- $\#_2 T_f^\infty = 12$ if they are not.

Because from the computation of the characteristic polynomial of T_f^∞ in Section 2 and the sequence in (4.4) one gets that

$$\dim H^2(X_t')_1 = \sum_{\lambda^6=1} \dim H^2(X_t)_\lambda = 58,$$

(where X_t' is defined in the proof of (5.3)). On the other hand, from Theorem (5.5) one gets that:

$$\#(T^{-6})_1 = \dim H^2(X_0') = 46 + \delta,$$

where $\delta = \dim H^3(X_0')$, X_0' being the hypersurface in $\mathbf{P}^3$ defined by $x_0^6 = f_6$, i.e., the 6-fold cyclic covering of $\mathbf{P}^2$ branched along the curve X^∞. The possible values of δ are known to be 2 if the six cusps are on a conic or 0 if they are not (cf. [27, VIII, Sect. 3]), and then the result follows.

APPENDIX: On the local invariant cycle theorem

by R. García López and J.H.M. Steenbrink

In this note all cohomology groups will be assumed to have coefficients in the field Q of rational numbers. We prove the following two theorems:

THEOREM 1. *Let X be a complex analytic space which can be embedded in a projective variety as an open analytic subset. Let $\pi: X \to D$ be a flat projective holomorphic map onto the unit disk D in the complex plane. Let Z be the singular locus of X, set $Y = \pi^{-1}(0)$ and assume that $Z \subset Y$. Let X_t be the generic fiber of π. Let $k \in \mathbf{N}$ and let $T \in \mathrm{Aut}(H^k(X_t))$ be the monodromy transformation of π around the critical value 0. Then the sequence*

$$H^k(X - Z) \to H^k(X_t) \xrightarrow{T-\mathrm{Id}} H^k(X_t)$$

is exact.

REMARKS. 1. The first map in the sequence above is the restriction map.

2. If $Z = \emptyset$, the theorem is due to Katz in the setting of l-adic cohomology and to Clemens and Schmid in the Kähler case ([3]).

3. The hypothesis $Z \subset \pi^{-1}(0)$ is equivalent to the generic fiber of π being smooth.

Proof. After possibly shrinking D, we may assume that the restriction of π over the punctured disk $D - \{0\}$ is a C^∞- fiber bundle and that the inclusion $Y \hookrightarrow X$ is a homotopy equivalence. Let then $\tilde{X}$ be the limit fiber of π, defined as $\tilde{X} = X \times_D \mathbf{H}$, where $\mathbf{H}$ is the universal covering space of $D - \{0\}$. We recall that X_t and $\tilde{X}$ are of the same homotopy type. In the sequence

$$H^k(X - Z) \xrightarrow{\alpha} H^k(X - Y) \xrightarrow{\beta} H^k(\tilde{X})$$

one has $\mathrm{Im}(\beta) = \mathrm{Ker}(T - \mathrm{Id})$ by the Wang sequence. The terms in this sequence carry mixed Hodge structures (MHS) such that α and β become morphisms of MHS. We use Saito's formalism of mixed Hodge modules ([18]).

- For $H^k(\tilde{X})$ one has the limit MHS ([20], [23]) given by $H^k(\tilde{X}) \simeq \mathbf{H}^k(Y, \Psi_f \mathbf{Q}_X^H)$.
- Let $C \subset Y$ be any closed analytic subset, let $i: Y \hookrightarrow X$ and $j: X - C \hookrightarrow X$ be the inclusion maps. Then

  $$H^k(X - C) \simeq \mathbf{H}^k(Y, i^* Rj_* j^* \mathbf{Q}_X^H)$$

  gives $H^k(X - C)$ a MHS.

By [20], $\mathrm{Ker}(T - \mathrm{Id})$ has weight $\leq k$. Hence it suffices to show that $W_k H^k(X - Y) = \alpha(W_k H^k(X - Z))$, where $W_\bullet$ denotes the corresponding weight filtration. One has the exact sequence of MHS

$$H^k(X - Z) \to H^k(X - Y) \to H^{k+1}(X - Z, X - Y).$$

Fix a projective variety W containing X as an open analytic subset. Without loss of generality we can assume that $W - Z$ is smooth. By excision we have an isomorphism of MHS $H^{k+1}(W - Z, W - Y) \simeq H^{k+1}(X - Z, X - Y)$. We also have the exact sequence of MHS

$$H^k(W - Z) \to H^k(W - Y) \to H^{k+1}(W - Z, W - Y) \to H^{k+1}(W - Z).$$

Now $W_k H^{k+1}(W - Z) = 0$ as $W - Z$ is smooth, moreover $W_k H^k(W - Z) = \mathrm{Im}(H^k(W) \to H^k(W - Z))$ and similarly for $W_k H^k(W - Y)$, so $W_k H^k(W - Z) \to W_k H^k(W - Y)$ is surjective. We conclude that $W_k H^{k+1}(W - Z, W - Y) = 0$. Hence $\alpha : W_k H^k(X - Z) \to W_k H^k(X - Y)$ is surjective. □

REMARK. M. Saito has informed us that the theorem above follows also from the results in [19]. Actually, if $IH^*(X)$ denotes the intersection cohomology of X then, with the notations above one has a factorization

$$IH^k(X) \to H^k(X - Z) \to H^k(X_t)$$

and Theorem 1 follows then from [19, (3.8)].

If the central fiber has only isolated complete intersection singularities (icis) then we have:

THEOREM 2. *In addition to the hypothesis of Theorem 1 and with the same notations, assume that* $Y = \pi^{-1}(0)$ *has only icis and set* $\dim(X) = n + 1$. *Then there is an isomorphism:*

$$\frac{\ker[T - Id : H^n(X_t) \to H^n(X_t)]}{\mathrm{im}[sp^* : H^n(Y) \to H^n(X_t)]} \simeq H_Z^{n+1}(X),$$

where sp^* *denotes the morphism induced in cohomology by the specialization map.*

REMARKS. (1) The isomorphism above is also an isomorphism of mixed Hodge structures.

(2) In the applications in Section 5–6 of the paper above, X is a hypersurface with isolated singularities. Given $p \in Z$, let $g_p\colon (\mathbf{C}^{n+2}, 0) \to (\mathbf{C}, 0)$ be a map germ defining the germ (X, p) and let F_p, T_p be the corresponding Milnor fiber and local monodromy acting on $H^{n+1}(F_p)$. Then we recall that there is an isomorphism:

$$H^{n+1}_{\{p\}}(X) \simeq \operatorname{coker}[T_p - \mathrm{Id}\colon H^{n+1}(F_p) \to H^{n+1}(F_p)].$$

Proof. We claim first that there is an isomorphism $W_n H^n(X-Z) \simeq W_n H^n(X-Y)$. One can prove as in the proof of Theorem 1 that $W_n H^{n+1}(X-Z, X-Y) = 0$, so from the exact sequence of the pair $(X - Z, X - Y)$ it follows that in order to prove the claim it is enough to show that the map $H^{n-1}(X - Y) \to H^n(X - Z, X - Y)$ is surjective. Since the singularities of Y are icis, it follows from the long exact sequence of vanishing cycles that the monodromy acts as the identity on $H^k(\tilde{X})$ for $k \neq n$. Assume that $n \geqslant 2$. Then the map above fits in a commutative diagram with exact row:

$$\begin{array}{ccccc} & & H^{n-2}(\tilde{X})(-1) & & \\ & & \downarrow & \searrow^{\gamma} & \\ H^{n-1}(X-Z) & \longrightarrow & H^{n-1}(X-Y) & \longrightarrow & H^n(X-Z, X-Y) \end{array}$$

and the MHS of $H^{n-2}(\tilde{X})(-1)$ is pure of weight n. Since the singularities of the total space X are also icis, we have that $H^{n-1}(X - Z) \simeq H^{n-1}(X) \simeq H^{n-1}(Y)$ and since Y is complete the weights of $H^{n-1}(Y)$ are $\leq n - 1$. It follows then that the map γ above is injective. On the other hand, one has isomorphisms:

$$\begin{aligned} H^n(X - Z, X - Y) &\simeq H^{n-2}(Y - Z)(-1) \\ &\simeq H^{n-2}(Y)(-1) \simeq H^{n-2}(\tilde{X})(-1). \end{aligned}$$

The first is a Thom isomorphism, the second follows from the fact that the singularities of Y are icis (so $H^{n-2}_Z(Y) = H^{n-1}_Z(Y) = 0$) and the third is induced by the specialization map. So $\dim H^n(X - Z, X - Y) = \dim H^{n-2}(\tilde{X})$, thus γ is an isomorphism and the claim follows. The case $n = 1$ is similar and left to the reader.

Since $Y \hookrightarrow X$ is a homotopy equivalence, from the exact sequence of the couple $(X, X - Z)$ we get the exact sequence:

$$H^n(Y) \xrightarrow{\delta} W_n H^n(X - Z) \to W_n H^{n+1}_Z(X) \to W_n H^{n+1}(Y).$$

Since the singularities of Y and X are isolated, it follows from [24], [12] that $W_n H^{n+1}(Y) = 0$ and $W_n H^{n+1}_Z(X) \simeq H^{n+1}_Z(X)$. So we have:

$$\begin{array}{ccccccc} & & H^n(Y) & & & & \\ & & \delta \Big\downarrow & \searrow{\scriptstyle sp^*} & & & \\ 0 & \longrightarrow & W_nH^n(X-Z) & \longrightarrow & H^n(\tilde{X}) & \xrightarrow{T-\mathrm{Id}} & H^n(\tilde{X}) \end{array}$$

with coker(δ) $\simeq$ $H^{n+1}_Z(X)$. The horizontal sequence comes from the Wang sequence and is exact by the claim above and the fact that the weights of ker($T-\mathrm{Id}$) are $\leq n$. The theorem follows then from an easy diagram-chase. □

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