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An extension of the Kakutani–Bohnenblust characterization of L^p-spaces to $p \in (0, \infty)$

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Abstract

For $p \in [1, \infty)$, S. Kakutani and H.F. Bohnenblust have given characterizations of L^p as a Banach lattice. We generalize that result to $p \in (0, \infty)$. In particular, we show that a quasi-Banach lattice $(E, \rrbracket \llbracket)$ that satisfies $\rrbracket u + v \llbracket^p = \rrbracket u \llbracket^p + \rrbracket v \llbracket^p$ if $u \wedge v = 0$, is isometrically Riesz isomorphic to L^p.

Keywords L^p · Characterization · Locally solid Riesz space

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1 Introduction

For a measure space $(S, \mathscr{A}, \mu)$ and $p \in (0, \infty)$ we recall that $\mathscr{L}^p(S, \mathscr{A}, \mu)$ or shortly $\mathscr{L}^p(\mu)$ is the vector space of all $\mathscr{A}$-measurable functions $x : S \to \mathbb{R}$ for which

$$\|x\|_p := \left[\int |x|^p d\mu \right]^{1/p} < \infty.$$

Identifying functions that are μ-almost everywhere equal, we obtain $L^p(\mu)$.

For $p \in [1, \infty)$, L^p equipped with $\| \|_p$ is a Banach space [1]. For $p \in (0, 1)$, $\| \|_p$ is no longer a norm, but it is a quasi-norm:

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Definition 1.1 (*quasi-norm*) Let E be a vector space. A functional $\rrbracket\,\llbracket : E \to [0,\infty)$ is called a quasi-norm if there exists a $K \in (0,\infty)$ such that for all $x, y \in E$ and $r \in \mathbb{R}$:

$$\begin{cases} (i) & \rrbracket x \llbracket = 0 \text{ if and only if } x = 0; \\ (ii) & \rrbracket rx \llbracket = |r| \rrbracket x \llbracket ; \\ (iii) & \rrbracket x+y \llbracket \le K\,[\,\rrbracket x \llbracket + \rrbracket y \llbracket\,]. \end{cases}$$

For a quasi-norm $\rrbracket\,\llbracket$, we use expressions such as "$\rrbracket\,\llbracket$-Cauchy", "$\rrbracket\,\llbracket$-convergence", "$\rrbracket\,\llbracket$-limit", and "$\rrbracket\,\llbracket$-continuous" in a natural way.

A vector space $(E, \rrbracket\,\llbracket)$ is called a quasi-Banach space if every $\rrbracket\,\llbracket$-Cauchy sequence has a $\rrbracket\,\llbracket$-limit in E.

Let $(S, \mathscr{A}, \mu)$ be a measure space. For $p \in (0,\infty)$ (not only for $p \ge 1$), $L^p(\mu)$ is a Riesz space [2] and, under $\|\;\|_p$, a quasi-Banach space [1,3]; indeed, $\|x+y\|_p \le 2^{1/p}\left(\|x\|_p + \|y\|_p\right)$ $(x, y \in L^p(\mu))$.

In 1940, Bohnenblust [4] gave a characterization of the spaces $L^p(\mu)$ with $p \in [1,\infty)$ and independently Kakutani [5] did so for $p = 1$. In their characterization they not only took into account the linear and metric structure of L^p-spaces, but also their structure as a Riesz space [2]. In Theorem 1.3, we repeat a version of their characterization of L^p as a Banach lattice [2, p.71].

We explain some technicalities after the formulation, but here and later on we will assume some elementary notions from Riesz space theory to be known to the reader (such as "order bounded", "Dedekind complete", "Archimedean"). These can be found in books on Riesz spaces (e.g., [2,6]) or online (e.g., [3]).

Definition 1.2 A Banach lattice is a Riesz space E provided with a norm $\|\;\|$ that turns E into a Banach space and such that

$$|x| \le |y| \Rightarrow \|x\| \le \|y\| \quad (x, y \in E).$$

Two Banach lattices, E and F, are called isometrically isomorphic if there exists an isometric linear bijection $\Omega : E \to F$ with

$$x \le y \Longleftrightarrow \Omega(x) \le \Omega(y) \quad (x, y \in E).$$

Theorem 1.3 (The Kakutani–Bohnenblust characterization of L^p) *Let E be a Banach lattice and let $p \in [1,\infty)$. Assume $\|x+y\|^p = \|x\|^p + \|y\|^p$ whenever $x, y \in E$ and $x \wedge y = 0$. Then there exists a measure space $(S, \mathscr{A}, \mu)$ such that, as a Banach lattice, E is isomorphic with $L^p(\mu)$.*

Our purpose is to extend the Kakutani–Bohnenblust Theorem 1.3 by weakening the conditions on $\|\;\|$ and allowing values of p that are in $(0, 1)$. Mostly we deal with the following situation.

Definition 1.4 Condition (S)

$$(S)\begin{cases} E \text{ is a Riesz space, } \rrbracket\,\llbracket \text{ is a functional } E \to [0,\infty) \\ \text{satisfying, for all } x, y \in E: \\ (i) \quad \rrbracket x \llbracket = 0 \text{ if and only if } x = 0 \quad \text{(separating);} \\ (ii) \quad \rrbracket rx \llbracket = r \rrbracket x \llbracket \text{ if } r \in (0,\infty) \quad \text{(positive homogeneity);} \\ (iii) \quad |x| \le |y| \text{ implies } \rrbracket x \llbracket \le \rrbracket y \llbracket \quad \text{(Riesz property).} \end{cases}$$

Condition (S) implies that E is Archimedean.

Note 1.5 (*on Condition* (S)) Of course, (S) is satisfied if $\rrbracket\,\llbracket$ is a Riesz norm or Riesz quasi-norm (i.e., a norm or quasi-norm satisfying (iii) of 1.4).
However, many functionals $\rrbracket\,\llbracket$ satisfy (S) that are no (quasi-)norms. Although such functionals $\rrbracket\,\llbracket$ induce notions of (sequential) convergence and continuity, they may not render E a topological vector space. For an example, let E be $C[0,1]$, take any $p \in (0,\infty)$ and define $\rrbracket\,\llbracket$ by

$$\rrbracket f \llbracket := \begin{cases} \left[\int_0^1 |f|^p\right]^{1/p} & \text{if } f \text{ has a zero,} \\ \left[\int_0^1 |f| + |f(1)|\right]^{1/p} & \text{otherwise.} \end{cases}$$

For this $\rrbracket\,\llbracket$, the addition is not continuous. Indeed, with

$$g_n(t) = t^n, \quad h_n(t) = n^{-1} \quad (n \in \mathbb{N},\ t \in [0,1]),$$

we have $\rrbracket g_n \llbracket \to 0$ and $\rrbracket h_n \llbracket \to 0$, but not $\rrbracket g_n + h_n \llbracket \to 0$.

Definition 1.6 If E and $\rrbracket\,\llbracket$ are as in (S), we will say that $\rrbracket\,\llbracket$ is p-additive if

$$x, y \in E, x \wedge y = 0 \text{ implies } \rrbracket x + y \llbracket^p = \rrbracket x \llbracket^p + \rrbracket y \llbracket^p.$$

An example of a p-additive quasi-norm is $\|\ \|_p$.

We prove the following two theorems, which were announced earlier in [7] (see 1.8 for terminology).

Theorem 1.7 (Generalization of the Kakutani–Bohnenblust characterization of L^p, Theorem 1.3) *Assume we have the situation described in* (S). *Let* $p \in (0,\infty)$ *and let* $\rrbracket\,\llbracket$ *be* p*-additive. Suppose* E *is* $\rrbracket\,\llbracket$*-complete. Then there exist a measure space* $(S, \mathscr{A}, \mu)$ *and a Riesz isomorphism* $\Omega : E \to L^p(\mu)$ *such that* $\|\Omega(x)\|_p = \rrbracket x \llbracket$ $(x \in E)$.

Definition 1.8 (σ*-Levi*, $\rrbracket\,\llbracket$*-complete*) Let E and $\rrbracket\,\llbracket$ be as in (S).
E is said to be $\rrbracket\,\llbracket$-complete in case

$$\left.\begin{array}{l} x_1, x_2, \ldots \in E, \\ \rrbracket x_n - x_m \llbracket \to 0 \quad (n, m \to \infty) \end{array}\right\} \implies \begin{array}{l} \text{there exists an } x \in E \text{ with} \\ \quad \rrbracket x - x_n \llbracket \to 0 \quad (n \to \infty). \end{array}$$

E is σ-Levi (with respect to $\rfloor\!\rfloor\ \lfloor\!\lfloor$) if every sequence $0 \le u_1 \le u_2 \le u_3 \ldots$ in E^+ with $\sup_n \rfloor\!\rfloor u_n \lfloor\!\lfloor < \infty$ has a supremum in E.

Theorem 1.9 (σ-Levi characterization of L^p-spaces) *Assume we have the situation described in* (S). *Let $p \in (0, \infty)$ and let $\rfloor\!\rfloor\ \lfloor\!\lfloor$ be p-additive. Suppose E is σ-Levi. Then there exist a measure space $(S, \mathscr{A}, \mu)$ and a Riesz isomorphism $\Omega : E \to L^p(\mu)$ such that $\|\Omega(x)\|_p = \rfloor\!\rfloor x \lfloor\!\lfloor$ $(x \in E)$.*

Although condition (S) does not imply that the vector addition is continuous, it does imply the following compatibility between order convergence and $\rfloor\!\rfloor\ \lfloor\!\lfloor$-convergence (familiar from the context of locally solid Riesz spaces, see Theorem 5.6 (iii) [2]):

Lemma 1.10 *Assume (S). Let $x_1 \le x_2 \le \cdots$ in E, $x \in E$, and $\rfloor\!\rfloor x - x_n \lfloor\!\lfloor \to 0$. Then $x = \sup_n x_n$.*

Proof From the fact that $\rfloor\!\rfloor\ \lfloor\!\lfloor$ is separating and Riesz we get
(i) x is an upper bound of $\{x_1, x_2, \ldots\}$:
Let $n \in \mathbb{N}$. For all $m > n$:

$$(x - x_n)^- \le (x - x_m)^- + \underbrace{(x_m - x_n)^-}_{=0} \le |x - x_m| \,.$$

Hence $\rfloor\!\rfloor (x - x_n) \lfloor\!\lfloor^- \le \liminf_m \rfloor\!\rfloor x - x_m \lfloor\!\lfloor = 0$, i.e. $(x - x_n)^- = 0$.
(ii) x is the least of the upper bounds of $\{x_1, x_2, \ldots\}$:
Let $w \ge x_n$ (all n). Then for all n:

$$(w - x)^- \le \underbrace{(w - x_m)^-}_{=0} + (x_m - x)^- \le |x_m - x|.$$

Thus, $\Big\rfloor\!\!\Big\rfloor (w - x)^- \Big\lfloor\!\!\Big\lfloor \le \liminf_m \rfloor\!\rfloor x - x_m \lfloor\!\lfloor = 0$. □

The following lemma illustrates the relation of the σ-Levi property to $\rfloor\!\rfloor\ \lfloor\!\lfloor$-completeness:

Lemma 1.11 (σ-Levi) *Let E be a Riesz space, provided with a Riesz norm $\|\ \|$.*

(i) If E is σ-Levi then E is $\|\ \|$-complete.
(ii) If E is $\|\ \|$-complete and $\|\ \|$ is additive on E^+, then E is σ–Levi. Moreover, an increasing $\|\ \|$-bounded sequence in E^+ is $\|\ \|$-convergent to its supremum.

Proof We first prove (i). Let E be σ-Levi. It suffices to prove:

$$x_n \in E, \quad \sum_{i=1}^{\infty} \|x_i\| < \infty \quad \Longrightarrow \quad \sum_n x_n \text{ is } \|\ \| - \text{convergent.}$$

We may assume $x_n \ge 0$ for all n.

Put $u_n := x_1 + \cdots + x_n$. Then $\sup_n \|u_n\| \le \sum_{i=1}^{\infty} \|x_i\| < \infty$. By the σ-Levi property, $u := \sup_n u_n$ exists.

Now take natural numbers $k_1 \leq k_2 \leq \cdots$ with $k_n \to \infty$ such that $\sum_1^\infty k_n \,\|x_n\| < \infty$.

Then $z_n := \sum_{i=1}^n k_i x_i$ has $\|z_n\| \leq \sum_{i=1}^\infty k_i \,\|x_i\| < \infty$, so $z := \sup_n z_n$ exists.

We are done if we show that $\|u - u_n\| \to 0$. If $m \geq n$, then $0 \leq u_m - u_{n-1} = \sum_{i=n}^m x_i \leq k_n^{-1}(\sum_{i=n}^m k_i x_i) \leq k_n^{-1} z$. Then, $0 \leq u - u_{n-1} \leq k_n^{-1} z$ for $n \in \mathbb{N}$. Therefore, $\|u - u_n\| \leq k_n^{-1} \,\|z\| \to 0$.

(ii) Let E be $\|\ \|$-complete and let $\|\ \|$ be additive on E^+. Take $0 \leq u_n \uparrow$ with $\sup_n \|u_n\| < \infty$. By the additivity of $\|\ \|$ on E^+, we have for $m \geq n$ that $\|u_m - u_n\| = \|u_m\| - \|u_n\|$. As $\sup_n \|u_n\| < \infty$, we have that $u_1, u_2, \ldots$ is a $\|\ \|$-Cauchy sequence and thus there exists a $u \in E$ with $\|u - u_n\| \to 0$. The triangle inequality implies that $\lim_n \|u_n\| = \|u\|$ and, by the above Lemma 1.10, $u = \sup_n u_n$. □

Note 1.12 The converse to Lemma 1.11 (ii) does not hold. For instance, the space of all functions $f : \mathbb{R} \to \mathbb{R}$ with $\lim_{|t|\to\infty} f(t) = 0$ is $\|\ \|_\infty$-complete but not σ-Levi.

Note 1.13 The name of the σ-Levi property is somewhat misleading as the σ-Levi property does not mean that

$$0 \leq u_1 \leq u_2 \leq \cdots \text{ and } \sup_n \|u_n\| < \infty \quad \Longrightarrow \quad \left[\begin{array}{l} \sup_n u_n \text{ exists}, \\ \|u_n\| \uparrow \left\|\sup_n u_n\right\|, \end{array}\right.$$

holds.

As a counterexample, observe that $E = l^\infty$ with $\|\!|x|\!\| := \|x\|_\infty + \limsup_n |x_n|$ is σ-Levi, but $u_n := (1, 1, \ldots, 1, 0, 0, 0, \ldots)$ (a leading part of n 1s followed by 0s) is increasing and order bounded in E^+, while $\|\!|u_n|\!\| = 1$ for all n and $\sup_n u_n = (1, 1, 1, \ldots)$ has $\left\|\!\left|\sup_n u_n\right|\!\right\| = 2$.

Note 1.14 In the original formulations, the Riesz property ($|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$) was not used in Bohnenblust's characterization, while Katutani's characterization was confined to $p = 1$, used additivity on positive elements (instead of only disjoint positive elements), and used the additional condition that $\|x + y\| = \|x - y\|$ whenever $x \wedge y = 0$.

Line of argumentation

Before proceeding, we first give an overview of how we prove Theorems 1.7 and 1.9. We start with Theorem 1.9. Then we have the conditions described in (S), where $\|\!|\ |\!\|$ is a p-additive quasi-norm and E is σ-Levi. We will first prove that E is Dedekind complete. Then by the Maeda–Ogasawara–Vulikh Representation Theorem there exists an extremally disconnected compact Hausdorff space S such that E is Riesz isomorphic to an order dense Riesz ideal of $C^\infty(S)$. In the following we assume that E itself is such an order dense Riesz ideal.

The line of argumentation is illustrated by the following diagram:

$$\begin{array}{ccc} (E,\ \rrbracket\,\llbracket) & \xrightarrow{\ \simeq\ } & \big(L^p(\mu),\ \|\ \|_p\big) \\ \Phi_p\big\downarrow & & \Phi_{1/p}\big\uparrow \\ (E_1,\ \|\ \|) & \xrightarrow[\Psi]{} & \big(L^1(\mu),\ \|\ \|_1\big) \end{array}$$

We show that E can be mapped onto a Riesz ideal E_1 of $C^\infty(S)$ and correspondingly $\rrbracket\,\llbracket$ is mapped to a Riesz norm that is additive on E_1^+. Then we construct a measure μ on a certain σ-algebra $\mathscr{A}$ of subsets of S and a linear order preserving bijection Ψ of $C^\infty(S)$ onto the space $L^0(\mu)$ of all μ-equivalence classes of $\mathscr{A}$-measurable functions, while Φ_p and $\Phi_{1/p}$ are (non-linear) order preserving bijections $C^\infty(S) \to C^\infty(S)$ and $L^0(\mu) \to L^0(\mu)$, respectively. All these maps respect the relevant quasi-norms, i.e., from $E \to E_1$ and from $L^1(\mu) \to L^p(\mu)$ respectively. The composition $\Phi_{1/p} \circ \Psi \circ \Phi_p$ is the desired isometric isomorphism $E \to L^p(\mu)$. Finally, Theorem 1.7 follows, because we will prove that the $\rrbracket\,\llbracket$-completeness implies the σ-Levi property.

The following Sect. 2 will collect preparatory results to execute this line of argumentation and Sect. 3 will conclude with the final proofs.

2 Preparatory results

Suppose we can prove that E is Dedekind complete and that $\|x\| := \rrbracket x \llbracket^p$ is a Riesz norm that is additive on E^+ (the proofs of these will follow later on in Sects. 2.1–2.3). Then we can detail the line of argumentation via Step 0 to 4 below.

Step 0: *We may assume that E is as a Riesz ideal of an $C^\infty(S)$ where S is an extremally disconnected compact Hausdorff space.*

This observation is justified by the following Theorem [8,9].

Theorem 2.1 (Maeda–Ogasawara–Vulikh Representation Theorem) *If E is a Dedekind complete Riesz space, then there exists an extremally disconnected compact Hausdorff space S such that E is Riesz isomorphic with an order dense Riesz ideal of $C^\infty(S)$.*

Step 1: $\Phi_p : (E,\ \rrbracket\,\llbracket) \to (E_1,\ \|\ \|)$.

Elements of E can then be seen as extended real-valued functions and we can take the pth powers of these (pointwise) to get a subspace E_1 of $C^\infty(S)$ that can be proven to be a the Riesz ideal as well. By our assumption, $\|\ \| := \rrbracket\,\llbracket^p$ is an additive Riesz norm on E_1.

Step 2: $\Psi : (E_1,\ \|\ \|) \longrightarrow \big(L^1(\mu),\ \|\ \|_1\big)$.

This identification follows from:

Theorem 2.2 *Let S be an extremally disconnected compact Hausdorff space and E an order dense Riesz ideal of $C^\infty(S)$, carrying a Riesz norm $\|\ \|$ that is additive on E^+. Suppose $\|\ \|$ either renders E a Banach lattice or, equivalently (see 1.11), makes E σ-Levi. Then there exists a measure μ on the Borel σ-algebra $\mathscr{B}$ of S such that*

$$X \text{ is } \mu\text{-negligible} \iff X \text{ is meager} \quad (X \subset S),$$

and the map Ψ from $C^{\infty}(S)$ to the vector space of μ-equivalence classes of $\mathscr{B}$-measurable functions $L^0(\mu)$ defined by

$$f \mapsto [f] := \{g | g \text{ is } \mathscr{B}\text{-measurable and } g = f \text{ a.e.}\}$$

is a linear order preserving bijection. In addition, the restriction of Ψ to E induces a Banach lattice isomorphism $\Psi : E \to L^1(\mu)$.

In the above, recall that a subset T in a topological space (S, τ) is called meager if it is a union of countably many nowhere dense subsets. A set K is called nowhere dense if for every open subset U, $K \cap U$ is not dense in U. For functions $f, g : S \to [-\infty, \infty]$ we write $f = g$ n.e. ("nearly everywhere") if $\{s \in S : f(s) \neq g(s)\}$ is meager.

Proof of Theorem 2.2 in several parts.

(I) If $f, g \in C^{\infty}(S)$ and $f = g$ n.e. then $f = g$.

(II) Let H be the set of all functions $h : S \to \mathbb{R}$ for which there exists a (necessarily unique) $f \in C^{\infty}(S)$ with $h = f$ n.e., and let $\mathscr{A}$ be the set of all subsets A of S for which $\mathbb{1}_A \in H$. For $A \subset S$ we have $A \in \mathscr{A}$ if and only if there is a clopen $U \subset S$ with $\mathbb{1}_A = \mathbb{1}_U$ n.e.

(III) It follows that $\mathscr{A}$ is a σ-algebra and H is the set of all $\mathscr{A}$-measurable functions $S \to \mathbb{R}$. In particular, H is a Riesz space.

(IV) For $h \in H$, define $N(h) \in [0, \infty]$ by:

$$\begin{aligned} N(h) &:= \|f\| \quad \text{if } f \in E, h = f \ \ n.e., \\ N(h) &:= \infty \quad \text{if there is no } f \in E \text{ with } h = f \ \ n.e. \end{aligned}$$

Then the following properties hold:
If $h, j \in H$ and $|h| \leq |j|$, then $N(h) \leq N(j)$.
For $h \in H$ we have $N(h) = 0$ if and only if $h = 0$ n.e.
N is additive on H^+.

(V) Let $h, h_1, h_2, \ldots \in H^+$. If $0 \leq h_1 \leq h_2 \leq \cdots$ and $h_n(s) \uparrow h(s)$ $(s \in S)$, then $N(h_n) \uparrow N(h)$.
For a proof, we distinguish two cases. The case $\sup_n N(h_n) = \infty$ is automatic, because $N(h_n) \leq N(h)$. Suppose now $\sup_n N(h_n) < \infty$. Then there are $g_n \in E$, with $g_n = h_n$ n.e. Define $f_n := g_1 \vee \ldots \vee g_n$ then $f_1 \leq f_2 \leq \cdots$. Then $f_n = h_n$ n.e. and $N(h_n) = \|f_n\|$ (all n). If E is $\| \, \|$-complete, then by the additivity of $\| \, \|$ on E^+ and the first conclusion of Lemma 1.11(ii), we have that E is σ-Levi. And when E is σ-Levi, we can apply the second conclusion in Lemma 1.11(ii).

(VI) Define $\mu : \mathscr{A} \to [0, \infty]$ by $\mu(A) = N(\mathbb{1}_A)$. Then μ is σ-additive. For $A \in \mathscr{A}$ we have $\mu(A) = 0$ if and only if A is meager.

(VII) If $h \in H^+$, then $N(h) = \int |h| d\mu$.
This is clear if h is an $\mathscr{A}$-step function, and the general case follows from (V).

(VIII) As a consequence, $E \subset \mathscr{L}^1(\mu)$ with $N(f) = \int |f| d\mu$ $(f \in E)$ and the embedding $E \to \mathscr{L}^1(\mu)$ induces a Banach lattice isomorphism $E \to L^1(\mu)$. This embedding is injective because $N(h) = 0$ iff $h = 0$ and surjective because for $h \in H^+$: $\int h d\mu = N(h) < \infty$ iff $h \in E^+$. □

By now, in terms of the diagram

$$\begin{array}{ccc} (E,\ |\!|\!|\ |\!|\!|) & \xrightarrow{\ \simeq\ } & \big(L^p(\mu),\ \|\ \|_p\big) \\ \Phi_p \big\downarrow & & \big\uparrow \Phi_{1/p} \\ (E_1,\ \|\ \|) & \xrightarrow[\Psi]{} & \big(L^1(\mu),\ \|\ \|_1\big) \end{array}$$

we have provided Φ_p (in step 1) and Ψ (in step 2) above . As concerns $\Phi_{1/p}$, we use:
Step 3: $\Phi_{1/p} : \big(L^1(\mu),\ \|\ \|_1\big) \longrightarrow \big(L^p(\mu),\ \|\ \|_p\big)$.
Intuitively, this embedding is achieved simply by taking $(1/p)$th powers, i.e. apply the order isomorphism $x \mapsto sign(x) \cdot |x|^{1/p}$ on the extended real line (this is formally defined in Sect. 3, proof of theorem 1.9, part *II*).
Step 4: $\Phi_{1/p} \circ \Psi \circ \Phi_p$ *is a Riesz isomorphism.*
For this we need the following result:

Lemma 2.3 *Let E be an Archimedean Riesz space, $a, b \in E^+$. Then*

$$a + b = \inf\{s^{-1}a \vee t^{-1}b :\ s, t \in (0, 1),\quad s + t = 1\}.$$

Proof For a subset X of a Riesz space D we write D-inf X to denote the largest element of D that is a lower bound of X (if such an element exists). Put $M := \{s^{-1}a \vee t^{-1}b : s, t \in (0, 1), s + t = 1\}$.

(I) The identity $a + b = \inf M$ is elementary in case $E = \mathbb{R}$.
This follows by finding extremes using differentiation for real-valued functions on $\mathbb{R}$.

(II) If E is the space $C(S)$ of all continuous functions on a topological space S, then it follows from (I) that $a + b$ is the pointwise infimum of M, whence $a + b = C(S)$-inf M.

(III) If E is an order dense Riesz subspace of such a $C(S)$, then $a + b = C(S)$-inf $M = E$-inf M.

(IV) In general, let E_0 be the Riesz ideal of E generated by $a + b$. Then $a, b \in E_0$ and $M \subset E_0$. It follows from (III) and from Yosida's Representation Theorem [10] that $a + b = E_0$-inf M. Hence, $a + b = E$-inf M. □

As a consequence of the above lemma, we get

Lemma 2.4 *Let E, F be Archimedan Riesz spaces, $\Omega : E^+ \to F^+$ a lattice isomorphism with $\Omega(rx) = r\Omega(x)$, $(x \in E^+, r \in [0, \infty))$. Then Ω is additive, hence extends uniquely to a Riesz isomorphism $E \to F$.*

The above machinery to prove Theorems 1.7 and 1.9 works if we can prove that E is Dedekind complete and that $\|x\| := |\!|\!|x|\!|\!|^p$ is a Riesz norm on E that is additive on E^+. To show the latter, we use something called the weak-Freudenthal property.

In the following subsections we will be discuss when E is Dedekind complete (Sect. 2.1), what the weak-Freudenthal property is (Sect. 2.2) and why the weak-Freudenthal property makes the vector addition continuous (Sect. 2.3).

2.1 Dedekind completeness

If (S) holds, then every increasing order bounded sequence in E^+ is $\rfloor\!\rfloor\ \lfloor\!\lfloor$-bounded, so:

Lemma 2.5 *Assume* (S). *If* E *is* σ*-Levi then* E *is* σ*-Dedekind complete.*

Adding p-additivity yields even more:

Lemma 2.6 *Assume* (S) *and let* $\rfloor\!\rfloor\ \lfloor\!\lfloor$ *be* p*-additive for some* $p \in (0, \infty)$. *If* E *is* σ*-Levi, then* E *is Dedekind complete (and even super-Dedekind complete, i.e., every set bounded from above has a supremum and contains a countable subset with this supremum).*

Proof In steps:

(I): E has the principal projection property.

By the previous Lemma 2.5, E is σ-Dedekind complete, hence has the Principal Projection Property [11, 25.1].

(II): $\rfloor\!\rfloor\ \lfloor\!\lfloor$ is strictly increasing on E^+.

Let $0 \le a < b$ in E, we show that $\rfloor\!\rfloor a \lfloor\!\lfloor < \rfloor\!\rfloor b \lfloor\!\lfloor$. As $a < b$ and E is Archimedean, we cannot have $b - a < b/n$ for all $n \in \mathbb{N}$. Consequently, we can choose an $r \in (0, 1)$ with $rb \not\le a$, so that $0 < (rb - a)^+ =: c$. Let P_c be the projection onto the band generated by c. Because $(rb - a) \le rb \le b$ we have $c = (rb - a)^+ \le b$ and thus $P_c(b) \ge P_c(c) = c > 0$. Thus, $\rfloor\!\rfloor P_c(b) \lfloor\!\lfloor > 0$. Also, $a - rb \le (rb - a)^-$ and the latter is disjoint with c, so $P_c(a - rb) \le 0$, $P_c(a) \le P_c(rb)$, and $\rfloor\!\rfloor P_c(a) \lfloor\!\lfloor \le r \rfloor\!\rfloor P_c(b) \lfloor\!\lfloor$.

The p-additivity of $\rfloor\!\rfloor\ \lfloor\!\lfloor$ implies that

$$
\begin{aligned}
\rfloor\!\rfloor a \lfloor\!\lfloor^p &= \rfloor\!\rfloor P_c(a) \lfloor\!\lfloor^p + \rfloor\!\rfloor (I - P_c)(a) \lfloor\!\lfloor^p \\
&\le (r \rfloor\!\rfloor P_c(b) \lfloor\!\lfloor)^p + \rfloor\!\rfloor (I - P_c)(b) \lfloor\!\lfloor^p \\
&< \rfloor\!\rfloor P_c(b) \lfloor\!\lfloor^p + \rfloor\!\rfloor (I - P_c)(b) \lfloor\!\lfloor^p = \rfloor\!\rfloor b \lfloor\!\lfloor^p,
\end{aligned}
$$

whence $\rfloor\!\rfloor a \lfloor\!\lfloor < \rfloor\!\rfloor b \lfloor\!\lfloor$.

(III): E is Dedekind complete, and even super-Dedekind complete.

Let $U_0 \subset E^+$ be nonempty and bounded from above. Set $U := \{\sup_k u_k : u_1, u_2, \ldots \in U_0\}$ be the set of suprema of sequences in U_0 and observe that $U \supset U_0$ has the same upper bounds as U_0, that $u_1 \vee u_2 \in U$ whenever $u_1, u_2 \in U$ and that $\sup_n u_n \in U$ whenever $u_n \in U$ (all n).

Realizing that $\rho := \sup\{\rfloor\!\rfloor u \lfloor\!\lfloor : u \in U\} < \infty$, we choose inductively $u_1, u_2, \ldots$ in U such that $0 \le u_n \uparrow$ and $\rfloor\!\rfloor u_n \lfloor\!\lfloor \uparrow \rho$. Then $u_\infty := \sup_n u_n \in U$, and $\rfloor\!\rfloor u_\infty \lfloor\!\lfloor = \rho$: indeed, $\rfloor\!\rfloor u_\infty \lfloor\!\lfloor \le \rho$ because $u_\infty \in U$.

We now prove that $u_\infty = \max U = \sup U = \sup U_0$. To that end, let $u \in U$. Then $u \vee u_\infty \in U$, so $\rfloor\!\rfloor u \vee u_\infty \lfloor\!\lfloor \le \rho = \rfloor\!\rfloor u_\infty \lfloor\!\lfloor$. It follows that from (II) above that $u_\infty = u \vee u_\infty \ge u$. □

Note 2.7 One might reasonably ask whether the p-additivity really has something to do with the conclusion of the above lemma. However, the Riesz space of all bounded Borel measurable functions on $\mathbb{R}$ with the Riesz norm $\|\ \|_\infty$ is σ-Levi but not Dedekind complete.

2.2 The weak-Freudenthal property

We will introduce the weak-Freudenthal property and investigate for what compact Hausdorff spaces S the Riesz space $C(S)$ has this property.

Definition 2.8 (*weak-Freudenthal*) A Riesz space E is called weak-Freudenthal if for all $a, e \in E^+$ with $0 \le a \le e$, and all $\epsilon \in (0, \infty)$ there exist disjoint $e_1, \dots, e_n \in E^+$ and scalars $\alpha_1, \dots, \alpha_n \in [0, \infty)$ such that

$$\sum_{i=1}^{n} e_i = e\,, \text{ and } 0 \le a - \sum_{i=1}^{n} \alpha_i e_i \le \epsilon e \qquad (1)$$

or equivalently,

$$\sum_{i=1}^{n} e_i = e, \quad \sum_{i=1}^{n} \alpha_i e_i \le a \le \sum_{i=1}^{n} (\alpha_i + \epsilon) e_i \qquad (2)$$

Note 2.9 E is weak-Freudenthal if and only if, in the terminology of Lavrič [12], "the weak form of Freudenthal's spectral theorem holds in E".

For $x, e \in E$, x is called a component of e if x is disjoint to $e - x$. The elements $e_1, \dots, e_n \in E^+$ in the above definition are thus components of e.

Theorem 2.10 (Freudenthal's Spectral Theorem) *A Riesz space with the Principal Projection Property is weak-Freudenthal.*

Proof See [11, 40.2] □

Lemma 2.11 *Let E be weak-Freudenthal. Let $a, b \in E^+$, $\epsilon > 0$. Then there exist disjoint nonzero $e_1, \dots, e_n$ in E^+ and $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ in $[0, \infty)$ such that*

$$\sum_{i=1}^{n} e_i = a + b, \; \text{and} \; \begin{cases} \sum_{i=1}^{n} \alpha_i e_i \le a \le \sum_{i=1}^{n} (\alpha_i + \epsilon) e_i, \\ \sum_{i=1}^{n} \beta_i e_i \le b \le \sum_{i=1}^{n} (\beta_i + \epsilon) e_i. \end{cases}$$

Proof Put $e := a+b$. Take $e_1, \dots, e_n$ and $\alpha_1, \dots, \alpha_n$ as in (2) of 2.8. Writing $b = e-a$ one obtains

$$\sum_{i=1}^{n} (1 - \alpha_i - \epsilon)\, e_i \le b \le \sum_{i=1}^{n} (1 - \alpha_i)\, e_i\,.$$

Set $\beta_i := (1 - \alpha_i - \epsilon)^+$ $(i = 1, \dots, n)$. Then $1 - \alpha_i \le \beta_i + \epsilon$. The formula

$$(\tau_1, \dots, \tau_n) \mapsto \sum_{i=1}^{n} \tau_i e_i \qquad (\tau_1, \dots, \tau_n \in \mathbb{R})$$

describes a Riesz homomorphism $\mathbb{R}^n \to E$. Thus,

$$\sum_{i=1}^{n} \beta_i e_i = \left(\sum_{i=1}^{n}(1-\alpha_i-\epsilon)e_i\right)^+ \leq b \leq \sum_{i=1}^{n}(1-\alpha_i)\, e_i \leq \sum_{i=1}^{n}(\beta_i+\epsilon)e_i.$$

□

Lemma 2.12 *Let E be a Riesz space and assume that for all a, e in E^+ with $a \leq e$ there exist disjoint $u, v \in E^+$ and $\sigma, \tau \in [0, \infty)$ with*

$$u + v = e, \quad 0 \leq a - (\sigma u + \tau v) \leq \frac{2}{3}e.$$

Then E is weak-Freudenthal.

Proof (I) Let $a \leq e$ in E^+. Suppose we have $\delta \in [0, \infty)$, disjoint $e_1, \ldots, e_n$ in E^+ and $\alpha_1, \ldots, \alpha_n$ in $[0, \infty)$ with

$$\sum_{i=1}^{n} e_i = e\,, \quad 0 \leq a - \sum_{i=1}^{n} \alpha_i e_i \leq \delta e.$$

Then $\sum_i (a \wedge e_i) = \sup_i (a \wedge e_i) = a \wedge \sup_i e_i = a \wedge e = a$, so we have

$$0 \leq \sum_i (a \wedge e_i - \alpha_i e_i) \leq \delta \sum_i e_i\,.$$

From the disjointness of $e_1, \ldots, e_n$ it follows that for each i

$$0 \leq a \wedge e_i - \alpha_i e_i \leq \delta e_i.$$

Hence, by the given property of E, for each i there exist disjoint u_i, v_i in E^+ and σ_i, τ_i in $[0, \infty)$ with

$$u_i + v_i = \delta e_i\,, \quad 0 \leq (a \wedge e_i - \alpha_i e_i) - (\sigma_i u_i + \tau_i v_i) \leq \frac{2}{3}\delta e_i.$$

Setting $u_i' := u_i/\delta$ and $v_i' := v_i/\delta$, we obtain disjoint $u_1', \ldots, u_n', v_1', \ldots, v_n'$ whose sum is e with

$$0 \leq a - \sum_{i=1}^{n} \big((\alpha_i + \delta\sigma_i)u_i' + (\alpha_i + \delta\tau_i)v_i'\big) \leq \sum_{i=1}^{n} \frac{2}{3}\delta e_i = \frac{2}{3}\delta e.$$

(II) It is clear how an induction argument concludes the proof. □

Theorem 2.13 *Let S be a compact Hausdorff space, $C(S)$ the Banach lattice of all continuous functions on S. Then $C(S)$ is weak-Freudenthal if and only if S is zerodimensional (i.e., the topology of S has a topological basis of clopen subsets).*

Proof (I) Suppose $C(S)$ is weak-Freudenthal. By the implication $(i) \Rightarrow (ii)$ of Corollary 2.8 in [12], $C(S)$ is a zerodimensional Riesz space. Then by Proposition 2.9 of [12], S is zerodimensional as a topological space.

(II) Suppose S is zerodimensional. Let $0 \le f \le g$ in $C(S)$. It follows from Lemma 2.12 that we are done if we can find $u, v \in C(S)$ and $\alpha, \beta \in [0, \infty)$ such that

$$u \wedge v = 0, \quad u + v = g, \quad 0 \le f - (\alpha u + \beta v) \le \frac{2}{3}g.$$

The set $[g > 0] := \{s \in S : g(s) > 0\}$ is σ-compact and open, so there exist clopen subsets $U_1, U_2, \ldots$ with $[g > 0] = \bigcup_i U_i$, and we can arrange for them to be disjoint. If δ_i is the largest value of g on U_i $(i \in \mathbb{N})$, then $\lim_i \delta_i = 0$. U_i is the union of the open sets $U_i \cap [f < \frac{2}{3}g]$ and $U_i \cap [f > \frac{1}{3}g]$, so there exist disjoint clopen A_i, B_i with

$$A_i \cup B_i = U_i, \quad A_i \subset \left[f < \frac{2}{3}g\right], \quad B_i \subset \left[f > \frac{1}{3}g\right].$$

Now $\| g\mathbb{1}_{A_i} \|_\infty \le \delta_i \to 0$ and the sets A_i are disjoint, so $u := \sum_i g\mathbb{1}_{A_i}$ is continuous; so is $v := \sum_i g\mathbb{1}_{B_i}$. Then $u \wedge v = 0$, $u + v = \sum_i g\mathbb{1}_{U_i} = g$. Take $\alpha := 0$, $\beta := \frac{1}{3}$, so $\alpha u + \beta v = \frac{1}{3}v$.

Let $s \in S$. If $s \in \bigcup_i A_i$, then $v(s) = 0$, $0 \le f(s) < \frac{2}{3}g(s)$; if $s \in \bigcup_i B_i$, then $v(s) = g(s)$, $\frac{1}{3}g(s) < f(s) \le g(s)$; for all other s we have $f(s) = g(s) = v(s) = 0$. It follows that $0 \le f(s) - \frac{1}{3}v(s) \le \frac{2}{3}g(s)$ $(s \in S)$, i.e., $0 \le f - (\alpha u + \beta v) < \frac{2}{3}g$. □

Note 2.14 It follows that, even for uniformly complete spaces, the weak-Freudenthal property does not imply the Principal Projection Property. For a counterexample, take $S := \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, which is basically the example put forward in [12] 3.9(iii).

Note 2.15 A weak-Freudenthal Riesz space may not be Archimedean, for example, consider the lexicographic plane, which is $\mathbb{R}^2$ ordered by

$$(x_1, x_2) \ge 0 \iff x_1 > 0 \text{ or } x_1 = 0, x_2 \ge 0.$$

Then we have: the lexicographic plane is weak-Freudenthal.

Proof Let $0 \le a \le e$, $\epsilon > 0$. We obtain an $\alpha \in [0, \infty)$ with $\alpha e \le a \le (\alpha + \epsilon)e$.

- (I) Suppose $e_1 = 0$. Then $e_2 \ge 0$ and we may assume $e_2 > 0$. We have $0 \le a_1 \le e_1$, so $a_1 = 0$, and $0 \le a_2 \le e_2$. Take $\alpha := \frac{a_2}{e_2}$. Then $a = \alpha e$.
- (II) Suppose $e_1 \ne 0$, so $e_1 > 0$. We may assume $e_1 = 1$.
- (IIa) If $a_1 > 0$: choose $\alpha \in \mathbb{R}$ with $0 < \alpha < a_1 < \alpha + \epsilon$. Then $\alpha e_1 < a_1 < (\alpha + \epsilon)e_1$, so $\alpha e \le a \le (\alpha + \epsilon)e$.
- (IIb) If $a_1 = 0$. Choose $\alpha := 0$. Then $a < (\alpha + \epsilon)e$ since $a_1 < \epsilon = (\alpha + \epsilon)e_1$, while $0 \le a$ by assumption. □

2.3 $\| \|_p$ -inequalities

We now turn to the question (see the text preceding Sect. 2.1) why the weak-Freudenthal property and p-additivity make the vector addition continuous. The pivot is that they imply what we will call $\| \|_p$ -inequalities. As a prelude to these we state the following.

Lemma 2.16 *Let $n \in \mathbb{N}$, $x, y \in (\mathbb{R}^n)^+$, and let $p \in (0, \infty)$.*

$$\text{If } p \geq 1, \text{ then } \|x+y\|_p \leq \|x\|_p + \|y\|_p, \text{ and } \|x+y\|_p^p \geq \|x\|_p^p + \|y\|_p^p. \quad (1)$$

$$\text{If } p \leq 1, \text{ then } \|x+y\|_p \geq \|x\|_p + \|y\|_p, \text{ and } \|x+y\|_p^p \leq \|x\|_p^p + \|y\|_p^p. \quad (2)$$

Proof For $p \geq 1$, the first inequality of (1) is the classical triangle inequality for the norm $\| \|_p$, whereas the second reduces to

$$\sum_{i=1}^n x_i^p + \sum_{i=1}^n y_i^p \leq \sum_{i=1}^n (x_i + y_i)^p.$$

Since $t^p \leq t$ $(t \in [0,1])$ for $p \in [1, \infty)$, we have that $(\frac{x}{x+y})^p + (\frac{y}{x+y})^p \leq 1$; In other words, $(x+y)^p \geq x^p + y^p$ $(x, y \in (0, \infty))$ and the desired inequality follows.

Now let $p \leq 1$. The second inequality of (2) says that

$$\sum_{i=1}^n (x_i + y_i)^p \leq \sum_{i=1}^n x_i^p + \sum_{i=1}^n y_i^p$$

Since $t^p \geq t$ $(t \in [0,1])$ for $p \in (0, 1]$, we have that $(\frac{x}{x+y})^p + (\frac{y}{x+y})^p \geq 1$, in other words, $(x+y)^p \leq x^p + y^p$ $(x, y \in (0, \infty))$.

In the first part of (2), put $q := 1/p$, $u_i := x_i^p$, $v_i := y_i^p$ $(i = 1, \ldots, n)$. Then $x_i = u_i^q$, $y_i = v_i^q$, so $\|x\|_p = \left(\sum_i x_i^p\right)^{1/p} = \left(\sum_i u_i\right)^q$, and the inequality may be rewritten as

$$\left(\sum_i (x_i + y_i)^p\right)^{1/p} \geq \left(\sum_i u_i\right)^q + \left(\sum_i v_i\right)^q,$$

or, by taking qth roots (i.e., pth powers):

$$\sum_i \left(u_i^q + v_i^q\right)^{1/q} \geq \left(\left(\sum_i u_i\right)^q + \left(\sum_i v_i\right)^q\right)^{1/q}.$$

But in terms of the norm $|| \ ||_q$ on $\mathbb{R}^2$ this inequality says

$$\sum_i ||(u_i, v_i)||_q \geq || \left(\sum_i u_i, \sum_i v_i\right) ||_q,$$

which is true by the triangle inequality of $|| \ ||_q$. □

Lemma 2.17 ($\| \ \|_p$-inequalities) *Assume* (S) *and suppose* E *is weak-Freudenthal,* $p \in (0, \infty)$, $\rrbracket \llbracket$ *p-additive. Let* $a, b \in E^+$. *Then:*

$$\text{If } p \geq 1, \text{ then } \rrbracket a + b \llbracket \ \leq \ \rrbracket a \llbracket \ + \ \rrbracket b \llbracket\ , \text{ and } \rrbracket a + b \llbracket^p \geq \rrbracket a \llbracket^p + \rrbracket b \llbracket^p.$$
$$\text{If } p \leq 1, \text{ then } \rrbracket a + b \llbracket \ \geq \ \rrbracket a \llbracket \ + \ \rrbracket b \llbracket\ , \text{ and } \rrbracket a + b \llbracket^p \leq \rrbracket a \llbracket^p + \rrbracket b \llbracket^p.$$

More concisely put:

$$\rrbracket a + b \llbracket^{1 \wedge p} \leq \rrbracket a \llbracket^{1 \wedge p} + \rrbracket b \llbracket^{1 \wedge p}, \qquad (1)$$
$$\rrbracket a + b \llbracket^{1 \vee p} \geq \rrbracket a \llbracket^{1 \vee p} + \rrbracket b \llbracket^{1 \vee p}. \qquad (2)$$

Proof We prove (2). The proof of (1) is similar but slightly simpler.

(I) We already know that for every $n \in \mathbb{N}$ and $x, y \in (\mathbb{R}^n)^+$:

$$\text{If } p \geq 1, \text{ then } \|x + y\|_p \ \leq \ \|x\|_p \ + \ \|y\|_p\ , \text{ and } \| x + y \|_p^p \ \geq \ \| x \|_p^p \ + \ \| y \|_p^p .$$
$$\text{If } p \leq 1, \text{ then } \|x + y\|_p \ \geq \ \|x\|_p \ + \ \|y\|_p\ , \text{ and } \| x + y \|_p^p \ \leq \ \| x \|_p^p \ + \ \| y \|_p^p .$$

In other words:

$$\|a + b\|_p{}^{1 \wedge p} \leq \ \|a\|_p{}^{1 \wedge p} + \ \|b\|_p{}^{1 \wedge p}, \qquad (1p)$$
$$\|a + b\|_p{}^{1 \vee p} \geq \ \|a\|_p{}^{1 \vee p} + \ \|b\|_p{}^{1 \vee p}. \qquad (2p)$$

(II) Let $\epsilon > 0$. As E is weak-Freudenthal, Lemma 2.11 gives us disjoint non-zero $e_1, \ldots, e_n \in E^+$ and nonnegative numbers $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ with

$$\sum_{i=1}^{n} e_i = a + b, \text{ and } \begin{cases} \sum_{i=1}^n \alpha_i e_i \leq a \leq \sum_{i=1}^n (\alpha_i + \epsilon) e_i, \\ \sum_{i=1}^n \beta_i e_i \leq b \leq \sum_{i=1}^n (\beta_i + \epsilon) e_i. \end{cases} \qquad (*)$$

Define elements $\bar{e}, \bar{a}, \bar{b}$ of $\mathbb{R}^n$ by

$$\bar{e} := (\rrbracket e_1 \llbracket, \ldots, \rrbracket e_n \llbracket),$$
$$\bar{a} := (\alpha_1 \rrbracket e_1 \llbracket, \ldots, \alpha_n \rrbracket e_n \llbracket),$$
$$\bar{b} := (\beta_1 \rrbracket e_1 \llbracket, \ldots, \beta_n \rrbracket e_n \llbracket).$$

If $\gamma_1, \ldots, \gamma_n \in [0, \infty)$ and $\bar{c} = (\gamma_1 \rrbracket e_1 \llbracket, \ldots, \gamma_n \rrbracket e_n \llbracket) \in \mathbb{R}^n$, then by the p-additivity of the functional $\rrbracket \llbracket$ we have $\rrbracket \sum_i \gamma_i e_i \llbracket = \|\bar{c}\|_p$. Thus, from $(*)$ and the monotonicity of $\rrbracket \llbracket$ it follows that

$$\left.\begin{array}{ll} \|\bar{e}\|_p = \rrbracket e \llbracket, & \\ \|\bar{a}\|_p \leq \rrbracket a \llbracket \leq \|\bar{a} + \varepsilon \bar{e}\|_p, & \|\bar{b}\|_p \leq \rrbracket b \llbracket \leq \|\bar{b} + \varepsilon \bar{e}\|_p, \\ \|\bar{a} + \bar{b}\|_p \leq \rrbracket a + b \llbracket. & \end{array}\right] \qquad (\pitchfork)$$

Thus,

$$\begin{aligned} \rrbracket a \llbracket^{1\vee p} + \rrbracket b \llbracket^{1\vee p} &\overset{(\pitchfork)}{\leq} ||\bar{a}+\epsilon\bar{e}||_p^{1\vee p} + ||\bar{b}+\epsilon\bar{e}||_p^{1\vee p} \\ &\overset{(2p)}{\leq} ||\left(\bar{a}+\epsilon\bar{e}\right)+\left(\bar{b}+\epsilon\bar{e}\right)||_p^{1\vee p} \\ &= ||\left(\bar{a}+\bar{b}\right)+2\epsilon\bar{e}||_p^{1\vee p} \\ &= \left[||\left(\bar{a}+\bar{b}\right)+2\epsilon\bar{e}||_p^{1\wedge p}\right]^{\frac{1\vee p}{1\wedge p}} \\ &\overset{(1p)}{\leq} \left[||\bar{a}+\bar{b}||_p^{1\wedge p} + ||2\epsilon\bar{e}||_p^{1\wedge p}\right]^{\frac{1\vee p}{1\wedge p}} \\ &\overset{(\pitchfork)}{\leq} \left[\rrbracket a+b \llbracket^{1\wedge p} + (2\epsilon \rrbracket e \llbracket)^{1\wedge p}\right]^{\frac{1\vee p}{1\wedge p}}. \end{aligned}$$

(III) The resulting inequality holds for every positive ϵ. This proves (2). □

Corollary 2.18 *Assume* (S). *Suppose* E *is weak-Freudenthal and* $\rrbracket\llbracket$ *is* p*-additive for a* $p \in (0,\infty)$. *Then the vector addition is continuous. Moreover,* $\rrbracket\llbracket$ *is a Riesz quasi-norm. For* $p \geq 1$, *it is a Riesz norm for* $p = 1$ *it is additive on* E^+.

Proof For $p = 1$, 1-additivity implies additivity on E^+ by applying the above lemma. Let now $x, y \in E$. Set $a := |x|$ and $b := |y|$. Then $|x + y| \leq a + b$, so $\rrbracket x + y \llbracket \leq \rrbracket a + b \llbracket$.

If $p \geq 1$, we obtain that $\rrbracket x + y \llbracket \leq \rrbracket a + b \llbracket \leq \rrbracket a \llbracket + \rrbracket b \llbracket = \rrbracket x \llbracket + \rrbracket y \llbracket$, i.e., $\rrbracket\llbracket$ is a Riesz norm.

If $p \leq 1$, we have $\rrbracket x + y \llbracket^p \leq \rrbracket a + b \llbracket^p \leq \rrbracket a \llbracket^p + \rrbracket b \llbracket^p \leq (\rrbracket a \llbracket + \rrbracket b \llbracket)^p + (\rrbracket a \llbracket + \rrbracket b \llbracket)^p = 2\,(\rrbracket a \llbracket + \rrbracket b \llbracket)^p = 2\,(\rrbracket x \llbracket + \rrbracket y \llbracket)^p$, so $\rrbracket x+y \llbracket \leq 2^{1/p}\,(\rrbracket x \llbracket + \rrbracket y \llbracket)$, and $\rrbracket\llbracket$ is a quasi-norm. □

3 Proof of characterization Theorems 1.7 and 1.9

Proof *(of Theorem 1.9)* Suppose condition (S) holds, and that $\rrbracket\llbracket$ is p-additive and E is σ-Levi.

(I) Applying the Maeda–Ogasawara–Vulikh representation theorem we may view E as an order dense Riesz subspace of $C^\infty(S)$ for some extremally disconnected compact Hausdorff space S. By Lemma 2.6, E is Dedekind complete and E is a Riesz ideal in $C^\infty(S)$.

(II) The map $\overline{\phi}_p : [-\infty,\infty] \to [-\infty,\infty]$ defined by

$$\overline{\phi}_p(t) := \begin{cases} t^p & \text{if } t \in [0,\infty); \\ -|t|^p & \text{if } t \in (-\infty,0); \\ +\infty & \text{if } t = \infty; \\ -\infty & \text{if } t = -\infty, \end{cases}$$

is an order preserving homeomorphism $[-\infty, \infty] \to [-\infty, \infty]$. As a result the formula

$$\Phi_p(f) = f^p := \overline{\phi}_p \circ f \ \in C^\infty(S) \quad (f \in C^\infty(S))$$

defines an order isomorphism of $C^\infty(S)$ onto itself. Observe that for every $f \in C^\infty(S)$

$$\Phi_p(-f) = -\Phi_p(f), \qquad \Phi_p(rf) = r^p \Phi(f) \quad (r \in (0, \infty).$$

(III) Put $E_1 := \Phi_p(E)$. Then E_1 is an Riesz ideal in $C^\infty(S)$.
If $g \in C^\infty(S)$, $h \in E_1$, $|g| \leq |h|$ then $|\Phi_p^{-1}(g)| = \Phi_p^{-1}(|g|) \leq \Phi_p^{-1}(|h)| = |\Phi_p^{-1}(h)| \in E$, so $\Phi_p^{-1}(g) \in E$ and $g \in E_1$. Clearly, $g \in E_1, r \in \mathbb{R}$ implies $rg \in E_1$. Now let $g, h \in E_1, g \geq 0, h \geq 0$. Then $0 \leq \Phi_p^{-1}(g + h) \leq \Phi_p^{-1}(2g \vee 2h) = 2^{1/p}\Phi_p^{-1}(g) \vee 2^{1/p}\Phi_p^{-1}(h) \in E$, so that $g + h \in E_1$. As a result, E_1 is a Riesz ideal.

(IV) The formula $\rVert \Phi_p(g) \lVert_1$ $(g \in E)$ defines a Riesz norm on E_1 that is additive on E_1^+.
The p-additivity of $\rVert\, \lVert$ on E translates to 1-additivity of $\rVert\, \lVert_1$ on E_1, and the Riesz property is conserved. Because E is Dedekind complete, so is E_1, and E_1 is weak-Freudenthal. For $r \in (0, \infty)$,
$\rVert r\Phi_p(g) \lVert_1 = \rVert \Phi_p(r^{1/p} g) \lVert_1 = \rVert r^{1/p} g \lVert^p = \left(r^{1/p} \rVert f \lVert\right)^p = r \rVert \Phi(f) \lVert_1$, so the pair $(E_1, \rVert\, \lVert_1)$ satifies condition (S). By Corollary 2.18, (IV) follows.

(V) $(E_1, \rVert\, \lVert_1) \simeq (L^1(\mu), |\,|\ \ |\,|_1)$.
Because E_1 is a Riesz ideal of $C^\infty(S)$ and (IV), E_1 inherits the σ-Levi property from E and we can apply Theorem 2.2 to obtain a isometric Riesz isomorphism $\Psi : E_1 \to L^1(\mu)$.

(VI) $L^p(\mu)$ is order isomorphic to $L^1(\mu)$
Define $\Phi_{1/p} : L^1(\mu) \to L^p(\mu)$ by $\Phi_{1/p}(f) = f^{1/p} := \phi_{1/p} \circ f$ $(f \in L^1(\mu))$, where $\phi_{1/p}$ is the restriction of $\overline{\phi}_{1/p}$ to $\mathbb{R}$, which is an order preserving homeomorphism $\mathbb{R} \to \mathbb{R}$. Then $\Phi_{1/p}$ is an order isomorphism and $\| f^{1/p} \|_p^p = \|f\|_1$ $(f \in L^1(\mu))$.

(VII) $\Omega := \Phi_{1/p}\Psi\Phi_p$ is an isometric Riesz isomorphism $(E, \rVert\, \lVert) \to (L^p(\mu), \|\ \|_p)$.
As a composition of order isomorphisms that preserve $\rVert\, \lVert$, $|\,|\ \ |\,|_1$, and $\|\ \|_p$, Ω is an isometric lattice isomorphism. Also, $\Omega(rx) = r\Omega(x)$ for $x \in E$ and $r \in [0, \infty)$. By applying Lemma 2.4, we have that Ω is a Riesz isomorphism. □

Theorem 1.7 follows from Theorem 1.9 once we have proven the following lemma.

Lemma 3.1 *Assume (S). Let E be $\rVert\, \lVert$-complete and let $\rVert\, \lVert$ be p-additive for some $p \in (0, \infty)$. Then E is σ-Levi.*

Proof (I) E is conditionally σ-laterally complete.
Let $u_1, u_2, \ldots$ be disjoint in E^+ and bounded from above by u. The latter implies that $\sum_1^\infty \rVert u_n \lVert^p$ is finite, because it is bounded from above by $\rVert u \lVert^p$. Then $x_n := u_1 + \cdots + u_n, n = 1, 2, \ldots$, is increasing and $\sup_n \rVert x_n \lVert^p = \sum_1^\infty \rVert u_n \lVert^p < \infty$, so its supremum exists because it is a $\rVert\, \lVert$-Cauchy sequence: $\rVert x_n - x_m \lVert^p \leq \sum_{i=m}^\infty \rVert u_i \lVert^p \to 0$ if $n \geq m \to \infty$. Let $x = \lim_n x_n$. Then $x = \sup_n x_n$ by Lemma 1.10.

(II): Hence E has the principal projection property [11, 25.1] and is weak-Freudenthal [11, 40.2].

(III): E is σ-Levi.

Let $0 \leq u_1 \leq u_2 \leq \cdots$ in E^+ with $\sup_n |\!|\!| u_n |\!|\!| < \infty$. We show that $(u_n)_n$ is $|\!|\!|\ |\!|\!|$ Cauchy; then by $|\!|\!|\ |\!|\!|$-completeness and by Lemma 1.10, it has a supremum. It follows from Lemma 2.17 that E satisfies the $\|\ \|_p$-inequalities. Hence for $m > n$, we have

$$|\!|\!| u_m - u_n |\!|\!| \leq |\!|\!| u_m |\!|\!|^{1\vee p} - |\!|\!| u_n |\!|\!|^{1\vee p} \leq \sup_{k\in\mathbb{N}} |\!|\!| u_k |\!|\!|^{1\vee p} - \sup_{k\leq n} |\!|\!| u_k |\!|\!|^{1\vee p}.$$

Thus, $(u_n)_n$ is $|\!|\!|\ |\!|\!|$-Cauchy. □

References

1. Köthe, G.: Topological Vector Spaces I. Grundlehren der mathematischen Wissenschaften, vol. 159. Springer, New York (1983)
2. Aliprantis, C.D., Burkinshaw, O.: Locally Solid Riesz Spaces. Academic Press, New York (1978)
3. Teerenstra, S.: Characterizations of L^p-spaces with $p \in (0, \infty)$. Ph.D. thesis, Katholieke Universiteit Nijmegen, Nijmegen, The Netherlands (2004). http://hdl.handle.net/2066/60703. Accessed 11 May 2017
4. Bohnenblust, H.F.: An axiomatic characterization of L^p-spaces. Duke Math. J. **6**, 627–640 (1940)
5. Kakutani, S.: Concrete representation of abstract L-spaces and the mean ergodic theorem. Ann. Math. **42**, 523–537 (1941)
6. Meyer-Nieberg, P.: Banach Lattices. Springer, Berlin (1991)
7. Teerenstra, S.: An extension of Kakutani's theorem on abstract L^p-spaces to the case of any positive p. Positivity **4**, 289–291 (2000)
8. Maeda, F., Ogasawara, T.: Representation of vector lattices. J. Sci. Hiroshima Univ. Ser. A. **12**, 17–35 (1942)
9. Vulikh, B.Z.: Introduction to the Theory of Partially Ordered Spaces. Gorden and Breach, New York (1967)
10. Yosida, K.: On the representation of the vector lattice. Proc. Imp. Acad. Tokyo **18**, 339–342 (1942)
11. Luxemburg, W.A.J., Zaanen, A.C.: Riesz Spaces, vol. I. North-Holland Publishing Company, Amsterdam (1971)
12. Lavrič, B.: On Freudenthal's spectral theorem. Indag. Math. **48**, 411–421 (1986)