Prosperity properties of TU-games

J.R.G. van Gellekom, J.A.M. Potters and J.H. Reijnierse

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Abstract

An important open problem in the theory of TU-games is to determine whether a game has a stable core (Von Neumann-Morgenstern solution). This seems to be a rather difficult combinatorial problem. There are many sufficient conditions for core-stability. Convexity is probably the best known of these properties. Other properties implying stability of the core are subconvexity and largeness of the core (two properties introduced by Sharkey (1982)) and a property that we have baptized extendability and is introduced by Kikuta and Shapley (1986). These last three properties have a feature in common: if we start with an arbitrary TU-game and increase only the value of the grand coalition, these properties arise at some moment and are kept if we go on with increasing the value of the grand coalition. We call such properties *prosperity properties*. In this paper we investigate the relations between several prosperity properties and their relation with core-stability. By counter examples we show that all the prosperity properties we consider are different.

1 Introduction

The main subject of this paper is stability of the core of a TU-game (N, v). The core of a TU-game is called stable if it is a stable set or Von Neumann-Morgenstern solution. One of the goals is to order the sufficient conditions for core stability that can be found in the literature. We start with

Sharkey (1982): in this paper two properties are introduced, namely subconvexity and largeness of the core (formal definitions will be given in the next sections). It is proved that subconvexity implies largeness of the core and that largeness of the core implies core-stability and exactness of the totally balanced cover of the game. Sharkey calls the last property 'essential completeness'. Sharkey thinks to have counterexamples for the converse statements. His examples are obtained by increasing the value of the grand coalition $v(N)$ of Lucas' 10-person game (1968) starting with $v(N) = 5$. Sharkey writes

> *The following assertions may be verified by routine, but exhaustive calculations. The game is balanced and essentially complete if $v(N) \geq 5$, it has a stable core if $v(N) \geq 7$, the core is large if $v(N) \geq 8$ and the game is subconvex if $v(N) \geq 14$.*

In fact all these bounds are wrong. As we will show in one of the subsequent sections, if $v(N) \geq 11$ (and not earlier), the core is large and stable and the game is essentially complete. If $v(N) \geq 16$ (and not earlier), the game is subconvex. There are, however, games which have a large core without being subconvex, games which have a stable core without having a large core and games which are 'essential complete' without having a large core, as we shall show by examples.

Kikuta and Shapley (1986): in this unpublished manuscript the authors introduce a new property, which we call *extendability*. A TU-game has this property if any core element of any subgame can be extended into a core element of the game. They prove that largeness of the core implies extendability and that extendability implies stability of the core. They conjecture that extendability is equivalent with core-stability. We shall show that this is not true.

For special classes of TU-games we mention:

Kulakovskaja (1979): the paper gives for *four-person* games a set of linear inequalities for core-stability. These conditions can be read as necessary conditions for exactness.

Biswas et al. (1996, 1998): these papers show that, for *symmetric* games and for games with *four or less players*, largeness of the core, stability of the core and essential completeness are equivalent. The first paper (1996) gives, moreover, an easy way to check these properties for symmetric games. For $n \geq 5$ the second paper (1998) gives examples of TU-games which are essentially complete without having a stable or large core. Summarizing these results we find the following scheme: (where KS stands for Kikuta and Shapley and B stands for Biswas et al.)

$$\text{Subconvexity} \underset{\text{Sharkey}}{\Longrightarrow} \text{Largeness} \underset{\text{KS}}{\Longrightarrow} \text{Extendability} \underset{\text{KS}}{\Longrightarrow} \text{Core-Stability} \qquad (*)$$

$$\text{Largeness} \underset{\text{Sharkey}}{\Longrightarrow} \text{Core-Stability and Essential completeness}$$

For symmetric games and for games with four or less players we have:

$$\text{Largeness} \underset{\text{B}}{\Longleftrightarrow} \text{Core-Stability} \underset{\text{B}}{\Longleftrightarrow} \text{Essential completeness}$$

The following counterexamples are known:

$$\text{Essential completeness} \underset{\text{B}}{\not\Longrightarrow} \text{Core-Stability or Largeness } (n \geq 5)$$

In Section 3 we shall give formal definitions of the mentioned properties of TU-games. The relations between the properties are studied by considering the set of values for $v(N)$ for which a TU-game has a certain property. It will appear that, for most of the properties, this set is of the form $[\alpha, \infty)$. Section 4 gives counterexamples of the converse statements of $(*)$.

2 Preliminaries

Definition 2.1: A *TU-game* is a pair (N, v), where N is the set of *players*, $n := |N|$. A subset $S \subseteq N$ is called a *coalition.* $v : 2^N \to \mathbb{R}$ assigns to every coalition a real number with the convention that $v(\emptyset) = 0$. v is called the *characteristic function* and $v(S)$ is the *value* of coalition S.

The restriction of v to $2^N \backslash \{N\}$ is denoted by v^0. (N, v^0) is called an *incomplete TU-game.* The *subgame* (S, v_S) is defined by

$$v_S(T) := v(T) \quad \text{for all } T \subseteq S.$$

◇

Definition 2.2: Let (N, v) be a TU-game. We write for short $x(S) := \sum_{i \in S} x_i$, where $x \in \mathbb{R}^n$ and $S \subseteq N$. The *imputation set*, $I(v)$, is defined by

$$I(v) := \{x \in \mathbb{R}^n \mid x(N) = v(N), \quad x_i \geq v(i) \quad \forall i \in N\}.$$

and the set of *upper vectors*, $U(v)$, by

$$U(v) := \{x \in \mathbb{R}^n \mid x(S) \geq v(S) \quad \forall S \subset N\}.$$

The *core* of (N, v) is the intersection of these two sets:

$$Core(N, v) := \{x \in \mathbb{R}^n \mid x(N) = v(N), \quad x(S) \geq v(S) \quad \forall S \subset N\}.$$

◇

We want to arrange properties on TU-games which are necessary or sufficient conditions for stability of the core. The idea is to do this in the following way: Let P be a property on TU-games and let (N, v^0) be an incomplete game. For which numbers β does the TU-game $(N, v^0, v(N) = \beta)$ have property P? Define

$$B_P(v^0) := \{\beta \geq \sum_{i \in N} v(i) \mid (N, v^0, v(N) = \beta) \text{ has property } P\}.$$

If $B_P(v^0)$ is contained in $B_{SC}(v^0)$ for every incomplete TU-game (N, v^0) then P is a sufficient condition for stability of the core; if $B_P(v^0)$ contains $B_{SC}(v^0)$ for every incomplete TU-game (N, v^0) then P is a necessary condition for stability. We are especially interested in properties P for which every incomplete game (N, v^0) has a number $\alpha_P(v^0)$ such that $B_P(v^0) = [\alpha_P(v^0), \infty)$, i.e., a TU-game has property P if and only if the value of the grand coalition is 'large' enough. Such properties are called 'prosperity properties'. Unfortunately we do not know whether stability of the core is a prosperity property. Therefore we also consider a weaker kind of prosperity property. Definition 2.3 gives the formal definitions.

Definition 2.3: A property P on TU-games is called a *(strong) prosperity property* if for every incomplete game (N, v^0) there exists a number $\alpha_P(v^0) \geq \sum_{i \in N} v(i)$ such that

$$(N, v^0, v(N)) \text{ has property P if and only if } v(N) \geq \alpha_P(v^0).$$

P is called a *weak prosperity property* if for every incomplete game (N, v^0) there exists a number $\alpha_P(v^0) \geq \sum_{i \in N} v(i)$ such that the following conditions hold:

- $(N, v^0, v(N))$ has property P if $v(N) > \alpha_P(v^0)$,
- for every number $\sum_{i \in N} v(i) \leq \beta < \alpha_P(v^0)$ there exists a number $\gamma \in (\beta, \alpha_P(v^0))$ with $(N, v^0, v(N) = \gamma)$ does *not* have property P.

◇

It is clear that every strong prosperity property is a weak prosperity property. Two more properties, 'monotonicity' and 'closedness' bridge the gap between weak and strong prosperity.

Definition 2.4: A property P on TU-games is called *monotone* if for every incomplete game (N, v^0) and every $\beta, \gamma \geq \sum_{i \in N} v(i)$ with $\gamma \geq \beta$ we have

$$\beta \in B_P(v^0) \quad \Rightarrow \quad \gamma \in B_P(v^0).$$

P is called *closed* if for every incomplete game (N, v^0) and every sequence $\beta_1, \beta_2, \ldots \to \beta$ we have

$$\beta_j \in B_P(v^0) \text{ for all } j \in \mathbb{N} \quad \Rightarrow \quad \beta \in B_P(v^0).$$

Note that for *monotone* properties it is sufficient to check the last condition only for sequences $\beta_1, \beta_2, \ldots \downarrow \beta$.

◇

From the definitions we get

Proposition 2.1: *If P is a strong prosperity property, then P is a weak prosperity property.*

If P is a weak prosperity property and P is, in addition, monotone and closed then P is a strong prosperity property.

Proposition 2.2: *Let P_1 and P_2 be two properties on TU-games. If P_1 is a weak prosperity property and P_1 implies P_2 then P_2 is also a weak prosperity property and $\alpha_{P_1}(v^0) \leq \alpha_{P_2}(v^0)$ for every incomplete game (N, v^0).*

Proof: Suppose that P_1 is a weak prosperity property and that P_1 implies P_2. Let (N, v^0) be an incomplete TU-game. There exists a number $\alpha_{P_1}(v^0) \geq \sum_{i \in N} v(i)$ such that the two conditions of weak prosperity property are satisfied. Now define

$$\alpha_{P_2}(v^0) := \inf\{\beta \geq \sum_{i \in N} v(i) \mid \text{ if } \gamma > \beta \text{ then } (N, v^0, v(N) = \gamma) \text{ has property } P_2\}.$$

Note that the given set is non-empty: it contains $\alpha_{P_1}(v^0)$. It is readily seen that $\alpha_{P_2}(v^0)$ satisfies the conditions of weak prosperity property. □

3 Prosperity properties

We now examine some properties of TU-games with respect to 'prosperity'. It will appear that the functions α_{P_1} and α_{P_2} can be ordered for most prosperity properties. The ordering can be refined by considering the totally balanced cover of a TU-game (Section 3.7).

3.1 Balancedness

Definition 3.1: A TU-game (N, v) is called *balanced* if

$$v(N) \geq \max\{\sum_{T \subset N} \lambda_T v^0(T) \mid \sum_{T \subset N} \lambda_T e_T = e_N \text{ and } \lambda_T \geq 0 \ (T \subset N)\}.$$

$\diamond$

It is immediately clear that we can take $\alpha_{Bal}(v^0)$ equal to the right-hand side of this inequality, hence balancedness is a prosperity property. As balancedness is equivalent with non-emptiness of the core (Bondareva (1963)/Shapley (1967)) we have that non-emptiness of the core is a prosperity property.

A TU-game (N, v) is called *totally balanced* if every subgame (S, v_S), $S \subseteq N$ is balanced. Obviously totally balancedness is not a (weak) prosperity property: if a proper subgame is not balanced, this can not be cured by increasing $v(N)$. For completeness we define: an incomplete TU-game (N, v^0) is called *totally balanced* if every proper subgame (S, v_S), $S \subset N$ is balanced.

3.2 Largeness of the core

Definition 3.2: [Sharkey (1982)] Let (N, v) be a TU-game with a non-empty core. The core is called *large* if for every upper vector $y \in U(v)$ with $y(N) > v(N)$ there is an allocation $x \in Core(N, v)$ such that $x \leq y$. $\diamond$

The following theorem connects largeness of the core with the extreme points of $U(v)$. From this theorem it follows that largeness of the core is a prosperity property.

Theorem 3.1: *Let (N, v) be a balanced game. Then*
(N, v) has a large core $\iff$ $z(N) \leq v(N)$ for all extreme points z of $U(v)$.

Proof: We start with the recession cone of the polyhedral set $U(v)$, i.e. the set

$$\{u \in \mathbb{R}^n \mid y + tu \in U(v) \text{ for all } y \in U(v), t \geq 0\}.$$

It is easy to see that the recession cone of $U(v)$ is equal to $\mathbb{R}^n_+$, a pointed cone. By the structure theorem for polyhedral sets we have

$$U(v) = Ch(Extr(U(v))) + \mathbb{R}^n_+,$$

where Ch means 'the convex hull of' and Extr means 'the extreme points of'.

$\Rightarrow$: Suppose that the core of (N, v) is large and nevertheless $z(N) > v(N)$ for some $z \in Extr(U(v))$. Then $z = x + u$ for some $x \in Core(N, v)$ and $u \geq 0$. Then $z + u \in U(v)$ and $z - u = x \in U(v)$ and $z \notin Extr(U(v))$.

$\Leftarrow$: Suppose that $z(N) \leq v(N)$ for all $z \in Extr(U(v))$. Take any upper vector $y \in U(v)$, $y(N) > v(N)$. Then $y = x + u$ for some $x \in Ch(Extr(U(v)))$ and $u \geq 0, u \neq 0$. Then $x(N) \leq v(N) < y(N)$ and $x + tu \in Core(v)$ where $t = \frac{v(N)-x(N)}{u(N)} \in [0, 1)$. □

Corollary 3.2: *Largeness of the core is a prosperity property and*

$$\alpha_{LC}(v^0) := \max\{z(N) \mid z \text{ is an extreme point of } U(v)\}. \tag{3.1}$$

Equation (3.1) is not very suitable to find the value of $\alpha_{LC}(v^0)$ in practical situations, because it is difficult to find *all* extreme points of $U(v)$. Some extreme points are given by the vectors x^π, defined below.

Definition 3.3: Let (N, v) be a TU-game and let $\pi : N \to \{1, \ldots, n\}$ be a bijective map; π is called an *enumeration* of N. π can be interpreted as follows: the players enter a room one by one; the first player who enters is $\pi^{-1}(1)$, the second $\pi^{-1}(2)$,..., the last player who enters is $\pi^{-1}(n)$. The set of *predecessors* of $i \in N$ is defined by

$$P_i^\pi := \{j \mid \pi(j) < \pi(i)\}.$$

Now we define the vector $x^\pi \in \mathbb{R}^n$ by

$$x_i^\pi := \max\{v(Q \cup i) - x^\pi(Q) \mid Q \subseteq P_i^\pi, Q \cup i \neq N\}.$$

Note that $Q \cup i \neq N$ is only restrictive if $\pi(i) = n$. ◇

Proposition 3.3: *Let (N, v) be a TU-game and let π be an enumeration of N. Then x^π is an extreme point of $U(v)$.*

Proof: We first show that $x^\pi \in U(v)$. Let $S \subset N$. Let i^* be the 'last' player in S who enters, i.e.

$$\pi(i^*) = \max\{\pi(i) \mid i \in S\}.$$

To compute $x_{i^*}^\pi$, a subset of $P_{i^*}^\pi$ must be chosen. One possibility is $Q = S \backslash i^*$. This gives the following inequality

$$x^\pi(S) = x_{i^*}^\pi + x^\pi(S \backslash i^*) \geq [v(S \backslash i^* \cup i^*) - x^\pi(S \backslash i^*)] + x^\pi(S \backslash i^*) = v(S).$$

So $x^\pi \in U(v)$. Take for every player i a subset of its predecessors $Q_i \subseteq P_i^\pi$ such that $x^\pi = v(Q_i \cup i) - x(Q_i)$. Then the set of tight coalitions of the vector x^π contains the set $\{Q_i \cup i \mid i \in N\}$ which is a complete system. Hence x^π is an extreme point of $U(v)$. □

The vectors x^π are extreme points of $U(v)$, but in general there are more extreme points than the vectors x^π (see Example 6, Section 4.2). x^π is a core element of the game $(N, v^0, v(N) = x^\pi(N))$. This, together with Theorem 3.1 implies that the following inequality holds for every incomplete game (N, v^0):

$$\alpha_{Bal}(v^0) \leq \max_\pi x^\pi(N) \leq \alpha_{LC}(v^0).$$

3.3 Stability of the core

For the sake of completeness we repeat the definition of stability of the core.

Definition 3.4: Let (N, v) be a TU-game with a non-empty core. The core is called *stable* if for every imputation $y \in I(v) \backslash Core(N, v)$ there exists a vector $x \in Core(N, v)$ and a coalition $S \subset N$ such that

$$x(S) = v(S), \quad x_i > y_i \quad \forall i \in S.$$

We say that x *dominates* y *(via* S*).* If the core is stable then it is the unique von Neumann-Morgenstern solution. ◇

Sharkey (1982) proved the following relation between games with a large core and a stable core:

Theorem 3.4: [Sharkey (1982)] *Let (N, v) be a balanced game. If (N, v) has a large core, then (N, v) has a stable core.*

It follows from Theorem 3.4, Proposition 2.2 and the fact that largeness of the core is a prosperity property, that stability of the core is a weak prosperity property and $\alpha_{SC}(v^0)$ satisfies

$$\alpha_{Bal}(v^0) \leq \alpha_{SC}(v^0) \leq \alpha_{LC}(v^0).$$

Unfortunately we were not able to determine whether stability of the core is a *strong* prosperity property. We can prove neither monotonicity nor closedness.

Conjecture: stability of the core is a strong prosperity property.

We had already

$$\alpha_{Bal}(v^0) \leq \max_{\pi} x^{\pi}(N) \leq \alpha_{LC}(v^0).$$

There is no order relation between $\alpha_{SC}(v^0)$ and $\max_{\pi} x^{\pi}(N)$ as will be shown by Examples 1 and 6 in Section 4.2. In general the converse of Theorem 3.4 does not hold either. In Section 4.2 we shall give a counterexample (Example 1).

3.4 Exactness

Definition 3.5: A TU-game (N, v) is called *exact* if

$$\forall S \subset N \quad \exists x \in Core(N, v) : x(S) = v(S).$$

◇

Exactness is not a (weak) prosperity property, because exactness of a game implies totally balancedness and increasing $v(N)$ does not make a game totally balanced. The following variant, which has to do with exactness of a totally balanced game related to (N, v), the so-called 'totally balanced cover', yields a prosperity property.

Definition 3.6: Let (N, v) be a TU-game. The *totally balanced cover* $(N, \bar{v})$ is defined by

$$\bar{v}(S) := \max \left\{ \sum_{T \subseteq S} \lambda_T v(T) \mid \lambda_T \geq 0 \quad (T \subseteq N) \text{ and } \sum_{T \subseteq S} \lambda_T e_T = e_S \right\}.$$

$\diamond$

The totally balanced cover is the unique game which is totally balanced, has values larger or equal to $v(S)$ for all $S \subseteq N$ and is minimal with respect to these two properties. Note that $\bar{v}(N) = v(N)$ for balanced games.

Definition 3.7: A TU-game (N, v) is called *TB-Exact* if the totally balanced cover $(N, \bar{v})$ is exact (Sharkey called it 'essential completeness'). $\diamond$

Proposition 3.5: *TB-Exactness is a prosperity property.*

Proof: Let (N, v^0) be an incomplete game. Let $S \subset N$. The subgame $(S, \bar{v}(S))$ of $(N, \bar{v}^0)$ is balanced and every core element of $(S, \bar{v}(S))$ can be extended to an element x of $U(v)$ with $x(N)$ 'large enough'. If $v(N) \geq x(N)$ then $(N, \bar{v}^0)$ is exact with respect to S. So TB-Exactness is a monotone weak prosperity property. If we can show that it is closed, then it follows from Proposition 2.1 that it is a strong prosperity property. Let $\beta_1, \beta_2, \ldots \downarrow \beta, \quad \beta_j \in B_P(v^0) \quad \forall j \in \mathbb{N}$. Let $S \subset N$. $(N, \bar{v}, \bar{v}(N) = \beta_j)$ is exact with respect to S for all $j \in \mathbb{N}$, so there is a sequence x^j of core elements of $(N, \bar{v}, \bar{v}(N) = \beta_j)$ $(j \in \mathbb{N})$. The sequence is contained in the compact set $U(v) \cap \{y \in \mathbb{R}^n \mid y(N) \leq \beta_1\}$. Hence it has a convergent subsequence with limit, say x. Then $x(N) = \beta, \quad x(S) = \bar{v}(S)$ and $x(T) \geq \bar{v}(T) \quad (T \subset N)$, i.e. $(N, \bar{v}, \bar{v}(N) = \beta)$ is exact with respect to S. $\square$

For every totally balanced incomplete game (N, v^0) we have $\alpha_{TB\text{-}Ex}(v^0) \leq \alpha_{LC}(v^0)$ which follows from the following proposition of Sharkey (1982):

Proposition 3.6: *A totally balanced TU-game with a large core is exact.*

We can have both $\alpha_{TB\text{-}Ex}(v^0) < \alpha_{SC}(v^0)$ and $\alpha_{TB\text{-}Ex}(v^0) > \alpha_{SC}(v^0)$. Biswas et al. (1998) give an example of the first case and Example 1 in Section 4.2 shows the second possibility.

3.5 Extendability

Definition 3.8: (N, v) is called *extendable* if for every non-empty coalition $S \subset N$ and every core element y of the subgame (S, v_S) there is a core element x of (N, v) such that $x_i = y_i$ for all $i \in S$. Hence if (N, v) is extendable then any core element of any subgame can be extended to a core element of (N, v). $\diamond$

Kikuta and Shapley (1986) have proved the following relations between extendability, large core and stable core:

Theorem 3.7: [Kikuta and Shapley (1986)] *Let (N, v) be a balanced TU-game. If the core of (N, v) is large, then (N, v) is extendable; if (N, v) is extendable, then the core of (N, v)*

is stable.

From this theorem we find, by Proposition 2.2 and the fact that largeness of the core is a prosperity property, that extendability is a weak prosperity property. Extendability is a monotone and closed property (the proof is the same as for TB-Exactness), hence (Proposition 2.1) extendability is a strong prosperity property and

$$\alpha_{Bal}(v^0) \leq \alpha_{SC}(v^0) \leq \alpha_{Ext}(v^0) \leq \alpha_{LC}(v^0).$$

In fact, extendability is the weakest sufficient condition for core-stability that is known at the moment. It follows immediately from the definitions that for totally balanced incomplete TU-games $\alpha_{TB\text{-}Ex}(v^0) \leq \alpha_{Ext}(v^0)$. But $\alpha_{TB\text{-}Ex}(v^0) > \alpha_{Ext}(v^0)$ is possible if (N, v^0) is not totally balanced (Example 2 in Section 4.2).

3.6 Convexity

Definition 3.9: A TU-game (N, v) is called *convex* if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \text{ for all } S, T \subseteq N.$$

◇

Convexity is not a (weak) prosperity property, because a violated inequality not concerning the grand coalition remains violated if $v(N)$ is increased.

The following weaker kind of convexity is a strong prosperity property.

Definition 3.10: [Sharkey (1982)] Let $\mathbf{P} := \{P_1, \ldots, P_k\}$ be an arbitrary partition of N and let $\mathbf{Q} := \{Q_1, \ldots, Q_k\}$ be a collection of coalitions such that

$$Q_i \subseteq \bigcup_{j=0}^{i-1} P_j \text{ and } P_k \cup Q_k \neq N, \text{ where } P_0 = \emptyset.$$

The game (N, v) is said to be *subconvex* if for all such collections $\mathbf{P}$ and $\mathbf{Q}$ it is true that

$$\sum_{i=1}^{k} [v(P_i \cup Q_i) - v(Q_i)] \leq v(N).$$

◇

From this definition it is immediately clear that subconvexity is a prosperity property and $\alpha_{Subc}(v^0)$ is equal to the maximum over all $\mathbf{P}$ and $\mathbf{Q}$ of the left-hand side of the last inequality. We show that Sharkey's definition of subconvexity can be simplified to a system of $n!$ inequalities. This gives another way to compute $\alpha_{Subc}(v^0)$.

Definition 3.11: Let (N, v) be a TU-game and let π be an enumeration of N. We define the vector $y^\pi \in \mathbb{R}^n$ by

$$y_i^\pi := \max\{v(Q \cup i) - v(Q) \mid Q \subseteq P_i^\pi, Q \cup i \neq N\}.$$

y_i^π is the maximal marginal of player i with respect to a subset of his predecessors. $\diamond$

Theorem 3.8: *(N,v) is subconvex $\iff$ $y^\pi(N) \leq v(N)$ for all enumerations π of N.*

Proof: $\Rightarrow$: Suppose that (N,v) is subconvex. Let π be an enumeration of N and consider y^π. For every $i \in N$ let $Q_{\pi(i)}$ be such that $y_i^\pi = v(Q_{\pi(i)} \cup i) - v(Q_{\pi(i)})$. Then $Q_{\pi(i)} \subseteq P_i^\pi$ and $\pi^{-1}(n) \cup Q_n \neq N$. Define a partition $\mathbf{P} = \{P_1, \ldots P_n\}$ by $P_i := \{\pi^{-1}(i)\}$. Then we have

$$Q_i \subseteq \bigcup_{j=0}^{i-1} P_j \text{ and } P_n \cup Q_n \neq N.$$

So

$$y^\pi(N) = \sum_{i=1}^{n} y_{\pi^{-1}(i)}^\pi = \sum_{i=1}^{n} \left[v(\pi^{-1}(i) \cup Q_i) - v(Q_i)\right] = \sum_{i=1}^{n} [v(P_i \cup Q_i) - v(Q_i)] \leq v(N).$$

$\Leftarrow$: Suppose that $y^\pi(N) \leq v(N)$ for all enumerations π of N. Take $P_1, \ldots, P_k, Q_1 \ldots, Q_k$ as in the definition of subconvexity and let $p_i := \#P_i, i = 0, \ldots, k$. Let π be an enumeration of N such that $P_i = \{\pi^{-1}(p_1 + \cdots + p_{i-1} + 1), \ldots, \pi^{-1}(p_1 + \cdots + p_i)\}, i = 1, \ldots, k$. By adding the players in P_i to Q_i one at a time we get $v(P_i \cup Q_i) - v(Q_i) \leq y^\pi_{\pi^{-1}(p_1+\ldots+p_{i-1}+p_i)} + \cdots + y^\pi_{\pi^{-1}(p_1+\ldots+p_{i-1}+1)} = y^\pi(P_i)$. So

$$\sum_{i=1}^{k} [v(P_i \cup Q_i) - v(Q_i)] \leq \sum_{i=1}^{k} y^\pi(P_i) = y^\pi(N) \leq v(N),$$

from which subconvexity follows. $\square$

Corollary 3.9:

$$\alpha_{Subc}(v^0) = \max_\pi y^\pi(N).$$

The vectors y^π are upper vectors, as the following proposition shows, but y^π is only an *extreme* point of $U(v)$ if $y^\pi = x^\pi$.

Proposition 3.10: *$y^\pi \in U(v)$ and $x^\pi \leq y^\pi$ for all enumerations π of N.*

Proof: Let $S \subset N$. There exist numbers $i_1 < i_2 < \ldots < i_s$, $(s = |S|)$, such that $S = \{\pi^{-1}(i_1), \ldots, \pi^{-1}(i_s)\}$. Then for every $j \in \{1, \ldots, s\}$ we have

$$y^\pi_{\pi^{-1}(i_j)} \geq v(\{\pi^{-1}(i_1), \ldots, \pi^{-1}(i_j)\}) - v(\{\pi^{-1}(i_1), \ldots, \pi^{-1}(i_{j-1})\}).$$

Summing over j gives $y^\pi(S) \geq v(S)$, i.e. $y^\pi \in U(v)$. Recall that for all $i \in N$

$$x_i^\pi := \max\{v(Q \cup i) - x^\pi(Q) \mid Q \subseteq P_i^\pi, Q \cup i \neq N\}$$

and

$$y_i^\pi := \max\{v(Q \cup i) - v(Q) \mid Q \subseteq P_i^\pi, Q \cup i \neq N\}.$$

From Proposition 3.3 we know that $x^\pi \in U(v)$, hence $x^\pi(Q) \geq v(Q)$ for all $Q \subset N$, from which we get $x_i^\pi \leq y_i^\pi$ for all $i \in N$. □

$\alpha_{LC}(v^0) \leq \alpha_{Subc}(v^0)$ follows from

Theorem 3.11: [Sharkey (1982)] *If (N,v) is subconvex, then the core of (N,v) is large.*

3.7 Totally balanced cover

We have found thus far

$$\alpha_{Bal}(v^0) \leq \alpha_{SC}(v^0) \leq \alpha_{Ext}(v^0) \leq \alpha_{LC}(v^0) \leq \alpha_{Subc}(v^0) = \max_\pi y^\pi(N),$$

$$\alpha_{Bal}(v^0) \leq \alpha_{TB\text{-}Ex}(v^0) \leq \alpha_{LC}(v^0)$$

and

$$\alpha_{Bal}(v^0) \leq \max_\pi x^\pi(N) \leq \alpha_{LC}(v^0).$$

We can refine the first row of inequalities by considering the totally balanced cover of a game as defined in Section 3.4. The following proposition gives the (weak) prosperity properties P for which the value $\alpha_P(v^0)$ does not change if we take the totally balanced cover of the game.

Proposition 3.12: *Let (N,v^0) be an incomplete game. Then*

1. $\alpha_{Bal}(v^0) = \alpha_{Bal}(\bar{v}^0)$,
2. $\alpha_{LC}(v^0) = \alpha_{LC}(\bar{v}^0)$,
3. $\alpha_{SC}(v^0) = \alpha_{SC}(\bar{v}^0)$,
4. $\alpha_{TB\text{-}Ex}(v^0) = \alpha_{TB\text{-}Ex}(\bar{v}^0)$.

Proof:

1. This follows from the equality $U(v) = U(\bar{v})$, which we prove now. $U(\bar{v}) \subseteq U(v)$ because $\bar{v}(S) \geq v(S)$ for all $S \subset N$. Now let $y \in U(v)$ and $S \subset N$. There are numbers $\lambda_T \geq 0$ such that $\sum_{T\subseteq S} \lambda_T e_T = e_S$ and $\bar{v}(S) = \sum_{T\subseteq S} \lambda_T v(T)$. Then

   $$y(S) = \sum_{T\subseteq S} \lambda_T y(T) \geq \sum_{T\subseteq S} \lambda_T v(T) = \bar{v}(S).$$

   So $y \in U(\bar{v})$ and the proof is complete.

2. Follows from $U(v) = U(\bar{v})$ (see proof of item 1) and Equation (3.1).

3. The equality follows from the following equivalence for balanced games:

$$(N, v) \text{ has a stable core } \iff (N, \bar{v}) \text{ has a stable core.}$$

Proof: $\Rightarrow$: Suppose that (N, v) has a stable core. Let $y \in I(\bar{v}) \backslash Core(N, \bar{v}) = I(v) \backslash Core(N, v)$ (because $U(v) = U(\bar{v})$, $v(N) = \bar{v}(N)$ and $\bar{v}(i) = v(i)$ $(i \in N)$). There exists a non-empty subset $S \subset N$ and a vector $x \in Core(N, v)$ such that

$$x(S) = v(S) \quad x_i > y_i \text{ for all } i \in S.$$

x restricted to S is a core element of the subgame (S, v_S), hence $\bar{v}(S) = v(S)$. So x dominates y via S in the game $\bar{v}$.

$\Leftarrow$: Suppose $(N, \bar{v})$ has a stable core. Let $y \in I(v) \backslash Core(N, v) = I(\bar{v}) \backslash Core(N, \bar{v})$. There exist a non-empty subset $S \subset N$ and a vector $x \in Core(N, \bar{v})$ such that

$$x(S) = \bar{v}(S) \quad x_i > y_i \text{ for all } i \in S.$$

If $\bar{v}(S) = v(S)$ then x dominates y via S in the game v. If $\bar{v}(S) > v(S)$, then there exist numbers $\lambda_T \geq 0$ such that

$$\sum_{T \subseteq S} \lambda_T e_T = e_S, \sum_{T \subseteq S} \lambda_T v(T) = \bar{v}(S).$$

Then $x(T) = v(T)$ for all $T \subseteq S$ with $\lambda_T > 0$. Take $T^* \subseteq S$: $x(T^*) = v(T^*)$. Then $x_i > y_i$ for all $i \in T^*$, hence x dominates y via T^* in the game v.

4. By definition.

$\square$

For the remaining two prosperity properties which we have considered, we have just one inequality:

Proposition 3.13: *Let (N, v^0) be an incomplete game. Then*

1. $\alpha_{Ext}(v^0) \leq \alpha_{Ext}(\bar{v}^0)$,

2. $\alpha_{Subc}(\bar{v}^0) \leq \alpha_{Subc}(v^0)$.

Proof:

1. Follows immediately from the definitions.

2. According to Theorem 3.8 it is sufficient to show that for every enumeration π and every player i we have $\bar{y}_i^\pi \leq y_i^\pi$, where $\bar{y}_i^\pi$ is computed with respect to the game $\bar{v}$.

   Let Q^* be a subset of P_i^π, $Q^* \cup i \neq N$ such that

$$\bar{y}_i^\pi = \bar{v}(Q^* \cup i) - \bar{v}(Q^*).$$

There exists a collection $\mathcal{T} \subseteq 2^{Q^* \cup i}$ and numbers $\lambda_T \geq 0 \quad \forall T \in \mathcal{T}$ such that

$$\bar{v}(Q^* \cup i) = \sum_{T \in \mathcal{T}} \lambda_T v(T) \quad \text{and} \quad \sum_{T \in \mathcal{T}} \lambda_T e_T = e_{Q^* \cup i}.$$

In particular $\sum_{T \in \mathcal{T}} \lambda_T e_{T \setminus i} = e_{Q^*}$, hence $\bar{v}(Q^*) \geq \sum_{T \in \mathcal{T}} \lambda_T v(T \setminus i)$. From this we get

$$\begin{aligned} \bar{y}_i^\pi = \bar{v}(Q^* \cup i) - \bar{v}(Q^*) &\leq \sum_{T \in \mathcal{T}} \lambda_T (v(T) - v(T \setminus i)) \\ &= \sum_{T \in \mathcal{T}: i \in T} \lambda_T (v(T) - v(T \setminus i)) \\ &\leq \max_{T \in \mathcal{T}: i \in T} \{v(T) - v(T \setminus i)\} \\ &\leq \max_{Q \subseteq P_i^\pi,\ Q \cup i \neq N} \{v(Q \cup i) - v(Q)\} = y_i^\pi, \end{aligned} \tag{3.2}$$

where the second inequality follows from the fact that $\sum_{T \in \mathcal{T}: i \in T} \lambda_T = 1$.

□

The values $\alpha_P(v^0)$ for the (weak) prosperity properties we have considered up to now, except those of TB-Exactness, can be ordered linearly

$$\alpha_{Bal}(v^0) \leq \alpha_{SC}(v^0) \leq \alpha_{Ext}(v^0) \leq \alpha_{Ext}(\bar{v}^0) \leq \alpha_{LC}(v^0) \leq \alpha_{Subc}(\bar{v}^0) \leq \alpha_{Subc}(v^0).$$

Section 4.2 gives for each of the cases an example where strict inequality holds. It shows the power of some of the propositions that we have proved.

4 Examples

4.1 Lucas game

To show that stability of the core does not imply large core, Sharkey (1982) considers a game (N, v) based on Lucas' 10-person game (1968). The characteristic function of this game is as follows: (we write 0 instead of 10)

$v(12) = v(34) = v(56) = v(78) = v(90) = 1,$
$v(137) = v(139) = v(157) = v(159) = v(357) = v(359) = 2,$
$v(1479) = v(2579) = v(3679) = 2,$
$v(1379) = v(1579) = v(3579) = 3,$
$v(13579) = 4,$
$v(S) = 0$ for other $S \subset N$.

If $v(N) = 5$ then we get Lucas' example. Sharkey claims that if $v(N) \geq 7$ then the core is stable and if $v(N) \geq 8$ then the core is large. But computing x^π for the enumeration:

$$\begin{matrix} i \\ \pi(i) \end{matrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 \\ 1 & 7 & 5 & 8 & 6 & 2 & 3 & 9 & 4 & 0 \end{pmatrix}$$

i.e. the players enter in the order $1, 6, 7, 9, 3, 5, 2, 4, 8, 0$, gives $x^\pi = (0, 1, 3, 2, 3, 0, 0, 1, 0, 1)$ and $x^\pi(N) = 11$. As x^π is an extreme point of $U(v)$ it follows from Theorem 3.1 that if the core is large then $v(N) \geq 11$. The proof of the following proposition can be found in Appendix A.

Proposition 4.1: $\alpha_{LC}(v^0) = 11$.

By Theorem 3.4 we have, as a consequence, that the core is stable if $v(N) \geq 11$. The following proposition shows that the condition is also necessary for stability of the core.

Proposition 4.2: *The core is stable* $\iff v(N) \geq 11$.

Proof: We shall give an undominated imputation outside the core for every value of the grand coalition in the interval $[5, 11)$. We first consider the interval $[8, 11)$ and then $[5, 8)$.

$[8, 11)$:

Let $\delta \in (0, 1]$ and consider the allocation $y = (0, 1, 3-\delta, 2-\delta, 3-\delta, 0, 0, 1, 0, 1)$. y is an imputation of the TU-game with $v(N) = 11 - 3\delta$ and there are 3 coalitions for which the core condition is violated: $(1479), (1379), (1579)$. We show that none of these coalitions can be used to dominate y. Suppose that there exists a core allocation x which dominates y via $S = (1379)$. Then in particular $x(1379) = 3$ and $x(3) > 3 - \delta$, from which it follows that $x(4) > 2-\delta$ (because $x(1479) \geq 2$) and $x(5) > 3-\delta$ (because $x(1579) \geq 3$). And this gives a contradiction: $11-3\delta = x(N) = x(12)+x(3)+x(4)+x(5)+x(6)+x(78)+x(90) > 1+3-\delta+2-\delta+3-\delta+0+1+1 = 11-3\delta$. By symmetry we have that also x dominates y via $S = (1579)$ is impossible. Now suppose that y is dominated by a core allocation x via $S = (1479)$. Then $x(1479) = 2, x(4) > 2-\delta$, hence $x(3) > 3-\delta$ and $x(5) > 3-\delta$. This gives the same contradiction: $11-3\delta = x(N) = x(12)+x(3)+x(4)+x(5)+x(6)+x(78)+x(90) > 1+3-\delta+2-\delta+3-\delta+0+1+1 = 11-3\delta$.

$[5, 8)$:

Again let $\delta \in (0, 1]$ and consider the allocation $y = (2-\delta, 0, 2-\delta, 0, 2-\delta, 0, 0, 1, 0, 1)$. If, in addition, $\delta > \frac{2}{3}$ then $y(13579) < v(13579)$, but assuming that y is dominated by x via (13579) gives a contradiction as before: $8-3\delta = x(N) = x(1)+x(2)+x(3)+x(4)+x(5)+x(6)+x(78)+x(90) > 2-\delta+0+2-\delta+0+2-\delta+0+1+1 = 8-3\delta$. If $\delta \in (\frac{1}{2}, 1]$ then $y(S) < v(S) \quad \forall S \in \{(1379), (1579), (3579)\}$. If x dominates y via one of these 3 coalitions then $8-3\delta = x(N) = x(1)+x(2)+x(3)+x(4)+x(5)+x(6)+x(78)+x(90) > 2-\delta+0+2-\delta+0+2-\delta+0+1+1 = 8-3\delta$. Finally if y is dominated by x via $(1479), (2579)$ or (3679) then $8-3\delta = x(N) = x(14)+x(25)+x(36)+x(78)+x(90) > 2-\delta+2-\delta+2-\delta+1+1 = 8-3\delta$. □

Another consequence of Proposition 4.1 is that if $v(N) \geq 11$ then the totally balanced cover of the game is exact. As $\bar{v}(1679) = 0$, x^π shows that $\alpha_{TB\text{-}Ex}(\bar{v}^0) = 11$.

Conclusion: the core is stable $\iff v(N) \geq 11 \iff$ the core is large $\iff v$ is TB-exact.

Sharkey claims that the game is subconvex $\iff v(N) \geq 14$. But the following π gives a vector y^π with $y^\pi(N) = 16$:

$$\begin{matrix} i \\ \pi(i) \end{matrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 \\ 1 & 6 & 2 & 7 & 4 & 8 & 3 & 9 & 5 & 0 \end{pmatrix},$$

$$\begin{array}{llll} y_1^\pi = v(1) & = 0 & y_3^\pi = v(13) - v(1) & = 0 \\ y_7^\pi = v(137) - v(13) & = 2 & y_5^\pi = v(157) - v(17) & = 2 \\ y_9^\pi = v(13579) - v(1357) & = 4 & y_2^\pi = v(2579) - v(579) & = 2 \\ y_4^\pi = v(1479) - v(179) & = 2 & y_6^\pi = v(3679) - v(379) & = 2 \\ y_8^\pi = v(78) - v(7) & = 1 & y_0^\pi = v(90) - v(9) & = 1 \end{array}$$

The proof of the following proposition can be found in Appendix A.

Proposition 4.3: $\alpha_{Subc}(v^0) = 16$.

4.2 Other examples

Example 1: Stability of the core and extendability

The core of the following game is stable, but the game is not extendable. As a consequence the core is stable, but not large. This game is also an example of a game for which $\max_\pi x^\pi > \alpha_{SC}(v^0)$ and $\alpha_{TB\text{-}Ex}(v^0) > \alpha_{SC}(v^0)$.

Let $\delta \in (0, 1]$. $N = \{1, \ldots, 6\}$ and v is given by

$$v(12) = v(13) = v(45) = v(46) = 1, \quad v(N) = 4 - \delta, \quad v(S) = 0 \text{ otherwise.}$$

The game is not extendable because e.g. the unique core element of the subgame with players (14) can not be extended to a core element of (N, v): $v(14) = 0$ and there is not a core element x of (N, v) with $x_1 = x_4 = 0$.

We now show that the core is stable. Take $y \in I(v) \backslash Core(N, v)$. Then $y \geq 0$ and $y(N) = 4 - \delta$. Without loss of generality we assume that $y(12) < v(12) = 1$. We show that y is dominated by the allocation

$$x := (y_1 + \varepsilon, y_2 + \varepsilon, y_2 + \varepsilon, 3 - \delta - y_2 - \varepsilon, 0, 0),$$

where $\varepsilon > 0$ such that $x_1 + x_2 = 1$.

$x_4 = (3 - \delta) - (y_2 + \varepsilon) \geq 2 - 1 = 1$, in particular $x \geq 0$; $x(N) = 4 - \delta$, $x(12) = x(13) = 1, x(45) = x(46) = x_4 \geq 1$, so $x \in Core(N, v)$; $x_1 > y_1$ and $x_2 > y_2$. Hence x dominates y. Conclusion: the core of (N, v) is stable.

If we let the players enter in the order (123456) then we find $x^\pi = (0, 1, 1, 0, 1, 1)$, hence $\max_\pi x^\pi \geq 4 > \alpha_{SC}(v^0)$. As $\bar{v}(14) = 0$ we also have $\alpha_{TB\text{-}Ex}(v^0) \geq 4 > \alpha_{SC}(v^0)$.

Example 2: Extendability and the totally balanced cover

The following game is extendable, while the totally balanced cover is not extendable.

Let $\delta \in (0,1]$. $N = \{1,\ldots,6\}$, $v(i) = 0$ for all $i \in N$, $v(N) = 4-\delta$, $v(12) = v(13) = v(45) = v(46) = 1$, $v(S) = -1$ otherwise. The totally balanced cover of this game gives Example 1 and is not extendable, as we have seen.

The game is extendable:

- if $v(S) = -1$ then the core of the subgame (S, v_S) is empty;
- $\{(0,1,1,1,0,0),(1,0,0,0,1,1)\} \subseteq U(v)$, hence for every $i \in N$ there exists a core element x of (N,v) with $x_i = 0$;
- a core element $(t, 1-t)$ of - w.l.o.g. - the subgame with players (12) can be extendend to the following core element of (N,v): $(t, 1-t, 2-\delta, 1, 0, 0)$.

Example 3: Extendability and large core

The core of (the totally balanced cover of) the following game is not large, but the totally balanced cover is extendable.

$N = \{1,\ldots,7\}$, $v(N) = 7$, $v(17) = v(47) = 2$, $v(127) = v(137) = v(457) = v(467) = 3$, $v(S) = 0$ otherwise.

The core is not large, because if the players enter in the order $(7,1,4,2,3,5,6)$ then $x^\pi = (2,1,1,2,1,1,0)$. The proof that $(N,\bar{v})$ is extendable is straightforward.

Example 4: Large core and subconvexity

The core of the following game is large (because the game is exact and $n=4$), but the game is not subconvex.

$N = \{1,\ldots,4\}; v(N) = 5$, $v(S) = 2$ if $|S| = 3$ or $S \in \{(12),(13),(24)\}$, $v(S) = 0$ otherwise.

The game is not subconvex, because if the players enter in the order $(1,2,3,4)$ then we find $y^\pi = (0,2,2,2)$.

Example 5: Subconvexity and the totally balanced cover

Consider the following 5-person symmetric game (N,v):

$\|S\|$	1	2	3	4	5
$v(\|S\|)$	0	1	1	3	4

For every enumeration π we have $y^\pi(N) = 0+1+1+2+2 = 6$. Hence $\alpha_{Subc}(v^0) = 6$.

Taking the totally balanced cover of this game yields

$\|S\|$	1	2	3	4	5
$v(\|S\|)$	0	1	1.5	3	4

Now we find for every enumeration π: $y^\pi(N) = 0+1+1+1.5+1.5 = 5$. Hence $\alpha_{Subc}(\bar{v}^0) = 5$.

Example 6: Large core and vectors x^π

The (totally balanced cover of the) game of the previous example shows that $\alpha_{LC}(v^0)$ can be strictly larger than $\max_\pi x^\pi(N)$: for every enumeration π we have $x^\pi(N) = 0+1+1+1+1 = 4$, while $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\frac{1}{2}, 1\frac{1}{2})$ is an extreme point of $U(v) = U(\bar{v})$.

5 Summary of relations between properties

Let (N, v) be a balanced TU-game. The following implications hold:

$$
\begin{array}{ccc}
 & (N,v) \text{ is subconvex} & \\
 & \Downarrow \quad \not\Uparrow \,(n=5, \text{Ex.}5) & \\
 & (N,\bar{v}) \text{ is subconvex} & \\
 & \Downarrow \quad \not\Uparrow \,(n=4, \text{Ex.}4) & \\
 & \text{the core of } (N,v) \text{ is large} & \iff \text{the core of } (N,\bar{v}) \text{ is large} \\
 & \Downarrow \quad \not\Uparrow \,(n=7, \text{Ex.}3) & \\
 & (N,\bar{v}) \text{ is extendable} & \\
 & \Downarrow \quad \not\Uparrow \,(n=6, \text{Ex.}2) & \\
 & (N,v) \text{ is extendable} & \\
 & \Downarrow \quad \not\Uparrow \,(n=6, \text{Ex.}1) & \\
 & \text{the core of } (N,v) \text{ is stable} & \iff \text{the core of } (N,\bar{v}) \text{ is stable}
\end{array}
$$

Appendix A: Proofs

A.1 Proof of Proposition 4.1: $\alpha_{LC}(v^0) = 11$

Proof: As Sharkey already mentioned, the proof that the core is large from a certain level of $v(N)$ is long and tedious. We have constructed a proof by finding all extreme points of $U(v)$. If x is an extreme point of $U(v)$ then the (characteristic vectors of the) tight coalitions are a complete system in $\mathbb{R}^{10}$. In particular there must be a tight coalition containing player 2, implying that at least one of the following coalitions is tight: $(2), (12)$ or (2579) (It is sufficient to consider only the coalitions S with $v(S) > 0$ or $|S| = 1$, because if e.g. $S = (2369)$ is tight, then also $(2), (3), (6)$ and (9) are tight). There are also 3 possibilities for both players 4 and 6 and 2 for both players 8 and 0. By distinguishing 36 cases we have found that $x(N) \leq 11$ for every extreme point of $U(v)$. We confine ourselves to the following case: $(12), (1479), (6), (78)$ and (90) are tight. Suppose that x is an extreme point of $U(v)$ and that these five coalitions are tight.

- If in addition (8) and (0) are tight, then $x_6 = x_8 = x_0 = 0, x_7 = x_9 = 1, x_1 = x_4 = 0$ ((1479) is tight), $x_2 = 1$ ((12) is tight). $x_3 + x_4 \geq 1$ and $x_5 + x_6 \geq 1$ hence $x_3 \geq 1$ and $x_5 \geq 1$. There is at least one tight coalition containing player 3. In all possible cases $((34), (137), (139), (1379)$ and $(13579))$ x_3 must be equal to 1. Similarly we find $x_5 = 1$. So $x = (0, 1, 1, 0, 1, 0, 1, 0, 1, 0)$.

- If in addition (8) is tight and (0) is not tight, then $x_6 = x_8 = 0$ and $x_7 = 1, x_1 + x_4 + x_7 + x_9 = 2$, hence $x_1 + x_9 \leq 1$.

  - if $x_1 + x_9 = 1$, then $x_4 = 0$; every possible tight coalition containing player 3 gives $x_3 = 1$ and every possible tight coalition containing player 5 gives $x_5 = 1$; if $x_9 > 0$ then the tight coalitions are (4), (6), (8), (12), (34), (56), (78), (90), (139), (159), (1479), (1379), (1579), (13579), which is not a complete system; so $x_9 = 0, x_1 = 1, x_2 = 0, x_0 = 1$ and the extreme point becomes $x = (1, 0, 1, 0, 1, 0, 1, 0, 0, 1)$.
  - if $x_1 + x_9 < 1$, then $x_3 \geq 3 - x_1 - x_7 - x_9 > 1$ and $x_5 \geq 3 - x_1 - x_7 - x_9 > 1$; the possible tight coalitions are (1), (6), (8), (9), (12), (78), (90), (139), (159), (1479), (1379), (1579); if $x_1 > 0$ or $x_9 > 0$ then the set of tight coalitions is not a complete system, hence $x_1 = x_9 = 0, x_2 = x_0 = x_4 = 1, x_3 = x_5 = 2$ ((139) or (1379) is tight and (159) or (1579) is tight) and $x = (0, 1, 2, 1, 2, 0, 1, 0, 0, 1)$;

- If (8) is not tight and (0) is tight, then we find in the same way two extreme points: $x = (1, 0, 1, 0, 1, 0, 0, 1, 1, 0)$ and $x = (0, 1, 2, 1, 2, 0, 0, 1, 1, 0)$.

- If (8) and (0) are both not tight, then $x_6 = 0$; if $x_3 > 3 - x_1 - x_7 - x_9$ then none of the coalitions containing player 3 is tight, hence $x_3 = 3 - x_1 - x_7 - x_9$; similarly $x_5 = 3 - x_1 - x_7 - x_9$; if $x_4 = 0$ then $x_1, x_7, x_9 > 0$ (because $x_1 \leq 1$, $x_7, x_9 < 1$ and $x_1 + x_7 + x_9 = 2$) and the possible tight coalitions are (6), (12), (78), (90), (1479), (1379), (1579), (13579), (4), (34), (56), (2), (2579), (3679), (3579) which is not a complete system; so $x_4 > 0$ and the possible tight coalitions are $(6), (12), (78), (90), (1479), (1379), (1579), (1), (2), (7), (9), (2579), (3679)$; if $x_7 > 0$ or $x_9 > 0$ then the set of tight coalitions is not complete, from which it follows that $x_7 = x_9 = 0$, $x_8 = x_0 = 1$ and $x = (x_1, 1-x_1, 3-x_1, 2-x_1, 3-x_1, 0, 0, 1, 0, 1)$; this gives two extreme points of $U(v)$: $x = (1, 0, 2, 1, 2, 0, 0, 1, 0, 1)$ and $x = (0, 1, 3, 2, 3, 0, 0, 1, 0, 1)$.

□

A.2 Proof of Proposition 4.3: $\alpha_{Subc}(v^0) = 16$

Proof: Let π be an enumeration of N. We show that $y^\pi(N) \leq 16$. Then Theorem 3.8 completes the proof. Because of symmetry in the coalition values we may assume, without loss of generality, that $\pi(1) < \pi(3) < \pi(5)$ and $\pi(7) < \pi(9)$. For any enumeration satisfying these conditions we can estimate the co-ordinates of y^π from above by A.1

$$\begin{array}{ll} y_1^\pi \leq 2 & y_6^\pi \leq 2 \\ y_2^\pi \leq 2 & y_7^\pi \leq 2 \\ y_3^\pi \leq 3 & y_8^\pi \leq 1 \\ y_4^\pi \leq 2 & y_9^\pi \leq 4 \\ y_5^\pi \leq 3 & y_0^\pi \leq 1 \end{array} \tag{A.1}$$

With the assumptions on π, the odd players can enter in 10 different orders. By merging some orders we distinguish 7 cases:

$\pi(5) < \pi(7)$: The odd players enter in the order 13579. Then $y_1^\pi + y_2^\pi \leq 2$ because if $\pi(1) < \pi(2)$ then $y_1^\pi = v(1) - v(\emptyset) = 0$ and $y_2^\pi \leq 2$ (if $\pi(2) < \pi(9)$ then $y_2^\pi = v(12) - v(2) = 1$, if $\pi(9) < \pi(2)$ then $y_2^\pi = v(2579) - v(579) = 2$) and if $\pi(2) < \pi(1)$ then $y_2^\pi = v(2) - v(\emptyset) = 0$ and $y_1^\pi = v(12) - v(1) = 1$. Similarly we have $y_3^\pi + y_4^\pi \leq 2$ and $y_5^\pi + y_6^\pi \leq 2$. Further $y_7^\pi = v(137) - v(13) = 2$ and $y_9^\pi = v(13579) - v(1357) = 4$. So

$$y^\pi(N) = (y_1^\pi + y_2^\pi) + (y_3^\pi + y_4^\pi) + (y_5^\pi + y_6^\pi) + y_7^\pi + y_8^\pi + y_9^\pi + y_0^\pi \leq 14$$

In all 7 cases we estimate the co-ordinates which are not mentioned explicitly, by A.1.

$\pi(3) < \pi(7) < \pi(5) < \pi(9)$: Now the odd players enter in the order 13759. Then $y_1^\pi + y_2^\pi \leq 2$ and $y_3^\pi + y_4^\pi \leq 2$ as in the previous case. $y_5^\pi = v(157) - v(17) = 2$, $y_7^\pi = v(137) - v(13) = 2$ and $y_9^\pi = v(13579) - v(1357) = 4$. Then we get

$$y^\pi(N) = (y_1^\pi + y_2^\pi) + (y_3^\pi + y_4^\pi) + y_5^\pi + y_6^\pi + y_7^\pi + y_8^\pi + y_9^\pi + y_0^\pi \leq 16$$

$\pi(7) < \pi(3), \pi(5) < \pi(9)$: There are 2 possibilities for the order of the odd players: 17359 or 71359. In both cases we have $y_1^\pi + y_2^\pi \leq 2, y_3^\pi = 2, y_5^\pi = 2, y_7^\pi + y_8^\pi = 1, y_9^\pi = 4$. Estimating the other co-ordinates by A.1 gives

$$y^\pi(N) = (y_1^\pi + y_2^\pi) + y_3^\pi + y_4^\pi + y_5^\pi + y_6^\pi + (y_7^\pi + y_8^\pi) + y_9^\pi + y_0^\pi \leq 16$$

$\pi(3) < \pi(7), \pi(9) < \pi(5)$: The odd players enter in the order 13795.

$$y^\pi(N) = (y_1^\pi + y_2^\pi) + (y_3^\pi + y_4^\pi) + y_5^\pi + y_6^\pi + y_7^\pi + y_8^\pi + y_9^\pi + y_0^\pi \leq 15$$

$\pi(7) < \pi(3) < \pi(9) < \pi(5)$: Order of the odd players: 17395 or 71395.

$$y^\pi(N) = (y_1^\pi + y_2^\pi) + y_3^\pi + y_4^\pi + y_5^\pi + y_6^\pi + (y_7^\pi + y_8^\pi) + y_9^\pi + y_0^\pi \leq 15$$

$\pi(1) < \pi(9) < \pi(3)$: Order of the odd players: 17935 or 71935.

$$y^\pi(N) = (y_1^\pi + y_2^\pi) + y_3^\pi + (y_4^\pi + y_9^\pi + y_0^\pi) + y_5^\pi + y_6^\pi + (y_7^\pi + y_8^\pi) \leq 14$$

$y_4^\pi + y_9^\pi + y_0^\pi \leq 3$ because if $\pi(4) < \pi(9)$ then $y_4^\pi = 0, y_9^\pi = v(1479) - v(147) = 2, y_0^\pi \leq 1$ and if $\pi(9) < \pi(4)$ then $y_4^\pi = v(1479) - v(179) = 2, y_9^\pi + y_0^\pi \leq 1$.

$\pi(9) < \pi(1)$: Order of the odd players: 79135.

$$y^\pi(N) = (y_1^\pi + y_2^\pi + y_4^\pi) + y_3^\pi + y_5^\pi + y_6^\pi + (y_7^\pi + y_8^\pi) + (y_9^\pi + y_0^\pi) \leq 14$$

$y_1^\pi + y_2^\pi + y_4^\pi \leq 4$ because if $\pi(4) < \pi(1)$ then $y_1^\pi = 2, y_2^\pi \leq 1, y_4^\pi = 0$; if $\pi(1) < \pi(4)$ then $y_1^\pi + y_2^\pi \leq 2, y_4^\pi = 2$.

□

References

[1] **Biswas, A., G. Ravindran and T. Parthasarathy (1996)** *Stability and Largeness of the Core of Symmetric Games* Technical Report No. 31, Statistical Quality Control & Operations Research Unit, ISI Madras

[2] **Biswas, A., T. Parthasarathy, J.A.M. Potters and M. Voorneveld (1998)** *Large Cores and Exactness* To appear.

[3] **Bondareva, O. N. (1963)** *Some Applications of Linear Programming Methods to the Theory of Cooperative Games (In Russian)* Problemy Kibernet. 10, 119-139.

[4] **Kikuta, K. and L.S. Shapley (1986)** *Core Stability in n-Person Games* Unpublished paper

[5] **Kulakovskaja T. E. (1979)** *The Solution of a Class of Cooperative Four-Person Games with Nonempty Core* Vestnik Leningrad Univ. Math. 19, 42-27.

[6] **Lucas, W. F. (1968)** *A Game with no Solution* Bulletin of the American Math. Society 74, 237-239.

[7] **Neumann, J. von and O. Morgenstern (1944)** *Theory of Games and Economic Behavior*

[8] **Shapley, L.S. (1967)** *On balanced Sets and Cores* Naval Research Logistics Quarterly 14, 453-460.

[9] **Sharkey, W. W. (1982)** *Cooperative games with Large Cores* International Journal of Game Theory 11, 175-182.