A NEW CLASS OF NILPOTENT JACOBIANS IN ANY DIMENSION

ÁLVARO CASTAÑEDA AND ARNO VAN DEN ESSEN

Abstract. The classification of the nilpotent Jacobians with some structure has been an object of study because of its relationship with the Jacobian conjecture. In this paper we classify the polynomial maps in dimension n of the form $H = (u(x, y), u_2(x, y, x_3), \dots , u_{n-1}(x, y, x_n), h(x, y))$ with JH nilpotent. In addition we prove that the maps $X + H$ are invertible, which shows that for this kind of maps the Jacobian conjecture is verified.

1. Introduction

Let k be a field of characteristic zero and $k[X] = k[x_1, \cdots , x_n]$ the polynomial ring in n variables over k. Since the remarkable works of H. Bass *et al.* [1] and A.V. Yagzhev [16] concerning the Jacobian Conjecture, the study of polynomial maps $H : k^n \to k^n$ such that its Jacobian matrix JH is nilpotent has grabbed the attention of many authors. Although the previously mentioned works establish that, in order to study the conjecture, it is sufficient to focus on maps of the form $X+H$ where H is homogeneous of degree 3, the classification of maps with nilpotent Jacobian of any degree, even inhomogeneous, has an interest which goes beyond the Jacobian Conjecture. For example it led various authors to formulate the following problem:

(Homogeneous) Dependence Problem. Let $H = (H_1, \dots , H_n) \in k[X]^n$ (homogeneous of degree $d \geq 1$) such that JH is nilpotent and $H(0) = 0$. Does it follow that $H_1, \dots , H_n$ are linearly dependent over k or equivalently does it follow that the rows of JH are linearly dependent over k?

An affirmative answer was given in the following cases: rank $JH \leq 1$ in [1], hence if $n = 2$ and in case H is homogeneous of degree 3 when $n = 3$ by D. Wright in [15] (resp. when $n = 4$ by E. Hubbers in [14]). In dimension three an affirmative answer to the homogeneous dependence problem (in any degree) was given by M. de Bondt and A. van den Essen in [2]. On the other hand M. de Bondt in [3] constructed homogeneous examples in all dimensions ≥ 5 of nilpotent Jacobians with over k linearly *independent* rows.

Although the answer to the dependence problem turned out to be negative in general, studying this problem payed off in several ways. For example the assumption that the answer to the dependence problem would be positive led the authors in [11] to construct a large class of polynomial maps H such that JH is

Date: September 7, 2018.
2010 *Mathematics Subject Classification.* Primary: 14R15 14R10.
Key words and phrases. Jacobian Conjecture; Nilpotent Jacobian matrix; Polynomial maps.
The first author have been partially supported by FONDECYT Regular 1170968.

nilpotent. Several of these examples were subsequently used to find counterexamples to various conjectures, such as Meisters' Cubic Linearization Conjecture [9], the DMZ-Conjecture [12], the long standing Markus-Yamabe Conjecture and the Discrete Markus Yamabe Problem [6].

The first negative answer to the dependence problem was found by the second author in [8], namely

$$H = (y - x^2, z + 2x(y - x^2), -(y - x^2)^2).$$

Remarkably, searching for more negative examples in dimension three, the authors of [5] showed that, looking for such examples of the form

$$(u(x, y), v(x, y, z), h(u(x, u), v(x, y, z))$$

the above example is, apart from a linear coordinate change, essentially the only example. This example was generalized in Proposition 7.1.9 [10] to give nilpotent Jacobians in all dimensions, with over k linearly independent rows. It was shown in [4] that for these examples H and each $\lambda < 0$, the corresponding dynamical system $\dot{x} = F(x)$, where $F(x) = \lambda x + H(x)$, has orbits which escape to infinity, hence are counterexamples to the Markus-Yamabe Conjecture.

Recently, in [17], Dan Yan completely classified all H of the form

$$(u(x, y), v(x, y, z), h(x, y))$$

with nilpotent Jacobian and over k linearly independent rows. Again they all turned out to be linearly equivalent to the first example found by the second author. These results confirm a conjecture of the first author which asserts that if JH is nilpotent, with over $\mathbb{R}$ linearly independent rows, then the corresponding dynamical system $\dot{x} = F(x)$, where $F(x) = \lambda x + H(x)$ and $\lambda < 0$, has orbits which escape to infinity. To get more evidence for this last conjecture it is therefore natural to look for nilpotent Jacobians in dimensions ≥ 4.

In this paper we pursue this idea and generalize the recent result of Dan Yan to all dimensions $n \geq 3$. More precisely, we study maps of the form

$$H = (u(x, y), u_2(x, y, x_3), u_3(x, y, x_4), \cdots, u_{n-1}(x, y, x_n), h(x, y))$$

The main result of this paper, Theorem 1, completely classifies all such H, which Jacobian is nilpotent. Moreover, in the last section we give a very detailed description of these maps. This enables us to show that the corresponding maps $F = X + H$, which Jacobian determinant equal 1, are invertible. So we confirm the Jacobian Conjecture for these maps. A priori, from the construction of the H's it is not at all obvious why F should be invertible. The delicate proof we give below is, in our opinion, a strong indication that the Jacobian Conjecture might be true after all (inspite of several statements of the second author in the past). More evidence in favor of the Jacobian Conjecture can be found in the works of Zhao and his co-authors, in which the Jacobian Conjecture is firmly embedded in the framework of Mathieu-Zhao spaces (see [18], [19], [20], [13] and [7]).

2. The nilpotency of JH

In this section we establish a characterization of the nilpotency of JH with H a polynomial map of the form $H = (u(x, y), u_2(x, y, x_3), \ldots, u_{n-1}(x, y, x_n), h(x, y))$.

Proposition 1. *JH nilpotent if and only if*

$$u_x + u_{2y} = 0$$
$$u_x u_{2y} - u_y u_{2x} - u_{2x_3} u_{3y} = 0$$
$$u_{2x_3}(u_x u_{3y} - u_y u_{3x} - u_{3x_4} u_{4y}) = 0$$
$$u_{2x_3} u_{3x_4}(u_x u_{4y} - u_y u_{4x} - u_{4x_5} u_{5y}) = 0$$
$$\cdots$$
$$u_{2x_3} u_{3x_4} \cdots u_{n-1x_n}(u_x h_y - u_y h_x) = 0$$

Proof (started): Let S be a new variable and put $T := S^{-1}$. Then JH is nilpotent if and only if $-JH$ is nilpotent if and only if $det\,(SI_n + JH) = S^n$ if and only if $d(T) := det\,(I_n + TJH) = 1$. Since $d(T)$ is a polynomial in $k[x, y, \cdots, x_n][T]$ of degree n in T and $d(0) = 1$, the statement that $d(T) = 1$ is equivalent to the fact that for each $1 \leq i \leq n$ the coefficient of T^i in $d(T)$ is equal to zero. We will show that the coefficient of T^1 being zero gives the first equation, the coefficient of T^2 the second and so on. We use some linear algebra to see this. Therefore put $D_n := I_n + TJH$. For $1 \leq k \leq n$ denote by $D_{n(k)}$ the k-th column of D_n. Then

$$D_{n(1)} = T \begin{pmatrix} u_{1x} \\ \vdots \\ u_{nx} \end{pmatrix} + e_1, \ D_{n(2)} = T \begin{pmatrix} u_{1y} \\ \vdots \\ u_{ny} \end{pmatrix} + e_2,$$

and

$$D_{n(k)} = e_k + T u_{k-1x_k} e_{k-1}, \text{ for all } 3 \leq k \leq n$$

where e_i is the i-th standard basis vector in k^n.

Write $(a_1, \cdots, a_n)^t$ instead of $D_{n(1)}$ and $(b_1, \cdots, b_n)^t$ instead of $D_{n(2)}$ and put $c_i = T u_{ix_{i+1}}$, for $2 \leq i \leq n-1$. So $a_1 = 1 + Tu_x$, $a_i = Tu_{ix}$, for $2 \leq i \leq n$, $b_1 = Tu_y$, $b_2 = 1 + Tu_{2y}$ and $b_i = Tu_{iy}$ for $3 \leq i \leq n$.

Lemma 1. Let $d_n := det\,D_n$. Then

$$d_n = a_1 b_2 - a_2 b_1 + \sum_{k=2}^{n-1} (-c_2) \cdots (-c_k)(a_1 b_{k+1} - b_1 a_{k+1})$$

Proof. Using the Laplace expansion of d_n along the n-th column of D_n we get

$$d_n = d_{n-1} + (-c_{n-1}) det\,A_{n-1}$$

where A_{n-1} is the $(n-1) \times (n-1)$ matrix obtained from D_n by deleting the $(n-1)$-th row and n-th column. One easily verifies that $det\,A_{n-1} = (-c_2) \cdots (-c_{n-2})(a_1 b_n - b_1 a_n)$. So

$$d_n = d_{n-1} + (-c_2) \cdots (-c_{n-1})(a_1 b_n - b_1 a_n)$$

The result now follows by induction on n. □

Proof (finished). An easy calculation gives that

$$a_1 b_2 - a_2 b_1 = 1 + T(u_x + u_{2u}) + T^2(u_x u_{2y} - u_y u_{2x})$$

and if $2 \leq k \leq n-1$, then

$$(-c_2) \cdots (-c_k)(a_1 b_{k+1} - b_1 a_{k+1}) =$$
$$(-1)^k u_{2x_3} \cdots u_{kx_{k+1}} (T^k u_{k+1y} + T^{k+1}(u_x u_{k+1y} - u_y u_{k+1x}))$$

Using the previous lemma, it is left to the reader to deduce that, apart from a minus sign, the coefficient of T^k in d_n gives the k-th equation of Proposition 1, which concludes the proof.

Corollary 1. *Notations as in Proposition 1. If $u_{2x_3} = 0$, then JH is nilpotent if and only if there exist $\lambda_1, \lambda_2, c_1, c_2 \in k$ and $f(T) \in k[T]$ such that $u = \lambda_2 f(\lambda_1 x + \lambda_2 y) + c_1$ and $u_2 = -\lambda_1 f(\lambda_1 x + \lambda 2y) + c_2$.*

Proof. By Proposition 1 we get that JH is nilpotent if and only if $u_x + u_{2y} = 0$ and $u_x u_{2y} - u_y u_{2x} = 0$. Since $u_{2x_3} = 0$ the result follows from Theorem 7.2.25 [10]. □

So from now on *we may assume that* $u_{2x_3} \neq 0$. Since $u_{nx_{n+1}} = 0$, there exists $3 \leq r \leq n$ such that $u_{ix_{i+1}} \neq 0$ for all $2 \leq i \leq r-1$ and $u_{rx_{r+1}} = 0$. By Proposition 1, we have the following equations

$$u_x + u_{2y} = 0,$$

$$u_x u_{2y} - u_y u_{2x} = u_{2x_3} u_{3y},$$

$$u_{2x_3}(u_x u_{3y} - u_y u_{3x} - u_{3x_4} u_{4y}) = 0,$$

$$\vdots$$

$$u_{2x_3} \cdots u_{r-2x_{r-1}}(u_x u_{r-1y} - u_y u_{r-1x} - u_{r-1x_r} u_{ry}) = 0,$$

$$u_{2x_3} \cdots u_{r-1x_r}(u_x u_{ry} - u_y u_{rx}) = 0.$$

Since $u_{2x_3} \neq 0, \ldots, u_{r-1x_r} \neq 0$, these equations become

$$u_x + u_{2y} = 0 \qquad (1),$$

$$u_x u_{2y} - u_y u_{2x} = u_{2x_3} u_{3y} \qquad (2),$$

$$u_x u_{3y} - u_y u_{3x} = u_{3x_4} u_{4y} \qquad (3),$$

$$\vdots$$

$$u_x u_{r-1y} - u_y u_{r-1x} = u_{r-1x_r} u_{ry}, \qquad \text{(r-1)},$$

$$u_x u_{ry} - u_y u_{rx} = 0 \qquad \text{(r)}.$$

Corollary 2. *Let $u_{2x_3} \neq 0$ and r as above. If $u_y = 0$, then JH is nilpotent if and only if $u \in k$ and $u_{iy} = 0$ for all $2 \leq i \leq r$.*

Proof. The if-part follows from the equations $(1) \cdots (r)$. Conversely, assume that the equations $(1) \cdots (r)$ hold. Since $u_y = 0$ equation (r) gives $u_x u_{ry} = 0$. Assume $u_x \neq 0$. Then $u_{ry} = 0$. So equation $(r-1)$ implies that $u_{r-1y} = 0$. Continuing in this way we arrive at $u_{3y} = 0$ and then by (2) that $u_{2y} = 0$. This contradicts equation (1), since by assumption $u_x \neq 0$. Consequently $u_x = 0$, i.e. $u \in k$. It follows from (1) that $u_{2y} = 0$ and that equation (r) is satisfied. Furthermore, for each $2 \leq i \leq r-1$ equation (i) becomes $u_{ix_{i+1}} u_{i+1y} = 0$, from which the desired result follows. □

3. A lemma of Dan Yan

The following result was proved by Dan Yan (see [17, Lemma 2.1]) for the case that the field k is algebraically closed. We will extend her result to arbitrary fields of characteristic zero. To keep this paper self-contained we give a short proof.

Lemma 2. *Let k be a field of characteristic zero, $q \in k[x,y]$ and $0 \neq w(q) \in k[q]$ such that $q_y | w^{e_1} {q_x}^{e_2}$ for some $e_1, e_2 \geq 1$. If $p \nmid q_y$ for every $p \in k[x] \backslash k$, then $q = P(y + b(x))$, for some $P(T) \in k[T]$ and $b(x) \in k[x]$.*

Let $p \in k[x,y]$ be irreducible. If $0 \neq a \in k[x,y]$ we denote by $v_p(a)$ the number of factors p in a. So $v_p(a) \geq 0$ and one easily verifies that if $a, b \in k[x,y] \backslash \{0\}$, then $v_p(ab) = v_p(a) + v_p(b)$. If $p_y \neq 0$, then $p \nmid p_y$ (look at degrees). One easily deduces

$$\text{If } p_y \neq 0 \text{ and } d := v_p(g) \geq 1, \text{ then } v_p(g_y) = d - 1. \tag{3.1}$$

Proof. First assume that k is algebraically closed.
i) We show that $q_y | q_x$: let p be irreducible and $v_p(q_y) = e \geq 1$. Then $p_y \neq 0$, for if $p_y = 0$, then $p \in k[x] \backslash k$ divides q_y, contradicting the hypothesis. Also by the hypothesis $p | q_x$ or $p | w(q)$. We prove that in both cases $p^e | q_x$. Since this holds for all prime factors p of q_y we get $q_y | q_x$.

Case 1. $p | q_x$. Then $d := v_p(q_x) \geq 1$. So by (3.1) $v_p(q_{xy}) = d - 1$. Since $v_p(q_y) = e$ we get $v_p(q_{xy} \geq e - 1$. So $d \geq e$, whence $p^e | q_x$.

Case 2. $p | w(q)$. Since k is algebraically closed we can write $w(q)$ as a product of factors $q + c$, with $c \in k$. So $p | q + c$, for some $c \in k$. Then $d := v_p(q + c) \geq 1$. So by (3.1) $e = v_p(q_y) = d - 1$, i.e. $d = e + 1$. Hence $p^{e+1} | q + c$. So $p^e | q_x$.
ii) Let $r := deg_y q$. Then $r \geq 1$. Since $deg_y\, q_x \leq deg_y\, q_y + 1$, it follows from $q_y | q_x$ that $q_x = (c_1(x)y + c_0(x)) q_y$, for some $c_i \in k[x]$. The coefficient of y^r gives $q_r'(x) = c_1(x) r q_r(x)$. Hence $deg_x\, q_r(x) = 0$, i.e. $q_r \in k^*$. So $0 = c_1(x) r q_r$, whence $c_1(x) = 0$. So $q_x = c_0(x) q_y$, i.e. $(\partial_x - c_0(x)\partial_y) q = 0$. Let $b'(x) = c_0(x)$. Then $q \in k[y + b(x)]$, as desired.

iii) Now let k be an arbitrary field of characteristic zero. From linear algebra one knows that if $k \subseteq L$ is a field extension, then any system of non-homogeneous linear equations in n variables with coefficients in k, which has a solution in L^n, also has a solution in k^n. From this fact one readily deduces that if $a(x,y), b(x,y) \in k[x,y]$ are such that $b(x,y) | a(x,y)$ in $L[x,y]$, then also $b(x,y) | a(x,y)$ in $k[x,y]$.

Finally assume that the hypothesis of Dan Yan's lemma are satisfied for polynomials in $k[x,y]$. Then they are obviously satisfied in $\overline{k}[x,y]$, where $\overline{k}$ is an algebraic closure of k. It then follows from i) that $q_y | q_x$ in $\overline{k}[x,y]$. Hence, as observed above, $q_y | q_x$ in $k[x,y]$. Then, by the argument given in ii), which does *not* use the algebraically closedness condition, we get the desired result. $\square$

4. $u(x,y) = p(y + a(x))$

In this section we *assume the relations of Proposition 1* and show that $u(x,y) = p(y+a(x))$ for some $a(x) \in k[x]$ and $p(T) \in k[T]$.

So we have the following situation: $n \geq 3$, $u = u(x,y)$, $u_i = u_i(x,y,x_{i+1})$ for all $2 \leq i \leq n-1$ and $u_n = h(x,y)$. We define $u_{n+1} = 0$. Put

$$D_0 := u_y \partial_x - u_x \partial_y$$

Then $k[x,y]^{D_0} = k[q]$ for some $q \in k[x,y]$ (see [10, Theorem 1.2.25]). We may assume $q(0) = 0$. The equations in Proposition 1 can be written as

$$u_x + u_{2y} = 0 \tag{4.1}$$

$$-D_0(u_2) = u_{2x_3} u_{3y} \tag{4.2}$$

$$u_{2x_3} \cdots u_{i-1x_i}(-D_0(u_i) - u_{ix_{i+1}} u_{i+1y}) = 0, \text{ for all } 3 \leq i \leq n$$

We may assume that $u_y \neq 0$ and $u_{2x_3} \neq 0$.

Lemma 3. *Let $v = v_0(x,y) + \sum_{i=1}^d v_i(x)s^i$, with $v_d \neq 0$ and $d \geq 2$. If $v_{0y} \neq 0$ and there exists $w \in k[x,y,t]$ such that*

$$D_0(v) = -v_s w_y \tag{4.3}$$

then $v_d \in k^*$, $w_y = -\frac{1}{dv_d}v'_{d-1}(x)u_y$ and $v_y = Q(q)_y$ for some $Q(T) \in k[T]$ with $deg_T\, Q(T) \geq 1$.

Proof. The coefficient of s^d in (4.3) gives $v_d \in k^*$ and the coefficient of s^{d-1} gives $u_y v'_{d-1}(x) = -dv_d w_y$. So $w_y = -\frac{1}{dv_d}v'_{d-1}(x)u_y$. Then the coefficient of s^0 implies that $D_0(v_0) = \frac{1}{dv_d}v'_{d-1}(x)v_1(x)u_y$. Let $b(x) \in k[x]$ with $b'(x)) = \frac{1}{dv_d}v'_{d-1}(x)v_1(x)$. Then $D_0(v_0) = D_0(b(x))$. So $v_0 = b(x) + Q(q)$, for some $Q(T) \in k[T]$. So $v_y = v_{0y} = Q(q)_y$. Since $v_{0y} \neq 0$ we get $deg_T\, Q(T) \geq 1$. □

Let $3 \leq r \leq n$ be such that $u_{ix_{i+1}} \neq 0$ for all $i < r$ and $u_{rx_{r+1}} = 0$ (observe that $u_{nx_{n+1}} = h(x,y)_{x_{n+1}} = 0$, so such an r exists).

Proposition 2. *If u and the u_i satisfy the equations of Proposition 1, then $u = p(y+a(x))$, for some $p(T) \in k[T]$ with $deg_T\, p(T) \geq 1$ and $a(x) \in k[x]$.*

Proof. Let r be as above. Then $u_{rx_{r+1}} = 0$ and $u_{2x_3}, \cdots, u_{r-1x_r}$ are all non-zero. So the above equations become

$$u_x + u_{2y} = 0 \tag{4.4}$$

$$-D_0(u_i) = u_{ix_{i+1}} u_{i+1y}, \text{ for all } 2 \leq i \leq r-1 \tag{4.5}$$

$$D_0(u_r) = 0. \tag{4.6}$$

From (4.6) we get $u_r = H(q)$, for some $H(T) \in k[T]$. Also $u = p(q)$. So $u_y = p'(q)q_y \equiv 0\,(mod\, q_y)$. Since $-D_0(u_i) = u_x u_{iy} - u_y u_{ix}$ we get $-D_0(u_i) \equiv u_x u_{iy}\,(mod\, q_y)$. So by (4.5) we get

$$(4.7) \qquad u_x u_{iy} \equiv u_{i x_{i+1}} u_{i+1 y}\,(mod\, q_y),\ \text{for all } 2 \leq i \leq r-1.$$

Since $u_n = H(q)$ we get $u_{ny} = H'(q)q_y \equiv 0\,(mod\, q_y)$. So by (4.7) applied to $i = r-1$ we get $u_x u_{r-1y} \equiv 0\,(mod\, q_y)$. Then, multiplying (4.7) ($i = r-2$) by u_x, we get ${u_x}^2 u_{r-1y} \equiv 0\,(mod\, q_y)$. Continuing in this way we find that ${u_x}^{r-2} u_{2y} \equiv 0\,(mod\, q_y)$. Finally, (4.1) implies that ${u_x}^{r-1} \equiv 0\,(mod\, q_y)$. Since $u_x = p'(q)q_x$ we get that $q_y | {p'(q)}^{r-1} {q_x}^{r-1}$. Let $d := deg_y\, q$ and let $q_d(x)$ be the coefficient of y^d. In lemma 5 below we will show that $q_d(x) \in k^*$. So it follows from lemma 2 that $q = p(y + a(x))$, for some $p(T) \in k[T]$ with $deg_T\, p(T) \geq 1$ and $a(x) \in k[x]$, which completes the proof. □

In order to prove that $q_d \in k^*$ we need some preparations. By $\mathcal{T} \subseteq k[x,y]$ we denote the set of *terms* $x^i y^j$ with $i, j \geq 0$. On $\mathcal{T}$ we define the *lexicographical ordering* $>$ as follows

$$x^{i_1} y^{j_1} > x^{i_2} y^{j_2} \text{ if } j_1 > j_2 \text{ or, if } j_1 = j_2 \text{ if } i_1 > i_2$$

In other words, first look at the y-degree and in case of equality at the x-degree. This ordering is a total ordering. If $0 \neq f \in k[x,y]$ we can write f as a finite sum of the form $f = \sum_{t \in \mathcal{T}} c_t t$, with all $c_t \in k^*$. The greatest t appearing in f is called the *leading term of* f, denoted $lt(f)$. The corresponding coefficient c_t is called the *leading coefficient of* f, denoted $lc(f)$. The following easy result is crucial

Lemma 4. *Let $u, v \in k[x,y]$ with $lt(u) = x^{i_1} y^{j_1}$ and $lt(v) = x^{i_2} y^{j_2}$ be such that $i_1, j_1 \geq 1$, $i_2 \geq 0$ and $j_2 \geq 1$. Then*

$$lt(u_x v_y - u_y v_x) = x^{i_1 + i_1 - 1} y^{j_1 + j_2 - 1},\ \text{if } i_1 j_2 - i_2 j_1 \neq 0$$

Proof. The result follows easily from the fact that if $u = x^{i_1} y^{j_1}$ and $v = x^{i_2} y^{j_2}$ then $(u_x v_y - u_y v_x) = (i_1 j_2 - i_2 j_1) x^{i_1 + i_1 - 1} y^{j_1 + j_2 - 1}$. □

Lemma 5. $q_d \in k^*$.

Proof. i) Since $u_y \neq 0$ and $u = p(q)$ we get $q_y \neq 0$, so $d \geq 1$ and $N := deg_T\, p(T) \geq 1$. We must show that $s := deg_x\, q_d(x) = 0$. Therefore assume $s \geq 1$. We use the lexicographical order described above and compute the leading terms of the u_i, for all $1 \leq i \leq m+1$. First, from $u = p(q)$ it follows that $lt(u) = x^{sN} y^{dN}$. Then, by $(eq\, 1)$ we get $lt(u_2) = x^{sN-1} y^{dN+1}$.

First assume that $deg_{x_3}\, u_2 \geq 2$. It then follows from lemma 3 and (4.2) that $u_{2y} = Q(q)_y$ for some $Q(T) \in k[T]$ with $\rho := deg_T\, Q(T) \geq 1$. So $lt(u_{2y}) = x^{\rho s} y^{\rho d - 1}$. Consequently, $sN - 1 = \rho s$ and $dN + 1 = \rho d - 1$. Multiplying the first equation by d, the second by s and then subtracting these new equations we get $-dm - s = s$, a contradiction. So we may assume that $deg_{x_3}\, u_2 = 1$, i.e. $u_{2x_3} \in k^*$. So there exists $2 \leq m \leq n-1$, maximal such that $\lambda_2 := u_{2x_3} \in k^*, \cdots, \lambda_m := u_{m x_{m+1}} \in k^*$. Observe $m \leq r-1$. We claim that for all $2 \leq i \leq m+1$ we have

$$lt(u_i) = x^{(i-1)sN - (i-1)} y^{(i-1)dN + 1}$$

We use induction on i, the case $i = 2$ is already done. So assume the case is proved for $i < m + 1$. It follows from (4.5) that

$$u_x u_{iy} - u_y u_{ix} = \lambda_i u_{i+1y}. \tag{4.8}$$

It then follows from lemma 4 that the leading term of the left hand side is equal to $x^{isN-i}y^{idN}$. Then (4.8) gives that $lt(u_{i+1}) = x^{isN-i}y^{idN+1}$, which proves the claim.
ii) In particular we have $lt(u_{m+1}) = x^{msN-m}y^{mdN+1}$. On the other hand, by lemma 3, there exists $Q(T) \in k[T]$ such that $u_{m+1y} = Q(q)_y$. So if $deg_T Q(T) = \rho$, then we get $lt(u_{m+1y}) = x^{\rho r}y^{\rho d-1}$. Consequently $msN - m = \rho r$ and $mdN + 1 = \rho d - 1$. Multiplying the first equation by d, the second by s and then subtracting these new equations we get $-dm - s = s$, a contradiction. So $s = 0$, as desired. □

5. The main result

Now we will describe the main result of this paper. Recall that

$$H = (u(x, y), u_2(x, y, x_3), u_3(x, y, x_4), \ldots, u_{n-1}(x, y, x_n), u_n(x, y)). \tag{5.1}$$

By Corollary 1 and Corollary 2, in order to describe all H in (5.1) such that JH is nilpotent, we may assume that $u_{2x_3} \neq 0$ and $u_y \neq 0$. As seen before, it follows from $u_{2x_3} \neq 0$ that there exists $3 \leq r \leq n$ such that $u_{ix_{i+1}} \neq 0$ for all $2 \leq i \leq r-1$ and $u_{rx_{r+1}} = 0$. Let $d_i := deg_{x_{i+1}} u_i$, for all $2 \leq i \leq n-1$. So $d_i \geq 1$ if $2 \leq i \leq r-1$.

Definition 1. *$P(T) \in k[T]$ of degree $d \geq 1$ is called* nice *if the coefficient of T^{d-1} equals zero. The (leading) coefficient of T^d will be denoted by p_d.*

Theorem 1. *Let H be as in (5.1) with $u_{2x_3} \neq 0$, $u_y \neq 0$ and r as above. Then JH is nilpotent if and only if the following conditions hold*

(a)
$$u(x, y) = p(y + a(x)) \text{ and } u_2 = -a'(x)u + P_2(x_3 + \frac{1}{d_2 p_{d_2}} b_2(x)),$$
for some $p(T) \in k[T]$ with $deg_T p(T) \geq 1$, $a(x), b_2(x) \in k[x]$ and $P_2(T) \in k[T]$ nice of degree d_2. If $d_2 \geq 2$, then $a''(x) = 0$.

(b) *If $3 \leq i \leq r-1$ and $u_{i-1} = \sum_{j=1}^{l} c_{i-1,j}(x)u^j + P_{i-1}(x_i + \frac{1}{d_{i-1}p_{d_{i-1}}} b_{i-1}(x))$, with $c_{i-1,j}(x), b_{i-1}(x) \in k[x]$ and $P_{i-1}(T)$ nice of degree d_{i-1}, then*
$$u_i = -\frac{1}{d_{i-1}p_{d_{i-1}}} \big[\sum_{j=1}^{l} \frac{1}{j+1} c'_{i-1,j}(x)u^{j+1} + b'_{i-1}(x)u\big] + P_i(x_{i+1} + \frac{1}{d_i p_{d_i}} b_i(x))$$
for some $b_i(x) \in k[x]$ and $P_i(T) \in k[T]$, nice of degree d_i. If $d_{i-1} \geq 2$, then $c'_{i-1,j}(x) = 0$ for all j.

(c) *If $u_{r-1} = \sum_{j=1}^{l} c_{r-1,j}(x)u^j + P_{r-1}(x_r + \frac{1}{d_{r-1}p_{d_{r-1}}} b_{r-1}(x))$, with $c_{r-1,j}(x), b_{r-1}(x) \in k[x]$ and $P_{r-1}(T)$ nice of degree d_{r-1}, then*
$$u_r(x, y) = -\frac{1}{d_{r-1}p_{d_{r-1}}} \big[\sum_{j=1}^{l} \frac{1}{j+1} c'_{r-1,j}(x)u^{j+1} + b'_{r-1}(x)u\big] + b_r,$$

with $c'_{r-1,j} \in k$ *for all* $j \geq 1$ *and* $b_r \in k$, $b'_{r-1} \in k$. *If* $d_{r-1} \geq 2$, *then* $c'_{r-1,j} = 0$ *for all* j.

(d) *No extra conditions on* u_i *if* $i > r$.

To prove this theorem we need some preliminaries:

Theorem 2. *Let* $v = \sum_{i=1}^{l} c_i(x)u^i + P(s + \frac{1}{dp_d}b(x))$, *with* P *nice of degree* $d \geq 1$ *and* $b(x) \in k[x]$. *Let* v *and* w *satisfy*

$$D_0(v) = -v_s w_y \tag{5.2}$$

$$D_0(w) = -w_t g_y \tag{5.3}$$

for some $w \in k[x, y, t]$ *with* $e := deg_t\, w \geq 0$ *and* $g \in k[x, y, r]$.

i) *If* $e = 0$, *then*

$$w = -\frac{1}{dp_d}\big(\sum_{i=1}^{l} \frac{1}{i+1} c'_i(x)u^{i+1} + b'(x)u\big) + c(x)$$

with $b'(x), c(x) \in k$ *and* $c'_i(x) \in k$ *for all* i.

ii) *If* $e \geq 1$ *there exist* $c(x) \in k[x]$ *and* $Q(T) \in k[T]$, *nice of degree* e, *with leading coefficient* q_e *such that*

$$w = -\frac{1}{dp_d}\big(\sum_{i=1}^{l} \frac{1}{i+1} c'_i(x)u^{i+1} + b'(x)u\big) + Q(t + \frac{1}{eq_e}c(x))$$

iii) *Furthermore, if* $d \geq 2$, *then* $c'_i = 0$ *for all* i.

Proof. Write $v = v_0(x, y) + \sum_{i=1}^{d} v_i(x)s^i$ and $w = w_0(x, y) + W$, where $W = 0$ if $e = 0$ and $W = \sum_{i=1}^{e} w_i(x, y)t^i$, if $e \geq 1$. Then $v_d = p_d \in k^*$, $v_y = v_{0y}$, $w_y = w_{0y}$ (by (5.2)) and $w_e \in k^*$ (by (5.3)), if $e \geq 1$.

First assume $d \geq 2$. Then $w_y = -\frac{1}{dv_d}v'_{d-1}(x)u_y$ (by lemma 3). So $w_0 = -\frac{1}{dv_d}v'_{d-1}(x)u + c(x)$ for some $c(x) \in k[x]$. Put $b(x) = v_{d-1}(x)$. So, if $e = 0$, then $w = -\frac{1}{dp_d}b'(x)u + c(x)$ and if $e = 1$ then $w = -\frac{1}{dp_d}b'(x)u + c(x) + q_1 t$, where $q_1 := w_1$. Substituting these formulas in (5.2) we get $u_y \sum_{i=1}^{l} c'_i(x)u^i = 0$, which implies that all $c'_i = 0$, since u contains y. If $e = 0$, then $w_t = 0$, so 5.3 implies that $b'(x), c(x) \in k$. This proves the case $d \geq 2$, $e \leq 1$.

Now let $e \geq 2$. Then by lemma 3, applied to (5.3), we get $g_y = -\frac{1}{ew_e}w'_{e-1}(x)u_y$. Substituting this formula into (5.3) we get

$$u_x(-\frac{1}{dv_d}v'_{d-1}(x)u_y) - u_y(w_{0x} + \partial_x(W)) = -\frac{1}{ew_e}w'_{e-1}(x)u_y\partial_t(W)$$

Also, using the formula for w_0 obtained above, we have

$$w_{0x} = -\frac{1}{dv_d}v'_{d-1}(x)u_x + -\frac{1}{dv_d}v''_{d-1}(x)u + c'(x)$$

So, combining the last two formulas, we get

$$-u_y[-\frac{1}{dv_d}v''_{d-1}(x)u + c'(x) + \partial_x(W)] = u_y[-\frac{1}{ew_e}w'_{e-1}(x)\partial_t(W)]$$

Hence

$$\frac{1}{dv_d}v''_{d-1}(x)u - c'(x) = (\partial_x - \frac{1}{ew_e}w'_{e-1}(x)\partial_t)W \in k[x,t]$$

Since $u_y \neq 0$ we get $v''_{d-1}(x) = 0$. So $(\partial_x - \frac{1}{ew_e}w'_{e-1}(x)\partial_t)(c(x) + W) = 0$, whence $W = -c(x) + Q(t + \frac{1}{ew_e}w_{e-1}(x))$, for some $Q(T) \in k[T]$. Since $w = w_0 + W$ and $w_0 = -\frac{1}{dv_d}v'_{d-1}(x)u + c(x)$ we get the desired formula for w, using that $v_{d-1} = b(x)$ and $v_d = p_d$ and observing that $Q(T)$ is nice of degree e. The statement in iii) follows again from (5.2), using that $w_y = -\frac{1}{dv_d}v'_{d-1}(x)u_y$.

Now, assume $d = 1$. So $v = \sum_{i=1}^{l} c_i(x)u^i + p_1 s + b(x)$. Using (5.2) we get

$$-u_y(\sum_{i=1}^{l} c'_i(x)u^i + b'(x)) = p_1 w_y = p_1 w_{0y}$$

So

$$w_0 = -\frac{1}{p_1}\big(\sum_{i=1}^{l}\frac{1}{i+1}c'_i(x)u^{i+1} + b'(x)u\big) + c(x), \tag{5.4}$$

for some $c(x) \in k[x]$. So, if $e = 0$, 5.3 implies again that $b'(x), c(x) \in k$ and all $c'_i(x) \in k$. So this case is done. Also the case $e = 1$ is done, using that $w = w_0 + q_1 t$. So assume that $e \geq 2$. Then, as observed above $g_y = -\frac{1}{ew_e}w'_{e-1}(x)u_y$. By (5.3) and (5.4) we get

$$(-\frac{1}{p_1})\big[\sum_{i=1}^{l} c''_i(x)u^{i+1} + b''(x)u\big] + c'(x) = -(\partial_x - \frac{1}{ew_e}w'_{e-1}(x)\partial_t)(W) \in k[x,t]$$

Since u contains y we get that $b''(x) = 0$ and all $c''_i(x) = 0$. So

$$(\partial_x - \frac{1}{ew_e}w'_{e-1}(x)\partial_t)(W + c(x)) = 0$$

Hence $W = -c(x) + Q(t + \frac{1}{ew_e}w_{e-1}(x))$, for some $Q(T) \in k[T]$, which is nice of degree e. Then the formula for w follows from $w = w_0 + W$ and (5.4).

□

Now we prove the main result of this paper

Proof of Theorem 1: As seen above the proof of Corollary 2, the nilpotency of JH is equivalent to the following equations

$$u_x + u_{2y} = 0 \qquad (1),$$

$$u_x u_{2y} - u_y u_{2x} = u_{2x_3} u_{3y} \qquad (2),$$

$$u_x u_{3y} - u_y u_{3x} = u_{3x_4} u_{4y} \qquad (3),$$

$$\vdots$$

$$u_x u_{r-1y} - u_y u_{r-1x} = u_{r-1x_r} u_{ry}, \quad \text{(r-1)},$$

$$u_x u_{ry} - u_y u_{rx} = 0 \qquad \text{(r)}.$$

First assume that JH is nilpotent. So to prove the theorem we need to solve the r equations above. Let $2 \le j \le r-1$ and write

$$u_j = u_{j,0}(x,y) + \sum_{i=1}^{d_j} u_{j,i}(x,y)x_{j+1}^i$$

As $u_{j_{x_{j+1}}} \neq 0$, we obtain $d_j \geq 1$ and if $i \geq 1$ it follows from from (j) and $u_y \neq 0$ that $u_{j,i} = u_{j,i}(x)$. So $u_{j_y} = u_{j,0_y}$. Moreover we obtain from equation (j) that $u_{j,d_j} \in k^*$.

(a) By Proposition 2 we have that $u = p(y + a(x))$ for some $p(T) \in k[T]$ with $deg_T\, p(T) \geq 1$ and $a(x) \in k[x]$. From (1) we get $u_{2,0} = -a'(x)u + c(x)$, with $c(x) \in k[x]$. So if $d_2 = 1$, then u_2 has the desired form. If $d_2 \geq 2$, then $u_2 = -a'(x) + c(x) + U_2$, where $U_2 = \sum_{i=1}^{d_2} u_{2,i}(x)x_3^i$. It follows from (2) and lemma 3 that $u_{3_y} = -\frac{1}{d_2 p_{d_2}} b_2'(x)u_{2_y}$, for some $b_2(x) \in k[x]$. Substituting these formulas in (2), an easy calculation gives

$$a''(x)u - c'(x) = (\partial_x - \frac{1}{d_2 p_{d_2}} b_2'(x)\partial_{x_3})U_2 \in k[x, x_3]$$

Since u contains y we get $a''(x) = 0$ and hence

$$(\partial_x - \frac{1}{d_2 p_{d_2}} b_2'(x)\partial_{x_3})(U_2 + c(x)) = 0$$

So $U_2 = -c(x) + P_2(x_3 + \frac{1}{d_2 p_{d_2}} b_2(x))$, for some $P_2(T) \in k[T]$, nice of degree d_2. Since $u_2 = -a'(x) + c(x) + U_2$ it follows that u_2 has the desired form.

(b) This case follows directly from Theorem 2 ii) and iii)

(c) u_r is obtained by using Theorem 2 i).

(d) This follows immediately from the equations $(1), \cdots, (r)$, which do not contain u_i with $i > r$.

Conversely, it is left the reader to verify that the formulas obtained in $(a) \cdots (d)$ indeed satisfy the equations $(1) \cdots (r)$, which shows that the corresponding H has a nilpotent Jacobian matrix.

6. Invertibility

Throughout this section

$$H = (u(x,y), u_2(x,y,x_3), u_3(x,y,x_4), \cdots, u_{n-1}(x,y,x_n), u_n(x,y))$$

In the previous sections we completely described all such maps H with the property that JH is nilpotent. For the the corresponding maps $F = X + H$ we have that $det\, JF = 1$. So if the Jacobian Conjecture is true, F must be invertible. The main result of this section (Theorem 3 below) confirms this. More precisely we show that F is a product of elementary maps (see definition below), i.e.

Theorem 3. *If H is as above and JH is nilpotent, then $F \in E(k,n)$.*

Before we prove this result we make some preliminary remarks. Recall that a polynomial map is called *elementary* if it is of the form $(x_1, \cdots, x_{i-1}, x_i + a, x_{i+1}, \cdots, x_n)$ for some $a \in k[x]$ not containing x_i. We denote such a map for short as $(x_i + a)$. The subgroup of $Aut_k k[x_1, \cdots, x_n]$ generated by these elementary maps is denoted by $E(k,n)$. Two polynomial maps F and G are called *elementary equivalent* if there exist $E_1, E_2 \in E(k,n)$ such that $G = E_1 \circ F \circ E_2$. Since the E_i are invertible we have that F is invertible if and only if G is invertible. So to prove Theorem 3 it suffices to show that F is elementary equivalent to the identity map.

First we consider the case $u_{2x_3} = 0$, described in Corollary 1.

Proposition 3. *Notations as in Corollary 1. Then* $F \in E(k,n)$.

Proof. First let $n > 3$. By the description given in Corollary 1 we get

$$(F_1, F_2) = (x + \lambda_2 f(\lambda_1 x + \lambda_2 y) + c_1, y - \lambda_1 f(\lambda_1 x + \lambda_2 y) + c_2$$

$$F_i = x_i + u_i(x, y, x_{i+1}) \text{ for all } 3 \leq i \leq n-1 \text{ and } F_n = x_n + u_n(x,y)$$

Let T be the translation $(x - c_1, x - c_2, x_3, \cdots, x_n)$. Replacing F by $T \circ F$ we may assume that $c_1 = c_2 = 0$. Furthermore we may assume that $\lambda_1 = \lambda_2 = 0$: if for example $\lambda_1 \neq 0$ let S be the invertible linear map

$$(\lambda_1 x + \lambda_2 y, y, x_3, \cdots, x_n)$$

Then $S \circ F \circ S^{-1} = (x, y, F_2', \cdots, F_n')$, with $F_i' = x_i + \tilde{u}_i(x, y, x_{i+1})$ for all $3 \leq i < n$ and $F_n' = x_n + \tilde{u}_n(x,y)$. So we may assume

$$F = (x, y, x_3 + u_3(x, y, x_4), \cdots, x_{n-1} + u_{n-1}(x, y, x_n), x_n + u_n(x, y)$$

Finally, let $E_n = (x, y, \cdots, x_{n-1}, x_n - u_n(x,y))$. Then

$$E_n \circ F = (x, y, x_3 + u_3(x, y, x_4), \cdots, x_{n-1} + u_{n-1}(x, y, x_n), x_n)$$

Now one readily verifies that this map belongs to $E(k,n)$, which implies the proposition in case $n > 3$. The case $n = 3$ is left to the reader. □

Next we consider the case $u_{2x_3} \neq 0$ and $u_y = 0$, described in Corollary 2.

Proposition 4. *Notations as in Corollary 2. Then* $F \in E(k,n)$.

Proof. By the description of Corollary 2 we get

$$(F_1, \cdots, F_r) = (x + u, y + u_2(x, x_3), \cdots, x_{r-1} + u_{r-1}(x, x_r), x_r + u_r(x))$$

$F_n = x_n + u_n(x,y)$ and if there exists $r < i < n$, then $F_i = x_i + u_i(x, y, x_{i+1})$. Replacing F by $(x - u) \circ F$ we may assume that $F_1 = x$. Then, replacing F by $(x_r - u_r(x)) \circ F$, we may assume that $u_r = 0$. Next, replacing F by $(x_{r-1} - u_{r-1}(x, x_r)) \circ F$, we may assume that $u_{r-1} = 0$. Continuing in this way we arrive at $(F_1, \cdots, F_r) = (x, y, x_3, \cdots, x_r)$. So if $r = n$ we are done. Now let $r < n$. Then consider $(x_n - u_n(x,y)) \circ F$. So we may assume that $u_n = 0$. Next consider $(x_{n-1} - u_{n-1}(x, y, x_n)) \circ F$ etcetera. Finally we arrive at the identity map, which proves the proposition. □

So from Proposition 3 and Proposition 4 it follows, that in order to prove Theorem 3, we may assume from now on that $u_y \neq 0$ and $u_{2x_3} \neq 0$ and that we have an r as above. First we claim F is invertible if and only if $(F_1, \cdots, F_r)$ is invertible: if $r = n$ there is nothing to prove, so assume $r < n$. Using that $F_1, \cdots, F_r \in k[x_1, \cdots, x_r]$, $F_n = x_n + u_n(x,y)$ and $F_i = x_i + u_i(x, y, x_{i+1})$ for all $i > r$, it is an easy exercise to show that F is elementary equivalent to the map

$$(F_1, \cdots, F_r, x_{r+1}, \cdots, x_n)$$

Furthermore, since the polynomials $F_1, \cdots, F_r \in k[x_1, \cdots, x_r]$ it is well-known that $(F_1, \cdots, F_r, x_{r+1}, \cdots, x_n)$ is invertible if and only if $(F_1, \cdots, F_r)$ is. This implies our claim. So it suffices to show that $(F_1, \cdots, F_r) \in E(k, r)$.

Using the notations of Theorem 1 we introduce some new notations. First, if $2 \le i < r$ let l_i denote the coefficient of T^{d_i} in $P_i(T)$ and $L_i := d_i l_i$. Furthermore, put $d_1 := 2$, $L_1 := 1$, $l_r = 0$ and $b_1(x) := a(x)$. Let $s \ge 2$ be maximal such that $d_{s-1} \ge 2$. So $2 \le s \le r$ and $d_i = 1$ if $s \le i < r$. Hence $L_i = l_i$ if $s \le i < r$. Finally define

$$\gamma_{k,t} := L^{-1}_{s-1+(t-1)} \cdots L^{-1}_{s-1+(t-k)}, \text{ for all } 1 \le k \le t \le r-s+1$$

One readily verifies that

$$\gamma_{1,t} = L^{-1}_{s-1+t-1} \text{ and } \gamma_{k,t-1} = L_{s-1+t-1}\gamma_{k+1,t}, \text{ if } 1 \le k \le t-1 \quad (*)$$

Then the next result follows by induction on t, using Theorem 1 and (*).

Proposition 5. *If $1 \le t \le r-s+1$, then*

$$u_{s-1+t} = \sum_{k=1}^{t} (-1)^k \frac{1}{k!} \gamma_{k,t} b^{(k)}_{s-1+t-k}(x) u^k + l_{s-1+t} x_{s+t} + b_{s-1+t}(x)$$

with $b^{(r-s+2)}_{s-1} = \cdots = b^{(2)}_{r-1} = b^{(1)}_r = 0$.

Corollary 3. *Let $F = (x+u, x_2+u_2, \cdots, x_r+u_r)$. Then for every $1 \le t \le r-s$ there exists $E_t \in E(k,r)$ such that $F \circ E_t = (F_1, \cdots, F_{r-t-1}, \widetilde{F}_{r-t}, \widetilde{F}_{r-t+1}, \cdots, \widetilde{F}_r)$ where $\widetilde{F}_{r-i} = x_{r-i} + b_{r-i}(F_1) + l_{r-i}x_{r-i+1}$, for all $0 \le i < t$ and*

$$\widetilde{F}_{r-t} = \sum_{k=1}^{r-s-t+1} (-1)^k \gamma_{k,r-s-t+1} \Big[\frac{1}{k!} b^{(k)}_{r-t-k} u^k + \frac{1}{(k+1)!} b^{(k+1)}_{r-t-k} u^{(k+1)} + \cdots + \frac{1}{(k+t)!} b^{(k+t)}_{r-t-k} u^{k+t}\Big]$$

$$+ b_{r-t}(F_1) + l_{r-t}x_{r-t+1} + x_{r-t}$$

Proof. By induction on t. First the case $t = 1$. From Proposition 5 (with $t = r-s+1$) and $l_r = 0$ we get $F_r = x_r + [u_r] + b_r$, where

$$[u_r] := \sum_{k=1}^{r-s+1} (-1)^k \frac{1}{k!} \gamma_{k,r-s+1} b^{(k)}_{r-k}(x) u^k$$

with $b_r \in k$ and $b^{(k+1)}_{r-k}(x) = 0$ for all $1 \le k \le r-s+1$. From Proposition 5 (with $t = r-s$) we get

$$F_{r-1} = x_{r-1} + \sum_{k=1}^{r-s} (-1)^k \frac{1}{k!} \gamma_{k,r-s} b^{(k)}_{r-1-k}(x) u^k + l_{r-1}x_r + b_{r-1}(x)$$

Define $E_1 = (x_1, \cdots, x_{r-1}, x_r - [u_r])$. Observe that $[u_r] \in k[x, x_2]$ and $r > 2$. So $E_1 \in E(k,r)$. Furthermore $F \circ E_1 = (F_1, \cdots, F_{r-2}, \widetilde{F}_{r-1}, x_r + b_r)$, where

$$\widetilde{F}_{r-1} = x_{r-1} + \sum_{k=1}^{r-s} (-1)^k \frac{1}{k!} \gamma_{k,r-s} b^{(k)}_{r-1-k}(x) u^k + l_{r-1}x_r$$

$$+ \sum_{k=1}^{r-s+1} (-1)^{k+1} \frac{1}{k!} l_{r-1} \gamma_{k,r-s+1} b^{(k)}_{r-k}(x) u^k + b_{r-1}(x)$$

Now write

$$\sum_{k=1}^{r-s+1}(-1)^{k+1}\frac{1}{k!}l_{r-1}\gamma_{k,r-s+1}b_{r-k}^{(k)}(x)u^k = l_{r-1}\gamma_{1,r-s+1}b_{r-1}^{(1)}u$$

$$+\sum_{k=1}^{r-s}(-1)^k\frac{1}{(k+1)!}l_{r-1}\gamma_{k+1,r-s+1}b_{r-1-k}^{(k+1)}(x)u^{k+1}$$

and use that $l_{r-1}\gamma_{k+1,r-s+1} = \gamma_{k,r-s}$ and $l_{r-1}\gamma_{1,r-s+1} = 1$. Then we get

$$\widetilde{F}_{r-1} = x_{r-1} + \sum_{k=1}^{r-s}(-1)^k\gamma_{k,r-s}\big[\frac{1}{k!}b_{r-1-k}^{(k)}(x)u^k + \frac{1}{(k+1)!}b_{r-1-k}^{(k+1)}(x)u^{k+1}\big]$$

$$+l_{r-1}x_r + b_{r-1}^{(1)}(x)u + b_{r-1}(x)$$

Since by Proposition 5 $b_{r-1}^{(2)}(x) = 0$, it follows from Taylor's theorem that $b_{r-1}(F_1) = b_{r-1}(x+u) = b_{r-1}(x) + b_{r-1}^{(1)}(x)u$. This finishes the proof of the case $t = 1$

Now assume $t \geq 1$ and that we already know the existence of a map E_t, having the properties as described in the statement of this corollary. In particular we have $\widetilde{F}_{r-t} = x_{r-t} + [u_{r-t}] + b_{r-t}(F_1) + l_{r-t}x_{r-t+1}$. Observe that $[u_{r-t}] \in k[x, x_2]$ and define

$$E' := (x_1, \cdots, x_{r-t-1}, x_{r-t} - [u_{r-t}], x_{r-t+1}, \cdots, x_r)$$

Then a similar argument as given for the case $t = 1$ above, shows that $(F \circ E_t) \circ E'$ has the desired form.

□

Corollary 4. *Let* $F = (x+u, x_2+u_2, \cdots, x_r+u_r)$. *Then* F *is elementary equivalent to* $(F_1, \cdots, F_{s-1}, \widetilde{F}_s, x_{s+1}, \cdots, x_r)$, *where* $\widetilde{F}_s = x_s + L_{s-1}^{-1}b_{s-1}(x)$.

Proof. By Corollary 3, with $t = r - s$, there exists $E \in E(k, r)$ such that

$$F\circ E = (F_1, \cdots, F_{s-1}, \widetilde{F}_s, x_{s+1}+b_{s+1}(F_1)+l_{s+1}x_{s+2}, \cdots, x_{r-1}+b_{r-1}(F_1)+l_{r-1}x_r, x_r)$$

where

$$\widetilde{F}_s = x_s - L_{s-1}^{-1}\big[b_{s-1}^{(1)}(x)u + \frac{1}{2!}b_{s-1}^{(2)}(x)u^2 + \cdots + \frac{1}{(r-s+1)!}b_{s-1}^{(r-s+1)}(x)u^{r-s+1}\big]$$

$$+b_s(F_1) + l_sx_{s+1}$$

Since $b_{s-1}^{(r-s+2)}(x) = 0$, by Proposition 5, it follows from Taylor's theorem, using $F_1 = x + u$, that

$$b_{s-1}(F_1) = b_{s-1}(x) + b^{(1)}(x)u + \frac{1}{2!}b_{s-1}^{(2)}(x)u^2 + \cdots \frac{1}{(r-s+1)!}b_{s-1}^{(r-s+1)}(x)u^{r-s+1}$$

So

$$\widetilde{F}_s = x_s - L_{s-1}^{-1}\big[b_{s-1}(F_1) - b_{s-1}(x)\big] + b_s(F_1) + l_sx_{s+1}$$

So if we define

$$E' := (x_1, \cdots, x_{s-1}, x_s+L_{s-1}^{-1}b_{s-1}(x_1)-b_s(x_1), x_{s+1}-b_{s+1}(x_1), \cdots, x_{r-1}-b_{r-1}(x_1), x_r)$$

Then $E' \in E(k, r)$ and

$$E'\circ F\circ E = (F_1, \cdots, F_{s-1}, x_s+L_{s-1}^{-1}b_{s-1}(x)+l_sx_{s+1}, x_{s+1}+l_{s+1}x_{s+2}, \cdots, x_{r-1}+l_{r-1}x_r, x_r)$$

One readily verfies that $E' \circ F \circ E$ is elementary equivalent to

$$F' := (F_1, \cdots, F_{s-1}, x_s + L_{s-1}^{-1}b_{s-1}(x), x_{s+1}, \cdots, x_r)$$

which completes the proof. □

Now we are ready to prove

Proposition 6. *Let $F = (x + u, x_2 + u_2, \cdots, x_r + u_r)$. Then $F \in E(k, r)$.*

Proof. We use induction on $n(H)$:= the number of $d_i \geq 2$. Since $d_1 = 2$ we have $n(H) \geq 1$. First the case $n(H) = 1$. So $s = 2$. It follows from Corollary 4 that F is elementary equivalent to $(F_1, \widetilde{F}_2, x_3, \cdots, x_r)$, where $\widetilde{F}_2 = x_2 + a(x)$ (L_1=1 and $b_1(x) = a(x)$). Since $F_1 = x + p(x_2 + a(x))$, the case $n(H) = 1$ follows.

So let $n(H) > 1$. Then $s \geq 3$. Since $d_{s-1} \geq 2$ it follows from Theorem 1 that $u_{s-1} = [u_{s-1}] + P_{s-1}(x_s + L_{s-1}^{-1} b_{s-1}(x))$, where $[u_{s-1}] = \sum c_{s-1,j} u^j$, with $c_{s-1,j} \in k$ for all j. So by Corollary 4 F is elementary equivalent to

$$F' := (F_1, \cdots, F_{s-2}, x_{s-1} + [u_{s-1}] + P_{s-1}(x_s + L_{s-1}^{-1} b_{s-1}(x)), x_s + L_{s-1}^{-1} b_{s-1}(x), x_{s+1}, \cdots, x_r)$$

Now define the elementary map

$$E'' := (x_1, \cdots, x_{s-1}, x_s - L_{s-1}^{-1} b_{s-1}(x), x_{s+1}, \cdots, x_r)$$

Then

$$F' \circ E'' = (F_1, \cdots, F_{s-2}, x_{s-1} + [u_{s-1}] + P_{s-1}(x_s), x_s, \cdots, x_r)$$

Consequently, $F' \circ E''$ is elementary equivalent to $(F_1, \cdots, F_{s-2}, x_{s-1} + [u_{s-1}], x_s, \cdots, x_r)$. Finally put $\widetilde{H} := (u_1, \cdots, u_{s-2}, [u_{s-1}], 0, \cdots, 0)$. Then obviously $\widetilde{H}$ is special and $n(\widetilde{H}) = n(H) - 1$. It follows from Proposition 1 that $J(\widetilde{H})$ is nilpotent. So by the induction hypothesis we get that $F' \circ E'' \in E(k, r)$, which implies that $F \in E(k, r)$, as desired. □

References

[1] H. Bass, E. Connell, D. Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. **2** (1982) 287–330.

[2] M. de Bondt, A. van den Essen, *The Jacobian Conjecture: Linear triangularization for homogeneous polynomial maps in dimension three*, J. Algebra, **294** (2005) 294–306.

[3] M. de Bondt, *Quasi-translations and counterexamples to the homogeneous dependence problem*, Proc. Amer. Math. Soc. **134** (2006), 28492856.

[4] Á. Castañeda, V. Guíñez, *Some results about global asymptotic stability*, Qual. Theory Dyn. Syst., **12** (2013), 427–441.

[5] M. Chamberland, A. van den Essen, *Nilpotent Jacobians in dimension three*, J. Pure Appl. Algebra **205** (2006) 146–155.

[6] A. Cima, A. van den Essen, A. Gasull, E. Hubbers, F. Mañosas, *A polynomial counterexample to the Markus-Yamabe Conjecture*, Advances in Mathematics, **131** (1997), 453-457.

[7] H. Derksen, A. van den Essen and W. Zhao, *The Gaussian Moments Conjecture and the Jacobian Conjecture*, Israel J. of Math. **219** (2017) 917-928, arxiv.org/pdf/1506.05192.pdf.

[8] A. van den Essen, *Nilpotent jacobian matrices with independent rows*, Report 9603, University of Nijmegen, The Netherlands, 1996.

[9] —, *A counterexample to Meisters' Cubic-Linearization Conjecture*, Indagationes Mathematicae, New Series **9**, No. 3 (1998) 333-339.

[10] —, *Polynomial Automorphisms and the Jacobian Conjecture*, Progress in Mathematics, Vol. **190**, (2000), Birkhäuser, Basel.

[11] A. van den Essen, E. Hubbers, *A new lass of invertible polynomial maps*, J. of Algebra bf 187 (1997) 214-226.

[12] —, *Chaotic Polynomial Automorphisms: counterexamples to several conjecture*, Advances in Applied Mathematics, **18** (1997) 382-388.

[13] A. van den Essen, D. Wright, W. Zhao, *On the Image Conjecture*, J. Alg. **340** (2011), no.1, 211-224. See also arXiv:1008.3962 [math.RA].

[14] E. Hubbers, *The Jacobian conjecture: cubic homogeneous maps in dimension four*, Master's Thesis, University of Nijmegen, The Netherlands, 1994.
[15] D. Wright, *The Jacobian conjecture: linear triangularization for cubics in dimension three*, Linear Multilinear Algebra **34** (1993) 85–97.
[16] A.V. Yagzhev, *On Keller's problem*, Sib. Math. J **21** (1980) 747–754.
[17] D. Yan, *The classification of some polynomial maps with nilpotent Jacobians*, arXiv:1710.04210.
[18] W. Zhao, *Images of Commuting Differential Operators of Order One with Constant Leading Coefficients*, J. Algebra **324** (2010), no.2, 231-247. See also arXiv:0902.0210 [math.CV].
[19] —, *Generalizations of the Image Conjecture and the Mathieu Conjecture*, J. Pure Appl. Algebra **214** (2010), 1200-1216. See also arXiv:0902.0212 [math.CV].
[20] —, *Mathieu Subspaces of Associative Algebras*, J. Algebra **350** (2012), no.1, 245-272. See also arXiv: 1005.4260.

(Á. Castañeda) Departamento de Matemáticas, Facultad de Ciencias, Universidad de Chile, Casilla 653, Santiago, Chile

(A. van den Essen) Department of Mathematics, Radboud University Nijmegen, Postbus 9010, 6500 GL Nijmegen, The Netherlands

E-mail address: castaneda@uchile.cl, essen@math.ru.nl