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The Archimedean ℓ-Group Tensor Product

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Abstract. We introduce a construction (in ZF-set theory) for the Archimedean ℓ-group tensor product. We relate this tensor product to the existing ones in the theory of Archimedean vector lattices and ℓ-groups.

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In [6] Fremlin introduces the Archimedean Riesz space tensor product of Archimedean Riesz spaces. Fremlin remarks that the existence of the Archimedean Riesz space tensor product is not remarkable at all. Indeed, its existence is implicitly contained in an earlier paper [10], in which Martinez without much ado defines the ℓ-group tensor product of commutative ℓ-groups. To get the Fremlin tensor product one can take the Martinez tensor product modulo the infinitesimal kernel (see Theorem 6 in this paper). Nonetheless, several authors after Fremlin have reinvented the wheel. In [6] and [7] the main motive is to establish the existence without the use of representation theorems (and improve on some of the properties of the Archimedean Riesz space tensor product). In [6] the Axiom of Choice is used freely, in [7] only the Countable Axiom of Choice is used. In neither the Axiom of Choice is an issue. In this paper we will freely use representation theorems, but we avoid the Axiom of Choice altogether.

Our approach is the following. Using the knowledge of [4] we introduce an ordering on the algebraic tensor product $E \otimes F$ of Archimedean ℓ-groups E, F. We show that $E \otimes F$ has a universal mapping property with respect to a notion of Riesz homomorphism as introduced in [5] and then define $E\overline{\otimes}F$ (the Archimedean ℓ-group tensor product of E and F) to be the ℓ-group generated by $E \otimes F$ in the Archimedean vector lattice cover of $E \otimes F$ (see [5]). Our method has several advantages over the existing ones. First, our method is constructive. Second, we enlarge the setting from Archimedean vector lattices to Archimedean ℓ-groups. Third, we connect the vector lattice theory with the earlier work in ℓ-groups by Martinez and indicate answers to some of Martinez' questions explicitly. Finally, we remark that though some results in this paper, noticeably Theorem 9, are not new in the setting of vector lattices, it is remarkable that these results continue to hold in the ℓ-group setting.

Let E be an Archimedean ℓ-group. We denote by E_d the divisible hull of E, that is its group theoretic divisible hull, where $x \in E_d$ is called positive if $nx \in E^+$ for some $n \in \mathbb{N}$. We denote by $R[E]$ the Archimedean vector lattice cover (see [3] and [5]) of E. In particular, E generates $R[E]$ and E_d is a relatively uniformly dense ℓ-subgroup of $R[E]$. We will make frequent use of the following representation theorem (Theorem 1) which is a special case of Bernau's representation theorem (see [1]) if one does not object to the Axiom of Choice. We prove it by using the constructively valid Theorem 2.4 in [4], which implies that every finitely generated Riesz space is representable.

THEOREM 1. *Let E be a finitely generated Archimedean ℓ-group. Choose an order unit e in E. Let S be the set of all ℓ-group homomorphisms $\varphi : E \to \mathbb{R}$ with $\varphi(e) = 1$. For E define $\hat{x} : S \to \mathbb{R}$ by*

$$\hat{x}(\varphi) = \varphi(x) \quad (\varphi \in S)$$

Endow S with the weakest topology that makes every $\hat{x}$ $(x \in E)$ continuous. Then S is a compact Hausdorff space, $x \mapsto \hat{x}$ is an ℓ-group isomorphism of E onto an ℓ-subgroup $\hat{E}$ of $C(S)$.

Proof. $R[E]$ is a finitely generated Archimedean Riesz space with e as an order unit and containing E_d as a relatively uniformly dense ℓ-subgroup. Every ℓ-group homomorphism $E \to \mathbb{R}$ extends uniquely to a Riesz homomorphism $R[E] \to \mathbb{R}$ (See [5]). Conversely, if $\varphi : R[E] \to \mathbb{R}$ is a Riesz homomorphism, then its restriction to E is an ℓ-group homomorphism $E \to \mathbb{R}$. Now apply Theorem 2.4 in [4]. □

We are ready to start our construction of the Archimedean ℓ-group tensor product. Let E and F be Archimedean ℓ-groups. We denote by $E \otimes F$ the group tensor product of E and F. Since E and F are Archimedean ℓ-groups they are torsion free (see proposition 11.6 in [1]). The latter implies (see Theorem 6.2 in [9]) that $E_0 \otimes F_0$ is a subgroup of $E \otimes F$ if E_0 is a subgroup of E and F_0 is a subgroup of F. The latter fact will be used a number of times without special mention.

Let E and F be finitely generated Archimedean ℓ-groups. For $\varphi \in \mathrm{Hom}(E, \mathbb{R})$ and $\psi \in \mathrm{Hom}(F, \mathbb{R})$ let $\varphi \otimes \psi$ be the group homomorphism $E \otimes F \to \mathbb{R}$ induced by the bimorphism

$$(x, y) \mapsto \varphi(x)\psi(y) \quad ((x, y) \in E \times F).$$

We define

$$(E \otimes F)^+ = \{u \in E \otimes F \,|\, (\varphi \otimes \psi)(u) \geqslant 0 \text{ for all } \varphi \in \operatorname{Hom}(E, \mathbb{R}) \text{ and } \psi \in \operatorname{Hom}(F, \mathbb{R})\}.$$

The background of this definition is the following. Choose order units $e \in E$ and $f \in F$; let $S = \{\varphi \in \operatorname{Hom}(E,\mathbb{R}) | \varphi(e) = 1\}$ and $T = \{\psi \in \operatorname{Hom}(F,\mathbb{R}) | \psi(f) = 1\}$. The natural maps $E \to C(S)$, $F \to C(T)$ (see Theorem 1) and $C(S) \otimes C(T) \to C(S \times T)$ induce an injective group homomorphism $E \otimes F \to C(S \times T)$. Then $(E \otimes F)^+$ consists precisely of the elements of $E \otimes F$ that are mapped into $C(S \times T)^+$.

It follows that $(E \otimes F)^+$ determines a partial ordering in $E \otimes F$ rendering $E \otimes F$ an integrally closed, directed partially ordered group (and an ordered subgroup of $C(S \times T)$).

The following lemma will enable us to define $(E \otimes F)^+$ for not necessarily finitely generated E and F.

LEMMA 2. *Let E_1, F_1 be finitely generated Archimedean ℓ-groups with finitely generated ℓ-subgroups $E \subset E_1$ and $F \subset F_1$. Then*

$$(E \otimes F)^+ = (E \otimes F) \cap (E_1 \otimes F_1)^+.$$

Proof. The inclusion $(E \otimes F)^+ \subset (E \otimes F) \cap (E_1 \otimes F_1)^+$ is quite obvious, its reverse is not.

Let e, f, S, T be as before. We indicate the natural maps $E \to C(S)$ and $F \to C(T)$ by $\hat{}$ and we view $C(S) \otimes C(T)$ as a subset of $C(S \times T)$. Let α be the resulting homomorphism $E \otimes F \to C(S \times T)$, where

$$\alpha(u)(\varphi, \psi) = (\varphi \otimes \psi)(u) \quad (u \in E \otimes F,\ \varphi \in S,\ \psi \in T).$$

Put $\Phi = \{\varphi \in S | \varphi$ extends to an ℓ-group homomorphism $E_1 \to \mathbb{R}\}$, $\Psi = \{\psi \in S | \psi$ extends to an ℓ-group homomorphism $F_1 \to \mathbb{R}\}$. The ℓ-group homomorphisms $E_1 \to \mathbb{R}$ separate the points of E_1. It follows that Φ separates the points of E. As $\hat{E}_d$ is uniformly dense in $C(S)$ we see that Φ must be dense in S. Similarly, Ψ is dense in T, so that $\Phi \times \Psi$ is dense in $S \times T$.

Now take $u \in (E \otimes F) \cap (E_1 \otimes F_1)^+$. Then $\alpha(u)(\varphi, \psi) \geqslant 0$ for all $\varphi \in \Phi$, $\psi \in \Psi$, so $\alpha(u) \geqslant 0$ on all of $S \times T$. Then $u \in (E \otimes F)^+$. □

Let E and F be Archimedean ℓ-groups. We can define a subset $(E \otimes F)^+$ of $E \otimes F$ by requiring the identity

$$(E \otimes F)^+ \cap (E_0 \otimes F_0) = (E_0 \otimes F_0)^+ \qquad (*)$$

for all finitely generated ℓ-subgroups $E_0 \subset E$ and $F_0 \subset F$. By Lemma 2 this definition generalizes the one given earlier for finitely generated E and F. The formula

$$u \leqslant v \Leftrightarrow v - u \in (E \otimes F)^+$$

turns $E \otimes F$ into an integrally closed, directed partially ordered group. Furthermore, let $E_0 \subset E$ and $F_0 \subset F$ be ℓ-subgroups, not necessarily finitely generated. Then by a simple argument one obtains the identity (*) and thus $E_0 \otimes F_0$ is a partially ordered subgroup of $E \otimes F$. We will use this property frequently in the sequel. Also note that it easily follows that $E \otimes \mathbb{Z}$ is isomorphic to E, whereas $E \otimes \mathbb{Q}$ is isomorphic to E_d.

Let once again E and F by Archimedean ℓ-groups. A map φ from $E \times F$ to an Archimedean ℓ-group G is called a *bimorphism* if for each $a \in E^+$ and each $b \in F^+$ the maps $x \mapsto \varphi(x, b)$ $(x \in E)$ and $y \mapsto \varphi(a, y)$ $(y \in F)$ are ℓ-group homomorphisms. An Archimedean ℓ-tensor product for E and F is an Archimedean ℓ-group H together with a bimorphism $f : E \times F \to H$ having the following universal property. For any Archimedean ℓ-group G and every bimorphism $\varphi : E \times F \to G$ there exists an ℓ-group homomorphism $\Phi : H \to G$ with $\varphi = \Phi \circ f$:

$$\begin{array}{ccc} E \times F & \xrightarrow{f} & H \\ \varphi \downarrow & \swarrow_{\Phi} & \\ G & & \end{array}$$

So far we are in the following position. $E \otimes F$ is an integrally closed, directed partially ordered group. Of course, any bimorphism φ from $E \times F$ to a lattice ordered group G induces a group homomorphism $\Phi : E \otimes F \to G$. Since $E \otimes F$ need not be an ℓ-group it does not make sense to ask if Φ is an ℓ-group homomorphism.

On the other hand, it may have the property that

$$\Phi(h)^+ = \inf\{\Phi(j) | j \in (E \otimes F)^+, \, j \geqslant h\} \tag{**}$$

for all $h \in E \otimes F$. Such a map is called a Riesz homomorphism in [5]. That (**) indeed holds is the content of Lemma 3, with the restriction that E, F, and G are divisible.

LEMMA 3. *Let E, F, and G be divisible Archimedean ℓ-groups and $\varphi : E \times F \to G$ a bimorphism. Then the induced group homomorphism $\Phi : E \otimes F \to G$ is a Riesz homomorphism.*

Proof. (1) Suppose we have compact Hausdorff spaces S, T, W such that E, F, G are uniformly dense $\mathbb{Q}$-sublattices of $C(S)$, $C(T)$, and $C(W)$, respectively. Assume in addition that $1_S \in E$, $1_T \in F$, $1_W \in G$ and $\varphi(1_S, 1_T) = 1_W$.

Let $h \in E \otimes F$; we wish to prove that in G, or, equivalently in $C(W)$

$$\Phi(h)^+ = \inf\{\Phi(j) | j \in E \otimes F, \, j \geqslant 0, \, j \geqslant h\}.$$

Now $E \otimes F$ is uniformly dense in $C(S \times T)$. Hence, for every $\varepsilon \geqslant 0$ there exists $j \in E \otimes F$ with $j \geqslant 0$ and $j \geqslant h$ and $j \leqslant h^+ + \varepsilon 1_{S \times T}$. We are done if for such j

$$\Phi(j) \leqslant \Phi(h)^+ + \varepsilon 1_W.$$

This in turn, will be true if for every $w \in W$ there is an $(s,t) \in S \times T$ with

$$(\Phi(k))(w) = k(s,t) \quad (k \in E \otimes F).$$

For simplicity, put $w(e \otimes f) = (\Phi(e \otimes f))(w) = (\varphi(e,f))(w)$ if $e \in E, f \in F$. The function

$$e \mapsto w(e \otimes 1_T) = (\varphi(e, 1_T))(w) \quad (e \in E)$$

is a $\mathbb{Q}$-linear lattice homomorphism $E \to \mathbb{R}$, extending uniquely to a Riesz homomorphism $C(S) \to \mathbb{R}$. Its value at 1_S is 1, so there is an $s \in S$ with

$$w(e \otimes 1_T) = e(s) \quad (e \in E).$$

Similarly, there is a $t \in T$ such that

$$w(1_S \otimes f) = f(t) \quad (f \in F).$$

In the same fashion, for any $e \in E^+$ with $e(s) \neq 0$ there exists a $t_e \in T$ for which

$$\frac{w(e \otimes f)}{e(s)} = f(t_e) \quad (f \in F).$$

Then for $f \in F^+$ we have

$$e(s)f(t_e) = w(e \otimes f) \leqslant w(|e|1_S \otimes f) = \|e\|_\infty f(t).$$

from which it follows that $t_e = t$. Thus we see that

$$w(e \otimes f) = e(s)f(t)$$

whenever $e \in E^+$, $e(s) \neq 0$, and $f \in F^+$. By bilinearity the same identity holds for all $e \in E, f \in F$ and (**) follows.

(2) Now we turn to the general situation. Let $u \in E \otimes F$. We wish to prove that $\Phi(u)^+ \geqslant z$ whenever $z \in G^+$ is a lower bound of

$$\{\Phi(x) | \, x \in E \otimes F, \; x \geqslant 0, \; x \geqslant u\}.$$

Let z be such a lower bound in G^+. There exist finite sets $A \subset E^+$ and $B \subset F^+$, generating divisible ℓ-subgroups $E_0 \subset E$ and $F_0 \subset F$, such that $u \in E_0 \otimes F_0$. We may assume $u \leqslant e \otimes f$ where $e = \sum_{a \in A} a \in E_0$ and $f = \sum_{b \in B} b \in F_0$. Then e and f are order units in E_0 and F_0, respectively, and $z \leqslant \Phi(e \otimes f)$.

Let G_0 be the divisible ℓ-subgroup of G generated by $\{\Phi(a \otimes b) | \, a \in A, \; b \in B\} \cup \{z\}$. Then G_0 has $\Phi(e \otimes f)$ as an order unit. It is not difficult to see that φ maps $E_0 \times F_0$ into G_0.

We may assume the existence of compact Hausdorff spaces S, T, W such that E_0, F_0, G_0 are uniformly dense $\mathbb{Q}$-linear sublattices of $C(S), C(T), C(W)$ whereas

$e = 1_S$, $f = 1_T$, $\Phi(e \otimes f) = 1_W$. Then by part (1) the restriction of Φ is a Riesz homomorphism $E_0 \otimes F_0 \to G_0$, so that in G_0

$$\Phi(u)^+ = \inf\{\Phi(x) | x \in E_0 \otimes F_0,\ x \geqslant 0, x \geqslant u\}.$$

Now the element z of G_0 is a lower bound of $\{\Phi(x) | x \in E_0 \otimes F_0,\ x \geqslant 0, x \geqslant u\}$. Then $\Phi(u)^+ \geqslant z$. □

As an integrally closed directed partially ordered group $E \otimes F$ has an Archimedean vector lattice cover $R[E \otimes F]$ (see [5]). We now define $E\overline{\otimes}F$ to be the ℓ-subgroup of $R[E \otimes F]$ generated by $E \otimes F$. We remind the reader that Riesz homomorphisms (in the sense of (**) before Lemma 3) on $E \otimes F$ with values in an Archimedean Riesz space H extend (uniquely) to Riesz homomorphisms $R[E \otimes F] \to H$ (see [5]).

THEOREM 4. *Let E, F be Archimedean ℓ-groups. Let $E\overline{\otimes}F$ be the ℓ-subgroup of $R[E \otimes F]$ generated by $E \otimes F$. Then $E\overline{\otimes}F$ with the natural map $E \times F \to E\overline{\otimes}F$ is an Archimedean ℓ-tensor product of E and F.*

Proof. (I) Let $x \in E$, $b \in F^+$, $u \in (E \otimes F)^+$, $u \geqslant x \otimes b$. Then $u \geqslant x^+ \otimes b$. Indeed, let $E_0 \subset E$ and $F_0 \subset F$ be finitely generated ℓ-subgroups with $x \in E_0$, $b \in F_0$, and $u \in E_0 \otimes F_0$. Then for all $\psi \in \operatorname{Hom}(F_0, \mathbb{R})$ and all $\varphi \in \operatorname{Hom}(E_0, \mathbb{R})$ we have

$$(\varphi \otimes \psi)(u) \geqslant ((\varphi \otimes \psi)(x \otimes b))^+ = \varphi(x)^+\psi(b)$$

$$= \varphi(x^+)\psi(b) = (\varphi \otimes \psi)(x^+ \otimes b).$$

Thus $u - (x^+ \otimes b) \in (E_0 \otimes F_0)^+ \subset (E \otimes F)^+$ and $u \geqslant x^+ \otimes b$.
(II) Hence, if $b \in F^+$, then for all $x \in E$, $x^+ \otimes b$ is the smallest element of

$$\{u \in (E \otimes F)^+ | u \geqslant x \otimes b\}.$$

But obviously it also is the smallest element of

$$\{y \otimes b | y \in E^+,\ y \geqslant x\}.$$

Apparently, these two sets have the same lower bounds. This means that $x \mapsto x \otimes b$ is a Riesz homomorphism $E \mapsto E \otimes F$. The natural map $E \otimes F \to R[E \otimes F]$ is a complete Riesz homomorphism [5]. Then so is the map $E \otimes F \to E\overline{\otimes}F$. Therefore, for $b \in F^+$, the map $x \mapsto x \otimes b$ is a Riesz homomorphism $E \to E\overline{\otimes}F$ [5].

Similarly, if $a \in E^+$, then $y \mapsto a \otimes y$ is a Riesz homomorphism $F \to E\overline{\otimes}F$. Thus $(x, y) \mapsto x \otimes y$ is a bimorphism $E \times F \to E\overline{\otimes}F$.

(III) Let G be an Archimedean ℓ-group, $\varphi : E \times F \to G$ a bimorphism. φ extends to a bimorphism : $E_d \times F_d \to G_d$ and thus induces Riesz homomorphisms $\varphi^{\otimes} : E_d \otimes F_d \to G_d$ (Lemma 3) and $(\psi^{\otimes})^R : R[E_d \otimes F_d] \to R[G_d]$.

The natural isomorphisms $R[E_d \otimes F_d] \cong R[(E \otimes F)_d] \cong R[E \otimes F]$ yield a Riesz homomorphism $R[E \otimes F] \to R[G_d]$ whose restriction to $E\overline{\otimes}F$ maps the lattice $E\overline{\otimes}F$ into the lattice G and sends $x \otimes y$ to $\varphi(x, y)$ $(x \in E,\ y \in F)$. □

If E is an Archimedean Riesz space and F is an Archimedean ℓ-group then

$$E\overline{\otimes}F = R[E \otimes F].$$

Indeed, for every $b \in F^+$ the map $x \mapsto b \otimes x$ of E into $R[E \otimes F]$ is an order preserving group homomorphism and therefore is linear. It follows that $E \otimes F$ is a linear subspace of $R[E \otimes F]$. Then the ℓ-subgroup of $R[E \otimes F]$ it generates is closed for scalar multiplication, i.e., $E\overline{\otimes}F$ is a Riesz subspace of $R[E \otimes F]$. Then $E\overline{\otimes}F = R[E \otimes F]$.

In particular, if E and F are Archimedean Riesz spaces, then so is $E\overline{\otimes}F$. One now easily shows that $E\overline{\otimes}F$ is an Archimedean Riesz space tensor product in the sense of Fremlin:

THEOREM 5. *Let E and F be Archimedean Riesz space. Then $E\overline{\otimes}F$ $(= R[E \otimes F])$ is an Archimedean Riesz space. If G is an Archimedean Riesz space and $\varphi : E \times F \to G$ is a Riesz bimorphism then there exists a unique Riesz homomorphism $\varphi^{\overline{\otimes}} : E\overline{\otimes}F \to G$ for which $\varphi^{\overline{\otimes}} = \varphi(x, y)$ for all $x \in E$ and all $y \in F$.*

$$\begin{array}{ccc} E \times F & \xrightarrow{\otimes} & E\overline{\otimes}F \\ \varphi \downarrow & \swarrow & f^{\overline{\otimes}} \\ G & & \end{array}$$

The following proposition can be proved straightforwardly (For the third assertion use the fact that $R[E]$ has the universal property of $\mathbb{R}\overline{\otimes}E$).

PROPOSITION 6. *Let E be an Archimedean ℓ-group. The following properties hold:*

(1) $E\overline{\otimes}\mathbb{Z} \cong E$,
(2) $E\overline{\otimes}\mathbb{Q} \cong E_d$,
(3) $E\overline{\otimes}\mathbb{R} \cong R[E]$.

Martinez in [10] introduced a tensor product $E \otimes_\ell F$ for commutative (not necessarily Archimedean) ℓ-groups E and F. As a consequence, for Archimedean

ℓ-groups E and F, we now have two tensor products, $E \otimes_\ell F$ and $E\overline{\otimes}F$. The difference between $E \otimes_\ell F$ and $E\overline{\otimes}F$ is explained by the difference between their universal properties. In the universal property for $E \otimes_\ell F$ there is no restriction on the range ℓ-groups while in the universal property for $E\overline{\otimes}F$ the range ℓ-groups have to be Archimedean. As a result of Fremlin's counterexample 4.7 in [6] (which, admittedly, is not in ZF but can easily be adapted to be in ZF) it follows that $E\overline{\otimes}F$ and $E \otimes_\ell F$ are not necessarily isomorphic. To connect the two different tensor products we need the following notion.

Let E be a commutative ℓ-group (not necessarily Archimedean). Let $\mathcal{J}$ be the set of all order ideals $J \subset E$ for which E/J is Archimedean. By the *infinitesimal kernel* of E we mean the kernel of the natural ℓ-group homomorphism

$$E \to \prod_{J \in \mathcal{J}} E/J.$$

The quotient of E by its infinitesimal kernel is itself Archimedean.

We use the letter K to indicate the infinitesimal kernel of any commutative ℓ-group.

THEOREM 7. *If E and F are commutative ℓ-groups, then*

$$(E/K)\overline{\otimes}(F/K) \cong (E \otimes_\ell F)/K.$$

In particular, if E and F are Archimedean, then $(E \otimes_\ell F)/K \cong E\overline{\otimes}F$.

The proof is straightforward reasoning to show that $(E \otimes_\ell F)/K$ has the universal property of the Archimedean ℓ-tensor product of E/K and F/K.

Martinez in [10] also introduced the vector lattice cover of a commutative (not necessarily Archimedean) ℓ-group E, named $V(E)$. Again, as a consequence, we have two vector lattice covers for an Archimedean ℓ-group E, namely $V(E)$ and $R[E]$. Since Martinez has shown ([10]) that $E \otimes_\ell \mathbb{R} = V(E)$ it follows from Theorem 7 that $V(E)/K \cong R[E]$. One can then answer questions raised in [10] (like 2.9), which we leave to the reader.

PROPOSITION 8. *Let E and F be Archimedean ℓ-groups. Then*
(1) $R[E\overline{\otimes}F] \cong R[E \otimes F]$ *and* (2) $R[E]\overline{\otimes}R[F] \cong R[E\overline{\otimes}F]$.

Proof. $E\overline{\otimes}F$ is an ℓ-subgroup of $R[E \otimes F]$ and generates $R[E \otimes F]$ as a Riesz space. Then $R[E \otimes F]$ is the vector lattice cover of $E\overline{\otimes}F$. This proves (1). (2) follows from (1) and Proposition 6. □

The following theorem generalizes 4.5 in [6].

THEOREM 9. *Let E, F be Archimedean ℓ-groups; let $E_1 \subset E$ and $F_1 \subset F$ be ℓ-subgroups. Then the natural injections $E_1 \to E$ and $F_1 \to F$ induce an injective ℓ-group homomorphism $E_1 \overline{\otimes} F_1 \to E \overline{\otimes} F$. Thus, $E_1 \overline{\otimes} F_1$ is naturally isomorphic to the ℓ-subgroup of $E \overline{\otimes} F$ generated by $E \otimes F$.*

Proof. To avoid ambiguities let us denote the injections $E_1 \to E$ and $F_1 \to F$ by α and β and let $\otimes_1$ and $\otimes$ be the natural bimorphisms $E_1 \times F_1 \to E_1 \overline{\otimes} F_1$ and $E \times F \to E \overline{\otimes} F$. The map

$$(x, y) \mapsto \alpha(x) \otimes \beta(y)$$

is a bimorphism $E_1 \times F_1 \to E \overline{\otimes} F$ and induces an ℓ-group homomorphism $\omega : E_1 \overline{\otimes} F_1 \to E \overline{\otimes} F$ with

$$\omega(x \otimes_1 y) = \alpha(x) \otimes \beta(y) \quad (x \in E_1,\ y \in F_1).$$

Suppose $u \in E_1 \overline{\otimes} F_1, \omega(u) \leqslant 0$; we are done if it follows that $u \leqslant 0$. If $v \in E_1 \otimes F_1$ and $v \leqslant u$ then $\omega(v) \leqslant 0$ so $v \leqslant 0$. But $E_1 \otimes F_1$ is order dense in $E_1 \overline{\otimes} F_1$. Hence $u \leqslant 0$. □

As just one example how to apply Theorem 9 we have the following

COROLLARY 10. *Let S and T be compact Haudorff spaces. Let $E \subset C(S)$ and $F \subset C(T)$ be ℓ-subgroups. Then $E \overline{\otimes} F$ is isomorphic to the ℓ-subgroup of $C(S \times T)$ generated by $E \otimes F$.*

Proof. $C(S) \otimes C(T)$ is order dense in $C(S \times T)$. Hence $R[C(S) \otimes C(T)]$ (which is $C(S) \overline{\otimes} C(T)$) is just the Riesz subspace of $C(S \times T)$ generated by $C(S) \otimes C(T)$. Now apply Theorem 9. □

Our final result generalizes 4.2 (iii) in [6] to the setting of divisible Archimedean ℓ-groups.

THEOREM 11. *Let E and F be divisible Archimedean ℓ-groups. Then $E \otimes F$ is a relatively uniformly dense subset of $E \overline{\otimes} F$.*

Proof. (I) If E and F are uniformly dense ℓ-subgroups of $C(S)$ and $C(T)$ for certain compact Hausdorff spaces S and T, then $E \otimes F$ is uniformly dense in $C(S \times T)$, hence by Proposition 10 uniformly dense in $E \overline{\otimes} F$.
(II) For ℓ-subgroups $E_1 \subset E$ and $F_1 \subset F$, view $E_1 \overline{\otimes} F_1$ as an ℓ-subgroup of $E \overline{\otimes} F$. The union of all these ℓ-subgroups $E_1 \overline{\otimes} F_1$ for finitely generated E_1 and F_1 is an ℓ-subgroup of $E \overline{\otimes} F$ containing $E \otimes F$, hence is $E \overline{\otimes} F$ itself.

Take $u \in E\overline{\otimes}F$. By the above there exist finitely generated ℓ-subgroups $E_1 \subset E$ and $F_1 \subset F$ with $u \in E_1\overline{\otimes}F_1 \subset (E_1)_d\overline{\otimes}(F_1)_d$. It follows from Theorem 1 and (I) that u lies in the relative uniform closure of $(E_1)_d \otimes (F_1)_d$ in $(E_1)_d\overline{\otimes}(F_1)_d$. *A fortiori*, u lies in the relatively uniform closure of $E \otimes F$ in $E\overline{\otimes}F$. □

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