Characterization of the Owen set of linear production processes

J.R.G. van Gellekom[1], J.A.M. Potters,[1]
J.H. Reijnierse[1], S.H. Tijs[2] and M.C. Engel

Abstract

In this paper we study linear production processes. The maximal profit that can be made has to be divided among the agents in a 'fair' way. Owen(1975) assigned to every linear production process a cooperative game, a 'linear production game' and introduced a method to find a subset of the core of linear production games, which we call the 'Owen set'. In the first part of the paper we give an axiomatic characterization of the Owen set. In the second part we study the relation between the 'Shuffle' property (one of the axioms) and the core of the corresponding linear production game.

1 Introduction

In this paper we study a model of a production economy, in which the production process is linear and freely accessible for every group of agents. The situation is as follows: there is a finite set R of resources and these resources can be used to produce consumption goods. The set of consumption goods (or products) is denoted by P. The production technologies are given by a production matrix A, where A_{ij} is the amount of resource i necessary to produce one unit of product j. The products can be sold at a fixed market price, given by a vector c. It is assumed that the demand is large enough to sell all produced consumption goods. Then the maximal profit, which can be made from a resource bundle b is equal to the maximum of the following linear program:

$$\begin{aligned} x &\in \mathbb{R}^P_+, \\ Ax &\leq b, \\ \max\, &\langle c, x\rangle, \end{aligned} \tag{1.1}$$

where x_j denotes the amount of product j that is produced.

Further there is a finite set N of agents and each agent owns a bundle of resources. The agents try to maximize their profits. They can work on their own, but they are allowed

[1]Department of Mathematics, University of Nijmegen, Toernooiveld 1, 6525 ED Nijmegen, The Netherlands

[2]Department of Econometrics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands

to cooperate by pooling their resources. Pooling is favorable, because the maximal profit after pooling is always at least as high as the sum of the profits of the agents separately. The reason is that the agents can, when they cooperate, still make the products they could make on their own. Therefore it is assumed that all agents cooperate, yielding a maximal profit, say $v(N)$. The question arises how to divide $v(N)$ among the agents in a 'fair' way. The solution to this problem is not immediately clear, if the resource bundles of various agents are very different.

A possible way to divide $v(N)$ is to assign a TU-game to a linear production process and to apply game-theoretical solution rules, like the core, to allocate $v(N)$. Owen(1975) introduced such 'linear production games' (N, v), where $v(S)$ is the maximal profit that coalition S can obtain. He studied the core of linear production games and showed that it is nonempty. For an arbitrary TU-game it is a lot of work to find a core element, as the core condition has to be checked for every subset of agents. Owen(1975) has found a method to find a nonempty subset of the core of linear production games. We baptized this set the 'Owen set'. The Owen set is useful to find core elements, but it is also a division rule on its own. It gives for every linear production process a set of allocations of $v(N)$.

In this paper we study the Owen set as a division rule on the set of linear production processes (exact definitions are given in the next section). In Section 3 we characterize the Owen set. Section 4 pays special attention to one of the axioms (*Shuffle*) in relation with the core of linear production games.

2 Preliminaries

The linear program (1.1) needs, in general, not to be bounded. Therefore we put some conditions on A, B and c. Before we give a formal definition of a linear production process, we first introduce some notations with respect to matrices and vectors. Let A be an arbitrary matrix. Then

$$\begin{aligned} A_{ij} &:= \text{the element of } A \text{ in the } i\text{-th row and } j\text{-th column,} \\ A_{i\bullet} &:= \text{the } i\text{-th row of } A, \\ A_{\bullet j} &:= \text{the } j\text{-th column of } A, \\ A_{-i\bullet} &:= \text{the restriction of } A \text{ to all rows except the } i\text{-th row,} \\ A_{\bullet -j} &:= \text{the restriction of } A \text{ to all columns except the } j\text{-th column.} \end{aligned}$$

Let M and Q be two finite sets. The set of $M \times Q$-matrices with elements in a given set V is denoted by $\text{Mat}_{M,Q}(V)$.

Definition 2.1: The set of *agents* is denoted by N; the set of *resources* is denoted by R; and the set of *products* is denoted by P. We define $n := |N|$, $r := |R|$ and $p := |P|$. $\diamond$

For a *coalition* $S \subseteq N$ we define the N-dimensional *characteristic vector* e_S by: $(e_S)_k = 1$ if $k \in S$; $(e_S)_k = 0$ otherwise. The $N \times N$-identity matrix is denoted by I_N. We do not use a transpose sign to distinguish between row and column vectors. It will be clear from the context which one is meant.

Definition 2.2: A *linear production process* is a triple (A, B, c) satisfying the following conditions

- $A \in \mathrm{Mat}_{R,P}(\mathbb{R}_+),\ B \in \mathrm{Mat}_{R,N}(\mathbb{R}_+),\ c \in \mathbb{R}^P,\ R, P, N \neq \emptyset$,
- $Be_N > 0$,
- there is at least one product $j \in P$ with $c_j \geq 0$,
- if $c_j > 0$, then there is at least one resource $i \in R$ with $A_{ij} > 0$.

The set of linear production processes is denoted by $\mathcal{L}$. ◇

The interpretation is as follows: A is the *production matrix*: A_{ij} gives the amount of resource i necessary to produce one unit of product j. B gives the resource bundles owned by the agents: B_{ik} is the amount of resource i owned by agent k. c is the *price vector*: c_j is the market price for product j. The condition $Be_N > 0$ says that every resource is owned by at least one agent in a positive quantity. The last condition states that if the market price of a product is strictly positive, then some resources are needed to produce it; otherwise none of the linear programs of type (1.1) is bounded.

For every $(A, B, c) \in \mathcal{L}$ and every coalition S we can compute the maximal profit $v(S) := v^{(A,B,c)}(S)$ that the agents in S can obtain by pooling their resources. $v(S)$ is equal to the maximum of the following linear program:

$$LP(S): \quad \begin{aligned} x &\in \mathbb{R}^P_+, \\ Ax &\leq Be_S, \\ \max\ &\langle c, x\rangle, \end{aligned} \tag{2.1}$$

where x_j is the amount of product j that is produced. This linear program is feasible ($x = 0$ is a feasible vector) and the conditions put on (A, B, c) ensure that it is bounded. We prove this with the help of the dual linear program:

$$LP^*(S): \quad \begin{aligned} y &\in \mathbb{R}^R_+, \\ yA &\geq c, \\ \min\ &\langle y, Be_S\rangle. \end{aligned} \tag{2.2}$$

The following vector is a feasible point of the dual program:

$$y_i := \begin{cases} 0 & \text{if} A_{ij} = 0 \quad \forall j \in P \\ \max_{j \in P: A_{ij} > 0}\{\frac{c_j}{A_{ij}}, 0\} & \text{otherwise.} \end{cases}$$

So both linear programs are feasible. Applying the Duality Theorem of Linear Programming gives that both linear programs are bounded and that the maximum is equal to the minimum. An advantage of using the dual linear program for the computation of $v(S)$ is that the feasible region is the same for all coalitions; only the objective function changes. The pair $(N, v^{(A,B,c)})$ is called a *linear production game*. If there is no danger for confusion, we omit the superscript (A, B, c).

Pooling resources yields a profit $v(N)$, which has to be divided among the agents. We are looking for 'solution rules' as defined by

Definition 2.3: A *solution rule* φ *on* $\mathcal{L}$ is a map, which assigns to every linear production process $(A, B, c) \in \mathcal{L}$ a subset of $\mathbb{R}^N$. ◇

Note that it is allowed that $\varphi(A, B, c) = \emptyset$ for some (or all) $(A, B, c) \in \mathcal{L}$.

A well-known solution rule for TU-games is the *core*, introduced by Gillies(1959): $\text{Core}(N, v) := \{x \in \mathbb{R}^N \mid x(N) = v(N),\ x(S) \geq v(S) \text{ for all } S \subseteq N\}$. We define the core of a linear production process $(A, B, c) \in \mathcal{L}$ by $\text{Core}(A, B, c) := \text{Core}(N, v)$. In this paper we mainly consider the solution rule introduced in Owen(1975). Before we define the 'Owen set', we introduce some more notation.

Definition 2.4: Let $(A, B, c) \in \mathcal{L}$. The *feasible region* of $LP^*(N)$ is denoted by

$$F_{\text{min}}(A, B, c) := \{y \in \mathbb{R}^R_+ \mid yA \geq c\},$$

the *optimal value of* $LP^*(N)$ by

$$v_{\text{min}}(A, B, c) := \min\{\langle y, Be_N \rangle \mid y \in F_{\text{min}}(A, B, c)\}$$

and the set of *optimal solutions of* $LP^*(N)$ by

$$O_{\text{min}}(A, B, c) := \{y \in F_{\text{min}}(A, B, c) \mid \langle y, Be_N \rangle = v_{\text{min}}(A, B, c)\}.$$

◇

Note that the value $v_{\text{min}}(A, B, c)$ is the total profit that the agents in N can make, i.e. $v_{\text{min}}(A, B, c) = v(N)$. The total profit is independent of the division of the resources among the agents. It only depends on the total quantity of resources that are available. In particular, we have $v_{\text{min}}(A, B, c) = v_{\text{min}}(A, Be_N, c)$.

Now we can define the Owen set:

Definition 2.5: Let $(A, B, c) \in \mathcal{L}$. The *Owen set* of (A, B, c) is defined by

$$\text{Owen}(A, B, c) := \{yB \mid y \in O_{\text{min}}(A, B, c)\}.$$

◇

The vectors of $O_{\text{min}}(A, B, c)$ can be seen as shadow prices for the resources. The agents are paid for their resources according to the shadow price vector, which yields an Owen vector. For the special case $B = I_N$, we have $\text{Owen}(A, I_N, c) = O_{\text{min}}(A, I_N, c)$. In this case every agent owns exactly one unit of one resource and every resource is owned by exactly one agent.

Owen(1975) showed the following relation between the Owen set and the core:

Theorem 2.1: *Let* $(A, B, c) \in \mathcal{L}$. *Then*

$$\text{Owen}(A, B, c) \subseteq \text{Core}(A, B, c).$$

Remark 1: Generically the Owen set consists of one point and therefore it is 'often' a proper subset of the core.
Remark 2: As a consequence of Theorem 2.1 linear production games are totally balanced. They are nonnegative as well. The converse can also be shown, i.e. nonnegative totally balanced games are linear production games. In general the linear production process yielding a prescribed nonnegative totally balanced game is not unique. It is even possible that the Owen sets of two linear production processes generating the same linear production game are different. The following example illustrates this phenomenon.

Example 2.1: Consider the following linear production process $(A, B, c) \in \mathcal{L}$:

$$A := \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}, \; B := \begin{pmatrix} 0 & 4 \\ 8 & 3 \end{pmatrix}, \; c := (5, 6, 8).$$

We have $v(1) = 0,\ v(2) = \min\{4y_1 + 3y_2 \mid y_1 + y_2 \geq 5,\ 2y_1 + y_2 \geq 6,\ y_1 + 4y_2 \geq 8\} = 16,\ v(12) = 27$. Furthermore, $O_{\min}(A, B, c) = \{(4, 1)\}$ and $\text{Owen}(A, B, c) = \{(8, 19)\}$. The triple $(A', B', c') \in \mathcal{L}$, where

$$A' := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \; B' := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; c' := (16, 27),$$

gives the same linear production game, but the Owen set is different:
$\text{Owen}(A', B', c') = O_{\min}(A', B', c') = \{(y, 27 - y) \mid 0 \leq y \leq 16\}$. ◦

This means that the Owen set is not a game-theoretical solution, i.e. it is a solution rule on linear production processes, not on linear production games.

Let φ be a solution rule on $\mathcal{L}$. In this paper we characterize the Owen set with the following axioms:

***OnePersonEfficiency*:** $\varphi(A, e_R, c) = \{v_{\min}(A, e_R, c)\}$ for all $(A, e_R, c) \in \mathcal{L}$.

If there is only one agent owning one unit of all resources, then φ assigns to him the maximal profit that can be made from his resource bundle.

***Rescaling*:** $\varphi(DA, DB, c) = \varphi(A, B, c)$ for all *diagonal* matrices $D \in \text{Mat}_{R,R}(\mathbb{R}_+)$ with positive diagonal entries, for all $(A, B, c) \in \mathcal{L}$.

Rescaling means that the solution rule is independent of the units in which the resources are measured. The related property 'independence of changing the units in which the *products* are measured', i.e. $\varphi(AD, B, cD) = \varphi(A, B, c)$ also holds for the Owen set, but it is not needed to characterize the Owen set.

***Shuffle*:** $\varphi(A, BX, c) = \varphi(A, B, c)X$ for all matrices $X \in \text{Mat}_{N,M}(\mathbb{R}_+)$ with $Xe_M = e_N$, for all $(A, B, c) \in \mathcal{L}$, where $\varphi(A, B, c)X := \{yX \mid y \in \varphi(A, B, c)\}$.

This property says that if the resources are shuffled among the agents, then the solution rule changes in the same way. Examples:

Permutation of the bundles of agents 1 and 2:

$$X := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Merging the bundles of agents 1 and 2:

$$X := \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

Splitting and merging the bundles of agent 1 and 2:

$$X := \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Splitting bundles of an agent is always splitting into proportional parts.

***Consistency*:** For all $(A, I_N, c) \in \mathcal{L}$ with $n \geq 2$ and for all $y \in \varphi(A, I_N, c)$: $(A_{-i\bullet}, I_{N\setminus i}, \tilde{c}) \in \mathcal{L}$ and $y_{-i} \in \varphi(A_{-i\bullet}, I_{N\setminus i}, \tilde{c})$ for all $i \in N$, where $\tilde{c}_j := c_j - y_i A_{ij}$ for all $j \in P$.

This property has to do with the special case that every agent owns exactly one unit of exactly one resource and different agents own different resources. Suppose that the agents agree that the profit is divided according to a vector $y \in \varphi(A, I_N, c)$. Agent i takes y_i and leaves. His resource can be used by the other agents for a price of y_i per unit. This is the same as to say that the profit of a product decreases with y_i for every unit needed of this resource. A solution rule satisfies *Consistency* if the restriction of y to the remaining agents is a solution to the reduced linear production process.

***Deletion*:** For all $(A, I_N, c) \in \mathcal{L}$ and for all $J \subset P$: if $v_{\min}(A_{\bullet -J}, I_N, c_{-J}) = v_{\min}(A, I_N, c)$ then $\varphi(A, I_N, c) \subseteq \varphi(A_{\bullet -J}, I_N, c_{-J})$.

Deletion says that if a production technology is not needed to make maximal profit $v(N)$ we can delete these technologies. The outcomes of the old situation are also outcomes in the new situation.

It will appear that the five axioms imply the following axioms, which are useful in the proofs we give.

***Nonemptiness*:** $\varphi(A, B, c) \neq \emptyset$ for all $(A, B, c) \in \mathcal{L}$.

***Efficiency*:** $\langle y, e_N \rangle = v_{\min}(A, B, c)$ for all $y \in \varphi(A, B, c)$, for all $(A, B, c) \in \mathcal{L}$.

The agents can make a maximal total profit $v(N) = v_{\min}(A, B, c)$. Efficient solution rules divide exactly this amount among the agents.

3 Characterization of the Owen set on $\mathcal{L}$

Theorem 3.1 is the main theorem of this paper. It characterizes the Owen set on $\mathcal{L}$.

Theorem 3.1: *If φ is a solution rule on $\mathcal{L}$ then φ satisfies OnePersonEfficiency, Rescaling, Shuffle, Consistency and Deletion if and only if $\varphi(A,B,c) = \mathrm{Owen}(A,B,c)$ for all $(A,B,c) \in \mathcal{L}$.*

The theorem will follow from a series of lemmas and propositions. Proposition 3.2 shows that the Owen set satisfies the five axioms. Lemma 3.3 - Lemma 3.6 show that the Owen set is the unique solution rule satisfying the five axioms.

Proposition 3.2: *The Owen set satisfies*

a) *OnePersonEfficiency*

b) *Rescaling*

c) *Shuffle*

d) *Consistency*

e) *Deletion*

Proof:

a) *OnePersonEfficiency*: Let $(A, e_R, c) \in \mathcal{L}$. Then

$$\mathrm{Owen}(A, e_R, c) = \{ye_R \mid y \in O_{\min}(A, e_R, c)\} = \{v_{\min}(A, e_R, c)\}.$$

b) *Rescaling*: Let $(A,B,c) \in \mathcal{L}$ and let $D \in \mathrm{Mat}_{R,R}(\mathbb{R}_+)$ be a diagonal matrix with positive diagonal entries. First note that

$$\begin{aligned} v_{\min}(DA, DB, c) &= \min\{\langle y, DBe_N\rangle \mid y \in \mathbb{R}^R_+,\ yDA \geq c\} \\ &= \min\{\langle D^t y, Be_N\rangle \mid D^t y \in \mathbb{R}^R_+,\ (D^t y)^t A \geq c\} \\ &= \min\{\langle z, Be_N\rangle \mid z \in \mathbb{R}^R_+,\ zA \geq c\} \\ &= v_{\min}(A,B,c), \end{aligned}$$

where the third equality holds because D is a nonnegative invertible diagonal matrix. Using this again (the fourth equality) we have

$$\begin{aligned} \mathrm{Owen}(DA, DB, c) &= \{yDB \mid y \in O_{\min}(DA, DB, c)\} \\ &= \{yDB \mid y \in \mathbb{R}^R_+,\ yDA \geq c,\ \langle y, DBe_N\rangle = v_{\min}(DA, DB, c)\} \\ &= \{(D^t y)^t B \mid D^t y \in \mathbb{R}^R_+,\ (D^t y)^t A \geq c,\ \langle D^t y, Be_N\rangle = v_{\min}(DA, DB, c)\} \\ &= \{zB \mid z \in \mathbb{R}^R_+,\ zA \geq c,\ \langle z, Be_N\rangle = v_{\min}(A,B,c)\} \\ &= \{zB \mid z \in O_{\min}(A,B,c)\} \\ &= \mathrm{Owen}(A,B,c). \end{aligned}$$

c) *Shuffle*: Let $(A,B,c) \in \mathcal{L}$ and $X \in \mathrm{Mat}_{N,M}(\mathbb{R}_+),\ Xe_M = e_N$. The row sums of matrix X are equal to one, which implies $v_{\min}(A,BX,c) = v_{\min}(A,B,c)$. Then

$$\begin{aligned}
\mathrm{Owen}(A,BX,c) &= \{yBX \mid y \in O_{\min}(A,BX,c)\} \\
&= \{yBX \mid y \in \mathbb{R}^R_+,\ yA \geq c,\ \langle y, BXe_M\rangle = v_{\min}(A,BX,c)\} \\
&= \{yBX \mid y \in \mathbb{R}^R_+,\ yA \geq c,\ \langle y, Be_N\rangle = v_{\min}(A,B,c)\} \\
&= \{yB \mid y \in \mathbb{R}^R_+,\ yA \geq c,\ \langle y, Be_N\rangle = v_{\min}(A,B,c)\}X \\
&= \mathrm{Owen}(A,B,c)X.
\end{aligned}$$

d) *Consistency*: Let $(A, I_N, c) \in \mathcal{L},\ n \geq 2,\ y \in \mathrm{Owen}(A, I_N, c),\ i \in N$. We have to show that $(A_{-i\bullet}, I_{N\backslash i}, \tilde{c}) \in \mathcal{L}$ and $y_{-i} \in \mathrm{Owen}(A_{-i\bullet}, I_{N\backslash i}, \tilde{c})$ where $\tilde{c}_j := c_j - y_iA_{ij}$ for all $j \in P$. We first show that $(A_{-i\bullet}, I_{N\backslash i}, \tilde{c}) \in \mathcal{L}$. Therefore we have to prove that there is at least one product j^* with $\tilde{c}_{j^*} \geq 0$ and if $\tilde{c}_j > 0$ for some product j then there is at least one resource l with $A_{lj} > 0$. From $y \in \mathrm{Owen}(A, I_N, c) = O_{\min}(A, I_N, c)$ it follows that

$$\begin{aligned}
y &\in \mathbb{R}^R_+, \\
yA &\geq c, \\
\langle y, e_N\rangle &= v_{\min}(A, I_N, c).
\end{aligned} \tag{3.1}$$

If $\tilde{c}_j < 0$ for all products j then $c_j < y_iA_{ij}$ for all $j \in P$. $(A, I_N, c) \in \mathcal{L}$ implies that there is at least one product j^* with $c_{j^*} \geq 0$. Then $0 \leq c_{j^*} < y_iA_{ij^*}$, which implies that $y_i > 0$. For small $\varepsilon > 0$ we have $(1-\varepsilon)y_iA_{ij} > c_j$, i.e. $(1-\varepsilon)y_ie_i \in F_{\min}(A, I_N, c)$ for all $j \in P$. Then $v_{\min}(A, I_N, c) \leq (1-\varepsilon)y_i < y_i \leq \langle y, e_N\rangle = v_{\min}(A, I_N, c)$. Contradiction. So there is at least one product j^* with $\tilde{c}_{j^*} \geq 0$. Suppose that $\tilde{c}_j > 0$ for some product j. Then $\sum_{l=1}^r y_lA_{lj} \geq c_j > y_iA_{ij}$. So there is a resource $l \in R\backslash i$ with $A_{lj} > 0$. Then $(A_{-i\bullet})_{lj} = A_{lj} > 0$. This proves that $(A_{-i\bullet}, I_{N\backslash i}, \tilde{c}) \in \mathcal{L}$.

Next we show that $y_{-i} \in \mathrm{Owen}(A_{-i\bullet}, I_{N\backslash i}, \tilde{c})$, i.e.

$$\begin{aligned}
y_{-i} &\in \mathbb{R}^{R\backslash i}_+, \\
y_{-i}A_{-i\bullet} &\geq \tilde{c}, \\
\langle y_{-i}, e_{N\backslash i}\rangle &= v_{\min}(A_{-i\bullet}, I_{N\backslash i}, \tilde{c}).
\end{aligned} \tag{3.2}$$

$y_{-i} \in \mathbb{R}^{R\backslash i}_+$ follows from $y \in \mathbb{R}^R_+$. For all $j \in P$ we have $\sum_{l=1}^r y_lA_{lj} \geq c_j$ which implies $y_{-i}A_{-i\bullet}e_j = \sum_{l\neq i} y_lA_{lj} \geq c_j - y_iA_{ij} = \tilde{c}_j$, i.e. $y_{-i}A_{-i\bullet} \geq \tilde{c}$. Finally we show that $\langle y_{-i}, e_{N\backslash i}\rangle = v_{\min}(A_{-i\bullet}, I_{N\backslash i}, \tilde{c})$. From $y_{-i} \in F_{\min}(A_{-i\bullet}, I_{N\backslash i}, \tilde{c})$, it follows that $v_{\min}(A_{-i\bullet}, I_{N\backslash i}, \tilde{c}) \leq \langle y_{-i}, e_{N\backslash i}\rangle$. Suppose that $v_{\min}(A_{-i\bullet}, I_{N\backslash i}, \tilde{c}) < \langle y_{-i}, e_{N\backslash i}\rangle$. Choose $z \in F_{\min}(A_{-i\bullet}, I_{N\backslash i}, \tilde{c})$ such that $v_{\min}(A_{-i\bullet}, I_{N\backslash i}, \tilde{c}) = \langle z, e_{N\backslash i}\rangle$. Then $z \geq 0$ and $\sum_{l\neq i} z_lA_{lj} \geq c_j - y_iA_{ij}$, i.e. $(z, y_i) \in F_{\min}(A, I_N, c)$ and $\langle (z, y_i), e_N\rangle < \langle y, e_N\rangle$. This contradicts the fact that $y \in O_{\min}(A, I_N, c)$. So we have $v_{\min}(A_{-i\bullet}, I_{N\backslash i}, \tilde{c}) = \langle y_{-i}, e_{N\backslash i}\rangle$.

e) *Deletion*: Let $(A, I_N, c) \in \mathcal{L}$ and suppose that there exists a proper subset of products $J \subset P$ such that $v_{\min}(A_{\bullet -J}, I_N, c_{-J}) = v_{\min}(A, I_N, c)$. We have to show that

$$\text{Owen}(A, I_N, c) \subseteq \text{Owen}(A_{\bullet -J}, I_N, c_{-J}).$$

Take $y \in \text{Owen}(A, I_N, c)$. Then

$$\begin{aligned} y &\in \mathbb{R}_+^R, \\ yA &\geq c, \\ \langle y, e_N \rangle &= v_{\min}(A, I_N, c), \end{aligned} \tag{3.3}$$

which implies that y also satisfies

$$\begin{aligned} y &\in \mathbb{R}_+^R, \\ yA_{\bullet -J} &\geq c_{-J}, \\ \langle y, e_N \rangle &= v_{\min}(A_{\bullet -J}, I_N, c_{-J}), \end{aligned} \tag{3.4}$$

i.e. $y \in \text{Owen}(A_{\bullet -J}, I_N, c_{-J})$.

□

We shall use the following $R \times R$ diagonal matrix $\hat{D}$ several times in the proofs in this section (in combination with *Rescaling*):

$$\begin{aligned} \hat{D}_{ii} &= (e_i B e_N)^{-1} \quad \text{for all} \quad i \in R, \\ \hat{D}_{ij} &= 0 \qquad\qquad\quad \text{otherwise.} \end{aligned}$$

Note that $\hat{D}B$ is a matrix with all row sums equal to 1: $\hat{D}Be_N = e_R$.

The following lemma shows that a solution rule satisfies *Nonemptiness* and *Efficiency* if it satisfies *OnePersonEfficiency*, *Rescaling* and *Shuffle*. The properties *Nonemptiness* and *Efficiency* appear to be useful in the proofs of the lemmas.

Lemma 3.3: *If φ satisfies OnePersonEfficiency, Rescaling and Shuffle then φ satisfies Nonemptiness and Efficiency.*

Proof: Suppose that φ satisfies *OnePersonEfficiency*, *Rescaling* and *Shuffle* and take $(A, B, c) \in \mathcal{L}$. By *Rescaling* we have $\varphi(\hat{D}A, \hat{D}B, c) = \varphi(A, B, c)$. Applying *Shuffle* with $X := e_N$ gives $\varphi(\hat{D}A, \hat{D}Be_N, c) = \varphi(\hat{D}A, \hat{D}B, c)e_N$. Note that $(\hat{D}A, \hat{D}Be_N, c) \in \mathcal{L}$ is a linear production process with one agent, which possesses exactly one unit of each resource: $\hat{D}Be_N = e_R$. Applying *OnePersonEfficiency* gives $\varphi(A, B, c)e_N = \varphi(\hat{D}A, \hat{D}B, c)e_N = \varphi(\hat{D}A, \hat{D}Be_N, c) = \varphi(\hat{D}A, e_R, c) = \{v_{\min}(\hat{D}A, e_R, c)\}$, from which we get $\varphi(A, B, c) \neq \emptyset$. To prove that φ satisfies *Efficiency* it is sufficient to show that $v_{\min}(DA, e_R, c) = v_{\min}(A, B, c)$. This follows from

$$v_{\min}(\hat{D}A, e_R, c) = v_{\min}(A, (\hat{D})^{-1}e_R, c) = v_{\min}(A, Be_N, c) = v_{\min}(A, B, c),$$

where the first equality holds because the feasible region of $LP(N)$ does not change by rescaling, hence the optimal value of $LP^*(N)$, which is equal to the optimal value of $LP(N)$, does not change. □

To characterize the Owen set, we first consider linear production processes where $B = I_N$ (Lemmas 3.4 and 3.5). Lemma 3.6 gives a characterization of the Owen set on $\mathcal{L}$.

Lemma 3.4: *If φ satisfies OnePersonEfficiency, Rescaling, Shuffle and Consistency then $\varphi(A, I_N, c) \subseteq \mathrm{Owen}(A, I_N, c)$ for all $(A, I_N, c) \in \mathcal{L}$.*

Proof: The proof is by induction to n, the number of agents. The case $n = 1$ follows from Lemma 3.3: $\varphi(A, I_{\{1\}}, c) = \{v_{\min}(A, I_{\{1\}}, c)\} = \mathrm{Owen}(A, I_{\{1\}}, c)$.

Suppose that the lemma has been proved if the number of agents is less than $n \geq 2$. Take $(A, I_N, c) \in \mathcal{L}$ and $y \in \varphi(A, I_N, c) \neq \emptyset$ by Lemma 3.3. Applying this lemma again gives $\langle y, e_N \rangle = v_{\min}(A, I_N, c) \geq 0$. So there is an agent i with $y_i \geq 0$. *Consistency* w.r.t. this agent and the induction hypothesis give

$$y_{-i} \in \varphi(A_{-i\bullet}, I_{N\backslash i}, \tilde{c}) \subseteq \mathrm{Owen}(A_{-i\bullet}, I_{N\backslash i}, \tilde{c}),$$

where $\tilde{c}_j = c_j - y_i A_{ij}$ for all j. In particular $y_{-i} \geq 0$, hence $y \geq 0$. Further $y_{-i}A_{-i\bullet} \geq \tilde{c}_j = c_j - y_i A_{ij}$, i.e. $yA_{\bullet j} \geq c_j$ for all j. Summarizing we have

$$\begin{aligned} y &\in \mathbb{R}^R_+, \\ yA &\geq c, \\ \langle y, e_N \rangle &= v_{\min}(A, I_N, c), \end{aligned} \tag{3.5}$$

i.e. $y \in O_{\min}(A, I_N, c) = \mathrm{Owen}(A, I_N, c)$ as was to be shown. □

Lemma 3.5: *If φ satisfies OnePersonEfficiency, Rescaling, Shuffle, Consistency and Deletion then $\mathrm{Owen}(A, I_N, c) \subseteq \varphi(A, I_N, c)$ for all $(A, I_N, c) \in \mathcal{L}$.*

Proof: Let $(A, I_N, c) \in \mathcal{L}$ and take $y \in \mathrm{Owen}(A, I_N, c)$. Define $\overline{A} := [A\, I_N]$, $\overline{c} := [c\, y]$. Then $\mathrm{Owen}(\overline{A}, I_N, \overline{c}) = \{\overline{y} \in \mathbb{R}^N_+ \mid \overline{y}A \geq c,\ \overline{y} \geq y,\ \langle \overline{y}, e_N \rangle = v_{\min}(\overline{A}, I_N, \overline{c})\} = \{y\}$. From Lemma 3.4 we get $\emptyset \neq \varphi(\overline{A}, I_N, \overline{c}) \subseteq \mathrm{Owen}(\overline{A}, I_N, \overline{c}) = \{y\}$. Hence $\varphi(\overline{A}, I_N, \overline{c}) = \{y\}$. As $v_{\min}(\overline{A}, I_N, \overline{c}) = v_{\min}(A, I_N, c)$, we can apply *Deletion* which gives $\varphi(\overline{A}, I_N, \overline{c}) \subseteq \varphi(A, I_N, c)$. So $y \in \varphi(A, I_N, c)$. □

Lemma 3.6: *If φ satisfies OnePersonEfficiency, Rescaling, Shuffle, Consistency and Deletion then $\varphi(A, B, c) = \mathrm{Owen}(A, B, c)$ for all $(A, B, c) \in \mathcal{L}$.*

Proof: Take $(A, B, c) \in \mathcal{L}$. Then we have

$$\begin{aligned} \varphi(A, B, c) &= \varphi(\hat{D}A, \hat{D}B, c) \qquad (\textit{Rescaling}) \\ &= \varphi(\hat{D}A, I_N, c)\hat{D}B \qquad (\textit{Shuffle}) \\ &= \mathrm{Owen}(\hat{D}A, I_N, c)\hat{D}B \qquad (\text{Lemmas 3.4 and 3.5}) \\ &= \mathrm{Owen}(\hat{D}A, \hat{D}B, c) \qquad (\text{Prop. 3.2c}) \\ &= \mathrm{Owen}(A, B, c) \qquad (\text{Prop. 3.2b}). \end{aligned}$$

□

The following five examples show that the axioms are logically independent.

Example 3.1: $\varphi(A, B, c) := \emptyset$ for all $(A, B, c) \in \mathcal{L}$. φ satisfies *Rescaling*, *Shuffle*, *Consistency* and *Deletion*, but not *OnePersonEfficiency*. ∘

Example 3.2: Take $(A, B, c) \in \mathcal{L}$ and let $\hat{D}$ be the (unique) diagonal matrix such that $\hat{D}Be_N = e_R$ (as before). Define the solution rule φ by $\varphi(A, B, c) = \text{Owen}(A, \hat{D}B, c)$. φ satisfies *OnePersonEfficiency*, *Shuffle*, *Consistency* and *Deletion*, but not *Rescaling*. ∘

Example 3.3: For all $(A, B, c) \in \mathcal{L}$:

$$\varphi(A, B, c) := \begin{cases} \text{Owen}(A, B, c) & \text{if } n = 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

φ satisfies *OnePersonEfficiency*, *Rescaling*, (trivially) *Consistency* and *Deletion*, but not *Shuffle*. ∘

Example 3.4: The definition of the solution rule to show that *Consistency* is logically independent of the other four axioms is more complicated. The idea is to construct a solution rule, which equals the Owen set for 'a lot of' linear production processes, but which is a proper subset of the Owen set if

$$A' = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, B' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, c' = (1).$$

Then we hope that φ does not satisfy *Consistency* applied on a linear production process which yields this A', B' and c'. The Owen set of this triple is equal to the set $\{(t, 1-t) \mid 0 \leq t \leq 1\}$. We choose $\varphi(A', B', c') = \{(1, 0)\}$. This choice determines φ for a whole class of linear production processes, if we want that *OnePersonEfficiency*, *Rescaling*, *Shuffle* and *Deletion* are satisfied. It appears that this class is equal to the following subset of $\mathcal{L}$:

$$\begin{aligned} \mathcal{L}' := \{(A, B, c) \in \mathcal{L} \mid \quad & r = 2,\ \exists d_1, d_2 > 0\ \exists j^* : A_{ij^*} = d_i\ (i = 1, 2) \quad c_{j^*} = 1, \\ & e_i B e_N = d_i, \quad A_{ij} \geq d_i c_j\ \forall j \neq j^*\ (i = 1, 2)\}. \end{aligned}$$

$\mathcal{L}'$ consists of all linear production processes which can be obtained from (A', B', c') or yield (A', B', c') after applying *Rescaling*, *Shuffle* or *Deletion*. We define

$$\varphi(A, B, c) := \begin{cases} \{\frac{1}{e_1 B e_N}(B_{11}, \ldots, B_{1n})\} & \text{if } (A, B, c) \in \mathcal{L}', \\ \text{Owen}(A, B, c) & \text{otherwise.} \end{cases}$$

The proof that φ satisfies *OnePersonEfficiency*, *Rescaling*, *Shuffle* and *Deletion* is straightforward. φ does not satisfy *Consistency* by Theorem 3.1. ∘

Example 3.5: Define $\varphi(A, B, c) := \mathit{Extr}(O_{\min}(A, B, c))B$ for all $(A, B, c) \in \mathcal{L}$, where *Extr* is the set of extreme points. φ satisfies *OnePersonEfficiency*, *Rescaling*, *Shuffle* and *Consistency*, but not *Deletion*. ∘

4 The *Shuffle* property

For an arbitrary linear production process $(A,B,c) \in \mathcal{L}$ and matrix $X \in \text{Mat}_{N,M}(\mathbb{R}_+)$ with $Xe_M = e_N$, the equality '$\text{Core}(A,BX,c) = \text{Core}(A,B,c)X$' does not hold. Hence the core does not satisfy the *Shuffle* property. In this section we shall show that the equality is satisfied for all $X \in \text{Mat}_{N,M}(\mathbb{R}_+)$ with $Xe_M = e_N$ (only) if the linear production game (N,v) corresponding to (A,B,c) is an additive game, i.e. $v(S) = \sum_{k\in S} v(k)$ for all $S \subseteq N$. If we only consider matrices X with $X_{kl} \in \{0,1\}$ for all k,l (only permuting and merging bundles) then the equation is satisfied (only) if (N,v) is convex, i.e. for all $S,T \subseteq N$ $v(S)+v(T) \leq v(S\cup T)+v(S\cap T)$. These observations show that the property *Shuffle* is an important axiom in the characterization of the Owen set.

Theorem 4.1: *Let $(A,B,c) \in \mathcal{L}$. Then*

$$\text{Core}(A,BX,c) = \text{Core}(A,B,c)X \textit{ for all } X \in \text{Mat}_{N,M}(\mathbb{R}_+) \textit{ with } Xe_M = e_N$$
$$\Updownarrow$$
$$(N, v^{(A,B,c)}) \textit{ is additive}$$

Theorem 4.2: *Let $(A,B,c) \in \mathcal{L}$. Then*

$$\text{Core}(A,BX,c) = \text{Core}(A,B,c)X \textit{ for all } X \in \text{Mat}_{N,M}(\{0,1\}) \textit{ with } Xe_M = e_N$$
$$\Updownarrow$$
$$(N, v^{(A,B,c)}) \textit{ is convex}$$

Before we prove the theorems, we shall first repeat some definitions and theorems w.r.t. convex and additive games. For additive games we have

Proposition 4.3: *Let (N,v) be a superadditive game. Then*

$$(N,v) \textit{ is additive} \iff v(N) = \sum_{k\in N} v(k).$$

The proof is straightforward.

Definition 4.1: [Weber(1988)] Let (N,v) be a TU-game and let $\sigma : N \to N$ be a permutation of N. The *marginal vector* m^σ is defined by

$$\begin{aligned} m^\sigma_{\sigma(1)} &:= v(\sigma(1)) \\ m^\sigma_{\sigma(k)} &:= v(\sigma(1),\ldots,\sigma(k)) - v(\sigma(1),\ldots,\sigma(k-1)) \quad \forall k \in N\backslash 1. \end{aligned}$$

The *Weber set* $W(N,v)$ is the convex hull of the $n!$ marginal vectors. $\diamond$

We need the following characterization of convex games:

Theorem 4.4: [Shapley(1971)/Ichiishi(1981)/Derks(1992)]
For every TU-game (N,v) we have

$$(N,v) \textit{ is convex} \iff W(N,v) = \text{Core}(N,v).$$

Definition 4.2: [Schmeidler (1972)] A TU-game is called *exact* if for all $\emptyset \neq S \subset N$ there exists a vector $x \in \text{Core}(N, v)$ such that $x(S) = v(S)$. $\diamond$

Well-known is the following relation between convex games and exact games:

Proposition 4.5: *Every convex game is exact. If $n \leq 3$ then convexity and exactness are equivalent.*

Finally we make some remarks w.r.t. the matrices X:

Let $X \in \text{Mat}_{N,M}(\{0,1\})$ with $Xe_M = e_N$. The columns of X can be seen as characteristic vectors of subsets $S_1, \ldots, S_t$ of N. As $Xe_M = e_N$ these subsets form a 'partition' of N, where $S_l = \emptyset$ is allowed. The linear production game $v^{(A,BX,c)}$ is the restriction of $v^{(A,B,c)}$ to the sets $S_1, \ldots, S_t$ and their unions. From this we get immediately that for all matrices $X \in \text{Mat}_{N,M}(\{0,1\})$ with $Xe_M = e_N$ and all linear production processes $(A, B, c) \in \mathcal{L}$: $\text{Core}(A, B, c)X \subseteq \text{Core}(A, BX, c)$. If $X \in \text{Mat}_{N,M}(\{0,1\})$ with $Xe_M = e_N$ and $Y \in \text{Mat}_{M,Q}(\{0,1\})$ with $Ye_Q = e_M$, then $XYe_Q = Xe_M = e_N$. This together with the fact that the entries of XY are nonnegative integers gives $XY \in \text{Mat}_{N,Q}(\{0,1\})$.

In the proofs of the two theorems we shall use the following notations: (N, v) is the linear production game corresponding to (A, B, c); $(\overline{N}, \overline{v})$ is the linear production game corresponding to (A, BX, c), where X is given; we write $\overline{N} = \{\overline{1}, \ldots, \overline{n}\}$, where $\overline{n} := |\overline{N}|$.

Proof of Theorem 4.2:

$\Downarrow$: Suppose that the equation $\text{Core}(A, BX, c) = \text{Core}(A, B, c)X$ holds for all matrices $X \in \text{Mat}_{N,M}(\{0,1\})$ with $Xe_M = e_N$. We first show that (N, v) is exact. Let $\emptyset \neq S \subset N$. We have to prove that there exists a core element $y \in \text{Core}(A, B, c)$ such that $y(S) = v(S)$. Define the $n \times 2$ matrix X by

$$X := \begin{bmatrix} e_S & e_{N \setminus S} \end{bmatrix}.$$

Then $X_{kl} \in \{0,1\}$ for all k, l and $Xe_{\overline{N}} = e_N$ so $\text{Core}(A, BX, c) = \text{Core}(A, B, c)X$. In (A, BX, c) the bundles of the agents in S are merged to the bundle of agent $\overline{1}$ and the bundles of the agents in $N \setminus S$ to the bundle of agent $\overline{2}$. $(\overline{N}, \overline{v})$ is a linear production game and, hence, the core is nonempty. So $\overline{v}(\overline{1}) + \overline{v}(\overline{2}) \leq \overline{v}(\overline{N})$, which implies that $z := (\overline{v}(\overline{1}), \overline{v}(\overline{N}) - \overline{v}(\overline{1})) \in \text{Core}(A, BX, c) = \text{Core}(A, B, c)X$. Thus there exists a vector $y \in \text{Core}(A, B, c)$ such that $z = yX$. In particular $y(S) = yXe_{\overline{1}} = ze_{\overline{1}} = \overline{v}(\overline{1}) = v(S)$. Conclusion: (N, v) is an exact game.

Because exactness and convexity are equivalent for $n \leq 3$ (Proposition 4.5) we assume that $n \geq 4$ and show that $v(U) + v(W) \leq v(U \cup W) + v(U \cap W)$ for all $U, W \subseteq N$. Let $U, W \subseteq N$ and define $S := U \cup W$, $T := U \cap W$. Consider $(\overline{N}, \overline{v})$ for

$$X := \begin{bmatrix} e_T & e_{S \setminus T} & e_{N \setminus S} \end{bmatrix}.$$

The number of agents is equal to three and $\text{Core}(A, BXY, c) = \text{Core}(A, B, c)XY = \text{Core}(A, BX, c)Y$ for all $Y \in \text{Mat}_{\{1,2,3\},Q}(\{0,1\})$ with $Ye_Q = e_M$, so applying the first part of the proof gives that $(\overline{N},\overline{v})$ is convex. For convex games we know that all marginal vectors $\overline{m}^\sigma$ are in the core (Theorem 4.4). In particular for σ where the three agents of $\overline{N}$ 'enter' in the order $\overline{1},\overline{2},\overline{3}$. $\overline{m}^\sigma \in \text{Core}(A,BX,c) = \text{Core}(A,B,c)X$ implies that there exists a vector $z \in \text{Core}(A,B,c)$ such that

$$(\overline{m}^\sigma_{\overline{1}},\overline{m}^\sigma_{\overline{2}},\overline{m}^\sigma_{\overline{3}}) = zX.$$

In particular $z(T) = \overline{m}^\sigma_{\overline{1}} = \overline{v}(\overline{1}) = v(T)$, $z(S\backslash T) = \overline{m}^\sigma_{\overline{2}} = \overline{v}(\overline{1},\overline{2}) - \overline{v}(\overline{1}) = v(S) - v(T)$ and $z(S) = z(T)+z(S\backslash T) = v(S)$. Then $v(U)+v(W) \leq z(U)+z(W) = z(U\cup W)+z(U\cap W) = v(U\cup W) + v(U\cap W)$.

$\Uparrow$: Conversely suppose that $(A,B,c) \in \mathcal{L}$ and that the game (N,v) is convex. Let $X \in \text{Mat}_{N,\overline{N}}(\{0,1\})$ with $Xe_{\overline{N}} = e_N$. Then

$$X = [e_{S_1} \quad e_{S_2} \quad \dots \quad e_{S_t}],$$

for some subsets $S_1,\dots,S_t$ of N.

As we have already seen that $\text{Core}(A,B,c)X \subseteq \text{Core}(A,BX,c)$, it is sufficient to show that $\text{Core}(A,BX,c) \subseteq \text{Core}(A,B,c)X$. $(\overline{N},\overline{v})$ is convex (it is a restriction of v), which implies that the Weber set equals $\text{Core}(A,BX,c)$ (Theorem 4.4). We are done if we prove that all marginal vectors $\overline{m}^\sigma$ are an element of $\text{Core}(A,B,c)X$, which is a convex set. Let $\overline{m}^\sigma$ be a marginal vector of the game $(\overline{N},\overline{v})$; w.l.o.g. we assume that $\sigma(\overline{l}) = S_{\overline{l}}$ for all $\overline{l}$. Then

$$\overline{m}^\sigma_{\overline{l}} = \overline{v}(\overline{1},\dots,\overline{l}) - \overline{v}(\overline{1},\dots,\overline{l-1}) = v(S_1\cup\dots\cup S_l) - v(S_1\cup\dots\cup S_{l-1}).$$

Let us go back to (N,v). Let $\tau : N \to N$ be a ranking such that first the agents in S_1 enter (in arbitrary order), then the agents in S_2, …, and finally the agents in S_m. (N,v) is convex, so $m^\tau \in \text{Core}(A,B,c)$ and

$$(m^\tau X)_l = m^\tau(S_l) = v(S_1\cup\dots\cup S_l) - v(S_1\cup\dots\cup S_{l-1}) = \overline{m}^\sigma_l.$$

So $\overline{m}^\sigma = m^\tau X \in \text{Core}(A,B,c)X$. □

Next we prove that the core satisfies the property *Shuffle* if and only if the corresponding game is additive (Theorem 4.1).

Proof: $\Downarrow$: Suppose that

$$\text{Core}(A,BX,c) = \text{Core}(A,B,c)X \text{ for all } X \in \text{Mat}_{N,M}(\mathbb{R}_+) \text{ with } Xe_M = e_N. \quad (4.1)$$

It is sufficient to prove that $v(S\cup k) = v(S) + v(k)$ for all $k \in N$ and all $S \subseteq N\backslash k$. We shall show this for the first agent; the proof for other agents is similar. Consider the situation where the bundle of the first agent is split into two equal bundles, i.e. consider the following matrix

$$X := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix},$$

where the remaining entries are 0. We write $\overline{N} := \{\overline{1}, \overline{\overline{1}}, \overline{2}, \ldots, \overline{n}\}$.

Take $\overline{y} \in \text{Core}(A, BX, c)$. By Formula (4.1) we have $\overline{y} = yX$ for some $y \in \text{Core}(A, B, c)$. In particular $\overline{y}_{\overline{1}} = \frac{1}{2}y_1 = \overline{y}_{\overline{\overline{1}}}$. So $\overline{y}_{\overline{1}} = \overline{y}_{\overline{\overline{1}}}$ for all $\overline{y} \in \text{Core}(A, BX, c)$. Formula (4.1) also implies that $\text{Core}(A, BXY, c) = \text{Core}(A, BX, c)Y$ for all $Y \in \text{Mat}_{\overline{N},M}(\{0,1\})$ with $Ye_M = e_{\overline{N}}$. Applying Theorem 4.2 gives that $(\overline{N}, \overline{v})$ is a convex game. Hence $\text{Core}(A, BX, c)$ is equal to the Weber set.

Let $S \subseteq \overline{N} \backslash \{\overline{1}, \overline{\overline{1}}\}$. We consider two marginal vectors $\overline{m}^\sigma$ and $\overline{m}^\tau$. First consider the marginal vector corresponding to the following permutation σ of $\overline{N}$: the agents enter in the order S, $\overline{1}$, $\overline{\overline{1}}$, $\overline{N} \backslash (S \cup \overline{1} \cup \overline{\overline{1}})$. We have

$$\overline{v}(S \cup \overline{1}) - \overline{v}(S) = \overline{m}^\sigma_{\overline{1}} = \overline{m}^\sigma_{\overline{\overline{1}}} = \overline{v}(S \cup \overline{1} \cup \overline{\overline{1}}) - \overline{v}(S \cup \overline{1}),$$

from which it follows that

$$\overline{v}(S \cup \overline{1}) = \frac{1}{2}(\overline{v}(S \cup \overline{1} \cup \overline{\overline{1}}) + \overline{v}(S)) = \frac{1}{2}(v(S \cup 1) + v(S)).$$

Next consider τ : the agents enter in the order $\overline{1}$, S, $\overline{\overline{1}}$, $\overline{N} \backslash (S \cup \overline{1} \cup \overline{\overline{1}})$. Then

$$\overline{v}(\overline{1}) = \overline{m}^\tau_{\overline{1}} = \overline{m}^\tau_{\overline{\overline{1}}} = \overline{v}(S \cup \overline{1} \cup \overline{\overline{1}}) - \overline{v}(S \cup \overline{1}),$$

hence

$$v(S \cup 1) = \overline{v}(S \cup \overline{1} \cup \overline{\overline{1}}) = \overline{v}(\overline{1}) + \overline{v}(S \cup \overline{1}) = \frac{1}{2}v(1) + \frac{1}{2}v(S \cup 1) + \frac{1}{2}v(S),$$

i.e. $v(S \cup 1) = v(S) + v(1)$ for all $S \subseteq N \backslash 1$ as was to be shown.

$\Uparrow$: Let $X \in \text{Mat}_{N,\overline{N}}(\mathbb{R}_+)$ with $Xe_{\overline{N}} = e_N$. We shall prove that $(\overline{N}, \overline{v})$ is also an additive game. In the proof we need that every optimal solution of $LP^*(N)$ is also an optimal solution of $LP^*(\{k\})$ for all $k \in N$. We prove this first. Let $\hat{y}$ be an optimal solution of $LP^*(N)$. Then

$$v(N) = \langle \hat{y}, Be_N \rangle = \langle \hat{y}, \sum_{k \in N} Be_k \rangle = \sum_{k \in N} \langle \hat{y}, Be_k \rangle \geq \sum_{k \in N} v(k) = v(N).$$

So $v(k) = \langle \hat{y}, Be_k \rangle$ for all $k \in N$. Let $\overline{k} \in \overline{N}$. As the bundle $BXe_{\overline{k}}$ of this agent is a nonnegative linear combination of the bundles Be_k $(k \in N)$ we also have $\overline{v}(\overline{k}) = \langle \hat{y}, BXe_{\overline{k}} \rangle$. Then

$$\sum_{\overline{k} \in \overline{N}} \overline{v}(\overline{k}) = \sum_{\overline{k} \in \overline{N}} \langle \hat{y}, BXe_{\overline{k}} \rangle = \langle \hat{y}, BXe_{\overline{N}} \rangle = \langle \hat{y}, Be_N \rangle = v(N) = \overline{v}(\overline{N}).$$

Applying Proposition 4.3 gives that $(\overline{N}, \overline{v})$ is an additive game. Further $\text{Core}(A, BX, c) = \{(\overline{v}(\overline{1}), \ldots, \overline{v}(\overline{m}))\} = \{(v(1), \ldots, v(n))\}X = \text{Core}(A, B, c)X$, because

$$\overline{v}(\overline{k}) = \langle \hat{y}, BXe_{\overline{k}} \rangle = \langle \hat{y}, \sum_{l \in N} X_{l\overline{k}} Be_l \rangle = \sum_{l \in N} X_{l\overline{k}} \langle \hat{y}, Be_l \rangle = \sum_{l \in N} X_{l\overline{k}} v(l).$$

$\square$

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