Journal of Combinatorial Theory, Series A 178 (2021) 105362

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Dwork-type supercongruences through a creative q-microscope

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ARTICLE INFO

Article history:
Received 12 May 2020
Received in revised form 28 October 2020
Accepted 4 November 2020
Available online 26 November 2020

Keywords:
Hypergeometric series
Supercongruence
q-Congruence
Creative microscoping

ABSTRACT

We develop an analytical method to prove congruences of the type

$$\sum_{k=0}^{(p^r-1)/d} A_k z^k \equiv \omega(z) \sum_{k=0}^{(p^{r-1}-1)/d} A_k z^{pk} \pmod{p^{mr}\mathbb{Z}_p[[z]]}$$
$$\text{for } r=1,2,\dots,$$

for primes $p>2$ and fixed integers $m,d\geqslant 1$, where $f(z)=\sum_{k=0}^{\infty} A_k z^k$ is an 'arithmetic' hypergeometric series. Such congruences for $m=d=1$ were introduced by Dwork in 1969 as a tool for p-adic analytical continuation of $f(z)$. Our proofs of several Dwork-type congruences corresponding to $m\geqslant 2$ (in other words, supercongruences) are based on constructing and proving their suitable q-analogues, which in turn have their own right for existence and potential for a q-deformation of modular forms and of cohomology groups of algebraic varieties. Our method follows the principles of creative microscoping introduced by us to tackle $r=1$

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[1] The first author was partially supported by the National Natural Science Foundation of China (grant 11771175).

https://doi.org/10.1016/j.jcta.2020.105362

instances of such congruences; it is the first method capable of establishing the supercongruences of this type for general r.

1. Introduction

Extending his work on the rationality of the zeta function of an algebraic variety defined over a finite field, Dwork [2] considered a question of continuing analytical solutions $f(z)=\sum_{k=0}^{\infty}A_kz^k$ of linear differential equations p-adically. A general strategy was to verify that the truncated sums $f_r(z)=\sum_{k=0}^{p^r-1}A_kz^k$, where $r=0,1,2,\dots$, satisfy the so-called Dwork congruences [33]

$$\frac{f_{r+1}(z)}{f_r(z^p)}\equiv\frac{f_r(z)}{f_{r-1}(z^p)} \pmod{p^r\mathbb{Z}_p[[z]]} \quad \text{for } r=1,2,\dots \tag{1.1}$$

(see [2, Theorem 3] for a precise statement). Formally, one needs the condition $f_1(z^p)=\sum_{k=0}^{p-1}A_kz^{pk}\not\equiv 0 \pmod{p\mathbb{Z}_p[[z]]}$ to make sense of (1.1). Then the congruences imply the existence of a p-adic analytical function ('unit root') $\omega(z)$ such that

$$\omega(z)=\lim_{r\to\infty}\frac{f_r(z)}{f_{r-1}(z^p)};$$

in other words,

$$\omega(z)\equiv\frac{f_r(z)}{f_{r-1}(z^p)} \pmod{p^r\mathbb{Z}_p[[z]]} \quad \text{for } r=1,2,\dots.$$

Notice that the argument extends to the cases when $f_1(z^p)\equiv 0 \pmod{p\mathbb{Z}_p[[z]]}$ but $f_1(z^p)\not\equiv 0 \pmod{p^m\mathbb{Z}_p[[z]]}$ for some $m\geqslant 2$, provided the congruences (1.1) hold modulo a higher power of p, for example,

$$\frac{f_{r+1}(z)}{f_r(z^p)}\equiv\frac{f_r(z)}{f_{r-1}(z^p)} \pmod{p^{mr}\mathbb{Z}_p[[z]]} \quad \text{for } r=1,2,\dots. \tag{1.2}$$

It is this type of congruences that we refer to as Dwork-type supercongruences; other truncations of the initial power series are possible as well, usually of the type $f_r(z)=\sum_{k=0}^{(p^r-1)/d}A_kz^k$ for some fixed positive integer d. Whether the congruences (1.2) are 'super' ($m\geqslant 2$) or not ($m=1$), we conclude from them that

$$f_r(z)\equiv\omega(z)f_{r-1}(z^p) \pmod{p^{mr}\mathbb{Z}_p[[z]]} \quad \text{for } r=1,2,\dots. \tag{1.3}$$

This gives an equivalent — somewhat more transparent — way to state Dwork-type (super)congruences in the case of known unit root $\omega(z)$.

Our illustrative examples include

$$\sum_{k=0}^{(p^r-1)/2}(8k+1)\frac{\binom{4k}{2k}\binom{2k}{k}^2}{2^{8k}3^{2k}} \equiv p\left(\frac{-3}{p}\right)\sum_{k=0}^{(p^{r-1}-1)/2}(8k+1)\frac{\binom{4k}{2k}\binom{2k}{k}^2}{2^{8k}3^{2k}} \pmod{p^{3r}}, \tag{1.4}$$

$$\sum_{k=0}^{p^r-1}(8k+1)\frac{\binom{4k}{2k}\binom{2k}{k}^2}{2^{8k}3^{2k}} \equiv p\left(\frac{-3}{p}\right)\sum_{k=0}^{p^{r-1}-1}(8k+1)\frac{\binom{4k}{2k}\binom{2k}{k}^2}{2^{8k}3^{2k}} \pmod{p^{3r}}, \tag{1.5}$$

where $\left(\frac{-3}{\cdot}\right)$ denotes the Kronecker symbol, valid for any prime $p>3$ and integer $r\geqslant 1$ and corresponding to the truncation of the power series

$$\sum_{k=0}^{\infty}(8k+1)\binom{4k}{2k}\binom{2k}{k}^2\frac{z^k}{2^{8k}3^{2k}}$$

at $z=1$. We point out that not so many supercongruences of this type are recorded in the literature; the principal sources are the conjectures from Swisher's paper [44], in turn built on Van Hamme's list [47], and a geometric heuristics for hypergeometric series $f(z)$ outlined by Roberts and Rodriguez-Villegas in [36]. The only *proven* cases known (namely, weaker forms of Conjectures (C.3) and (J.3) from [44] together with their companions) for arbitrary $r\geqslant 1$ are due to the first author [17].

The principal goal of this paper is to extend the approach of [17] and establish general techniques for proving Dwork-type supercongruences using the method of creative microscoping, which we initiated in [23] for proving $r=1$ instances of such supercongruences. Observe that such $r=1$ cases of (1.4), (1.5) (known as Ramanujan-type supercongruences [50]) served as principal illustrations of how the creative microscope machinery works. It should be therefore not surprising that we place them again as principal targets. Here we prove Dwork-type supercongruences (1.4), (1.5) by establishing the following q-analogues of them.

Theorem 1.1. *Let $n>1$ be an integer coprime with 6 and let $r\geqslant 1$. Then, modulo $[n^r]\prod_{j=1}^{r}\Phi_{n^j}(q)^2$,*

$$\sum_{k=0}^{(n^r-1)/2}[8k+1]\frac{(q;q^2)_k^2(q;q^2)_{2k}}{(q^6;q^6)_k^2(q^2;q^2)_{2k}}q^{2k^2}$$
$$\equiv q^{(1-n)/2}[n]\left(\frac{-3}{n}\right)\sum_{k=0}^{(n^{r-1}-1)/2}[8k+1]_{q^n}\frac{(q^n;q^{2n})_k^2(q^n;q^{2n})_{2k}}{(q^{6n};q^{6n})_k^2(q^{2n};q^{2n})_{2k}}q^{2nk^2}, \tag{1.6}$$

$$\sum_{k=0}^{n^r-1}[8k+1]\frac{(q;q^2)_k^2(q;q^2)_{2k}}{(q^6;q^6)_k^2(q^2;q^2)_{2k}}q^{2k^2}$$
$$\equiv q^{(1-n)/2}[n]\left(\frac{-3}{n}\right)\sum_{k=0}^{n^{r-1}-1}[8k+1]_{q^n}\frac{(q^n;q^{2n})_k^2(q^n;q^{2n})_{2k}}{(q^{6n};q^{6n})_k^2(q^{2n};q^{2n})_{2k}}q^{2nk^2}. \tag{1.7}$$

Here and throughout the paper we adopt the standard q-notation: $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ is the q-shifted factorial (q-Pochhammer symbol), $[n]=[n]_q=(1-q^n)/(1-q)$ is the q-integer, and

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta_n^k),$$

is the n-th cyclotomic polynomial, where $\zeta_n=e^{2\pi i/n}$ is an n-th primitive root of unity. Also recall the ordinary shifted factorial $(a)_n=\Gamma(a+n)/\Gamma(a)=a(a+1)\cdots(a+n-1)$ for $n=0,1,2,\dots$. In what follows, the congruence $A_1(q)/A_2(q)\equiv 0 \pmod{P(q)}$ for polynomials $A_1(q),A_2(q),P(q)\in\mathbb{Z}[q]$ is understood as $P(q)$ divides $A_1(q)$ and is coprime with $A_2(q)$; more generally, $A(q)\equiv B(q)\pmod{P(q)}$ for rational functions $A(q),B(q)\in\mathbb{Z}(q)$ means $A(q)-B(q)\equiv 0\pmod{P(q)}$.

It is not hard to check (see [23,52] for related details of this computation) that, when $n=p$ is a prime and $q\to 1$, the q-supercongruences (1.6) and (1.7) reduce to (1.4) and (1.5), respectively.

Another family of Dwork-type supercongruences

$$\sum_{k=0}^{(p^r-1)/2}\frac{(\frac12)_k^3}{k!^3}(3k+1)2^{2k}\equiv p\sum_{k=0}^{(p^{r-1}-1)/2}\frac{(\frac12)_k^3}{k!^3}(3k+1)2^{2k} \pmod{p^{3r}}, \tag{1.8}$$

$$\sum_{k=0}^{p^r-1}\frac{(\frac12)_k^3}{k!^3}(3k+1)2^{2k}\equiv p\sum_{k=0}^{p^{r-1}-1}\frac{(\frac12)_k^3}{k!^3}(3k+1)2^{2k} \pmod{p^{4r-\delta_{p,3}}}, \tag{1.9}$$

expectedly valid for any prime $p>2$ and integer $r\geqslant 1$, originate from the *divergent* hypergeometric series

$$\sum_{k=0}^{\infty}\frac{(\frac12)_k^3}{k!^3}(3k+1)(2^2z)^k$$

at $z=1$. (Here $\delta_{i,j}$ is the usual Kronecker delta, $\delta_{i,j}=1$ if $i=j$ and $\delta_{i,j}=0$ otherwise.) The congruences (1.8) and (1.9) modulo p^3 merge into the single entry

$$\sum_{k=0}^{(p-1)/2}\frac{(\frac12)_k^3}{k!^3}(3k+1)2^{2k}\equiv p \pmod{p^3} \quad \text{for } p>2, \tag{1.10}$$

when $r=1$, because $(\frac12)_k\equiv 0 \pmod p$ for $(p-1)/2<k\leqslant p-1$; these 'divergent' Ramanujan-type supercongruences were proved by Guillera and the second author [5] (while independently observed numerically by Sun [42, Conjecture 5.1 (ii)]). The first author [12] gave a q-analogue of (1.10) and recorded (1.8), (1.9) as conjectures. In this paper we prove the supercongruences (1.8), (1.9) modulo p^{3r} by establishing the following q-counterparts.

Theorem 1.2. *Let $n > 1$ be odd and $r \geqslant 1$. Then, modulo $[n^r]\prod_{j=1}^{r}\Phi_{n^j}(q)^2$,*

$$\sum_{k=0}^{(n^r-1)/2}[3k+1]\frac{(q;q^2)_k^3 q^{-\binom{k+1}{2}}}{(q;q)_k^2(q^2;q^2)_k} \equiv q^{(1-n)/2}[n]\sum_{k=0}^{(n^{r-1}-1)/2}[3k+1]_{q^n}\frac{(q^n;q^{2n})_k^3 q^{-n\binom{k+1}{2}}}{(q^n;q^n)_k^2(q^{2n};q^{2n})_k}, \tag{1.11}$$

$$\sum_{k=0}^{n^r-1}[3k+1]\frac{(q;q^2)_k^3 q^{-\binom{k+1}{2}}}{(q;q)_k^2(q^2;q^2)_k} \equiv q^{(1-n)/2}[n]\sum_{k=0}^{n^{r-1}-1}[3k+1]_{q^n}\frac{(q^n;q^{2n})_k^3 q^{-n\binom{k+1}{2}}}{(q^n;q^n)_k^2(q^{2n};q^{2n})_k}. \tag{1.12}$$

Although q-supercongruences serve here as a principal tool for proving their non-q-counterparts, they have established themselves as an independent topic. For some recent developments on q-supercongruences we refer the reader to the papers [4,6,7,10,12–15,18–21,23,24,27,35,39,45,51].

Both hypergeometric identities and congruences for their truncations originate from their q-hypergeometric versions in a very natural way, through the asymptotics as $q \to 1$ for the former and as q approaches other roots of unity for the latter; it is this asymptotic analysis at roots of unity, which we refer to as 'q-microscopic'. Notice that proving a congruence $A(q) \equiv B(q) \pmod{\Phi_N(q)}$ is equivalent to verifying that $A(\zeta) = B(\zeta)$ for all primitive N-th roots of unity ζ. Furthermore, proofs of the congruences require 'creative' introduction of extra (generic) parameter a (and, possibly, some other); those parameters are often (but not always!) suggested by general forms of the underlying q-hypergeometric identities. The intermediate parametric supercongruences of the form $A(q,a) \equiv B(q,a)$ are verified to be true modulo polynomials $a-q^N$ and $1-aq^N$ (for particular choices of integers N) by showing that $A(q,q^N) = B(q,q^N)$ and $A(q,q^{-N}) = B(q,q^{-N})$; afterwards, the dependence on the parameter is eliminated via a careful analysis of degeneration as $a \to 1$. A plain overview of the method can be found in [52]. Quite remarkably, the strategy of creative q-microscoping makes it possible to prove many congruences that are not accessible to other techniques.

The exposition below is organized as follows. In Section 2 we provide detailed proofs of Theorems 1.1 and 1.2. The methodology set up in that section is further used in Section 3 to prove several other q-supercongruences whose limiting $q \to 1$ cases correspond to Dwork-type supercongruences, occasionally conjectured in the existing literature. Most of the results in Section 3 are supplied with sketches of their proofs. Finally, in Section 4 we leave several open problems about q-congruences behind Dwork-type (super)congruences (1.3) and discuss possible future of the q-setup.

In our proofs below we make use of transformation formulas of basic hypergeometric series [3]

$${}_{s+1}\phi_s\left[\begin{matrix}a_0, a_1, \dots, a_s\\ b_1, b_2, \dots, b_s\end{matrix}; q, z\right] = \sum_{k=0}^{\infty}\frac{(a_0,a_1,\dots,a_s;q)_k\, z^k}{(q,b_1,\dots,b_s;q)_k},$$

where the symbol $(a_0, a_1, \ldots, a_s; q)_k$ is a shortcut for $\prod_{\ell=0}^{s}(a_\ell; q)_k$.

2. Proof of the principal theorems

2.1. Proof of Theorem 1.1

We shall make use of the following q-congruences, which are special cases of [23, Theorem 1.4].

Lemma 2.1. *Let n be a positive integer coprime with 6. Then*

$$\sum_{k=0}^{(n-1)/2}[8k+1]\frac{(aq,q/a;q^2)_k(q;q^2)_{2k}}{(aq^6,q^6/a;q^6)_k(q^2;q^2)_{2k}}q^{2k^2} \equiv 0 \pmod{[n]},$$

$$\sum_{k=0}^{n-1}[8k+1]\frac{(aq,q/a;q^2)_k(q;q^2)_{2k}}{(aq^6,q^6/a;q^6)_k(q^2;q^2)_{2k}}q^{2k^2} \equiv 0 \pmod{[n]}.$$

We need the following q-series identity (see [23, Lemma 3.1]), which plays an important role in our proof of $r=1$ instances of (1.4) and (1.5).

Lemma 2.2. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{(n-1)/2}[8k+1]\frac{(q^{1-n},q^{1+n};q^2)_k(q;q^2)_{2k}}{(q^{6-n},q^{6+n};q^6)_k(q^2;q^2)_{2k}}q^{2k^2} = q^{(1-n)/2}[n]\left(\frac{-3}{n}\right). \tag{2.1}$$

In order to prove Theorem 1.1, we need to establish the following parametric generalization.

Theorem 2.3. *Let $n>1$ be an integer coprime with 6 and let $r\geqslant 1$. Then, modulo*

$$[n^r]\prod_{j=0}^{(n^{r-1}-1)/d}(1-aq^{(2j+1)n})(a-q^{(2j+1)n}),$$

we have

$$\sum_{k=0}^{(n^r-1)/d}[8k+1]\frac{(aq,q/a;q^2)_k(q;q^2)_{2k}}{(aq^6,q^6/a;q^6)_k(q^2;q^2)_{2k}}q^{2k^2}$$
$$\equiv q^{(1-n)/2}[n]\left(\frac{-3}{n}\right)\sum_{k=0}^{(n^{r-1}-1)/d}[8k+1]_{q^n}\frac{(aq^n,q^n/a;q^{2n})_k(q^n;q^{2n})_{2k}}{(aq^{6n},q^{6n}/a;q^{6n})_k(q^{2n};q^{2n})_{2k}}q^{2nk^2}, \tag{2.2}$$

where $d=1,2$.

Proof. By Lemma 2.1 with n replaced by n^r, we see that the left-hand side of (2.2) is congruent to 0 modulo $[n^r]$. On the other hand, replacing n by n^{r-1} and q by q^n in Lemma 2.1, we conclude that the summation on the right-hand side of (2.2) is congruent to 0 modulo $[n^{r-1}]_{q^n}$. Furthermore, since n is odd, it is easily seen that the q-integer $[n]$ is relatively prime to $1+q^k$ for any positive integer k, and so it is also relatively prime to the denominators of the sum on the right-hand side of (2.2) because

$$\frac{(q^n;q^{2n})_{2k}}{(q^{2n};q^{2n})_{2k}} = \begin{bmatrix}4k\\2k\end{bmatrix}_{q^n} \frac{1}{(-q^n;q^n)_{2k}^2},$$

where $\begin{bmatrix}2k\\k\end{bmatrix}_{q^n} = (q^n;q^n)_{2k}/(q^n;q^n)_k^2$ denotes the central q-binomial coefficient. This proves that the right-hand side of (2.2) is congruent to 0 modulo $[n][n^{r-1}]_{q^n} = [n^r]$; hence the q-congruence (2.2) is true modulo $[n^r]$.

To show it also holds modulo

$$\prod_{j=0}^{(n^{r-1}-1)/d} (1-aq^{(2j+1)n})(a-q^{(2j+1)n}), \tag{2.3}$$

we only need to prove that both sides of (2.2) are identical when we take $a=q^{-(2j+1)n}$ or $a=q^{(2j+1)n}$ for any j with $0 \leqslant j \leqslant (n^{r-1}-1)/d$, that is,

$$\sum_{k=0}^{(n^r-1)/d} [8k+1] \frac{(q^{1-(2j+1)n}, q^{1+(2j+1)n};q^2)_k (q;q^2)_{2k}}{(q^{6-(2j+1)n}, q^{6+(2j+1)n};q^6)_k (q^2;q^2)_{2k}} q^{2k^2}$$
$$= q^{(1-n)/2}[n]\left(\frac{-3}{n}\right) \sum_{k=0}^{(n^{r-1}-1)/d} [8k+1]_{q^n} \frac{(q^{-2jn}, q^{(2j+2)n};q^{2n})_k (q^n;q^{2n})_{2k}}{(q^{(5-2j)n}, q^{(2j+7)n};q^{6n})_k (q^{2n};q^{2n})_{2k}} q^{2nk^2}. \tag{2.4}$$

It is easy to see that $(n^r-1)/d \geqslant ((2j+1)n-1)/2$ for $0 \leqslant j \leqslant (n^{r-1}-1)/d$, and $(q^{1-(2j+1)n};q^2)_k = 0$ for $k > ((2j+1)n-1)/2$. By Lemma 2.2 the left-hand side of (2.4) is equal to

$$q^{(1-(2j+1)n)/2}[(2j+1)n]\left(\frac{-3}{(2j+1)n}\right).$$

Likewise, the right-hand side of (2.4) is equal to

$$q^{(1-n)/2}[n]\left(\frac{-3}{n}\right) \cdot q^{-jn}[2j+1]_{q^n}\left(\frac{-3}{2j+1}\right) = q^{(1-(2j+1)n)/2}[(2j+1)n]\left(\frac{-3}{(2j+1)n}\right).$$

This proves (2.4). Namely, the q-congruence (2.2) holds modulo (2.3). Since $[n^r]$ and (2.3) are relatively prime polynomials, the proof of (2.2) is complete. □

Proof of Theorem 1.1. It is not hard to see that the limit of (2.3) as $a \to 1$ has the factor

$$\begin{cases} \prod_{j=1}^{r} \Phi_{n^j}(q)^{2n^{r-j}} & \text{if } d=1, \\ \prod_{j=1}^{r} \Phi_{n^j}(q)^{n^{r-j}+1} & \text{if } d=2. \end{cases}$$

Note that the denominator of the left-hand side of (2.2) is a multiple of that of the right-hand side of (2.2). Since $\gcd(n,6)=1$, the factor related to a of the former is

$$(aq^6;q^6)_{(n^r-1)/d}(q^6/a;q^6)_{(n^r-1)/d},$$

whose limit as $a \to 1$ only has the factor

$$\begin{cases} \prod_{j=1}^{r} \Phi_{n^j}(q)^{2n^{r-j}-2} & \text{if } d=1, \\ \prod_{j=1}^{r} \Phi_{n^j}(q)^{n^{r-j}-1} & \text{if } d=2, \end{cases}$$

related to $\Phi_n(q), \Phi_{n^2}(q), \dots, \Phi_{n^r}(q)$. Hence, letting $a \to 1$ in (2.2) we conclude that (1.6) is true modulo $\prod_{j=1}^{r}\Phi_{n^j}(q)^3$, where one product $\prod_{j=1}^{r}\Phi_{n^j}(q)$ comes from $[n^r]$.

Finally, by [23, Theorem 1.1] we obtain

$$\sum_{k=0}^{(n-1)/d}[8k+1]\frac{(q;q^2)_k^2(q;q^2)_{2k}}{(q^2;q^2)_{2k}(q^6;q^6)_k^2}q^{2k^2} \equiv 0 \pmod{[n]} \quad \text{for } d=1,2.$$

Replacing n by n^r in the above congruences, we deduce that the left-hand sides of (1.6) and (1.7) are congruent to 0 modulo $[n^r]$, while letting $q \mapsto q^n$ and $n \mapsto n^{r-1}$ in the above congruences, we see that the right-hand sides of them are congruent to 0 modulo $[n][n^{r-1}]_{q^n}=[n^r]$ as well. This means that the q-congruences (1.6) and (1.7) hold modulo $[n^r]$. The proof then immediately follows from the fact that the least common multiple of $\prod_{j=1}^{r}\Phi_{n^j}(q)^3$ and $[n^r]$ is just $[n^r]\prod_{j=1}^{r}\Phi_{n^j}(q)^2$. □

2.2. Proof of Theorem 1.2

Similarly to what we did above, we need the following q-congruence and q-identity; they follow from the $b \to 0$ case of [23, Theorem 4.8].

Lemma 2.4. *Let n be a positive odd integer, and $d=1$ or 2. Then*

$$\sum_{k=0}^{(n-1)/d}[3k+1]\frac{(aq,q/a;q^2)_k(q;q^2)_k}{(aq,q/a;q)_k(q^2;q^2)_k}q^{-\binom{k+1}{2}} \equiv 0 \pmod{[n]}, \tag{2.5}$$

$$\sum_{k=0}^{(n-1)/2}[3k+1]\frac{(q^{1-n},q^{1+n};q^2)_k(q;q^2)_k}{(q^{1-n},q^{1+n};q)_k(q^2;q^2)_k}q^{-\binom{k+1}{2}} = q^{(1-n)/2}[n]. \tag{2.6}$$

For a real number x, we use the standard notation $\lfloor x\rfloor$ and $\lceil x\rceil$ for the floor (integer part) and ceiling functions; these integers satisfy $\lfloor x\rfloor \leqslant x \leqslant \lceil x\rceil$. We have the following parametric generalization of Theorem 1.2.

Theorem 2.5. *Let $n>1$ be an odd integer and $r\geqslant 1$. Then, modulo*

$$[n^r]\prod_{j=\lceil (n^{r-1}-1)/(2d)\rceil}^{(n^{r-1}-1)/d}(1-aq^{(2j+1)n})(a-q^{(2j+1)n}),$$

we have

$$\begin{aligned}\sum_{k=0}^{(n^r-1)/d}&[3k+1]\frac{(aq,q/a;q^2)_k(q;q^2)_k}{(aq,q/a;q)_k(q^2;q^2)_k}q^{-\binom{k+1}{2}}\\ &\equiv q^{(1-n)/2}[n]\sum_{k=0}^{(n^{r-1}-1)/d}[3k+1]_{q^n}\frac{(aq^n,q^n/a;q^{2n})_k(q^n;q^{2n})_k}{(aq^n,q^n/a;q^n)_k(q^{2n};q^{2n})_k}q^{-nk\binom{k+1}{2}},\end{aligned} \tag{2.7}$$

where $d=1,2$.

Proof. Replacing n by n^r in (2.5), we see that the left-hand side of (2.7) is congruent to 0 modulo $[n^r]$. Moreover, replacing n by n^{r-1} and q by q^n in (2.5) means that the right-hand side of (2.7) is congruent to 0 modulo $[n][n^{r-1}]_{q^n}=[n^r]$. That is, the q-congruence (2.7) holds modulo $[n^r]$.

To prove it is also true modulo

$$\prod_{j=\lceil (n^{r-1}-1)/(2d)\rceil}^{(n^{r-1}-1)/d}(1-aq^{(2j+1)n})(a-q^{(2j+1)n}), \tag{2.8}$$

it suffices to show that both sides of (2.7) are equal for all $a=q^{-(2j+1)n}$ and $a=q^{(2j+1)n}$ with $(n^{r-1}-1)/(2d)\leqslant j\leqslant (n^{r-1}-1)/d$, i.e.,

$$\begin{aligned}\sum_{k=0}^{(n^r-1)/d}&[3k+1]\frac{(q^{1-(2j+1)n},q^{1+(2j+1)n};q^2)_k(q;q^2)_k}{(q^{1-(2j+1)n},q^{1+(2j+1)n};q)_k(q^2;q^2)_k}q^{-\binom{k+1}{2}}\\ &=q^{(1-n)/2}[n]\sum_{k=0}^{(n^{r-1}-1)/d}[3k+1]_{q^n}\frac{(q^{-2jn},q^{(2j+2)n};q^{2n})_k(q^n;q^{2n})_k}{(q^{-2jn},q^{(2j+2)n};q^n)_k(q^{2n};q^{2n})_k}q^{-n\binom{k+1}{2}}.\end{aligned} \tag{2.9}$$

It is easy to see that $(n^r-1)/d\geqslant((2j+1)n-1)/2$ and $(2j+1)n>(n^r-1)/d$ for $\lceil (n^{r-1}-1)/2d\rceil\leqslant j\leqslant (n^{r-1}-1)/d$. Hence, the left-hand side of (2.9) is well-defined (the denominator is non-zero) and is equal to

$$\sum_{k=0}^{((2j+1)n-1)/2} [3k+1] \frac{(q^{1-(2j+1)n}, q^{1+(2j+1)n}; q^2)_k (q;q^2)_k}{(q^{1-(2j+1)n}, q^{1+(2j+1)n}; q)_k (q^2;q^2)_k} q^{-\binom{k+1}{2}}$$
$$= q^{(1-(2j+1)n)/2}[(2j+1)n]$$

by (2.6). Similarly, the right-hand side of (2.9) is equal to

$$q^{(1-n)/2}[n] \cdot q^{-jn}[2j+1]_{q^n} = q^{(1-(2j+1)n)/2}[(2j+1)n],$$

and so the identity (2.9) holds. Namely, the q-congruence (2.7) is true modulo (2.8). This completes the proof of (2.7). □

Proof of Theorem 1.2. This time the limit of (2.8) as $a \to 1$ has the factor

$$\begin{cases} \prod_{j=1}^{r} \Phi_{n^j}(q)^{n^{r-j}+1} & \text{if } d=1, \\ \prod_{j=1}^{r} \Phi_{n^j}(q)^{n^{r-j}+1-2\lfloor (n^{r-j}+1)/4 \rfloor} & \text{if } d=2, \end{cases}$$

where in the $d = 2$ case we use the fact the set $\{(2j+1)n : j = 0, \dots, \lfloor (n^{r-1}-3)/4 \rfloor\}$ contains exactly $\lfloor (n^{r-j}+1)/4 \rfloor$ multiples of n^j for $j = 1, \dots, r$.

On the other hand, the denominator of (the reduced form of) the left-hand side of (2.7) is a multiple of that of the right-hand side of (2.7). The factor related to a of the denominator is

$$\begin{cases} \dfrac{(aq, q/a; q)_{n^r-1}}{(aq, q/a; q^2)_{(n^r-1)/2}} = (aq^2, q^2/a; q^2)_{(n^r-1)/2} & \text{if } d=1, \\ \dfrac{(aq, q/a; q)_{(n^r-1)/2}}{(aq, q/a; q^2)_{\lceil (n^r-1)/4 \rceil}} = (aq^2, q^2/a; q^2)_{\lfloor (n^r-1)/4 \rfloor} & \text{if } d=2. \end{cases}$$

Its limit as $a \to 1$ only has the following factor

$$\begin{cases} \prod_{j=1}^{r} \Phi_{n^j}(q)^{n^{r-j}-1} & \text{if } d=1, \\ \prod_{j=1}^{r} \Phi_{n^j}(q)^{2\lfloor (n^{r-j}-1)/4 \rfloor} & \text{if } d=2, \end{cases}$$

related to $\Phi_n(q), \Phi_{n^2}(q), \dots, \Phi_{n^r}(q)$. Therefore, setting $a \to 1$ in (2.7), we conclude that (1.12) holds modulo $\prod_{j=1}^{r} \Phi_{n^j}(q)^3$, where one product $\prod_{j=1}^{r} \Phi_{n^j}(q)$ is from $[n^r]$.

Finally, along the lines of proof of Theorem 1.1, using the following q-congruences from [12]:

$$\sum_{k=0}^{(n-1)/d} [3k+1] \frac{(q;q^2)_k^3 q^{-\binom{k+1}{2}}}{(q;q)_k^2 (q^2;q^2)_k} \equiv 0 \pmod{[n]} \quad \text{for } d=1,2,$$

we can prove that the q-congruences (1.11) and (1.12) hold modulo $[n^r]$, thus completing the proof of the theorem. □

3. More Dwork-type q-congruences

Throughout this section, p always denotes an odd prime. Below we give q-analogues of some known or conjectural Dwork-type congruences. In particular, we completely confirm the supercongruence conjectures (B.3), (L.3) of Swisher [44] and also confirm the first cases of her conjectures (E.3) and (F.3).

3.1. Another q-analogue of (1.8) and (1.9)

From [10,25] we see that supercongruences may have different q-analogues. Here we show that the supercongruences (1.8) and (1.9) fall into this category and possess q-analogues different from those presented in Theorem 1.2.

Theorem 3.1. *Let $n>1$ be odd and let $r\geqslant 1$. Then, modulo $[n^r]\prod_{j=1}^{r}\Phi_{n^j}(q)^2$,*

$$\sum_{k=0}^{(n^r-1)/d}[3k+1]\frac{(q;q^2)_k^3(-1;q)_k q^k}{(q;q)_k^3(-q^2;q)_{2k}} \equiv \frac{1+q}{1+q^n}[n]\sum_{k=0}^{(n^{r-1}-1)/2}[3k+1]_q\frac{(q^n;q^{2n})_k^3(-1;q^n)_k q^{nk}}{(q^n;q^n)_k^3(-q^{2n};q^n)_{2k}}, \tag{3.1}$$

where $d=1,2$.

Sketch of proof. Letting $b=-1$ in [23, Theorem 4.8], we get the following q-congruence: modulo $[n](1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{(n-1)/d}[3k+1]\frac{(aq,q/a,q;q^2)_k(-1;q)_k}{(aq,q/a,q;q)_k(-q^2;q)_{2k}}q^k \equiv \frac{1+q}{1+q^n}[n], \tag{3.2}$$

where $d=1,2$. This means that the left-hand side of (3.2) is congruent to 0 modulo $[n]$, and also (when $a=q^n$) that

$$\sum_{k=0}^{(n-1)/2}[3k+1]\frac{(q^{1-n},q^{1+n},q;q^2)_k(-1;q)_k}{(q^{1-n},q^{1+n},q;q)_k(-q^2;q)_{2k}}q^k = \frac{1+q}{1+q^n}[n].$$

Thus, like in the proof of Theorem 1.2, we can establish the following parametric generalization of (3.1): modulo

$$[n^r]\prod_{j=\lceil (n^{r-1}-1)/(2d)\rceil}^{(n^{r-1}-1)/d}(1-aq^{(2j+1)n})(a-q^{(2j+1)n}),$$

we have

$$\sum_{k=0}^{(n^r-1)/d}[3k+1]\frac{(aq,q/a,q;q^2)_k(-1;q)_k}{(aq,q/a,q;q)_k(-q^2;q)_{2k}}q^k \equiv \frac{1+q}{1+q^n}[n]\sum_{k=0}^{(n^{r-1}-1)/d}[3k+1]_{q^n}\frac{(aq^n,q^n/a,q^n;q^{2n})_k(-1;q^n)_k}{(aq^n,q^n/a,q^n;q^n)_k(-q^{2n};q^n)_{2k}}q^{nk}, \tag{3.3}$$

where $d=1,2$.

Letting $a\to 1$ in (3.3), we conclude that the q-congruence (3.1) is true modulo $\prod_{j=1}^r\Phi_{n^j}(q)^3$. Note that the proof of [20, Theorem 6.1] also implies that (3.2) modulo $[n]$ holds for $a=1$. Applying this q-congruence on both sides of (3.1), we deduce that (3.1) are also true modulo $[n^r]$. □

3.2. Another 'divergent' Dwork-type supercongruence

Guillera and the second author [5] proved the following 'divergent' Ramanujan-type supercongruence:

$$\sum_{k=0}^{(p-1)/2}\frac{(\frac12)_k^3}{k!^3}(3k+1)(-1)^k2^{3k}\equiv p\left(\frac{-1}{p}\right)\pmod{p^3} \tag{3.4}$$

(see also [42, Conjecture 5.1(ii)]). The first author [12] gave a q-analogue of (3.4) and proposed the following conjecture on Dwork-type supercongruences:

$$\sum_{k=0}^{(p^r-1)/2}\frac{(\frac12)_k^3}{k!^3}(3k+1)(-1)^k2^{3k} \equiv p\left(\frac{-1}{p}\right)\sum_{k=0}^{(p^{r-1}-1)/2}\frac{(\frac12)_k^3}{k!^3}(3k+1)(-1)^k2^{3k}\pmod{p^{3r+\delta_{p,3}}}, \tag{3.5}$$

$$\sum_{k=0}^{p^r-1}\frac{(\frac12)_k^3}{k!^3}(3k+1)(-1)^k2^{3k}\equiv p\left(\frac{-1}{p}\right)\sum_{k=0}^{p^{r-1}-1}\frac{(\frac12)_k^3}{k!^3}(3k+1)(-1)^k2^{3k}\pmod{p^{3r}}. \tag{3.6}$$

In the spirit of Theorems 1.1 and 1.2, we have the following q-generalization of the above two supercongruences modulo p^{3r}.

Theorem 3.2. *Let $n>1$ be odd and let $r\geqslant 1$. Then, modulo $[n^r]\prod_{j=1}^r\Phi_{n^j}(q)^2$,*

$$\sum_{k=0}^{(n^r-1)/d}(-1)^k[3k+1]\frac{(q;q^2)_k^3(-q;q)_k}{(q;q)_k^3(-q^2;q^2)_k}q^{-\binom{k+1}{2}} \equiv q^{(1-n)/2}[n]\left(\frac{-1}{n}\right)\sum_{k=0}^{(n^{r-1}-1)/d}(-1)^k[3k+1]_{q^n}\frac{(q^n;q^{2n})_k^3(-q^n;q^n)_k}{(q^n;q^n)_k^3(-q^{2n};q^{2n})_k}q^{-n\binom{k+1}{2}}, \tag{3.7}$$

where $d = 1, 2$.

Sketch of proof. Letting $b = -1$ and $c \to 0$ in [20, Theorem 6.1] (see also [23, Conjecture 4.6]), we get the following q-congruence: modulo $[n](1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{(n-1)/d} (-1)^k [3k+1] \frac{(aq, q/a, q; q^2)_k (-q; q)_k}{(aq, q/a, q; q)_k (-q^2; q^2)_k} q^{-\binom{k+1}{2}} \equiv q^{(1-n)/2} [n] \left(\frac{-1}{n}\right), \qquad (3.8)$$

where $d = 1, 2$. Namely, the left-hand side of (3.7) is congruent to 0 modulo $[n]$, and

$$\sum_{k=0}^{(n-1)/2} (-1)^k [3k+1] \frac{(q^{1-n}, q^{1+n}, q; q^2)_k (-q; q)_k}{(q^{1-n}, q^{1+n}, q; q)_k (-q^2; q^2)_k} q^{-\binom{k+1}{2}} = q^{(1-n)/2} [n] \left(\frac{-1}{n}\right).$$

Thus, we may establish a parametric generalization of (3.7): modulo

$$[n^r] \prod_{j=\lceil (n^{r-1}-1)/(2d) \rceil}^{(n^{r-1}-1)/d} (1 - aq^{(2j+1)n})(a - q^{(2j+1)n}),$$

we have

$$\begin{aligned} &\sum_{k=0}^{(n^r-1)/d} (-1)^k [3k+1] \frac{(aq, q/a, q; q^2)_k (-q; q)_k}{(aq, q/a, q; q)_k (-q^2; q^2)_k} q^{-\binom{k+1}{2}} \\ &\quad \equiv q^{(1-n)/2} [n] \left(\frac{-1}{n}\right) \sum_{k=0}^{(n^{r-1}-1)/d} (-1)^k [3k+1]_{q^n} \\ &\qquad \times \frac{(aq^n, q^n/a, q^n; q^{2n})_k (-q^n; q^n)_k}{(aq^n, q^n/a, q^n; q^n)_k (-q^{2n}; q^{2n})_k} q^{-n\binom{k+1}{2}}, \end{aligned} \qquad (3.9)$$

where $d = 1, 2$.

Letting $a \to 1$ in (3.9), we know that (3.7) holds modulo $\prod_{j=1}^{r} \Phi_{n^j}(q)^3$. Applying the q-congruence (3.8) modulo $[n]$ with $a = 1$ on both sides of (3.7), we conclude that (3.7) is also true modulo $[n^r]$. $\square$

3.3. Two supercongruences of Swisher

Swisher's conjectural supercongruence (B.3) from [44] can be stated as follows:

$$\sum_{k=0}^{(p^r-1)/2} (-1)^k (4k+1) \frac{(\frac{1}{2})_k^3}{k!^3}$$

$$\equiv \begin{cases} p\left(\dfrac{-1}{p}\right) \displaystyle\sum_{k=0}^{(p^{r-1}-1)/2} (-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3} \pmod{p^{3r}} & \text{if } p \equiv 1 \pmod 4, \\ p^2 \displaystyle\sum_{k=0}^{(p^{r-2}-1)/2} (-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3} \pmod{p^{3r-2}} & \text{if } p \equiv 3 \pmod 4,\ r \geqslant 2. \end{cases}$$

In fact we find out that, more generally, for any prime $p > 2$,

$$\sum_{k=0}^{(p^r-1)/2} (-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3} \equiv p\left(\frac{-1}{p}\right) \sum_{k=0}^{(p^{r-1}-1)/2} (-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3} \pmod{p^{3r}}. \tag{3.10}$$

Observe that Swisher's supercongruence (B.3) for $p \equiv 3 \pmod 4$ follows from using (3.10) twice. It is natural to conjecture that the following companion supercongruence of (3.10) is also true:

$$\sum_{k=0}^{p^r-1} (-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3} \equiv p\left(\frac{-1}{p}\right) \sum_{k=0}^{p^{r-1}-1} (-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3} \pmod{p^{3r}}. \tag{3.11}$$

Here we prove the Dwork-type supercongruences (3.10) and (3.11) by establishing the following q-analogues.

Theorem 3.3. *Let $n > 1$ be odd and let $r \geqslant 1$. Then, modulo $[n^r]\prod_{j=1}^r \Phi_{n^j}(q)^2$,*

$$\sum_{k=0}^{(n^r-1)/d} (-1)^k[4k+1]\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}$$
$$\equiv q^{(1-n)/2}[n]\left(\frac{-1}{n}\right) \sum_{k=0}^{(n^{r-1}-1)/d} (-1)^k[4k+1]_{q^n}\frac{(q^n;q^{2n})_k^2(q^{2n};q^{4n})_k}{(q^{2n};q^{2n})_k^2(q^{4n};q^{4n})_k}, \tag{3.12}$$

where $d = 1, 2$.

Sketch of proof. Letting $c = -1$ in [23, Theorem 4.2], we obtain the following q-congruence for odd n: modulo $[n](1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{(n-1)/d} (-1)^k[4k+1]\frac{(aq,q/a;q^2)_k(q^2;q^4)_k}{(aq^2,q^2/a;q^2)_k(q^4;q^4)_k} \equiv q^{(1-n)/2}[n]\left(\frac{-1}{n}\right), \tag{3.13}$$

where $d = 1, 2$. That is to say, the left-hand side of (3.13) is congruent to 0 modulo $[n]$, and

$$\sum_{k=0}^{(n-1)/2} (-1)^k[4k+1]\frac{(q^{1-n},q^{1+n};q^2)_k(q^2;q^4)_k}{(q^{2-n},q^{2+n};q^2)_k(q^4;q^4)_k} = q^{(1-n)/2}[n]\left(\frac{-1}{n}\right).$$

Along the lines of our proof of Theorem 1.1, we can prove the following parametric version of (3.12): modulo

$$[n^r]\prod_{j=0}^{(n^{r-1}-1)/d}(1-aq^{(2j+1)n})(a-q^{(2j+1)n}),$$

we have

$$\sum_{k=0}^{(n^r-1)/d}(-1)^k[4k+1]\frac{(aq,q/a;q^2)_k(q^2;q^4)_k}{(aq^2,q^2/a;q^2)_k(q^4;q^4)_k}$$
$$\equiv q^{(1-n)/2}[n]\left(\frac{-1}{n}\right)\sum_{k=0}^{(n^{r-1}-1)/d}(-1)^k[4k+1]_{q^n}\frac{(aq^n,q^n/a;q^{2n})_k(q^{2n};q^{4n})_k}{(aq^{2n},q^{2n}/a;q^{2n})_k(q^{4n};q^{4n})_k}, \tag{3.14}$$

where $d=1,2$.

Letting $a\to 1$ in (3.14), we see that (3.12) is true modulo $\prod_{j=1}^{r}\Phi_{n^j}(q)^3$. Note that the proof of [23, Theorem 4.2] also indicates that the q-congruence (3.13) modulo $[n]$ hold for $a=1$. Applying this q-congruence on both sides of (3.12), we conclude that (3.12) is also true modulo $[n^r]$. □

Swisher [44, Conjecture (L.3)] conjectured that, for $r\geqslant 1$,

$$\sum_{k=0}^{(p^r-1)/2}(-1)^k(6k+1)\frac{(\frac12)_k^3}{k!^3 8^k}\equiv p\left(\frac{-2}{p}\right)\sum_{k=0}^{(p^{r-1}-1)/2}(-1)^k(6k+1)\frac{(\frac12)_k^3}{k!^3 8^k}\pmod{p^{3r}}. \tag{3.15}$$

Recently, the first author [6, Conjecture 4.5] made the following similar conjecture:

$$\sum_{k=0}^{p^r-1}(-1)^k(6k+1)\frac{(\frac12)_k^3}{k!^3 8^k}\equiv p\left(\frac{-2}{p}\right)\sum_{k=0}^{p^{r-1}-1}(-1)^k(6k+1)\frac{(\frac12)_k^3}{k!^3 8^k}\pmod{p^{3r}}. \tag{3.16}$$

We confirm the supercongruences (3.15) and (3.16) by establishing the following Dwork-type q-supercongruence.

Theorem 3.4. *Let $n>1$ be odd and let $r\geqslant 1$. Then, modulo $[n^r]\prod_{j=1}^{r}\Phi_{n^j}(q)^2$,*

$$\sum_{k=0}^{(n^r-1)/d}(-1)^k[6k+1]\frac{(q;q^2)_k^3(-q^2;q^4)_k}{(q^4;q^4)_k^3(-q;q^2)_k}q^{k^2}$$
$$\equiv q^{(1-n)/2}[n]\left(\frac{-2}{n}\right)\sum_{k=0}^{(n^{r-1}-1)/d}(-1)^k[6k+1]_{q^n}\frac{(q^n;q^{2n})_k^3(-q^{2n};q^{4n})_k}{(q^{4n};q^{4n})_k^3(-q^n;q^{2n})_k}q^{nk^2}, \tag{3.17}$$

where $d = 1, 2$.

Sketch of proof. Setting $b = -q^2$ in [23, Theorem 4.5], we are led to the following q-congruence: modulo $[n](1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{(n^r-1)/d} (-1)^k [6k+1] \frac{(aq, q/a, q; q^2)_k (-q^2; q^4)_k}{(aq^4, q^4/a, q^4; q^4)_k (-q; q^2)_k} q^{k^2} \equiv q^{(1-n)/2}[n]\left(\frac{-2}{n}\right). \tag{3.18}$$

Thus, we can prove the following parametric version of (3.12): modulo

$$[n^r] \prod_{j=0}^{(n^{r-1}-1)/d} (1-aq^{(2j+1)n})(a-q^{(2j+1)n}),$$

we have

$$\begin{aligned}\sum_{k=0}^{(n^r-1)/d} &(-1)^k [6k+1] \frac{(aq, q/a, q; q^2)_k (-q^2; q^4)_k}{(aq^4, q^4/a, q^4; q^4)_k (-q; q^2)_k} q^{k^2} \\ &\equiv q^{(1-n)/2}[n]\left(\frac{-2}{n}\right) \sum_{k=0}^{(n^{r-1}-1)/d} (-1)^k [6k+1]_{q^n} \frac{(aq^n, q^n/a, q^n; q^{2n})_k (-q^{2n}; q^{4n})_k}{(aq^{4n}, q^{4n}/a, q^{4n}; q^{4n})_k (-q^n; q^{2n})_k} q^{nk^2},\end{aligned} \tag{3.19}$$

where $d = 1, 2$. The proof of (3.17) modulo $\prod_{j=1}^r \Phi_{n^j}(q)^3$ then follows by taking the limit as $a \to 1$ in (3.19), and the proof of (3.17) modulo $[n^r]$ follows from the q-congruence (3.18) modulo $[n]$ with $a = 1$. □

3.4. Another two supercongruences from Swisher's list

In [44, Conjectures (E.3), (F.3)] Swisher proposed the following conjectures:

$$\sum_{k=0}^{(p^r-1)/3} \frac{(6k+1)(\frac{1}{3})_k^3}{k!^3(-1)^k} \equiv p \sum_{k=0}^{(p^{r-1}-1)/3} \frac{(6k+1)(\frac{1}{3})_k^3}{k!^3(-1)^k} \pmod{p^{3r}} \quad \text{for } p \equiv 1 \pmod 3, \tag{3.20}$$

$$\sum_{k=0}^{(p^r-1)/4} \frac{(8k+1)(\frac{1}{4})_k^3}{k!^3(-1)^k} \equiv p\left(\frac{-2}{p}\right) \sum_{k=0}^{(p^{r-1}-1)/4} \frac{(8k+1)(\frac{1}{4})_k^3}{k!^3(-1)^k} \pmod{p^{3r}} \quad \text{for } p \equiv 1 \pmod 4. \tag{3.21}$$

Here we confirm (3.20) and (3.21) by showing the following q-analogues.

Theorem 3.5. *Let $n > 1$ be an integer with $n \equiv 1 \pmod 6$ and let $r \geqslant 1$. Then, modulo $[n^r]_{q^2} \prod_{j=1}^r \Phi_{n^j}(q^2)^2$,*

$$\sum_{k=0}^{(n^r-1)/d}(-1)^k[6k+1]_{q^2}\frac{(q^2;q^6)_k^3(-q^3;q^6)_k}{(q^6;q^6)_k^3(-q^5;q^6)_k}q^k$$
$$\equiv q^{1-n}[n]_{q^2}\sum_{k=0}^{(n^{r-1}-1)/d}(-1)^k[6k+1]_{q^{2n}}\frac{(q^{2n};q^{6n})_k^3(-q^{3n};q^{6n})_k}{(q^{6n};q^{6n})_k^3(-q^{5n};q^{6n})_k}q^{nk}, \tag{3.22}$$

where $d=1,3$.

Sketch of proof. It is easy to see that [23, Theorem 4.2] can be generalized as follows. Modulo $[n](1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{(n-1)/d}[2mk+1]\frac{(aq,q/a,q/c,q;q^m)_k}{(aq^m,q^m/a,cq^m,q^m;q^m)_k}c^kq^{(m-2)k}$$
$$\equiv\frac{(c/q)^{(n-1)/m}(q^2/c;q^m)_{(n-1)/m}}{(cq^m;q^m)_{(n-1)/m}}[n]\quad\text{for } n\equiv 1 \pmod m, \tag{3.23}$$

where $d=1$ or m. Here we emphasize that, in order to prove (3.23) holds modulo $[n]$, we need to show that

$$\sum_{k=0}^{n-1}[2mk+1]\frac{(aq,q/a,q/c,q;q^m)_k}{(aq^m,q^m/a,cq^m,q^m;q^m)_k}c^kq^{(m-2)k}\equiv 0 \pmod{\Phi_n(q)}$$

is true for all integers $n>1$ with $\gcd(n,m)=1$. Then we use the same arguments as [23, Theorems 1.2 and 1.3] to deal with the modulus $[n]$ case.

We now put $m=3$, $q\mapsto q^2$ and $c=-q^{-1}$ in (3.23) to get

$$\sum_{k=0}^{(n-1)/d}(-1)^k[6k+1]_{q^2}\frac{(aq^2,q^2/a,q^2,-q^3;q^6)_k}{(aq^6,q^6/a,q^6,-q^5;q^6)_k}q^k$$
$$\equiv q^{1-n}[n]_{q^2}(-1)^{n-1} \pmod{\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})}\quad\text{for } n\equiv 1 \pmod 6,$$

where $d=1,3$. Using this q-congruence, we can produce a generalization of (3.22) with an extra parameter a: modulo

$$[n^r]_{q^2}\prod_{j=0}^{(n^{r-1}-1)/d}(1-aq^{(6j+2)n})(a-q^{(6j+2)n}),$$

we have

$$\sum_{k=0}^{(n^r-1)/d}(-1)^k[6k+1]_{q^2}\frac{(aq^2,q^2/a,q^2,-q^3;q^6)_k}{(aq^6,q^6/a,q^6,-q^5;q^6)_k}q^k$$

$$\equiv q^{1-n}[n]_{q^2}\sum_{k=0}^{(n^{r-1}-1)/d}(-1)^k[6k+1]_{q^{2n}}\frac{(aq^{2n},q^{2n}/a,q^{2n},-q^{3n};q^{6n})_k}{(aq^{6n},q^{6n}/a,q^{6n},-q^{5n};q^{6n})_k}q^{nk},$$

where $d=1,3$. □

It is easy to see that, when $n=p$ and $q\to 1$, the q-supercongruence (3.22) for $d=3$ reduces to (3.20), and it for $d=1$ confirms the first supercongruence in [8, Conjecture 5.3]. Moreover, letting $n=p$ and $q\to -1$ in (3.22), we obtain the following new Dwork-type supercongruence: for $p\equiv 1 \pmod 3$,

$$\sum_{k=0}^{(p^r-1)/d}(6k+1)\frac{(\frac13)_k^3(\frac12)_k}{k!^3(\frac56)_k}\equiv p\sum_{k=0}^{(p^{r-1}-1)/d}(6k+1)\frac{(\frac13)_k^3(\frac12)_k}{k!^3(\frac56)_k} \pmod{p^{3r}},$$

where $d=1,3$.

When r is even and $p>3$, we always have $p^2\equiv 1 \pmod{24}$. Thus, letting $n=p^2$, $r\mapsto r/2$ and $q\to 1$ in (3.22) we arrive at

$$\sum_{k=0}^{(p^r-1)/3}\frac{(6k+1)(\frac13)_k^3}{k!^3(-1)^k}\equiv p^2\sum_{k=0}^{(p^{r-2}-1)/3}\frac{(6k+1)(\frac13)_k^3}{k!^3(-1)^k} \pmod{p^{2r}} \quad \text{for } r\geqslant 2 \text{ even.} \tag{3.24}$$

This partially confirm the second case of [44, Conjecture (E.3)], which asserts that (3.24) holds modulo p^{3r-2} for $p\equiv 2 \pmod 3$.

Theorem 3.6. *Let $n>1$ be an integer with $n\equiv 1 \pmod 4$ and let $r\geqslant 1$. Then, modulo $[n^r]\prod_{j=1}^{r}\Phi_{n^j}(q)^2$,*

$$\sum_{k=0}^{(n^r-1)/d}(-1)^k[8k+1]\frac{(q;q^4)_k^3(-q^2;q^4)_k}{(q^4;q^4)_k^3(-q^3;q^4)_k}q^k$$
$$\equiv q^{(1-n)/2}[n]\left(\frac{-2}{n}\right)\sum_{k=0}^{(n^{r-1}-1)/d}(-1)^k[8k+1]_{q^n}\frac{(q^n;q^{4n})_k^3(-q^{2n};q^{4n})_k}{(q^{4n};q^{4n})_k^3(-q^{3n};q^{2n})_k}q^{nk}, \tag{3.25}$$

where $d=1,4$.

Sketch of proof. This time we take $m=4$ and $c=-q^{-1}$ in (3.23) to get

$$\sum_{k=0}^{(n-1)/d}(-1)^k[8k+1]\frac{(aq,q/a,q,-q^2;q^4)_k}{(aq^4,q^4/a,q^4,-q^3;q^4)_k}q^k$$
$$\equiv q^{(1-n)/2}[n]\left(\frac{-2}{n}\right) \pmod{\Phi_n(q)(1-aq^n)(a-q^n)} \quad \text{for } n\equiv 1 \pmod 4,$$

where $d = 1, 4$, and we use $(-1)^{(n-1)/4} = \left(\frac{-2}{n}\right)$ for $n \equiv 1 \pmod 4$. Applying this q-congruence, we can produce a generalization of (3.25) with an extra parameter a: modulo

$$[n^r] \prod_{j=0}^{(n^{r-1}-1)/d} (1 - aq^{(4j+1)n})(a - q^{(4j+1)n}),$$

we have

$$\begin{aligned}\sum_{k=0}^{(n^r-1)/d} &(-1)^k [8k+1] \frac{(aq, q/a, q, -q^2; q^4)_k}{(aq^4, q^4/a, q^4, -q^3; q^4)_k} q^k \\ &\equiv q^{(1-n)/2}[n]\left(\frac{-2}{n}\right) \sum_{k=0}^{(n^{r-1}-1)/d} (-1)^k [8k+1]_{q^n} \frac{(aq^n, q^n/a, q^n, -q^{2n}; q^{4n})_k}{(aq^{4n}, q^{4n}/a, q^{4n}, -q^{3n}; q^{2n})_k} q^{nk},\end{aligned}$$

where $d = 1, 4$. □

It is easy to see that, when $n = p$ and $q \to 1$, the q-supercongruence (3.25) reduces to (3.21) when $d = 4$, and confirms the third supercongruence in [8, Conjecture 5.3] when $d = 1$. Besides, letting $n = p^2$, $r \mapsto r/2$, and $q \to 1$ in (3.25) we obtain

$$\sum_{k=0}^{(p^r-1)/4} \frac{(8k+1)(\frac14)_k^3}{k!^3(-1)^k} \equiv p^2 \sum_{k=0}^{(p^{r-2}-1)/4} \frac{(8k+1)(\frac14)_k^3}{k!^3(-1)^k} \pmod{p^{2r}}$$

for $r \geqslant 2$ even. This confirms in part the second case of [44, Conjecture (F.3)], where the supercongruence is predicted to hold modulo p^{3r-2} for $p \equiv 3 \pmod 4$.

Finally, we should mention the recent work [48], which saw the light after a preliminary version of this work had appeared; there Wang and Yue gave generalizations of Theorems 3.3, 3.5 and 3.6.

3.5. Generalizations of Swisher-type supercongruences

The $m = 3$ case of [10, Conjecture 6.1] asserts that

$$\sum_{k=0}^{(p^r-1)/d} (-1)^k (4k+1)^3 \frac{(\frac12)_k^3}{k!^3} \equiv p\left(\frac{-1}{p}\right) \sum_{k=0}^{(p^{r-1}-1)/d} (-1)^k (4k+1)^3 \frac{(\frac12)_k^3}{k!^3} \pmod{p^{3r-2}}, \tag{3.26}$$

where $d = 1, 2$. Here we confirm this supercongruence by establishing its q-analogue. Although there is a q-analogue of (3.26) modulo p^3 for $r = 1$ in [10], we need a different one to accomplish the proof of (3.26).

Lemma 3.7. *Let $n>1$ be an odd integer and a an indeterminate. Then, modulo $\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})$,*

$$\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(aq^2,q^2/a;q^4)_k(q^4;q^8)_k}{(aq^4,q^4/a;q^4)_k(q^8;q^8)_k}q^{-4k} \equiv q^{1-n}[n]_{q^2}\left(\frac{-1}{n}\right)\left(1-\frac{(1+q^2)(1-aq^2)(1-q^2/a)}{(1+q^4)(1-q)^2}\right). \tag{3.27}$$

Proof. For $a=q^{-2n}$ or $a=q^{2n}$, the left-hand side of (3.27) is equal to

$$\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(q^{2-2n},q^{2+2n};q^4)_k(q^4;q^8)_k}{(q^{4-2n},q^{4+2n};q^4)_k(q^8;q^8)_k}q^{-4k} = {}_8\phi_7\left[\begin{matrix}q^2,\ q^5,\ -q^5,\ q^5,\ q^5,\ -q^2,\ q^{2+2n},\ q^{2-2n}\\ q,\ -q,\ q,\ q,\ -q^4,\ q^{4-2n},\ q^{4+2n}\end{matrix};q^4,\ -q^{-4}\right], \tag{3.28}$$

where the basic hypergeometric series ${}_{s+1}\phi_s$ is defined in the introduction. By Watson's ${}_8\phi_7$ transformation formula [3, Appendix (III.18)] with $q\mapsto q^4$, $a=q^2$, $b=c=q^5$, $d=-q^2$, $e=q^{2+2n}$ and $n\mapsto(n-1)/2$, we can write the right-hand side of (3.28) as

$$\frac{(q^6,-q^{2-2n};q^4)_{(n-1)/2}}{(-q^4,q^{4-2n};q^4)_{(n-1)/2}}\,{}_4\phi_3\left[\begin{matrix}q^{-4},\ -q^2,\ q^{2+2n},\ q^{2-2n}\\ q,\ q,\ -q^4\end{matrix};q^4,\ q^4\right] = q^{1-n}[n]_{q^2}\left(\frac{-1}{n}\right)\left(1-\frac{(1+q^2)(1-q^{2-2n})(1-q^{2+2n})}{(1+q^4)(1-q)^2}\right), \tag{3.29}$$

which is just the $a=q^{-2n}$ or $a=q^{2n}$ case of the right-hand side of (3.27). This proves that the congruence (3.27) holds modulo $1-aq^{2n}$ or $a-q^{2n}$.

Moreover, by [20, Lemma 3.1] it is easy to verify that, for $0\leqslant k\leqslant(n-1)/2$, the k-th and $((n-1)/2-k)$-th terms on the left-hand side of (3.27) modulo $\Phi_n(q^2)$ cancel each other. Therefore, the left-hand side of (3.27) is congruent to 0 modulo $\Phi_n(q^2)$, and (3.27) is also true modulo $\Phi_n(q^2)$. □

We are now able to give a complicated q-analogue of (3.26).

Theorem 3.8. *Let $n>1$ be an odd integer and $r\geqslant 2$. Then, modulo*

$$\begin{cases}[n^r]_{q^2}\Phi_n(-q)^2\prod_{j=2}^r\Phi_{n^j}(q^2)^2 & \text{if } n>3,\\ [n^r]_{q^2}\Phi_n(q^2)\Phi_{n^2}(q^2)\Phi_n(-q)\Phi_{n^2}(-q)\prod_{j=3}^r\Phi_{n^j}(q^2)^2 & \text{if } n=3,\end{cases}$$

we have

$$\sum_{k=0}^{(n^r-1)/d}(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(q^2;q^4)_k^2(q^4;q^8)_k}{(q^4;q^4)_k^2(q^8;q^8)_k}q^{-4k}$$
$$\equiv q^{2-2n}[n]_{q^2}\left(\frac{-1}{n}\right)\frac{(1+q+q^2)(1+q^{4n})}{(1+q^4)(1+q^n+q^{2n})}$$
$$\times\sum_{k=0}^{(n^{r-1}-1)/d}(-1)^k[4k+1]_{q^{2n}}[4k+1]_{q^n}^2\frac{(q^{2n};q^{4n})_k^2(q^{4n};q^{8n})_k}{(q^{4n};q^{4n})_k^2(q^{8n};q^{8n})_k}q^{-4nk}, \tag{3.30}$$

where $d=1,2$.

Sketch of proof. Applying (3.27), we can prove the following parametric version of (3.30): modulo

$$[n^r]_{q^2}\prod_{j=0}^{(n^{r-1}-1)/d}(1-aq^{(4j+2)n})(a-q^{(4j+2)n}), \tag{3.31}$$

we have

$$\sum_{k=0}^{(n^r-1)/d}(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(aq^2,q^2/a;q^4)_k(q^4;q^8)_k}{(aq^4,q^4/a;q^4)_k(q^8;q^8)_k}q^{-4k}$$
$$\equiv q^{1-n}[n]_{q^2}\left(\frac{-1}{n}\right)\left(1-\frac{(1+q^2)(1-aq^2)(1-q^2/a)}{(1+q^4)(1-q)^2}\right)$$
$$\times\left(1-\frac{(1+q^{2n})(1-aq^{2n})(1-q^{2n}/a)}{(1+q^{4n})(1-q^n)^2}\right)^{-1}$$
$$\times\sum_{k=0}^{(n^{r-1}-1)/d}(-1)^k[4k+1]_{q^{2n}}[4k+1]_{q^n}^2\frac{(aq^{2n},q^{2n}/a;q^{4n})_k(q^{4n};q^{8n})_k}{(aq^{4n},q^{4n}/a;q^{4n})_k(q^{8n};q^{8n})_k}q^{-4nk}, \tag{3.32}$$

where $d=1,2$.

Similarly as before, the limit of (3.31) as $a\to 1$ has the factor

$$\begin{cases}\prod_{j=1}^r\Phi_{n^j}(q^2)^{2n^{r-j}+1} & \text{if } d=1,\\ \prod_{j=1}^r\Phi_{n^j}(q^2)^{n^{r-j}+2} & \text{if } d=2,\end{cases} \tag{3.33}$$

where one product $\prod_{j=1}^r\Phi_{n^j}(q^2)$ comes from $[n^r]_{q^2}$. However, this time we should be careful of the factor related to a in the common denominator of the two sides of (3.32). But it is at most

$$\big((1+q^{4n})(1-q^n)^2-(1+q^{2n})(1-aq^{2n})(1-q^{2n}/a)\big)$$
$$\times(aq^4;q^4)_{(n^r-1)/d}(q^4/a;q^4)_{(n^r-1)/d},$$

of which the limit as $a \to 1$ only contains the factor

$$\begin{cases} \Phi_n(q)^2 \prod_{j=1}^r \Phi_{n^j}(q^2)^{2n^{r-j}-2} & \text{if } d=1 \text{ and } n>3, \\ \Phi_n(q)^2 \prod_{j=1}^r \Phi_{n^j}(q^2)^{n^{r-j}-1} & \text{if } d=2 \text{ and } n>3, \\ \Phi_n(q)^2\Phi_{n^2}(q) \prod_{j=1}^r \Phi_{n^j}(q^2)^{2n^{r-j}-2} & \text{if } d=1 \text{ and } n=3, \\ \Phi_n(q)^2\Phi_{n^2}(q) \prod_{j=1}^r \Phi_{n^j}(q^2)^{n^{r-j}-1} & \text{if } d=2 \text{ and } n=3, \end{cases}$$

related to $\Phi_n(q^2), \Phi_{n^2}(q^2), \dots, \Phi_{n^r}(q^2)$. Here we used the identity

$$(1+q^{4n})(1-q^n)^2 - (1+q^{2n})(1-q^{2n})^2 = -2q^n(1+q^n+q^{2n})(1-q^n)^2. \tag{3.34}$$

Thus, letting $a \to 1$ in (3.32), we see that the q-congruence (3.30) holds modulo

$$\begin{cases} \Phi_n(q^2)\Phi_n(-q)^2 \prod_{j=2}^r \Phi_{n^j}(q^2)^3 & \text{if } n>3, \\ \Phi_n(q^2)\Phi_{n^2}(q^2)^2\Phi_n(-q)^2\Phi_{n^2}(-q) \prod_{j=3}^r \Phi_{n^j}(q^2)^3 & \text{if } n=3. \end{cases}$$

On the other hand, letting $a \to 1$ in (3.27), we can easily deduce that the left-hand side of (3.30) is congruent to

$$-2q^{2-n}[n]_{q^2}\left(\frac{-1}{n}\right)\frac{1+q+q^2}{1+q^4} \pmod{\Phi_n(q^2)^3},$$

which indicates that it is congruent to 0 modulo $\Phi_n(q)^2$ when $n=3$. Namely, the q-congruence (3.30) holds modulo $\Phi_n(q)^2$ when $n=3$. Combining this with the previous argument, we conclude that the q-congruence (3.30) is true modulo

$$\begin{cases} \Phi_n(q^2)\Phi_n(-q)^2 \prod_{j=2}^r \Phi_{n^j}(q^2)^3 & \text{if } n>3, \\ \Phi_n(q^2)^2\Phi_{n^2}(q^2)^2\Phi_n(-q)\Phi_{n^2}(-q) \prod_{j=3}^r \Phi_{n^j}(q^2)^3 & \text{if } n=3. \end{cases}$$

Furthermore, based on (3.27), along the lines of the proof of [23, Theorem 1.2] we can show that

$$\sum_{k=0}^{(n-1)/d} (-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(q^2;q^4)_k^2(q^4;q^8)_k}{(q^4;q^4)_k^2(q^8;q^8)_k}q^{-4k} \equiv 0 \pmod{[n]_{q^2}} \tag{3.35}$$

for $d=1,2$. Utilizing this q-congruence, we can prove that both sides of (3.30) are congruent to 0 modulo $[n^r]_{q^2}$. □

It is not hard to see that, when $n=p$ and $q \to 1$, the q-supercongruence (3.30) reduces to (3.26) for $r \geqslant 2$ (the case $r=1$ of (3.26) is obviously true by [10] or (3.35)). Moreover, letting $n=p$ and $q \to -1$ in (3.30), we are led to (3.10) again.

Similarly, we can partially confirm another conjecture in [10]. Recall that the $m=3$ case of [10, Conjecture 6.2] may be stated as follows:

$$\sum_{k=0}^{(p^r-1)/d}(4k+1)^3\frac{(\frac12)_k^4}{k!^4}\equiv p\sum_{k=0}^{(p^{r-1}-1)/d}(4k+1)^3\frac{(\frac12)_k^4}{k!^4}\pmod{p^{4r-3}},\tag{3.36}$$

where $d=1,2$. Here we prove that (3.36) is true modulo p^{3r-2} using the following q-supercongruences.

Theorem 3.9. *Let $n>1$ be an odd integer and $r\geqslant 2$. Then, modulo*

$$[n^r]_{q^2}\Phi_n(-q)^2\prod_{j=2}^{r}\Phi_{n^j}(q^2)^2,$$

we have

$$\sum_{k=0}^{(n^r-1)/d}[4k+1]_{q^2}[4k+1]^2\frac{(q^2;q^4)_k^4}{(q^4;q^4)_k^4}q^{-4k}$$
$$\equiv q^{2-2n}[n]_{q^2}\frac{1+q^{2n}}{1+q^2}\sum_{k=0}^{(n^{r-1}-1)/d}[4k+1]_{q^{2n}}[4k+1]_{q^n}^2\frac{(q^{2n};q^{4n})_k^4}{(q^{4n};q^{4n})_k^4}q^{-4nk},\tag{3.37}$$

where $d=1,2$.

Sketch of proof. By [10, Theorem 4.1], we have

$$\sum_{k=0}^{(n-1)/2}[4k+1]_{q^2}[4k+1]^2\frac{(aq^2,q^2/a;q^4)_k(q^2;q^4)_k^2}{(aq^4,q^4/a;q^4)_k(q^4;q^4)_k^2}q^{-4k}$$
$$\equiv q^{1-n}[n]_{q^2}\left(1-\frac{(1-aq^2)(1-q^2/a)}{(1+q^2)(1-q)^2}\right)\pmod{\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})}.$$

Using this q-congruence, we can establish the following parametric generalization of (3.37): modulo

$$[n^r]_{q^2}\prod_{j=0}^{(n^{r-1}-1)/d}(1-aq^{(4j+2)n})(a-q^{(4j+2)n}),\tag{3.38}$$

we have

$$\sum_{k=0}^{(n^r-1)/d}[4k+1]_{q^2}[4k+1]^2\frac{(aq^2,q^2/a;q^4)_k(q^2;q^4)_k^2}{(aq^4,q^4/a;q^4)_k(q^4;q^4)_k^2}q^{-4k}$$

$$
\begin{aligned}
&\equiv q^{1-n}[n]_{q^2}\left(1-\frac{(1-aq^2)(1-q^2/a)}{(1+q^2)(1-q)^2}\right)\left(1-\frac{(1-aq^{2n})(1-q^{2n}/a)}{(1+q^{2n})(1-q^n)^2}\right)^{-1} \\
&\quad\times\sum_{k=0}^{(n^{r-1}-1)/d}[4k+1]_{q^{2n}}[4k+1]_{q^n}^2\frac{(aq^{2n},q^{2n}/a;q^{4n})_k(q^{2n};q^{4n})_k^2}{(aq^{4n},q^{4n}/a;q^{4n})_k(q^{4n};q^{4n})_k^2}q^{-4nk}. \qquad (3.39)
\end{aligned}
$$

Like before, the limit of (3.38) as $a\to 1$ has the factor (3.33). While the factor related to a in the common denominator of the two sides of (3.39) is at most

$$
\big((1+q^{2n})(1-q^n)^2-(1-aq^{2n})(1-q^{2n}/a)\big)(aq^4;q^4)_{(n^r-1)/d}(q^4/a;q^4)_{(n^r-1)/d},
$$

whose limit as $a\to 1$ only incorporates the factor

$$
\begin{cases}
\Phi_n(q)^2\prod_{j=1}^r\Phi_{n^j}(q^2)^{2n^{r-j}-2} & \text{if } d=1,\\
\Phi_n(q)^2\prod_{j=1}^r\Phi_{n^j}(q^2)^{n^{r-j}-1} & \text{if } d=2,
\end{cases}
$$

related to $\Phi_n(q^2),\Phi_{n^2}(q^2),\dots,\Phi_{n^r}(q^2)$. Here we utilized the relation

$$
(1+q^{2n})(1-q^n)^2-(1-q^{2n})^2=-2q^n(1-q^n)^2.
$$

Thus, taking the limit of (3.39) as $a\to 1$, we see that the q-congruence (3.37) holds modulo $\Phi_n(q^2)\Phi_n(-q)^2\prod_{j=2}^r\Phi_{n^j}(q^2)^3$. Finally, to show that both sides of (3.37) are also congruent to 0 modulo $[n^r]_{q^2}$, we only need to use the modulus $[n]_{q^2}$ case of [10, Theorem 1.4]. □

It is clear that, when $n=p$ and $q\to 1$, the q-supercongruence (3.37) becomes the modulus p^{3r-2} case of (3.36). Meanwhile, taking $n=p$ and $q\to -1$ in (3.37), we obtain the modulus p^{3r} case of (C.3) from [44]:

$$
\sum_{k=0}^{(p^r-1)/2}(4k+1)\frac{(\frac12)_k^4}{k!^4}\equiv p\sum_{k=0}^{(p^{r-1}-1)/2}(4k+1)\frac{(\frac12)_k^4}{k!^4}\pmod{p^{3r}}
$$

and its companion, already proved by the first author in [17].

3.6. Dwork-type supercongruences involving $(4k-1)$ *and* $(4k-1)^3$

The first author [10, Corollary 5.2] proved that, for $r\geqslant 1$,

$$
\begin{aligned}
\sum_{k=0}^{(p^r+1)/2}(4k-1)^3\frac{(-\frac12)_k^3}{k!^3}&\equiv 3p^r\left(\frac{-1}{p^r}\right)\pmod{p^{r+2}},\\
\sum_{k=0}^{p^r-1}(4k-1)^3\frac{(-\frac12)_k^3}{k!^3}&\equiv 3p^r\left(\frac{-1}{p^r}\right)\pmod{p^{r+2}}.
\end{aligned}
$$

We observe that these two supercongruences also possess the following Dwork-type generalizations:

$$\sum_{k=0}^{(p^r+1)/2}(4k-1)^3\frac{(-\frac{1}{2})_k^3}{k!^3}\equiv p\left(\frac{-1}{p}\right)\sum_{k=0}^{(p^{r-1}+1)/2}(4k-1)^3\frac{(-\frac{1}{2})_k^3}{k!^3}\pmod{p^{3r-2}}, \qquad (3.40)$$

$$\sum_{k=0}^{p^r-1}(4k-1)^3\frac{(-\frac{1}{2})_k^3}{k!^3}\equiv p\left(\frac{-1}{p}\right)\sum_{k=0}^{p^{r-1}-1}(4k-1)^3\frac{(-\frac{1}{2})_k^3}{k!^3}\pmod{p^{3r-2}}. \qquad (3.41)$$

In fact, these two supercongruences can be further generalized to the q-setting. We first give the following result similar to Lemma 3.7.

Lemma 3.10. *Let $n>1$ be an odd integer and a an indeterminate. Then, modulo $\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})$,*

$$\begin{aligned}&\sum_{k=0}^{(n+1)/2}(-1)^k[4k-1]_{q^2}[4k-1]^2\frac{(aq^{-2},q^{-2}/a;q^4)_k(q^{-4};q^8)_k}{(aq^4,q^4/a;q^4)_k(q^8;q^8)_k}q^{4k}\\&\equiv\frac{-2q^{-n-3}[n]_{q^2}(1+q^4)}{(1+aq^2)(1+q^2/a)}\left(\frac{-1}{n}\right)\left(1-\frac{(1+q^2)(1-aq^{-2})(1-q^{-2}/a)}{(1+q^4)(1-q)^2}q^4\right). \qquad (3.42)\end{aligned}$$

Sketch of proof. For $a=q^{-2n}$ or $a=q^{2n}$, the left-hand side of (3.42) can be written as

$$-q^{-4}\,{}_8\phi_7\left[\begin{matrix}q^{-2},\,q^3,\,-q^3,\,q^3,\,q^3,\,-q^{-2},\,q^{-2+2n},\,q^{-2-2n}\\ q^{-1},\,-q^{-1},\,q^{-1},\,q^{-1},\,-q^4,\,q^{4-2n},\,q^{4+2n}\end{matrix};q^4,\,-q^4\right]$$

By Watson's ${}_8\phi_7$ transformation formula [3, Appendix (III.18)] with $q\mapsto q^4$, $a=q^{-2}$, $b=c=q^3$, $d=-q^{-2}$, $e=q^{-2+2n}$, and $n\mapsto(n+1)/2$, the above expression is equal to

$$\begin{aligned}&-q^{-4}\frac{(q^2,-q^{6-2n};q^4)_{(n+1)/2}}{(-q^4,q^{4-2n};q^4)_{(n+1)/2}}\,{}_4\phi_3\left[\begin{matrix}q^{-4},\,-q^{-2},\,q^{-2+2n},\,q^{-2-2n}\\ q^{-1},\,q^{-1},\,-q^{-4}\end{matrix};q^4,\,q^4\right]\\&=\frac{-2q^{n-5}[n]_{q^2}(1+q^4)}{(1+q^{2n-2})(1+q^{2n+2})}\left(\frac{-1}{n}\right)\left(1-\frac{(1+q^2)(1-q^{-2+2n})(1-q^{-2-2n})}{(1+q^4)(1-q)^2}q^4\right),\end{aligned}$$

which is just the $a=q^{-2n}$ or $a=q^{2n}$ case of (3.42). This means that (3.42) is true modulo $(1-aq^{2n})(a-q^{2n})$. Moreover, in view of [10, eq. (5.3)] with $q\mapsto q^2$, we can show that (3.42) is also true modulo $\Phi_n(q^2)$. □

We are now able to give q-analogues of (3.40) and (3.41) as follows.

Theorem 3.11. *Let $n>1$ be an odd integer and let $r\geqslant 2$. Then, modulo*

$$\begin{cases}[n^r]_{q^2}\prod_{j=2}^r\Phi_{n^j}(q^2)^2 & \text{if } n>3,\\ [n^r]_{q^2}\Phi_n(q)\Phi_{n^2}(q^2)\Phi_{n^2}(-q)\prod_{j=3}^r\Phi_{n^j}(q^2)^2 & \text{if } n=3,\end{cases}$$

we have

$$\sum_{k=0}^{M_1}(-1)^k[4k-1]_{q^2}[4k-1]^2\frac{(q^{-2};q^4)_k^2(q^{-4};q^8)_k}{(q^4;q^4)_k^2(q^8;q^8)_k}q^{4k}$$
$$\equiv q^{2n-2}[n]_{q^2}\left(\frac{-1}{n}\right)\frac{(1+q+q^2)(1+q^{2n})^2}{(1+q^2)^2(1+q^n+q^{2n})}$$
$$\times\sum_{k=0}^{M_2}(-1)^k[4k-1]_{q^{2n}}[4k-1]_{q^n}^2\frac{(q^{-2n};q^{4n})_k^2(q^{-4n};q^{8n})_k}{(q^{4n};q^{4n})_k^2(q^{8n};q^{8n})_k}q^{4nk}, \tag{3.43}$$

where $(M_1,M_2)=((n^r+1)/2,(n^{r-1}+1)/2)$ *or* $(M_1,M_2)=(n^r-1,n^{r-1}-1)$.

Sketch of proof. We first consider the case $(M_1,M_2)=((n^r+1)/2,(n^{r-1}+1)/2)$. Utilizing (3.42), we can prove the following parametric version of (3.43): modulo

$$[n^r]_{q^2}\prod_{j=0}^{(n^{r-1}-1)/2}(1-aq^{(4j+2)n})(a-q^{(4j+2)n}), \tag{3.44}$$

we have

$$\sum_{k=0}^{(n^r+1)/2}(-1)^k[4k-1]_{q^2}[4k-1]^2\frac{(aq^{-2},q^{-2}/a;q^4)_k(q^{-4};q^8)_k}{(aq^4,q^4/a;q^4)_k(q^8;q^8)_k}q^{4k}$$
$$\equiv\frac{q^{3n-3}[n]_{q^2}(1+q^4)}{(1+aq^2)(1+q^2/a)}\left(\frac{-1}{n}\right)\left(1-\frac{(1+q^2)(1-aq^{-2})(1-q^{-2}/a)}{(1+q^4)(1-q)^2}q^4\right)$$
$$\times\frac{(1+aq^{2n})(1+q^{2n}/a)}{1+q^{4n}}\left(1-\frac{(1+q^{2n})(1-aq^{-2n})(1-q^{-2n}/a)}{(1+q^{4n})(1-q^n)^2}q^{4n}\right)^{-1}$$
$$\times\sum_{k=0}^{(n^{r-1}+1)/2}(-1)^k[4k-1]_{q^{2n}}[4k-1]_{q^n}^2\frac{(aq^{-2n},q^{-2n}/a;q^{4n})_k(q^{-4n};q^{8n})_k}{(aq^{4n},q^{4n}/a;q^{4n})_k(q^{8n};q^{8n})_k}q^{4nk}. \tag{3.45}$$

As in the previous considerations, the limit of (3.44) as $a\to 1$ has the factor $\prod_{j=1}^r\Phi_{n^j}(q^2)^{n^{r-j}+2}$. This time the factor related to a in the common denominator of the two sides of (3.45) is at most

$$\big((1+q^{4n})(1-q^n)^2-(1+q^{2n})(1-aq^{-2n})(1-q^{-2n}/a)q^{4n}\big)$$
$$\times(aq^4;q^4)_{(n^r+1)/2}(1-aq^{2n(n^{r-1}+1)})(q^4/a;q^4)_{(n^r+1)/2}(1-q^{2n(n^{r-1}+1)}/a),$$

whose limit as $a\to 1$ only contains the factor

$$\begin{cases} \Phi_n(q)^2\Phi_n(q^2)^2\prod_{j=1}^r\Phi_{n^j}(q^2)^{n^{r-j}-1} & \text{if } n>3,\\ \Phi_n(q)^2\Phi_{n^2}(q)\Phi_n(q^2)^2\prod_{j=1}^r\Phi_{n^j}(q^2)^{n^{r-j}-1} & \text{if } n=3, \end{cases}$$

related to $\Phi_n(q^2),\Phi_{n^2}(q^2),\dots,\Phi_{n^r}(q^2)$. Here we used the identity (3.34) again. Thus, letting $a\to 1$ in (3.45) we find out that the q-congruence (3.43) holds modulo

$$\begin{cases} \Phi_n(-q)\prod_{j=2}^r\Phi_{n^j}(q^2)^3 & \text{if } n>3,\\ \Phi_{n^2}(q^2)^2\Phi_n(-q)\Phi_{n^2}(-q)\prod_{j=3}^r\Phi_{n^j}(q^2)^3 & \text{if } n=3. \end{cases}$$

On the other hand, letting $a\to 1$ in (3.42) we can easily deduce that the left-hand side of (3.43) is congruent to

$$4q^{-n-2}[n]_{q^2}\left(\frac{-1}{n}\right)\frac{1+q+q^2}{(1+q^2)^2}\pmod{\Phi_n(q^2)^3},$$

which indicates that it is congruent to 0 modulo $\Phi_n(q)^2$ when $n=3$, and so (3.43) is true modulo $\Phi_n(q)^2$ when $n=3$. From this we immediately deduce that the q-congruence (3.43) is true modulo

$$\begin{cases} \Phi_n(q^2)\prod_{j=2}^r\Phi_{n^j}(q^2)^3 & \text{if } n>3,\\ \Phi_n(q)^2\Phi_{n^2}(q^2)^2\Phi_n(-q)\Phi_{n^2}(-q)\prod_{j=3}^r\Phi_{n^j}(q^2)^3 & \text{if } n=3. \end{cases}$$

Furthermore, based on (3.42), along the lines of the proof of [23, Theorem 1.2] we can show that

$$\sum_{k=0}^{(n+1)/2}(-1)^k[4k-1]_{q^2}[4k-1]^2\frac{(q^{-2};q^4)_k^2(q^{-4};q^8)_k}{(q^4;q^4)_k^2(q^8;q^8)_k}q^{4k}\equiv 0\pmod{[n]_{q^2}}.$$

With the help of this q-congruence, we deduce that both sides of (3.43) are congruent to 0 modulo $[n^r]_{q^2}$. This proves (3.43) for $(M_1,M_2)=((n^r+1)/2,(n^{r-1}+1)/2)$.

For $(M_1,M_2)=(n^r-1,n^{r-1}-1)$, the proof follows from the same argument. In this case the corresponding parametric generalization holds modulo

$$[n^r]_{q^2}\prod_{j=0}^{n^{r-1}-2}(1-aq^{(4j+2)n})(a-q^{(4j+2)n}).$$

At the same time, the factor related to a in the common denominator of the two sides is at most

$$\begin{aligned}&\big((1+q^{4n})(1-q^n)^2-(1+q^{2n})(1-aq^{-2n})(1-q^{-2n}/a)q^{4n}\big)\\&\quad\times(aq^4;q^4)_{n^r-1}(q^4/a;q^4)_{n^r-1}.\end{aligned}$$

Therefore, we are led to the same modulus when we take the limit as $a \to 1$. $\square$

It is not hard to see that (3.40) and (3.41) follow from (3.43) by taking $n = p$ and $q \to 1$. In addition, we obtain the following supercongruences by setting $n = p$ and $q \to -1$ in (3.43):

$$\sum_{k=0}^{(p^r+1)/2} (4k-1)\frac{(-\frac{1}{2})_k^3}{k!^3} \equiv p\left(\frac{-1}{p}\right) \sum_{k=0}^{(p^{r-1}+1)/2} (4k-1)\frac{(-\frac{1}{2})_k^3}{k!^3} \pmod{p^{3r-2}},$$

$$\sum_{k=0}^{p^r-1} (4k-1)\frac{(-\frac{1}{2})_k^3}{k!^3} \equiv p\left(\frac{-1}{p}\right) \sum_{k=0}^{p^{r-1}-1} (4k-1)\frac{(-\frac{1}{2})_k^3}{k!^3} \pmod{p^{3r-2}},$$

which are related to the supercongruences in [10, Corollary 5.3].

3.7. Generalizations of Rodriguez-Villegas' supercongruences

Mortenson [31,32] proved the following four supercongruences conjectured by Rodriguez-Villegas [37, eq. (36)]:

$$\sum_{k=0}^{p-1} \frac{1}{16^k}\binom{2k}{k}^2 \equiv \left(\frac{-1}{p}\right) \pmod{p^2} \quad \text{for } p > 2, \tag{3.46}$$

$$\sum_{k=0}^{p-1} \frac{1}{27^k}\binom{3k}{2k}\binom{2k}{k} \equiv \left(\frac{-3}{p}\right) \pmod{p^2} \quad \text{for } p > 3, \tag{3.47}$$

$$\sum_{k=0}^{p-1} \frac{1}{64^k}\binom{4k}{2k}\binom{2k}{k} \equiv \left(\frac{-2}{p}\right) \pmod{p^2} \quad \text{for } p > 2, \tag{3.48}$$

$$\sum_{k=0}^{p-1} \frac{1}{432^k}\binom{6k}{3k}\binom{3k}{k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2} \quad \text{for } p > 3. \tag{3.49}$$

For an elementary proof of (3.46)–(3.49), we refer the reader to [41]; for a recent generalization of them, see [28]. Some q-analogues of (3.46)–(3.49) can be found in [11,18,22,35]. In particular, the first author [11, Corollary 1.4] proved that, for positive integers m, n and s with $\gcd(m,n) = 1$, we have

$$\sum_{k=0}^{n-1} \frac{2(q^s, q^{m-s}; q^m)_k q^{mk}}{(q^m; q^m)_k^2 (1+q^{mk})} \equiv (-1)^{\langle -s/m\rangle_n} \pmod{\Phi_n(q)^2}, \tag{3.50}$$

where $\langle x\rangle_n$ denotes the least nonnegative residue of x modulo n.

Here we give a Dwork-type generalization of (3.50) for $m = 2$ and $s = 1$.

Theorem 3.12. *Let $n>1$ be an odd integer and let $r \geqslant 1$. Then, modulo $\prod_{j=1}^{r}\Phi_{n^j}(q)^2$,*

$$\sum_{k=0}^{(n^r-1)/d} \frac{2(q;q^2)_k^2 q^{2k}}{(q^2;q^2)_k^2(1+q^{2k})} \equiv \left(\frac{-1}{n}\right) \sum_{k=0}^{(n^{r-1}-1)/d} \frac{2(q^n;q^{2n})_k^2 q^{2nk}}{(q^{2n};q^{2n})_k^2(1+q^{2nk})}, \tag{3.51}$$

where $d=1,2$.

Sketch of proof. By [11, Corollary 1.4], we have

$$\sum_{k=0}^{(n-1)/2} \frac{2(aq,q/a;q^2)_k q^{2k}}{(q^2;q^2)_k^2(1+q^{2k})} \equiv \left(\frac{-1}{n}\right) \pmod{(1-aq^n)(a-q^n)}.$$

This enables us to establish the following parametric generalization of (3.50): modulo

$$\prod_{j=0}^{(n^{r-1}-1)/d} (1-aq^{(2j+1)n})(a-q^{(2j+1)n}),$$

we have

$$\sum_{k=0}^{(n^r-1)/d} \frac{2(aq,q/a;q^2)_k q^{2k}}{(q^2;q^2)_k^2(1+q^{2k})} \equiv \left(\frac{-1}{n}\right) \sum_{k=0}^{(n^{r-1}-1)/d} \frac{2(aq^n,q^n/a;q^{2n})_k q^{2nk}}{(q^{2n};q^{2n})_k^2(1+q^{2nk})}. \qquad \square$$

Letting $n=p$ and $q\to 1$ in (3.51) we obtain the following Dwork-type supercongruence:

$$\sum_{k=0}^{(p^r-1)/d} \frac{1}{16^k}\binom{2k}{k}^2 \equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{(p^{r-1}-1)/d} \frac{1}{16^k}\binom{2k}{k}^2 \pmod{p^{2r}}, \tag{3.52}$$

where $d=1,2$. This confirms, for the first time, predictions of Roberts and Rodriguez-Villegas from [36].

Numerical calculation suggests that (3.47)–(3.49) have similar generalizations modulo p^{2r}. It seems that these supercongruences even have neat q-analogues as follows.

Conjecture 3.13. *Let m and s be positive integers with $s<m$. Let $n>1$ be an odd integer with $n\equiv \pm 1 \pmod m$. Then, for $r\geqslant 2$, modulo $\prod_{j=1}^{r}\Phi_{n^j}(q)^2$,*

$$\sum_{k=0}^{n^r-1} \frac{2(q^s,q^{m-s};q^m)_k q^{mk}}{(q^m;q^m)_k^2(1+q^{mk})} \equiv (-1)^{\langle -s/m\rangle_n} \sum_{k=0}^{n^{r-1}-1} \frac{2(q^{sn},q^{mn-sn};q^{mn})_k q^{mnk}}{(q^{mn};q^{mn})_k^2(1+q^{mnk})}. \tag{3.53}$$

Note that (3.51) with $d=1$ is just the $(m,s)=(2,1)$ case of (3.53). Although there is a parametric generalization of (3.53) for $r=1$ (see [11, Corollary 1.4]), we are not aware of a parametric extension for $r \geqslant 2$. After appearance of preliminary version of this paper, Ni [34] managed to prove the $n \equiv 1 \pmod{m}$ case of Conjecture 3.13 using the method of creative microscoping. However, we believe that the remaining $n \equiv -1 \pmod{m}$ case should still be very difficult.

4. Open problems and concluding remarks

4.1. Open problems

First we give some related open problems for further study. Recall that Swisher's conjectural supercongruence (A.3) for $p \equiv 1 \pmod 4$ can be stated as follows:

$$\sum_{k=0}^{(p^r-1)/2} (-1)^k(4k+1)\frac{(\frac12)_k^5}{k!^5} \equiv -p\Gamma_p(1/4)^4 \sum_{k=0}^{(p^{r-1}-1)/2} (-1)^k(4k+1)\frac{(\frac12)_k^5}{k!^5} \pmod{p^{5r}}, \tag{4.1}$$

where $\Gamma_p(x)$ denotes the p-adic gamma function and $p > 5$. Swisher [44] proves herself (4.1) for $r = 1$. We find the following partial q-analogue of (4.1).

Conjecture 4.1. *Let $n > 1$ be an integer with $n \equiv 1 \pmod 4$ and let $r \geqslant 1$. Then, modulo $[n^r]\prod_{j=1}^{r}\Phi_{n^j}(q)^2$,*

$$\begin{aligned}
&\sum_{k=0}^{(n^r-1)/d} (-1)^k[4k+1]\frac{(q;q^2)_k^4(q^2;q^4)_k}{(q^2;q^2)_k^4(q^4;q^4)_k}q^k \\
&\quad\equiv \frac{(q^2;q^4)_{(n^r-1)/4}^2(q^{4n};q^{4n})_{(n^{r-1}-1)/4}^2}{(q^4;q^4)_{(n^r-1)/4}^2(q^{2n};q^{4n})_{(n^{r-1}-1)/4}^2}[n] \\
&\quad\quad\times \sum_{k=0}^{(n^{r-1}-1)/d} (-1)^k[4k+1]_{q^n}\frac{(q^n;q^{2n})_k^4(q^{2n};q^{4n})_k}{(q^{2n};q^{2n})_k^4(q^{4n};q^{4n})_k}q^{nk}.
\end{aligned} \tag{4.2}$$

Note that the case $r = 1$ of (4.2) has been proved by the first author [16]. Therefore, the left-hand side of (4.1) is congruent to 0 modulo p^r (including $p = 5$). To see (4.2) is indeed a q-analogue of (4.1) modulo p^{3r}, one needs to check that

$$\frac{(\frac12)_{(p^r-1)/4}^2(1)_{(p^{r-1}-1)/4}^2}{(1)_{(p^r-1)/4}^2(\frac12)_{(p^{r-1}-1)/4}^2} \equiv -\Gamma_p(1/4)^4 \pmod{p^{2r}}$$

for any prime $p \equiv 1 \pmod 4$. This is similar to the case $r = 1$ treated by Van Hamme in [46, Theorem 3].

We also have the following complete q-analogues of (3.10) and (3.11).

Conjecture 4.2. *Let $n > 1$ be an odd integer and let $r \geqslant 1$. Then, modulo $[n^r] \times \prod_{j=1}^{r} \Phi_{n^j}(q)^2$,*

$$\sum_{k=0}^{(n^r-1)/d} (-1)^k [4k+1] \frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3} q^k \equiv \frac{[n]_{q^2} (-q^3;q^4)_{(n^r-1)/2} (-q^{5n};q^{4n})_{(n^{r-1}-1)/2}}{(-q^5;q^4)_{(n^r-1)/2} (-q^{3n};q^{4n})_{(n^{r-1}-1)/2}} (-q)^{(1-n)/2} \times \sum_{k=0}^{(n^{r-1}-1)/d} (-1)^k [4k+1]_{q^n} \frac{(q^{2n};q^{4n})_k^3}{(q^{4n};q^{4n})_k^3} q^{nk}, \tag{4.3}$$

where $d = 1, 2$.

Note that the case $r = 1$ of (4.3) was proved by the authors in [25]. However, using the creative microscoping method in a usual manner, we cannot prove Conjectures 4.1 and 4.2 for $r > 1$ in general.

Based on [25, Theorem 1.1] we formulate a partial q-analogue of Swisher's (H.3) supercongruence [44].

Conjecture 4.3. *Let $n > 1$ be an integer with $n \equiv 1 \pmod 4$ and let $r \geqslant 1$. Then, modulo $\prod_{j=1}^{r} \Phi_{n^j}(q)^2$,*

$$\sum_{k=0}^{(n^r-1)/d} \frac{(1+q^{4k+1})\,(q^2;q^4)_k^3}{(1+q)\,(q^4;q^4)_k^3} q^k \equiv \frac{[n]_{q^2} (q^3;q^4)_{(n^r-1)/2} (q^{5n};q^{4n})_{(n^{r-1}-1)/2}}{(q^5;q^4)_{(n^r-1)/2} (q^{3n};q^{4n})_{(n^{r-1}-1)/2}} q^{(1-n)/2} \times \sum_{k=0}^{(n^{r-1}-1)/d} \frac{(1+q^{(4k+1)n})\,(q^{2n};q^{4n})_k^3}{(1+q^n)\,(q^{4n};q^{4n})_k^3} q^{nk},$$

where $d = 1, 2$.

We also have the following partial q-analogues of (3.5) and (3.6).

Conjecture 4.4. *Let $n > 1$ be an odd integer and let $r \geqslant 1$. Then, modulo $[n^r]\Phi_{n^r}(q) \times \prod_{j=1}^{r} \Phi_{n^j}(q)$,*

$$\sum_{k=0}^{(n^r-1)/d} (-1)^k [3k+1] \frac{(q;q^2)_k^3}{(q;q)_k^3} \equiv q^{((n^r-1)^2 - n(n^{r-1}-1)^2)/4} [n] \left(\frac{-1}{n}\right) \sum_{k=0}^{(n^{r-1}-1)/d} (-1)^k [3k+1]_{q^n} \frac{(q^n;q^{2n})_k^3}{(q^n;q^n)_k^3}, \tag{4.4}$$

where $d = 1, 2$.

We point out that the case $r = d = 1$ of (4.4) was established by the first author in [12], while the case $r = 1$, $d = 2$ of (4.4) was confirmed by the authors in [23].

Similarly, we have the following partial q-analogues of (3.10) and (3.11). The proof of the case $r = 1$ can be found in [7,23].

Conjecture 4.5. *Let $n > 1$ be an odd integer and let $r \geqslant 1$. Then, modulo $[n^r]\Phi_{n^r}(q) \times \prod_{j=1}^{r} \Phi_{n^j}(q)$,*

$$\sum_{k=0}^{(n^r-1)/d} (-1)^k [4k+1] \frac{(q;q^2)_k^3}{(q^2;q^2)_k^3} q^{k^2}$$
$$\equiv q^{((n^r-1)^2-n(n^{r-1}-1)^2)/4} [n] \left(\frac{-1}{n}\right) \sum_{k=0}^{(n^{r-1}-1)/d} (-1)^k [4k+1]_{q^n} \frac{(q^n;q^{2n})_k^3}{(q^{2n};q^{2n})_k^3} q^{nk^2},$$

where $d = 1, 2$.

We also have a q-analogue of (3.52) modulo p^{r+1}, which seems difficult to prove; for the case $r = 1$, see [22].

Conjecture 4.6. *Let $n > 1$ be an odd integer and let $r \geqslant 1$. Then, modulo $\Phi_{n^r}(q) \times \prod_{j=1}^{r} \Phi_{n^j}(q)$,*

$$\sum_{k=0}^{(n^r-1)/d} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} \equiv q^{(1-n)(1+n^{2r-1})/4} \left(\frac{-1}{n}\right) \sum_{k=0}^{(n^{r-1}-1)/d} \frac{(q^n;q^{2n})_k^2}{(q^{2n};q^{2n})_k^2},$$

where $d = 1, 2$.

The authors [23, Theorem 4.14] utilized Andrews' q-analogue of Gauss' ${}_2F_1(-1)$ sum (see [3, Appendix (II.11)]) to prove that, for $n \equiv 3 \pmod 4$,

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2}{(q^2;q^2)_k (q^4;q^4)_k} q^{2k} \equiv 0 \pmod{\Phi_n(q)^2}.$$

Using the same method, we can show that, for $n \equiv 1 \pmod 4$,

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2}{(q^2;q^2)_k (q^4;q^4)_k} q^{2k} \equiv \left(\frac{-2}{n}\right) q^{(n-1)(n+3)/8} \frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}} \pmod{\Phi_n(q)^2}.$$

We have the following Dwork-type generalizations of the above q-congruence.

Conjecture 4.7. *Let $n>1$ be an integer with $n\equiv 1 \pmod 4$ and let $r\geqslant 1$. Then, modulo $\Phi_{n^r}(q)\prod_{j=1}^r\Phi_{n^j}(q)$,*

$$\sum_{k=0}^{(n^r-1)/d}\frac{(q;q^2)_k^2}{(q^2;q^2)_k(q^4;q^4)_k}q^{2k}$$
$$\equiv\left(\frac{-2}{n}\right)q^{((n^r-1)(n^r+3)-n(n^{r-1}-1)(n^{r-1}+3))/8}\frac{(q^2;q^4)_{(n^r-1)/4}(q^{4n};q^{4n})_{(n^{r-1}-1)/4}}{(q^4;q^4)_{(n^r-1)/4}(q^{2n};q^{4n})_{(n^{r-1}-1)/4}}$$
$$\times\sum_{k=0}^{(n^{r-1}-1)/d}\frac{(q^n;q^{2n})_k^2}{(q^{2n};q^{2n})_k(q^{4n};q^{4n})_k}q^{2nk},$$

where $d=1,2$.

For the case where n is a prime and q tends to 1, the following stronger Dwork-type supercongruences seem to be true: for any prime $p\equiv 1 \pmod 4$ and $d=1,2$,

$$\sum_{k=0}^{(p^r-1)/d}\frac{1}{32^k}\binom{2k}{k}^2\equiv\left(\frac{-2}{p}\right)\frac{(\frac12)_{(p^r-1)/4}(1)_{(p^{r-1}-1)/4}}{(1)_{(p^r-1)/4}(\frac12)_{(p^{r-1}-1)/4}}\sum_{k=0}^{(p^{r-1}-1)/d}\frac{1}{32^k}\binom{2k}{k}^2 \pmod{p^{2r}}.$$

Note that the $r=1$ case was first proved by Sun [40].

Recently, the first author [9] proved the q-congruence

$$\sum_{k=0}^{n-1}\frac{q^k}{(-q;q)_k}{2k \brack k}_q\equiv\left(\frac{-1}{n}\right)q^{(n^2-1)/4} \pmod{\Phi_n(q)^2}, \tag{4.5}$$

conjectured earlier by Tauraso [45] for n an odd prime. The first author also conjectured that

$$\sum_{k=0}^{n-1}q^k{2k \brack k}\equiv\left(\frac{-3}{n}\right)q^{(n^2-1)/3} \pmod{\Phi_n(q)^2},$$

which was confirmed by Liu and Petrov [29]. We indicate the following Dwork-type q-generalizations of them.

Conjecture 4.8. *Let $n>1$ be an odd integer and let $r\geqslant 1$. Then, modulo $\Phi_{n^r}(q)^{2-d}\times\prod_{j=1}^r\Phi_{n^j}(q)$,*

$$\sum_{k=0}^{(n^r-1)/d}\frac{q^k}{(-q;q)_k}{2k \brack k}_q\equiv q^{(n-1)(1+n^{2r-1})/4}\left(\frac{-1}{n}\right)\sum_{k=0}^{(n^{r-1}-1)/d}\frac{q^{nk}}{(-q^n;q^n)_k}{2k \brack k}_{q^n},$$
$$\sum_{k=0}^{(n^r-1)/d}q^k{2k \brack k}_q\equiv q^{(n-1)(1+n^{2r-1})/3}\left(\frac{-3}{n}\right)\sum_{k=0}^{(n^{r-1}-1)/d}q^{nk}{2k \brack k}_{q^n},$$

where $d = 1, 2$. When $d = 1$, the second q-congruence still holds for even integers n.

Sun [43, Conjecture 3 (ii),(iii)] conjectured that

$$\sum_{k=0}^{p^r-1} \frac{1}{2^k}\binom{2k}{k} \equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p^{r-1}-1} \frac{1}{2^k}\binom{2k}{k} \pmod{p^{2r}} \quad \text{for } p > 2, \tag{4.6}$$

$$\sum_{k=0}^{p^r-1} \binom{2k}{k} \equiv \left(\frac{-3}{p}\right) \sum_{k=0}^{p^{r-1}-1} \binom{2k}{k} \pmod{p^{2r}}, \tag{4.7}$$

and these expectations were recently confirmed by Zhang and Pan in [49]. The supercongruences (4.6) and (4.7) are somewhat different from the other ones discussed in this paper, because already for $r = 1$ they are valid for the truncations at $p - 1$ but not at $(p - 1)/2$. Apart from what is stated in Conjecture 4.8, we could not succeed in finding complete q-analogues for the pair of supercongruences.

Although the method of creative microscoping — in particular, its version developed in this paper — is an adequate tool in dealing with the congruences conjectured above, the difficulty of finding appropriate *parametric* q-congruences and q-hypergeometric sums seems to be a principal obstacle. The underlying identities require a human touch, and this fact makes it impossible to predict when resolutions of (some of these) conjectures take place.

4.2. Dwork-type q-congruences

Dwork-type (super)congruences (1.3) we address in this paper all correspond to the choice $z = 1$ and a specific shape of the unit root $\omega(z)$, namely, associated with a Dirichlet quadratic character. Nevertheless, there is experimental evidence for existence of q-congruences of the type

$$\sum_{k=0}^{(n^r-1)/d} A_k(q) \equiv \omega(q) \sum_{k=0}^{(n^{r-1}-1)/d} A_k(q^n) \tag{4.8}$$

modulo $\prod_{j=1}^{r} \Phi_{n^j}(q)$, say, for a suitable choice of q-hypergeometric term $A_k(q)$, in which the 'q-unit root' $\omega(q)$ has a more sophisticated structure than just $q^N\big(\frac{-D}{n}\big)$. One such example for truncations of the q-series

$$\sum_{k=0}^{\infty} \frac{(q;q^2)_k^4}{(q^2;q^2)_k^4}\, q^{2k}$$

is suggested by Conjectures 4.1–4.3 in [17], though an explicit form of $\omega(q)$ remains unclear. A significance of this particular example is due to the connection of its $q \to 1$ limit with the Dwork-type supercongruence

$$\sum_{k=0}^{p^r-1} \frac{(\frac{1}{2})_k^4}{k!^4} \equiv \omega_p \sum_{k=0}^{p^{r-1}-1} \frac{(\frac{1}{2})_k^4}{k!^4} \pmod{p^{3r}} \quad \text{for } p > 2,\ r = 1, 2, \dots,$$

conjectured in [36], with $r = 1$ instance established earlier by Kilbourn [26] (see also [30]). Here the unit root ω_p is the p-adic zero, not divisible by p, of quadratic polynomial $T^2 - a(p)T + p^3$, where the traces of Frobenius $a(p)$ originate from the modular form $\sum_{m=1}^{\infty} a(m)q^m = q\,(q^2;q^2)_\infty^4 (q^4;q^4)_\infty^4$. The congruence is remarkably related to a modular Calabi–Yau threefold [1], and we expect that its q-analogue will shed light on a q-deformation of the modular form and of the cohomology groups of the threefold [38].

It is certain that q-congruences of the type (4.8) not only provide us with an efficient method for proving their $q \to 1$ specializations but also have their own right to exist.

Acknowledgments

We thank the two anonymous referees for their valuable and enthusiastic feedback. The first author also thanks Zhi-Wei Sun for helpful comments on (4.6) and (4.7).

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