Two Definite Integrals That Are Definitely (and Surprisingly!) Equal

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This column is a place for those bits of contagious mathematics that travel from person to person in the community, because they are so elegant, surprising, or appealing that one has an urge to pass them on.

Contributions are most welcome.

➢ Submissions should be uploaded to http://tmin.edmgr.com or sent directly to **Sergei Tabachnikov**, e-mail: tabachni@math.psu.edu

To our good friend Gert Almkvist (1934–2018)
In Memoriam.

PROPOSITION 1. *For real $a > b > 0$ and nonnegative integer n, the following beautiful and surprising identity holds:*

$$\int_0^1 \frac{x^n(1-x)^n}{((x+a)(x+b))^{n+1}}\,dx = \int_0^1 \frac{x^n(1-x)^n}{((a-b)x+(a+1)b)^{n+1}}\,dx.$$

PROOF. Fix a and b, let $L(n)$ and $R(n)$ be the integrals on the left and right sides respectively, and let $F_1(n,x)$ and $F_2(n,x)$ be the corresponding integrands, so that $L(n) = \int_0^1 F_1(n,x)\,dx$ and $R(n) = \int_0^1 F_2(n,x)\,dx$. We cleverly construct the rational functions

$$R_1(x) = \frac{x(x-1)((a+b+1)x^2+2abx-ab)}{(x+b)(x+a)}$$

and

$$R_2(x) = \frac{x(x-1)\big((a-b)x^2+2b(a+1)x-(a+1)b\big)}{(a-b)x-(a+1)b}$$

motivated by the fact that (check!)

$$\begin{aligned}&(n+1)F_1(n,x)-(2n+3)(2ba+a+b)F_1(n+1,x)\\&\quad+(a-b)^2(n+2)F_1(n+2,x)\\&\quad=\frac{d}{dx}(R_1(x)F_1(n,x))\end{aligned}$$

and

$$\begin{aligned}&(n+1)F_2(n,x)-(2n+3)(2ba+a+b)F_2(n+1,x)\\&\quad+(a-b)^2(n+2)F_2(n+2,x)\\&\quad=\frac{d}{dx}(R_2(x)F_2(n,x)).\end{aligned}$$

Integrating both identities from $x=0$ to $x=1$ and noting that the right-hand sides vanish, we have

$$\begin{aligned}&(n+1)L(n)-(2n+3)(2ba+a+b)L(n+1)\\&\quad+(a-b)^2(n+2)L(n+2)=0\end{aligned}$$

and

$$\begin{aligned}&(n+1)R(n)-(2n+3)(2ba+a+b)R(n+1)\\&\quad+(a-b)^2(n+2)R(n+2)=0.\end{aligned}$$

Since $L(0)=R(0)$ and $L(1)=R(1)$ (check!), the proposition follows by mathematical induction. □

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REMARK 1. This beautiful identity is equivalent to an identity buried in Bailey's classic book [3, Section 9.5, formula (2)], but you need an expert (like the third-named author) to realize that!

REMARK 2. Our proof was obtained by the first-named author by running a Maple program[1] written by the second-named author that implements the Almkvist–Zeilberger algorithm [2] designed by Zeilberger and our good mutual friend Gert Almkvist, to whose memory this note is dedicated.

REMARK 3. Our integrals are not taken from a pool of no-one-cares-about-them analytic creatures: the right-hand side covers a famous sequence of rational approximations to $\log(1+(a-b)/((a+1)b))$ [1], and hence the left-hand side does, too.

REMARK 4. We thank Greg Egan for spotting a sign error in an earlier version.

REMARK 5. Watch Greg Egan's beautiful animation.[2]

REMARK 6. To our surprise, the identity turned out to be not as surprising as we had believed. Mikael Sundquist noticed that the change of variable $x = b(1-u)/(b+u)$ gives a "Calculus 1 proof." Indeed,

$$dx = -\frac{b(1+b)}{(b+u)^2}\,du,$$

and we have

$$\begin{aligned}
&\int_0^1 \frac{x^n(1-x)^n}{((x+a)(x+b))^{n+1}}\,dx \\
&= \int_0^1 \frac{(b(1-u)/(b+u))^n(1-(b(1-u)/(b+u)))^n}{(b(1-u)/(b+u)+a)(b(1-u)/(b+u)+b))^{n+1}} \\
&\quad\times \frac{b(1+b)}{(b+u)^2}\,du \\
&= \int_0^1 \frac{(1-u)^n u^n b^{n+1}(1+b)^{n+1}}{(b(1-u)+a(b+u))(b(1-u)+b(b+u)))^{n+1}}\,du \\
&= \int_0^1 \frac{u^n(1-u)^n}{((a-b)u+(a+1)b)^{n+1}}\,du.
\end{aligned}$$

Note that n does not have to be an integer.

REMARK 7. Alin Bostan has two further insightful proofs.[3]

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[1] K. Alladi and M. L. Robinson. Legendre polynomials and irrationality. *J. Reine Angew. Math.* 318 (1980), 137–155.

[2] Gert Almkvist and Doron Zeilberger. The method of differentiating under the integral sign. *J. Symbolic Computation* 10 (1990), 571–591.

[3] W. N. Bailey. *Generalized Hypergeometric Series*. Cambridge University Press, 1935.

[1]Availble online at http://sites.math.rutgers.edu/~zeilberg/tokhniot/EKHAD.txt.
[2]Available online at https://twitter.com/gregeganSF/status/1192309179119104000.
[3]See https://specfun.inria.fr/bostan/publications/EZZ.pdf and https://specfun.inria.fr/bostan/publications/EZZ2.pdf.