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Inverse degrees and the Jacobian conjecture

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Abstract

Bass, Connell and Wright proved that if the Jacobian Conjecture is true, then there exists a number $C(d)$ such that for every k -algebra A and every invertible polynomial map $F : A^n \rightarrow A^n$ with $\det(J(F)) = 1$ the degree of F^{-1} is bounded by $C(d)$. A year later Bass proved the converse. In this paper we give a short proof of this last result.

In this paper k is a field of characteristic 0. If $F = (F_1, \dots, F_n) : k^n \rightarrow k^n$ is a polynomial map, then we define the degree of F as the maximum degree of the polynomials F_1, \dots, F_n . This paper is about the equivalence of the following two conjectures:

Conjecture 1 $JC(n)$: *If $F : k^n \rightarrow k^n$ is a polynomial map with $\det(J(F)) = 1$ then F is invertible, e.g. there exists a polynomial map $G : k^n \rightarrow k^n$ such that $G \circ F = F \circ G = \text{id}$.*

This conjecture is known as the Jacobian conjecture (c.f. [3, 4, 6]), and it was first formulated by O.H. Keller in [5].

Conjecture 2 $BI(n)$: *For every finite dimensional k -algebra A the following holds: If $F : A^n \rightarrow A^n$ is an invertible polynomial map such that $\det(J(F)) = 1$, then the degree of F^{-1} is bounded by a constant $C(d)$, depending only on d (and n), and not on the choice of A .*

If A is a field, then one can take $C(d) = d^{n-1}$. This result is due to Gabber (c.f. [2, Ch. I, Cor. (1.4).]). if A is a reduced ring, then it is easy to reduce it to the case that A is a field. For a reduced ring A we can also take $C(d) = d^{n-1}$ as the upper bound. If A is allowed to have nilpotent elements then it is not known whether there can be given an upper bound for the degree of the inverse. In [2, Ch. I, Prop. (1.2).] Bass, Connell and Wright proved that $JC(n)$ implies $BI(n)$. Bass proved in [1] the converse. We will now give a short proof of this implication:

Theorem 1 *If the conjecture $BI(n)$ is true then the jacobian conjecture $JC(n)$ is true.*

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Proof: Assume $\text{BI}(n)$ holds and assume that $F : k^n \rightarrow k^n$ is a polynomial map satisfying $\det(J(F)) = 1$. Without loss of generality we may assume that $F(0) = 0$. Now F is locally invertible in the neighbourhood of 0, e.g. there exist $G_1, G_2, \dots, G_n \in k[[X_1, \dots, X_n]]$ such that $F_i(G_1, \dots, G_n) = X_i$ for all i . Write

$$F = F_{(1)} + F_{(2)} + \dots + F_{(d)} \text{ and } G = G_{(1)} + G_{(2)} + G_{(3)} + \dots$$

where $F_{(i)}$ and $G_{(i)}$ are homogeneous of degree d for all i . Let us define $\widehat{F} : k[t]^n \rightarrow k[t]^n$ by

$$\widehat{F} = t^{-1}F(tX) = F_{(1)} + tF_{(2)} + \dots + t^{d-1}F_{(d)}$$

and likewise define

$$\widehat{G} = t^{-1}G(tX) = G_{(1)} + tG_{(2)} + t^2G_{(3)} + \dots$$

Since $J(\widehat{F}) = J(t^{-1}F(tX)) = J(F)(tX)$, we have $\det(J(\widehat{F})) = \det(J(F))(tX) = 1$. Now \widehat{G} is the formal inverse of \widehat{F} because

$$\widehat{F} \circ \widehat{G} = t^{-1}F(t^{-1}G(tX)) = t^{-1}F(G(tX)) = t^{-1}tX = X$$

Choose $l > C(d)$ arbitrary. We now calculate modulo t^l . Define $\overline{F} : (k[t]/(t^l))^n \rightarrow (k[t]/(t^l))^n$ and \overline{G} to be \widehat{F} respectively \widehat{G} modulo t^l . Again we get $\det(J(\overline{F})) = 1$ and $\overline{F} \circ \overline{G} = X$. So \overline{G} is the inverse of \overline{F} , hence the degree of \overline{G} is bounded by $C(d)$. Since

$$\overline{G} = G_{(1)} + tG_{(2)} + \dots + t^{l-1}G_{(l)}$$

it follows that $t^{l-1}G_{(l)} = 0$. The fact that $t^{l-1} \neq 0$ (in $k[t]/(t^l)$) forces $G_{(l)} = 0$. We can conclude that $G_{(l)} = 0$ for all $l > C(d)$. So G is also a polynomial map and this proves that F is invertible. \square

References

- [1] H. Bass, The Jacobian conjecture and inverse degrees, *Arithmetic and geometry*, *Prog. Math.* 36, 65-75 (1983).
- [2] H. Bass, H. Connell and D. Wright, The Jacobian conjecture: reduction of degree, and formal expansions of the inverse, *Bull. Amer. Math. Soc.* (1982).
- [3] L. Drużkowski, The jacobian conjecture, preprint 492, *Inst. of Math., Polish Academy of Sciences, Warsaw*, 1991.
- [4] A. van den Essen, Polynomial maps and the Jacobian conjecture, *Computational aspects of Lie group representations and related topics, of the 1990 computational algebra seminar*, in *C.W.I. Tract.* 84 (1991), 29-24.
- [5] O.H. Keller, Ganze Cremona-Transformationen, *Monats. Math. Physik* 47 (1939), 299-306.
- [6] K. Rusek, Polynomial automorphisms (preprint 1989).

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