_{\ker}^*(X; L^2(E))^G$ and $g \in G$, $\widetilde{\text{TR}}(\kappa)(g)$ is the operator with compactly supported Schwartz kernel given by

$$(\widetilde{\text{TR}}(\kappa)(g))(x, x') = \chi(x)\chi(x')g\kappa(g^{-1}x, x'),$$

for $x, x' \in X$. We will use the following fact in the proofs of Lemma 5.7 and Proposition 5.9.

Proposition 5.10. *There is a unitary isomorphism $\eta: \mathcal{H} \xrightarrow{\cong} L^2(E)$ such that for every conjugation-invariant map $\tau: L^1(G) \rightarrow \mathbb{C}$, the following diagram commutes:*

$$\begin{array}{ccc} C_{\ker}^*(X; L^2(E))^G & \xrightarrow{\oplus 0} & C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G \\ \widetilde{\text{TR}} \downarrow & & \cong \downarrow \\ L^1(G) \otimes \mathcal{K}(L^2(E)) & & L^1(G) \otimes \mathcal{K}(\mathcal{H}) \\ \tau \otimes 1 \downarrow & & \tau \otimes 1 \downarrow \\ \mathcal{K}(L^2(E)) & \xleftarrow{\eta_*} & \mathcal{K}(\mathcal{H}) \end{array}$$

This is essentially Proposition 4.10 in [30]. There a specific map τ is used, but its only property used in the proof of this proposition is conjugation invariance.

Consider the isometric embedding $j_\chi: L_T^2(E)^G \rightarrow L^2(E)$ given by pointwise multiplication by χ . It induces $(j_\chi)_*: \mathcal{K}(L_T^2(E)^G) \rightarrow \mathcal{K}(L^2(E))$, given by

$$(j_\chi)_*(T)s = \begin{cases} \chi T \sigma & \text{if } s = \chi \sigma \text{ for } \sigma \in L_T^2(E)^G; \\ 0 & \text{if } s \in (\chi L_T^2(E)^G)^\perp, \end{cases}$$

for all $T \in \mathcal{K}(L_T^2(E)^G)$ and $s \in L^2(E)$. Consider the map

$$I \otimes 1: L^1(G) \otimes \mathcal{K}(L^2(E)) \rightarrow \mathcal{K}(L^2(E)).$$

Lemma 5.11. *For all $\kappa \in C_{\ker}^*(M; L^2(E))^G$,*

$$(I \otimes 1) \circ \widetilde{\text{TR}}(\kappa) = (j_\chi)_*(\text{Av}(T_\kappa)).$$

Proof. Let $\kappa \in C_{\ker}^*(X; L^2(E))^G$, and write

$$T := (I \otimes 1)(\widetilde{\text{TR}}(\kappa)).$$

Write

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with respect to the decomposition $L^2(E) = \chi L_T^2(E)^G \oplus (\chi L_T^2(E)^G)^\perp$.

If $s \in L^2(E)$ and $x \in X$, then it follows from the definitions that

$$(5.5) \quad (Ts)(x) = \chi(x) \int_X \int_G \chi(x') g \kappa(g^{-1}x, x') s(x') dg dx'.$$

For $x \in X$, set

$$\sigma(x) := \int_X \int_G \chi(x') g \kappa(g^{-1}x, x') s(x') dg dx'.$$

It follows from left invariance of dg that σ is a G -invariant section of E . And $\chi\sigma \in L^2(E)$, so $\sigma \in L_T^2(E)^G$. So the image of T lies inside $\chi L_T^2(E)^G$, which means that $c = d = 0$.

Because I and $\widetilde{\text{TR}}$ are $*$ -homomorphisms,

$$\begin{pmatrix} a^* & 0 \\ b^* & 0 \end{pmatrix} = T^* = (I \otimes 1)(\widetilde{\text{TR}}(\kappa^*)).$$

Since the image of $(I \otimes 1)(\widetilde{\text{TR}}(\kappa^*))$ lies inside $\chi L_T^2(E)^G$ by the same argument as for κ , we find that $b^* = 0$, so $b = 0$.

If $\sigma \in L_T^2(E)^G$, then it follows from (5.5), G -invariance of κ , σ and dm , and from (2.6) that

$$T\chi\sigma = \chi \text{Av}(\kappa)\sigma.$$

So $a = j_\chi \circ \text{Av}(\kappa) \circ j_\chi^{-1}$. □

Proof of Lemma 5.7 and Proposition 5.9. Proposition 5.10, with $\tau = I$, and Lemma 5.11 imply that the diagram

$$\begin{array}{ccc} C_{\ker}^*(X; L^2(E))^G & \xrightarrow{\oplus 0} & C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G \\ \text{Av} \downarrow & & \downarrow \\ \mathcal{K}(L_T^2(E)^G) & & L^1(G) \otimes \mathcal{K}(\mathcal{H}) \\ (j_\chi)_* \downarrow & & I \otimes 1 \downarrow \\ \mathcal{K}(L^2(E)) & \xleftarrow{\eta_*} & \mathcal{K}(\mathcal{H}) \end{array}$$

commutes. This implies both Lemma 5.7 and Proposition 5.9. □

5.4. Averaging operators on Hilbert C^* -modules. As in Subsection 3.1, we write $A := C_{\max}^*(X; L^2(E))_{\text{loc}}^G$. By (2.10) and Lemmas 5.1 and 5.8, we obtain a surjective $*$ -homomorphism

$$\text{Av}: A \rightarrow \mathcal{K}(L_T^2(E))^G.$$

This extends uniquely to multiplier algebras, giving

$$(5.6) \quad \widetilde{\text{Av}}: \mathcal{L}_A(A) \rightarrow \mathcal{B}(L_T^2(E))^G.$$

We return to the setting where $X = M$ is a Riemannian manifold, as in Subsection 3.1.

For clarity, we will use subscripts A and $L^2(E)$ to denote functional calculus of operators on the Hilbert module A and on the Hilbert space $L^2(E)$, as in the following lemma.

Lemma 5.12. *Let $f \in C_b(\mathbb{R})$ such that $f \in C_0(\mathbb{R})$, or $f(x) = \mathcal{O}(x)$ as $x \rightarrow 0$. By functional calculus on the Hilbert A -module A , we can form the operator*

$$f(D)_A \in \mathcal{L}_A(A).$$

Via the usual functional calculus, we can form the operator

$$f(D)_{L^2(E)} \in \mathcal{B}(L^2(E))^G.$$

Let $\kappa \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$. Let $T_\kappa \in \mathcal{B}(L^2(E))^G$ be the operator with Schwartz kernel κ . Then the operator $f(D)_{L^2(E)} \circ T_\kappa$ has a smooth kernel, and so does

$$f(D)_A(\kappa).$$

These two smooth kernels are equal.

Proof. For an operator $S \in \mathcal{B}(L^2(E))^G$ with a smooth kernel with finite propagation, and for all $n \in \mathbb{N}$, the operator $D^n f(D)S = f(D)D^n S$ is a bounded operator on $L^2(E)$. (Indeed, $D^n S$ has a smooth, G -invariant kernel with finite propagation, so it defines a bounded operator since M/G is compact.) Since D is elliptic, it follows that the image of $f(D)S$ lies in the smooth sections, so that this operator also has a smooth kernel. Let $f(D)_{L^2(E)}\kappa$ be the smooth kernel of $f(D)_{L^2(E)} \circ T_\kappa$.

Next, suppose that $f(x) = (x \pm i)^{-1}$. The unbounded operator $D \pm i$ on A is given by applying $D \pm i$ to the first coordinate of a smooth kernel. The element $f(D)_A(\kappa)$ lies in the domain of this operator, and

$$(D \pm i)(f(D)_A(\kappa)) = \kappa = (D \pm i)(f(D)_{L^2(E)}\kappa).$$

So the Schwartz kernels of $f(D)_A(\kappa)$ and $f(D)_{L^2(E)}\kappa$ are equal. Since the functions $x \mapsto (x \pm i)^{-1}$ generate $C_0(\mathbb{R})$, the claim follows for all $f \in C_0(\mathbb{R})$.

Now suppose that $f \in C_b(\mathbb{R})$, and $f(x) = \mathcal{O}(x)$ as $x \rightarrow 0$. Then $g(x) = f(x)/x$ defines a function $g \in C_0(\mathbb{R})$. The preceding arguments imply that

$$f(D)_A(\kappa) = Dg(D)_A(\kappa) = Dg(D)_{L^2(E)}\kappa = f(D)_{L^2(E)}\kappa.$$

□

We will use the following relation between the averaging maps in (5.4) and (5.6) in the proof of Theorem 3.9.

Lemma 5.13. *Let $b \in C_b(\mathbb{R})$ be such that $b(x) = xg(x)$ for all $x \in \mathbb{R}$, where $g \in C_0(\mathbb{R})$ has compactly supported Fourier transform. Then*

$$\widetilde{\text{Av}}(b(D)_A) = \text{Av}_{L^2}(b(D)_{L^2(E)}) \in \mathcal{K}(L_T^2(E))^G.$$

Proof. The map (5.6) is uniquely determined by the property that for all $\kappa \in A$ and all $S \in \mathcal{L}_A(A)$,

$$\text{Av}(\kappa S) = \text{Av}(\kappa) \widetilde{\text{Av}}(S) \quad \text{and} \quad \text{Av}(S\kappa) = \widetilde{\text{Av}}(S) \text{Av}(\kappa).$$

In fact, $\widetilde{\text{Av}}$ is already determined by these properties for κ in the dense subalgebra $C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$. So the claim is that for all $\kappa \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$,

$$(5.7) \quad \begin{aligned} \text{Av}(\kappa b(D)_A) &= \text{Av}(\kappa) \text{Av}_{L^2}(b(D)_{L^2}); \\ \text{Av}(b(D)_A \kappa) &= \text{Av}_{L^2}(b(D)_{L^2}) \text{Av}(\kappa). \end{aligned}$$

The second equality in (5.7) is true, because by Lemmas 5.5 and 5.12,

$$\text{Av}_{L^2}(b(D)_{L^2}) \text{Av}(\kappa) = \text{Av}(b(D)_{L^2} \kappa) = \text{Av}(b(D)_A \kappa).$$

The element $\kappa b(D)_A$ is defined as

$$L_\kappa \circ b(D)_A \in \mathcal{K}_A(A) \cong A.$$

Here L_κ is left composition with κ . Lemma 5.12 implies that for all $\kappa, \kappa' \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$, the element $\kappa b(D)_A \kappa'$ has a smooth kernel, equal to the composition of the kernels of κ and of $b(D)_{L^2} \kappa'$. By associativity, that equals the composition of the smooth kernels of $\kappa b(D)_{L^2}$ and κ' . So $\kappa b(D)_A \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$, and its kernel is the smooth kernel of $\kappa b(D)_{L^2}$. Hence, by Lemma 5.5,

$$\text{Av}(\kappa b(D)_A) = \text{Av}(\kappa b(D)_{L^2}) = \text{Av}(\kappa) \text{Av}_{L^2}(b(D)_{L^2}).$$

□

6. THE INVARIANT INDEX

In this section, we use the averaging maps from Section 5 to prove Theorem 3.9.

6.1. The index of $\widetilde{\text{Av}}(b(D))$. Let b be a normalising function as below Proposition 3.3. That proposition implies that

$$\widetilde{\text{Av}}(b(D))^2 - 1 = \text{Av}(b(D)^2 - 1) \in \mathcal{K}(L_T^2(E))^G.$$

So $\widetilde{\text{Av}}(b(D))$ is a Fredholm operator.

We will prove Theorem 3.9 by proving Propositions 6.1 and 6.2 below. Here, to be precise, by the index of the odd-graded operator $\widetilde{\text{Av}}(b(D))$ on $L_T^2(E)^G$, we mean the index in the graded sense; i.e. the index of its restriction $\widetilde{\text{Av}}(b(D))_+$ to even-graded sections.

Proposition 6.1.

$$\text{index}(\widetilde{\text{Av}}(b(D))) = \text{index}(D)^G.$$

Proposition 6.2.

$$\text{index}(\widetilde{\text{Av}}(b(D))) = I_*(\text{index}_G^{\text{loc}}(D)).$$

Proof. Consider the boundary maps

$$\begin{aligned} \partial_{\mathcal{B}}: K_1(\mathcal{B}(L_T^2(E)^G)/\mathcal{K}(L_T^2(E)^G)) &\rightarrow K_0(\mathcal{K}(L_T^2(E)^G)); \\ \partial_A: K_1(\mathcal{L}_A(A)/\mathcal{K}_A(A)) &\rightarrow K_0(\mathcal{K}_A(A)). \end{aligned}$$

Naturality of boundary maps with respect to $*$ -homomorphisms implies that

$$\text{index}(\widetilde{\text{Av}}(b(D))) = \partial_{\mathcal{B}}[\widetilde{\text{Av}}(b(D))_+] = \text{Av}(\partial_A[b(D)_+])$$

Here we used the fact that $\widetilde{\text{Av}}(b(D))_+ = \widetilde{\text{Av}}(b(D)_+)$. Proposition 5.9 and (2.10) now imply that the right hand side equals

$$I_*(\partial_A[b(D)_+] \oplus 0) = I_*(\text{index}_G^{\text{loc}}(D)).$$

□

To prove Theorem 3.9, it remains to prove Proposition 6.1.

6.2. The index of $\widetilde{\text{Av}}(b(D))$ and the invariant index. In this subsection, we prove Proposition 6.1. The main point of the proof is dealing with the fact that sections in $L_T^2(E)^G$ are not square integrable in general.

We may choose the normalising function b so that $b(t) = \mathcal{O}(t)$ as $t \rightarrow 0$. Then the operator

$$S := \frac{b(D)}{D}$$

on $L^2(E)$ is bounded.

Lemma 6.3. *For all $\sigma \in \Gamma_c^\infty(E)$ and $s \in \Gamma^\infty(E) \cap L_T^2(E)^G$, the inner product $(Ds, S\sigma)_{L^2(E)}$ is well-defined and equals*

$$(\widetilde{\text{Av}}(b(D))s, \sigma)_{L^2(E)}.$$

Proof. As in Subsection 5.4, we use subscripts A and L^2 to distinguish functional calculus of operators on the Hilbert A -module A and on $L^2(E)$.

Let $(b_j)_{j=1}^\infty$ be a sequence in $C_b(\mathbb{R})$ converging to b in the sup-norm, such that for each j , and all $x \in \mathbb{R}$, $b_j(x) = xg_j(x)$, where $g_j \in C_0(\mathbb{R})$ has compactly supported Fourier transform. Then $g_j(D)_{L^2}$, and hence $b_j(D)_{L^2}$, has finite propagation. So by Lemma 5.6, for all $s \in \Gamma_{\text{tc}}^G$ and $\sigma \in \Gamma_c(E)$,

$$(6.1) \quad |(\text{Av}_{L^2}(b_j(D)_{L^2})s, \sigma)_{L^2(E)}| \leq \|b_j(D)_{L^2}\| |(s, \sigma)_{L^2(E)}|.$$

We also have

$$(6.2) \quad \|b_j(D)_{L^2}\| \leq \|b_j(D)_A\|_{\mathcal{L}_A(A)}$$

To see that this is true, note that by Lemma 5.12,

$$b_j(D)_{L^2} = b_j(D)_A =: \kappa \in C_{\text{ker}}^*(M; L^2(E))_{\text{loc}}^G.$$

And

$$\|\kappa\|_{\mathcal{B}(L^2(E))} = \|\kappa \oplus 0\|_{\mathcal{B}(L^2(E) \otimes L^2(G))} \leq \|\kappa \oplus 0\|_{\text{max}} = \|\kappa\|_A = \|\kappa\|_{\mathcal{L}_A(A)}.$$

The inequality is true, because the defining representation of $\mathcal{B}(L^2(E) \otimes L^2(G))$ in $L^2(E) \otimes L^2(G)$ trivially restricts to a $*$ -representation of $C_{\text{ker}}^*(M; L^2(E))_{\text{loc}}^G$. So (6.2) follows.

By the first point of Theorem 3.2,

$$b_j(D) \rightarrow b(D) \in \mathcal{L}_A(A).$$

By Lemma 5.13,

$$\widetilde{\text{Av}}(b_j(D)) = \text{Av}_{L^2}(b_j(D)) \in \mathcal{B}(L_T^2(E))^G.$$

This equality, together with (6.1) and (6.2) implies that

$$|(\widetilde{\text{Av}}(b_j(D))s, \sigma)_{L^2(E)}| \leq \|b_j(D)\|_{\mathcal{L}_A(A)} |(s, \sigma)_{L^2(E)}|$$

and hence

$$(\widetilde{\text{Av}}(b(D))s, \sigma)_{L^2(E)} = \lim_{j \rightarrow \infty} (\widetilde{\text{Av}}(b_j(D))s, \sigma)_{L^2(E)}.$$

This means it suffices to prove the claim for each finite-propagation approximant $b_j(D)$, namely that

$$(\widetilde{\text{Av}}(b_j(D))s, \sigma)_{L^2(E)} = (\text{Av}_{L^2}(b_j(D))s, \sigma)_{L^2(E)} = (Ds, S_j\sigma)_{L^2(E)}.$$

To prove the latter equality, let $r_j > 0$ be the propagation of $b_j(D)$, and let the partition of unity $\{\psi_k\}_{k=1}^\infty$ be as in (5.4). Suppose these functions are real-valued. For $\sigma \in \Gamma_c^\infty(E)$, only finitely many of the functions ψ_k have supports closer than r_j to $\text{supp}(\sigma)$. Let $K_\sigma \subset \mathbb{N}$ be the set of the corresponding indices k . Then for $s \in \Gamma^\infty(E) \cap L_T^2(E)^G$, since $b_j(D)$ commutes with finite sums,

$$\begin{aligned} (\text{Av}(b_j(D))s, \sigma)_{L^2(E)} &= (b_j(D) \sum_{k \in K_\sigma} \psi_k s, \sigma)_{L^2(E)} \\ &= (s, \sum_{k \in K_\sigma} \psi_k b_j(D)\sigma)_{L^2(E)} = (s, b_j(D)\sigma)_{L^2(E)} = (Ds, S_j\sigma)_{L^2(E)}. \end{aligned}$$

□

Lemma 6.4.

$$\ker(\widetilde{\text{Av}}(b(D))) \subset \Gamma^\infty(E)$$

Proof. Let $s \in \ker(\widetilde{\text{Av}}(b(D)))$. Then

$$s = \text{Av}(1 - b(D)^2)s.$$

Here we used the fact that $1 - b(D)^2 \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$. That also implies that the right hand side is smooth, and hence so is s . □

Lemma 6.5. *The space $S(\Gamma_c^\infty(E))$ is dense in $L^2(E)$.*

Proof. Let $s \in L^2(E)$, and let $(s_j)_{j=1}^\infty$ be a sequence in $\Gamma_c^\infty(E)$ converging to s in L^2 -norm. Then

$$\|Ss - Ss_j\|_{L^2(E)} \rightarrow 0.$$

So $S(\Gamma_c^\infty(E))$ is dense in $\text{im}(S)$. Now if $t \in \Gamma_c^\infty(E)$, then $t \in \text{dom}(S^{-1})$. Hence $t = S(S^{-1}t) \in \text{im}(S)$. So $\text{im}(S) \subset L^2(E)$ is dense, which completes the proof. □

Proof of Proposition 6.1. By elliptic regularity, $\ker(D) \subset \Gamma^\infty(E)$. So Lemma 6.3 implies that

$$\ker(D) \cap L_T^2(E)^G \subset \ker(\widetilde{\text{Av}}(b(D))).$$

We claim that for all $s \in \ker(\widetilde{\text{Av}}(b(D))) \cap \Gamma^\infty(E)$ and $\sigma \in \Gamma_c^\infty(E)$,

$$(6.3) \quad (Ds, \sigma)_{L^2(E)} = 0.$$

Indeed, Let $\sigma \in \Gamma_c^\infty(E)$. By Lemma 6.5, there is a sequence $(s_j)_{j=1}^\infty$ in $\Gamma_c^\infty(E)$ such that $(Ss_j)_{j=1}^\infty$ converges to σ in L^2 -norm. For all $s \in \ker(\widetilde{\text{Av}}(b(D))) \cap \Gamma^\infty(E)$, Lemma 6.3 implies that $(Ds, Ss_j)_{L^2(E)} = 0$. So (6.3) follows.

By Lemma 6.4, (6.3) implies that

$$\ker(\widetilde{\text{Av}}(b(D))) \subset \ker(D) \cap L_T^2(E)^G.$$

So

$$\ker(\widetilde{\text{Av}}(b(D))) = \ker(D) \cap L_T^2(E)^G,$$

including gradings. □

7. QUANTISATION COMMUTES WITH REDUCTION

In this section, we use Theorem 3.9 to prove Theorem 3.11.

7.1. A localisation estimate. Let U be a relatively cocompact, G -invariant neighbourhood of $\mu^{-1}(0)$. Since U is relatively cocompact, we can enlarge Z , outside which the estimate (3.4) holds, if needed, so that its interior contains the closure of U .

Fix a G -invariant metric on $M \times \mathfrak{g} \rightarrow M$, where G acts on \mathfrak{g} via the adjoint action. Let $\|\mu\|^2$ be the square of the norm of the Spin^c -moment map μ with respect to this metric. Let v^μ be the vector field on M induced by the map $M \rightarrow \mathfrak{g}$ dual to μ with respect to this metric. Explicitly, if $m \in M$ and $X \in \mathfrak{g}$ is dual to $\mu(m) \in \mathfrak{g}^*$ for the metric at m , then

$$v^\mu(m) = X^M(m).$$

By G -invariance of the metric on $M \times \mathfrak{g}$, this vector field is G -invariant. Let $f \in C^\infty(M)^G$ be a nonnegative function with cocompact support,² such that $f \equiv 1$ on Z . For any G -invariant real-valued $h \in C^\infty(M)$, and any $p \in \mathbb{Z}_{\geq 0}$ and $t > 0$, consider the operator

$$D_{p,h,t} := \tilde{D}^{L^p} + t(h\Phi - ifc(v^\mu))$$

on $\Gamma^\infty(\mathcal{S} \otimes L^p)$.

Let $\chi \in C^\infty(M)$ be a cutoff function for the action by G on M , as in (2.6). For $s \in \Gamma_{\text{tc}}^\infty(\mathcal{S} \otimes L^p)^G$, define

$$\|\chi s\|_{W^1(\mathcal{S} \otimes L^p)}^2 := \|\chi \tilde{D}^{L^p} s\|_{L^2(\mathcal{S} \otimes L^p)}^2 + \|\chi s\|_{L^2(\mathcal{S} \otimes L^p)}^2$$

The proof of Theorem 3.11 is based on the following localisation property. This is an analogue of Theorem 2.1 in [47]. Let U' be a G -invariant neighbourhood of $\mu^{-1}(0)$ whose closure is contained in U .

Proposition 7.1. *There are constants $C, b > 0$, a function $h \in C^\infty(M)^G$, equal to a constant greater than or equal to 1 outside U and constant 0 inside U' , and a $p_0 \in \mathbb{Z}_{\geq 0}$, such that for all $p \geq p_0$, $t \geq 1$ and all $s \in \Gamma_{\text{tc}}^\infty(\mathcal{S} \otimes L^p)^G$ supported outside U ,*

$$(7.1) \quad \|\chi D_{p,h,t} s\|_{L^2(\mathcal{S} \otimes L^p)}^2 \geq C(\|\chi s\|_{W^1(\mathcal{S} \otimes L^p)}^2 + (t - b)\|\chi s\|_{L^2(\mathcal{S} \otimes L^p)}^2).$$

Similarly to [25, 26, 37, 47], the proof of Proposition 7.1 is based on an expression for squares of deformed operators, Proposition 7.2. This expression is deduced from an expression in [26].

For any $p \in \mathbb{Z}_{\geq 0}$ and any G -equivariant differential operator D on $\Gamma^\infty(\mathcal{S} \otimes L^p)$, let \hat{D} be the operator on $\chi \Gamma_{\text{tc}}^\infty(\mathcal{S} \otimes L^p)^G$ defined by

$$\hat{D}(\chi s) = \chi D s,$$

for all $s \in \Gamma_{\text{tc}}^\infty(\mathcal{S} \otimes L^p)^G$. Note that

$$\chi \Gamma_{\text{tc}}^\infty(\mathcal{S} \otimes L^p)^G \subset \Gamma_c^\infty(\mathcal{S} \otimes L^p).$$

Let \hat{D}^* be the formal adjoint of \hat{D} with respect to the L^2 -inner product.

²In earlier work on deformed Dirac operators of the form $D + ifc(v^\mu)$ on non-cocompact manifolds [8, 9, 25, 26, 28], the function f was required to grow at infinity in a suitable way. In our setting, we actually need f to vanish outside a cocompact set (this is used in the proof of Proposition 7.1). This is possible, because invertibility at infinity is guaranteed by the term Φ .

Proposition 7.2. *There is a G -equivariant vector bundle endomorphism B' of $\mathcal{S} \otimes L^p$, and for all $t > 0$, there is a G -equivariant vector bundle endomorphism B_t of $\mathcal{S} \otimes L^p$, which vanishes at points where f and df vanish, and which satisfies the fibrewise estimate $B_t \geq tB'$ for all $t \geq 1$, such that, on sections in $\chi\Gamma_{\text{tc}}^\infty(\mathcal{S} \otimes L^p)^G$ supported outside U ,*

$$(7.2) \quad \hat{D}_{p,h,t}^* \hat{D}_{p,h,t} = \widehat{\tilde{D}}^{L^p}{}^* \widehat{\tilde{D}}^{L^p} + ht\{\tilde{D}, \Phi\} + h^2 t^2 \Phi^2 + B_t + (2p+1)2\pi t f \|\mu\|^2.$$

(Here, as in (3.9), we omit ‘ $\otimes 1_{L^p}$ ’.)

Proof. Corollary 8.5 in [26] states that, on $\chi\Gamma_{\text{tc}}^\infty(\mathcal{S} \otimes L^p)^G$,

$$\hat{D}_{p,0,t}^* \hat{D}_{p,0,t} = \widehat{\tilde{D}}^{L^p}{}^* \widehat{\tilde{D}}^{L^p} + tB'' + (2p+1)2\pi t f \|\mu\|^2 + t^2 f^2 \|v^\mu\|^2,$$

for a G -equivariant vector bundle endomorphism B'' of $\mathcal{S} \otimes L^p$, which vanishes at points where f and df vanish. Because h is constant outside U , this implies that, on sections in $\chi\Gamma_{\text{tc}}^\infty(\mathcal{S} \otimes L^p)^G$ supported outside U , the left hand side of (7.2) equals

$$(7.3) \quad \widehat{\tilde{D}}^{L^p}{}^* \widehat{\tilde{D}}^{L^p} + tB'' + (2p+1)2\pi t f \|\mu\|^2 + t^2 f^2 \|v^\mu\|^2 + t^2 h^2 \Phi^2 + ht\{\tilde{D}, \Phi\} - iht^2 f \{\Phi, c(v^\mu)\}.$$

The fact that $\{\tilde{D}, \Phi\}$ is a vector bundle endomorphism implies that $\{\Phi, c(v^\mu)\} = 0$.

Set

$$\begin{aligned} B' &:= B'' + f^2 \|v^\mu\|^2; \\ B_t &:= tB'' + t^2 f^2 \|v^\mu\|^2; \end{aligned}$$

Then (7.2) follows. □

For $h \in C^\infty(M)^G$, $p \in \mathbb{N}$ and $t > 0$, write

$$A_{p,h,t} := \hat{D}_{p,h,t}^* \hat{D}_{p,h,t} - \widehat{\tilde{D}}^{L^p}{}^* \widehat{\tilde{D}}^{L^p}.$$

This is a vector bundle endomorphism by Proposition 7.2.

Lemma 7.3. *There are a constant $C \in (0, 1]$, a function $h \in C^\infty(M)^G$, equal to a constant greater than or equal to 1 outside U and constant 0 inside U' , and a $p_0 \in \mathbb{Z}_{\geq 0}$, such that for all $p \geq p_0$, $t \geq 1$, the vector bundle endomorphism $A_{p,h,t}$ satisfies the pointwise estimate*

$$(7.4) \quad A_{p,h,t} \geq Ct$$

on $M \setminus U$.

Proof. Let B' be as in Proposition 7.2. Because it is G -equivariant, it is bounded on cocompact sets. Let $h \in C^\infty(M)^G$ be nonnegative, such that $h|_{U'} = 0$, $h \geq 1$ outside U and, on the cocompact set $\text{supp}(f) \setminus U$,

$$(7.5) \quad hc^2/2 \geq \|B'\|.$$

On the cocompact set $Z \setminus U$, the positive function $\|\mu\|^2$ is bounded below by a positive constant. And the G -equivariant vector bundle endomorphism $\{\tilde{D}, \Phi\}$ is bounded on that set as well. Choose $p_0 \in \mathbb{Z}_{\geq 0}$ such that, on $Z \setminus U$,

$$(7.6) \quad (2p_0 + 1)2\pi \|\mu\|^2 \geq \|B'\| + h\|\{\tilde{D}, \Phi\}\| + 1.$$

Set $C := \min(c^2/2, 1)$, where c is as in (3.4). We claim that the estimate (7.4) holds for these choices of p_0 , h and C and all $t \geq 1$.

Let $t \geq 1$, and let B_t be as in Proposition 7.2. Then $B_t = 0$ on $M \setminus \text{supp}(f)$, so Proposition 7.2 implies that on that set,

$$A_{p,h,t} = ht\{\tilde{D}, \Phi\} + h^2t^2\Phi^2.$$

Since $ht \geq 1$ outside U , the right hand side is at least equal to tc^2 .

On $\text{supp}(f) \setminus Z$, we similarly have

$$\begin{aligned} A_{p,h,t} &\geq t^2h^2\Phi^2 + th\{\tilde{D}, \Phi\} + tB' + (2p+1)2\pi t f\|\mu\|^2 \geq \\ &t^2h\Phi^2 + t\{\tilde{D}, \Phi\} + h^{-1}tB' \geq t(c^2 - h^{-1}\|B'\|) \geq tc^2/2, \end{aligned}$$

by (7.5)

Finally, on $Z \setminus U$,

$$A_{p,h,t} \geq t(h\{\tilde{D}, \Phi\} + B' + (2p+1)2\pi\|\mu\|^2) \geq t,$$

if $p \geq p_0$ as in (7.6). □

Proof of Proposition 7.1. Let h , p_0 and C be as in Lemma 7.3. Let $p \geq p_0$ and $t \geq 1$. Let $s \in \Gamma_{\text{tc}}^\infty(\mathcal{S} \otimes L^p)^G$ be supported outside U . Then Lemma 7.3 implies that

$$\begin{aligned} \|\chi D_{p,h} s\|_{L^2(\mathcal{S} \otimes L^p)}^2 &\geq \|\chi \tilde{D}^{L^p} s\|_{L^2(\mathcal{S} \otimes L^p)}^2 + Ct\|\chi s\|_{L^2(\mathcal{S} \otimes L^p)}^2 \\ &\geq C(\|\chi s\|_{W^1(\mathcal{S} \otimes L^p)}^2 + (t - C^{-1})\|\chi s\|_{L^2(\mathcal{S} \otimes L^p)}^2). \end{aligned}$$

Here we used the fact that $C \leq 1$. □

7.2. Proof of Theorem 3.11.

Lemma 7.4. *Let D be any operator as in Subsection 3.1, where E is $\mathbb{Z}/2$ -graded. Let $S \in \text{End}(E)^G$ be an odd, fibrewise self-adjoint vector bundle endomorphism which is zero outside a cocompact set. Then $(D + S)^2$ has a uniform lower bound outside a cocompact set, and*

$$\text{index}_G^{\text{loc}}(D + S) = \text{index}_G^{\text{loc}}(D) \in K_0(C_{\max}^*(G)).$$

Proof. The operator $D + S$ is elliptic, and $(D + S)^2$ has a positive lower bound outside $Z \cup \text{supp}(S)$. Hence its index is well-defined. Since $\text{supp}(S)$ is cocompact, S is a bounded operator on $L^2(E)$. Hence the path of operators

$$t \mapsto b(D + tS)$$

is continuous in the operator norm, where b is a normalising function as in Subsection 3.1. This defines an operator homotopy showing that

$$[b(D + tS)_+] \in K_1(\mathcal{L}_A(A)/\mathcal{K}_A(A))$$

is independent of t . □

Lemma 7.5. *For all constants $h \geq 1$,*

$$\text{index}_G^{\text{loc}}(\tilde{D} + \Phi) = \text{index}_G^{\text{loc}}(\tilde{D} + h\Phi).$$

Proof. Set

$$\tilde{D}_t := \tilde{D} + (1 - t + th)\Phi$$

For all t , we have $(1 - t + th) \geq 1$, so that

$$\tilde{D}_t^2 = \tilde{D}^2 + (1 - t + th)(\tilde{D}\Phi + \Phi\tilde{D}) + (1 - t + th)^2\Phi^2 \geq (D + \Phi)^2.$$

This has a positive lower bound outside a fixed cocompact set. So, for a suitable normalising function b , we have an invertible element

$$b(D_t)_+ \in \mathcal{L}_A(A)/\mathcal{K}_A(A)$$

for all $t \in [0, 1]$. Because $\|\Phi\|$ is bounded, Lemma 4.11 implies that this path of operators is continuous in the operator norm. Hence the class

$$[b(D_t)_+] \in K_1(\mathcal{L}_A(A)/\mathcal{K}_A(A))$$

is independent of t . □

The following consequence of Proposition 7.1 is an important step in the proof of Theorem 3.11. It is an analogue of Lemma 6.12 in [25].

Lemma 7.6. *Let p_0 and h be as in Proposition 7.1. For all $\lambda > 0$, there exists $t_0 > 0$ such that for all $t \geq t_0$, the intersection of the interval $(-\infty, \lambda]$ with the spectrum of $D_{p,h,t}^2$ as an unbounded, self-adjoint operator on $L_T^2(E)^G$ is discrete, with finite-dimensional eigenspaces.*

Proof. This follows from Proposition 7.1 in exactly the same way that Lemma 6.12 in [25] follows from Proposition 6.3 in that paper. □

Proof of Theorem 3.11. Let μ be as in Subsection 7.1. Let h and p_0 be as in Proposition 7.1, and fix $p \geq p_0$. Let t_0 be as in Lemma 7.6, and fix $t \geq t_0$. Lemmas 7.4 and 7.5 imply that

$$\text{index}_G^{\text{loc}}(D_p) = \text{index}_G^{\text{loc}}(D_{p,h,t}).$$

So by Theorem 3.9,

$$I_*(\text{index}_G^{\text{loc}}(D_p)) = \text{index}(D_{p,h,t})^G.$$

From this point onwards, one proves that

$$\text{index}(D_{p,h,t})^G = \text{index}(D_{M_0}^{L^p})$$

exactly following Section 7 of [25], where Proposition 7.1 and Lemma 7.6 in this paper should be substituted for Proposition 6.3 and Lemma 6.12 in [25], respectively. Furthermore, we use the fact that on the set U' ,

$$D_{p,h,t} = \tilde{D}^{L^p} - itc(v^\mu),$$

which is exactly the operator $D_t^{L^p}$ in [25], by Lemma 2.3 in [25]. (The minus sign is caused by a different sign convention in the definition of vector fields induced by Lie algebra elements; compare (3.11) to (20) in [25].) □

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