A NOTE ON THE GELFAND–NAIMARK–SEGAL THEOREM

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ABSTRACT

We show that, modulo the Countable Axiom of Choice, the Ultrafilter Theorem suffices to prove the Gelfand–Naimark–Segal Theorem in *ZF* set theory.

The Gelfand–Naimark–Segal Representation Theorem (see [5, theorem 9.18]) states that every C^*-algebra can faithfully be represented as a C^*-algebra of operators on a Hilbert space. Takesaki [5, p. 54] remarks that all proofs of this fact require the Axiom of Choice. Indeed, the proofs available in textbooks use the Axiom of Choice, often by combining Alaoglu's Theorem with the Krein–Milman Theorem, which together imply the Axiom of Choice [4]. A standard proof of the theorem constructs a representation π_f for each state f and then takes a direct sum over the representations thus obtained. If f is a pure state then π_f is irreducible. The point where stronger set-theoretic axioms come in is where the existence of sufficiently many pure states is proved. Recently [3] we have investigated the role of the Ultrafilter Theorem in the theory of vector lattices. Some of our results there clarify how effective the existence of sufficiently many pure states really is.

To wit, let A be a C^*-algebra with unit 1 and $x \in A_{sa}$ where A_{sa} is the self-adjoint part of A and $x \neq 0$. A_{sa} is a partially ordered real algebra which is directed and Archimedean. As such it has a Dedekind completion E [2, p. 312]. E has 1 as its strong order unit. By the Kakutani Representation Theorem, which modulo the Countable Axiom of Choice, *CAC*, is equivalent to the Ultrafilter Theorem [3, theorem 4.1], we find a Riesz homomorphism $\tilde{f}$ on E with $\tilde{f}(1) = 1$ and $\tilde{f}(x) \neq 0$. Denote the restriction of $\tilde{f}$ to A_{sa} by f. We claim that the natural extension of f to A is a pure state on A. To prove that claim we have to show that if $g \in A_{sa}^*$ and $0 \leq g \leq f$, then $g = \lambda f$ for some $\lambda \in \mathbb{R}$. Suppose that $0 \leq g \leq f$. Let $y \in A_{sa}$ and let D be the self-adjoint part of the C^*-algebra generated by 1 and y in A. Then $f|_D$ is a Riesz homomorphism and $0 \leq g|_D \leq f|_D$. By theorem 3.13 of [1] it follows that $g|_D = g(1)f|_D$ and thus $g(y) = g(1)f(y)$.

Of course, if A happens to be a non-unital C^*-algebra we follow the above argument for the C^*-algebra obtained from A by adjoining a unit element. In this way we have obtained that (in $ZF + CAC$):

The Ultrafilter Theorem suffices to show that each C^*-algebra admits sufficiently many pure states.

References

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