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A survey of spectral models of gravity coupled to matter

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Dedicated to Alain Connes

Abstract

This is a survey of the historical development of the Spectral Standard Model and beyond, starting with the ground breaking paper of Alain Connes in 1988 where he observed that there is a link between Higgs fields and finite noncommutative spaces. We present the important contributions that helped in the search and identification of the noncommutative space that characterizes the fine structure of space-time. The nature and properties of the noncommutative space are arrived at by independent routes and show the uniqueness of the Spectral Standard Model at low energies and the Pati-Salam unification model at high energies.

Contents

1	Introduction	2
2	Early days of the spectral Standard Model	4
2.1	Noncommutative spaces and differential calculus	5
2.2	Two-sheeted spacetime	6
2.3	Constructions beyond the Standard Model	9
2.4	Coupling matter to gravity	11

3	The spectral action principle	12
3.1	Real structures on spectral triples	13
3.2	The spectral action principle	15
4	The spectral Standard Model	16
4.1	Classification of irreducible geometries	17
4.2	Noncommutative geometry of the Standard Model	18
4.3	The gauge and scalar fields as inner fluctuations	22
4.4	Spectral action	24
4.5	Fermionic action in KO-dimension 6	25
4.6	Phenomenological consequences	25
5	Beyond the Standard Model with noncommutative geometry	26
5.1	Resilience of the spectral Standard Model	28
5.2	Pati–Salam unification and first-order condition	30
5.3	Grand symmetry and twisted spectral triples	34
5.4	Algebraic constraints on the finite geometry	35
6	Volume quantization and uniqueness of SM	36
6.1	Higher form of Heisenberg’s commutation relations	36
6.2	Volume quantization	41
7	Outlook: towards quantization	45
A	Pati–Salam model: potential analysis	46

1 Introduction

In 1988, at the height of the string revolution, there appeared an alternative way to think about the structure of space-time, based on the breathtaking progress in the new field of noncommutative geometry. Despite the success of string theory in incorporating gravity, consistency of the theory depended on the existence of supersymmetry as well as six or seven extra dimensions. Enormous amount of research was carried out to obtain the Standard Model from string compactification, which even up to day did not materialize. Most compactifications start in ten dimensions with the Yang–Mills gauge group $E_8 \times E_8$ requiring a very large number of fields to become massive at high energies. In a remarkable paper, Alain Connes laid down the blue print of a new innovative approach to uncover the origin of the Standard

Model and its symmetries [28]. The foundation of this approach was based on noncommutative geometry, a field he founded few years before [27] (see also [29]). Alain realized that by making space slightly noncommutative by tensoring the four dimensional space with a space of two points, one gets a parallel universe where the distance between the two sheets is of the order of 10^{-16} cm, with the unexpected bonus of having the Higgs scalar field mediating between them. Although this looked similar to the idea of Kaluza–Klein, there were essential differences, mainly in avoiding the huge number of the massive tower of states as well as obtaining the Higgs field in a representation which is not the adjoint. Soon after this work inspired similar approaches also based on extending the four-dimensional space to become noncommutative [43, 44, 45, 46, 23].

In this survey we will review the key developments that allowed noncommutative geometry to deepen our understanding of the structure of space-time and explain from first principles why and how nature dictates the existence of the elementary particles and their fundamental interactions. In Section 2 we will start by reviewing the pioneering work of Alain Connes [28] introducing the basic mathematical definitions and structures needed to define a noncommutative space. We summarize the characteristic ingredients in the construction of the Connes–Lott model and later generalizations by others. We then consider how to develop the analogue of Riemannian geometry for non-commutative spaces, and to incorporate the gravitational field in the Connes–Lott model. In Section 3 we present a breakthrough in the development of noncommutative geometry with the introduction of the reality operator which led to the definition of KO dimension of a noncommutative space. With this it became possible to present the reconstruction theorem of Riemannian geometry from noncommutative geometry. Section 4 covers the formulation and applications of the spectral action principle where the spectrum of the Dirac operator plays a dominant role in the study of noncommutative spaces. This key development allowed to obtain the dynamics of the Standard Model coupled to gravity in a non-ambiguous way, and to study geometric invariants of noncommutative spaces. We then show that incorporating right-handed neutrinos with the fundamental fermions forces a change in the algebra of the noncommutative space and the use of real structures to impose simultaneously the reality and chirality conditions on fundamental states, singling out the KO dimension to be 6. We show in detail how the few requirements about KO dimension, Majorana masses for right-handed neutrinos and the first order condition on the Dirac operator, singles out the geometry of the Standard Model. In Section 5 we present

a classification of finite noncommutative spaces of KO dimension 6 showing the almost uniqueness of the Standard spectral model. In Section 6 we give a prescription of constructing spectral models from first principles and show that the spectral Standard Model agrees with the available experimental limits, provided that the scale giving mass to the right-handed neutrinos is promoted to a singlet scalar field. We then show that there exists a more general case where the first order condition on the Dirac operator is removed, the singlet scalar fields become part of a larger representation of the Pati–Salam model. The Standard Model becomes a special point in the spontaneous breaking of the Pati–Salam symmetries. In Section 6 we show that a different starting point where a Heisenberg like quantization condition in terms of the Dirac operator considered as momenta and two possible Clifford-algebra valued maps from the four-dimensional manifold to two four-spheres S^4 result in noncommutative spaces with quantized volumes. The Pati–Salam model and its various truncations are uniquely determined as the symmetries of the spaces solving the constraint. Section 7 contains the conclusions and a discussion of possible directions of future research.

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2 Early days of the spectral Standard Model

The first serious attempt to utilize the ideas of noncommutative geometry in particle physics was made by Alain Connes in 1988 in his paper "Essay on physics and noncommutative geometry" [28]. He observed that it is possible to change the structure of the (Euclidean) space-time so that the action functional gives the Weinberg-Salam model. The main emphasis was on the conceptual understanding of the Higgs field, which he calls, the black box of the standard model. The qualitative picture was taken to be of a two-sheeted Euclidean space-time separated by a distance of the order of 10^{-16} cm. In order to simplify the presentation, and to easily follow the historical development, we will use a uniform notation, representing old results in a new format. It is therefore more efficient to start with the basic definitions.

2.1 Noncommutative spaces and differential calculus

A noncommutative space is determined from the spectral data $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ where \mathcal{A} is an associative algebra with unit element 1 and involution $*$, \mathcal{H} a Hilbert space carrying a faithful representation π of the algebra, D is a self-adjoint operator on \mathcal{H} with $(D^2 + 1)^{-1}$ compact, γ is the unitary chirality operator and J an anti-unitary operator on \mathcal{H} , the reality structure. The operator J was introduced later in 1994 [30].

In the model proposed in 1988, there were ambiguities in defining the algebra and the action on the Hilbert space. These were rectified in the 1990 paper [33] with John Lott, in what became known as the Connes–Lott model. In order to appreciate the enormous progress made over the years, we will summarize this model in a simplified presentation. A more detailed account can be found in the early reviews [77, 66, 54, 55, 56, 57]. Note that at around the same time a derivation based differential calculus was introduced by others in [43, 44, 45, 46] with many similarities to the model proposed by Connes in 1988.

We first need to first introduce new ingredients. Given a unital involutive algebra \mathcal{A} , the universal differential algebra over \mathcal{A} is defined as

$$\Omega^*(\mathcal{A}) = \bigoplus_{n=0}^{\infty} \Omega^n(\mathcal{A})$$

where we set $\Omega^0(\mathcal{A}) = \mathcal{A}$, and take

$$\Omega^n(\mathcal{A}) = \left\{ \sum_i a_0^i da_1^i da_2^i \cdots da_n^i, \quad a_j^i \in \mathcal{A}, \quad \forall i, j \right\}, \quad n = 1, 2, \dots$$

Here da denotes an equivalence class of \mathcal{A} , modulo the following relations

$$d(a \cdot b) = da \cdot b + a \cdot db, \quad d1 = 0, \quad d^2 = 0$$

An element of $\Omega^n(\mathcal{A})$ is called a form of degree n . One forms can be considered as connections on a line bundle whose space of sections is given by the algebra \mathcal{A} . A one form $\rho \in \Omega^1(\mathcal{A})$ is expressed in the form

$$\rho = \sum_i a^i db^i, \quad a^i, b^i \in \mathcal{A}$$

and since $d1 = 0$, we may impose the condition $\sum_i a^i b^i = 1$, without any loss in generality. We say that (\mathcal{H}, D) is a Dirac K-cycle for \mathcal{A} if and only if there

exists an involutive representation π of \mathcal{A} on \mathcal{H} satisfying $\pi(a)^* = \pi(a^*)$ with the properties that $\pi(a)$ and $[D, \pi(a)]$ are bounded operators on \mathcal{H} for all $a \in \mathcal{A}$. The K-cycle is called even if there exists a chirality operator γ such that $\gamma D = -D\gamma$, $\gamma = \gamma^{-1} = \gamma^*$ and $[\gamma, \pi(a)] = 0$, otherwise it is odd. The action of π on $\Omega^*(\mathcal{A})$ is defined as

$$\pi \left(\sum_i a_0^i da_1^i \cdots da_n^i \right) = \sum_i \pi(a_0^i) [D, \pi(a_1^i)] \cdots [D, \pi(a_n^i)]$$

The space of auxiliary fields is defined by

$$\text{Aux} = \ker \pi + d \ker \pi$$

where

$$\ker \pi = \bigoplus_{n=0}^{\infty} \left\{ \sum_i a_0^i da_1^i \cdots da_n^i : \pi \left(\sum_i a_0^i da_1^i \cdots da_n^i \right) = 0 \right\}$$

and

$$d \ker \pi = \bigoplus_{n=0}^{\infty} \left\{ \sum_i da_0^i da_1^i \cdots da_n^i : \pi \left(\sum_i a_0^i da_1^i \cdots da_n^i \right) = 0 \right\}$$

The integral of a form $\alpha \in \Omega^*(\mathcal{A})$ over a noncommutative space of metric dimension d is defined by setting

$$\int \alpha = \text{Tr}_w \left(\pi(\alpha) D^{-d} \right)$$

where Tr_w is the Dixmier trace.

2.2 Two-sheeted spacetime

A simple extension of space-time is taken as a product of continuous four-dimensional manifold times a discrete set of two points. The algebra is $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ acting on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ where $\mathcal{A}_1 = C^\infty(M)$ and $\mathcal{A}_2 = M_2(\mathbb{C}) \oplus M_1(\mathbb{C})$, the algebra of 2×2 and 1×1 matrices. The Hilbert space is that of spinors of the form

$$L = \begin{pmatrix} l \\ e \end{pmatrix}$$

where l is a doublet and e is a singlet. The spinor L satisfies the chirality condition $\gamma_5 \otimes \Gamma_1 L = L$ where $\Gamma_1 = \text{diag}(1_2, -1)$ is a grading operator.

From this we deduce that l is a left-handed spinor and e is right handed, and we thus write $l = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$ and $e = e_R$. The Dirac operator is given by $D = D_1 \otimes 1 + \gamma_5 \otimes D_2$ where $D_1 = \gamma^\mu \partial_\mu$ and D_2 is the Dirac operator on \mathcal{A}_2 such that

$$D_l = \begin{pmatrix} \gamma^\mu \partial_\mu \otimes 1_2 \otimes 1_3 & \gamma_5 \otimes M_{12} \otimes k \\ \gamma_5 \otimes M_{21} \otimes k^* & \gamma^\mu \partial_\mu \otimes 1 \otimes 1_3 \end{pmatrix}$$

where $M_{21} = M_{12}^*$ and k is a 3×3 family mixing matrix representing Yukawa couplings for the leptons. The 1×2 matrix M_{12} turns out to be the vev of the Higgs field and is taken as $M_{12} = \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} = H_0$. The elements $a \in \mathcal{A}$ have the representation $a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ where a_1, a_2 are 2×2 and 1×1 unitary valued functions. A quick calculation shows that the self-adjoint one-form ρ has the representation

$$\pi_1(\rho) = \begin{pmatrix} A_1 \otimes 1_3 & \gamma_5 \otimes H \otimes k \\ \gamma_5 \otimes H^* \otimes k^* & A_2 \otimes 1_3 \end{pmatrix}$$

where

$$A_1 = \gamma^\mu \sum_i a_1^i \partial_\mu b_1^i, \quad A_2 = \gamma^\mu \sum_i a_2^i \partial_\mu b_2^i, \\ H = H_0 + \sum_i a_1^i H_0 b_2^i.$$

The quarks are introduced by taking for the finite space a bimodule structure relating two algebras \mathcal{A} and \mathcal{B} where the algebra \mathcal{B} is taken to be $M_1(\mathbb{C}) \oplus M_3(\mathbb{C})$ commuting with the action of \mathcal{A} . In addition, the mass matrices in the Dirac operator are taken to be zero when acting on elements of \mathcal{B} . The one-form $\eta \in \Omega^1(\mathcal{B})$ has the simple form $B_1 \text{diag}(1_2, 1)$ where B_1 is a gauge field associated with $M_1(\mathbb{C})$. The Hilbert space for the quarks is

$$Q = \begin{pmatrix} q_L \\ u_R \\ d_R \end{pmatrix}, \quad q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$$

The representation of $a \in \mathcal{A}$ is $a \rightarrow (a_1, a_2, \bar{a}_2)$ where a_1 and a_2 are a 2×2 and 1×1 complex valued functions. The Dirac operator acting on the quark Hilbert space is

$$D_q = \begin{pmatrix} \gamma^\mu (\partial_\mu + \dots) \otimes 1_2 \otimes 1_3 & \gamma_5 \otimes M_{12} \otimes k' & \gamma_5 \otimes \widetilde{M}_{12} \otimes k'' \\ \gamma_5 \otimes M_{12}^* \otimes k'^* & \gamma^\mu (\partial_\mu + \dots) \otimes 1_3 & 0 \\ \gamma_5 \otimes \widetilde{M}_{12}^* \otimes k''^* & 0 & \gamma^\mu (\partial_\mu + \dots) \otimes 1_3 \end{pmatrix}$$

where k' and k'' are 3×3 family mixing matrices and $\widetilde{M}_{12} = \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The one form in $\Omega^1(\mathcal{A})$ has then the representation

$$\pi_q(\rho) = \begin{pmatrix} A_1 \otimes 1_3 & \gamma_5 \otimes H \otimes k' & \gamma_5 \otimes \widetilde{H} \otimes k'' \\ \gamma_5 \otimes H^* \otimes k'^* & A_2 \otimes 1_3 & 0 \\ \gamma_5 \otimes \widetilde{H}^* \otimes k''^* & 0 & \overline{A}_2 \otimes 1_3 \end{pmatrix}$$

where $\widetilde{H}_a = \varepsilon_{ab} H^b$. When acting on the algebra \mathcal{B} the Dirac operator has zero mass matrices and the one-form η in $\Omega^1(\mathcal{B})$ has the representation $\pi_q(\eta) = B_2 \text{diag}(1_2, 1)$ where B_2 is the gauge field associated with $M_3(\mathbb{C})$. Imposing the unimodularity condition on the algebras \mathcal{A} and \mathcal{B} would then relate the $U(1)$ factors in both algebras so that $\text{tr}(A_1) = 0$, $A_2 = B_1 = -\text{tr}(B_2) \equiv \frac{i}{2} g_1 B$. With these we can then write

$$A_1 = -\frac{i}{2} g_2 \sigma^a A_a$$

$$B_2 = -\frac{i}{6} g_1 B - \frac{i}{2} g_3 V^i \lambda_i$$

where g_1 , g_2 and g_3 are the $U(1)$, $SU(2)$ and $SU(3)$ gauge coupling constants, σ^a and λ^i are the Pauli and Gell-Mann matrices respectively. The fermionic actions for the leptons and quarks are then given by

$$\langle L, (D + \rho + \eta) L \rangle = \int d^4x \sqrt{g} (\overline{L} (D_l + \pi_l(\rho) + \pi_l(\eta)) L)$$

$$\langle Q, (D + \rho + \eta) Q \rangle = \int d^4x \sqrt{g} (\overline{Q} (D_q + \pi_q(\rho) + \pi_q(\eta)) Q)$$

These terms can be easily checked to reproduced all the fermionic terms of the Standard Model.

The bosonic action is the sum of the square of curvatures in both the lepton and quark sectors. These are given by

$$I_l = \text{Tr} \left(C_l (\theta_\rho + \theta_\eta)^2 D_l^{-4} \right)$$

$$I_q = \text{Tr} \left(C_q (\theta_\rho + \theta_\eta)^2 D_q^{-4} \right)$$

where

$$\theta_\rho \equiv d\rho + \rho^2$$

is the curvature of ρ , and C_l and C_q are constant elements of the algebra. Since the representation π has a kernel, the auxiliary fields must be projected

out. This step mainly affects the potential. After some algebra one can show that the bosonic action given above reproduces all the bosonic interactions of the Standard Model with the same number of parameters. If one assumes that C_l and C_q belong to the center of the algebra, then one can get fixed values for the top quark mass and Higgs mass. The main advantage of the noncommutative construction of the Standard Model is that one gets a geometrical understanding of the origin of the Higgs field and a unification of the gauge and Higgs sectors. One sees that the Higgs fields are the components of the one form along discrete directions.

2.3 Constructions beyond the Standard Model

The early constructions of the Standard Model provided encouragements to look further into noncommutative spaces. The construction was also complicated with some ambiguities such as the independence of the lepton and quark sectors, the construction of the Higgs potential and projecting out the auxiliary fields. It was then natural to ask whether it is possible to go beyond the Standard Model. In particle physics the route taken was to consider larger groups such as $SU(5)$ or $SO(10)$ which contains $U(1) \times SU(2) \times SU(3)$ as a subgroup. The main advantage of GUT is that the fermionic fields are unified in one or two representations, the most attractive possibility being $SO(10)$ where the spinor representation 16_s contains all the known fermions in addition to the right-handed neutrino. The simplicity in the fermionic sector did not make the theory more predictive because of the arbitrariness of the Higgs sector. There are many possible Higgs representations that can break the symmetry spontaneously from $SO(10)$ to $SU(3) \times U(1)$. In the noncommutative construction the Higgs sector is more constrained which was taken as an encouragement to explore the possibility of considering larger matrix algebras. As an example if one arranges the leptons in the form $L = \begin{pmatrix} l_L \\ l_R \end{pmatrix}$ where $l = \begin{pmatrix} \nu \\ e \end{pmatrix}$ then the corresponding algebra will be $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$. A natural possibility is then to consider a discrete space of four points and where the fermions are arranged in the

format $\psi = \begin{pmatrix} l_L \\ l_R \\ l_L^c \\ l_R^c \end{pmatrix}$ and the representation π acting on \mathcal{A} is given by

$\pi(a) = \text{diag}(a_1, a_2, \bar{a}_1, \bar{a}_2)$ where a_1, a_2 are 2×2 complex matrices. The resulting model has $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ with the Higgs fields in the representations $(2, 2), (3, 1) + (1, 3)$ of $SU(2)_L \times SU(2)_R$. We can

summarize the steps needed to construct noncommutative particle physics models. First we specify the fermion representations then we choose the number of discrete points and the symmetry between them. From this we deduce the appropriate algebra and the map π acting on the Hilbert space of spinors. Finally we write down the Dirac operator acting on elements of the algebra and choose the mass matrices to correspond to the desired vacuum of the Higgs fields.

To illustrate these steps consider the chiral space-time spinors $P_+\psi$ to be in the 16_s representation of $SO(10)$, where P_+ is the $SO(10)$ chirality operator, and the number of discrete points to be four. The Hilbert space

is taken to be $\Psi = \begin{pmatrix} P_+\psi \\ P_+\psi \\ P_-\psi^c \\ P_-\psi^c \end{pmatrix}$ where $\psi^c = BC\bar{\psi}^T$, C being the charge

conjugation matrix while B is the $SO(10)$ conjugation matrix. The finite algebra is taken to be $\mathcal{A}_2 = P_+(\text{Cliff } SO(10))P_+$, and the finite Hilbert space $\mathcal{H}_2 = \mathbb{C}^{32}$. Let π_0 denote the representation of the algebra \mathcal{A} on the Hilbert space \mathcal{H} and let $\bar{\pi}_0$ denote the anti representation defined by $\bar{\pi}_0(a) = B\pi_0(\overline{a})B^{-1}$. We then define $\pi(a) = \pi_0(a) \oplus \pi_0(a) \oplus \bar{\pi}_0(a) \oplus \bar{\pi}_0(a)$. The Dirac operator is taken to be

$$\begin{pmatrix} \gamma^\mu \partial_\mu \otimes 1_{32} \otimes 1_3 & \gamma_5 \otimes M_{12} \otimes K_{12} & \gamma_5 \otimes M_{13} \otimes K_{13} & \gamma_5 \otimes M_{14} \otimes K_{14} \\ \gamma_5 \otimes M_{12}^* \otimes K_{12}^* & \gamma^\mu \partial_\mu \otimes 1_{32} \otimes 1_3 & \gamma_5 \otimes M_{23} \otimes K_{23} & \gamma_5 \otimes M_{24} \otimes K_{24} \\ \gamma_5 \otimes M_{13}^* \otimes K_{13}^* & \gamma_5 \otimes M_{23}^* \otimes K_{23}^* & \gamma^\mu \partial_\mu \otimes 1_{32} \otimes 1_3 & \gamma_5 \otimes M_{34} \otimes K_{34} \\ \gamma_5 \otimes M_{14}^* \otimes K_{14}^* & \gamma_5 \otimes M_{24}^* \otimes K_{24}^* & \gamma_5 \otimes M_{34}^* \otimes K_{34}^* & \gamma^\mu \partial_\mu \otimes 1_{32} \otimes 1_3 \end{pmatrix}$$

where the K_{mn} are 3×3 family mixing matrices commuting with $\pi(a)$. We may impose the exchange symmetries $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ so that $M_{12} = M_{12}^* = \mathcal{M}_0$, $M_{13} = M_{14} = M_{23} = M_{24} = \mathcal{N}_0$, $M_{34} = M_{34}^* = B\overline{\mathcal{M}}_0B^{-1}$. Computing $\pi(\rho)$ we get

$$\pi(\rho) = \begin{pmatrix} A & \gamma_5 \mathcal{M} K_{12} & \gamma_5 \mathcal{N} K_{13} & \gamma_5 \mathcal{N} K_{14} \\ \gamma_5 \mathcal{M} K_{12}^* & A & \gamma_5 \mathcal{N} K_{23} & \gamma_5 \mathcal{N} K_{24} \\ \gamma_5 \mathcal{N}^* K_{13}^* & \gamma_5 \mathcal{N}^* K_{23}^* & B\overline{A}B^{-1} & \gamma_5 B\overline{\mathcal{M}}B^{-1} K_{34} \\ \gamma_5 \mathcal{N}^* K_{14}^* & \gamma_5 \mathcal{N}^* K_{24}^* & \gamma_5 B\overline{\mathcal{M}}B^{-1} K_{34}^* & B\overline{A}B^{-1} \end{pmatrix}$$

where

$$\begin{aligned}
A &= P_+ \sum_i a^i \gamma^\mu \partial_\mu b^i P_+ \\
\mathcal{M} + \mathcal{M}_0 &= P_+ \sum_i a^i \mathcal{M}_0 b^i P_+ \\
\mathcal{N} + \mathcal{N}_0 &= P_+ \sum_i a^i \mathcal{N}_0 B \bar{b}^i B^{-1} P_-
\end{aligned}$$

One sees immediately that the Higgs fields \mathcal{M} and \mathcal{N} are in the $16_s \times 16_s$ and $16_s \times \bar{16}_s$ representations. Equating the action of A on ψ and ψ^c will reduce it to an $SO(10)$ gauge field. Specifying \mathcal{M}_0 and \mathcal{N}_0 determines the breaking pattern of $SO(10)$. One can then proceed to construct the bosonic sector and project out the auxiliary fields to determine the potential. There are very limited number of models one can construct. These models, however, will suffer the same problems encountered in the GUT construction, mainly that of low unification scale of 10^{14} GeV implying fast rate of proton decay which is ruled out experimentally.

2.4 Coupling matter to gravity

The dynamics of the gravitational force is based on Riemannian geometry. It is therefore natural to study the nature of the gravitational field in noncommutative geometry. The original attempt [24, 25] was based on generalizing the basic notions of Riemannian geometry, notably the theory of linear connections on differential forms. (Note that an alternative route that takes vector fields as a starting point ends with a derivation based differential calculus as in [43] (*cf.* [65])). In line with the Connes–Lott model, we will instead take differential forms as our starting point. For more details we also refer to the exposition in [60, Sect. 10.3]).

First one defines the metric as an inner product on a cotangent space. Then one shows that every cycle over \mathcal{A} yields a notion of cotangent bundle associated with \mathcal{A} and a Riemannian metric on the cotangent bundle $\Omega_D^1(\mathcal{A})$. With the connection ∇ the Riemann curvature of ∇ on $\Omega_D^1(\mathcal{A})$ is defined by $R(\nabla) := -\nabla^2$ and the torsion by $T = d - m \circ \nabla$ where m is the tensor product. Requiring ∇ to be unitary and the torsion to vanish we obtain the Levi–Civita connection. If $\Omega_D^1(\mathcal{A})$ is a finitely generated module, then it admits a basis e^A , $A = 1, 2, \dots, N$, and the connection $\omega_B^A \in \Omega_D^1(\mathcal{A})$ is defined by $\nabla e^A = -\omega_B^A \otimes e^B$. The components of the torsion $T(\nabla)$ are defined by $T^A = T(\nabla) e^A$ then $T^A \in \Omega_D^2(\mathcal{A})$ is given by

$$T^A = de^A + \omega_B^A e^B$$

Similarly, components of the curvature $R_B^A \in \Omega_D^2(\mathcal{A})$ satisfy the defining property that $R(\nabla)e^A = R_B^A \otimes e^B$ so that

$$R_B^A = d\omega_B^A + \omega_C^A \omega_B^C.$$

The analogue of the Einstein–Hilbert action is then

$$I(\nabla) := \kappa^{-2} \langle R_B^A e^B, e_A \rangle$$

where κ^{-1} is the Planck scale. Computing this action for the product space $M_4 \times Z_2$ one finds that

$$I(\nabla) = 2 \int_M d^4x \sqrt{g} (\kappa^{-2} r - 2 \partial_\mu \sigma \partial^\mu \sigma)$$

where r is the scalar curvature of the Levi–Civita connection of the Riemannian manifold M_4 coupled to a scalar field σ . Applying this construction to the Connes–Lott model is rather involved because the two sheets are not treated symmetrically, being associated with two different algebras. The complication arise because the projective module is not free and the basis e^A is constrained. The Einstein–Hilbert action in this case is given by

$$I(\nabla) = 2 \int_M d^4x \sqrt{g} \left(\kappa^{-2} \frac{3}{2} r - 2(3 + \lambda) \partial_\mu \sigma \partial^\mu \sigma + c(\lambda) e^{-2\sigma} \right)$$

where $\lambda = \text{Tr}(kk^*)^2 - 1$. To understand the significance of the field σ , we note that by examining the Dirac operator one finds that the field $\phi = e^{-\kappa\sigma}$ now replaces the weak scale. Thus quantum corrections to the classical potential will depend on σ , thus the vev of σ could be determined from the minimization equations.

3 The spectral action principle

Despite the success of the Connes–Lott model and the generalizations that followed in giving a geometrical meaning to the Higgs field and unifying it with the gauge fields, it was felt that the construction is not satisfactory. The first unpleasant feature was the use of the bimodule structure to introduce the $SU(3)$ symmetry and the second is the use of unimodularity condition to get the correct hypercharge assignments to the particles. Another major problem was the existence of mirror fermions as a consequence of the fact that the conjugation operator on fermions gives independent fields. In addition, there was arbitrariness in the construction of the potential in the bosonic sector associated with the step of eliminating the auxiliary fields.

3.1 Real structures on spectral triples

The first breakthrough came in 1995 with the publication of Alain Connes' paper "Noncommutative geometry and reality" [30]. In this paper, the notion of real structure is introduced, motivated by Atiyah's KR theory and Tomita's involution operator J . A hint for the necessity of the reality operator can be taken from physics. We have seen that space-time spinors, which are elements of the Hilbert space satisfy a chirality condition. The charge conjugation operator, when acting on these spinors, produces a conjugate element, which in general is independent. It is possible to replace the chirality condition, with a reality one, known as the Majorana condition which equates the two. Imposing both conditions, chirality and reality, simultaneously can only occur in certain dimensions. The action of the anti-linear isometry J on the algebra \mathcal{A} satisfies the commutation relation $[a, b^\circ] = 0$, $\forall a, b \in \mathcal{A}$ where

$$b^\circ = Jb^*J^{-1}, \quad \forall b \in \mathcal{A} \quad (1)$$

so that $b^\circ \in \mathcal{A}^\circ$. This gives a bimodule, using the representation of $\mathcal{A} \otimes \mathcal{A}^\circ$, given by

$$a \otimes b^\circ \rightarrow aJb^*J^{-1}, \quad \forall a, b \in \mathcal{A} \quad (2)$$

We define the fundamental class μ of the noncommutative space as a class in the KR -homology of the algebra $\mathcal{A} \otimes \mathcal{A}^\circ$ having the involution

$$\tau(a \otimes b^\circ) = b^* \otimes (a^*)^\circ, \quad \forall a, b \in \mathcal{A} \quad (3)$$

The KR -homology cycle implements the involution τ given by

$$\tau(w) = JwJ^{-1}, \quad \forall w \in \mathcal{A} \otimes \mathcal{A}^\circ \quad (4)$$

These imply that the KR -homology is periodic with period 8 and the dimension n modulo 8 is determined from the commutation rules

$$J^2 = \varepsilon, \quad JD = \varepsilon'DJ, \quad J\gamma = \varepsilon''\gamma J \quad (5)$$

where $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ are given as function of n modulo 8 according to the table

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

(6)

It is not surprising that this table agrees with the one obtained by classifying in which dimensions a spinor obey the Majorana and Weyl conditions. The

intersection form $K_*(\mathcal{A}) \times K_*(\mathcal{A}) \rightarrow \mathbb{Z}$ is obtained from the Fredholm index of D in $K_*(\mathcal{A} \otimes \mathcal{A}^o)$. Using the Kasparov intersection product, Poincare duality is formulated in terms of the invertibility of μ and that D is an operator of order one implies the condition

$$[[D, a], b^o] = 0, \quad \forall a, b \in \mathcal{A} \quad (7)$$

Next we consider automorphisms of the algebra \mathcal{A} denoted by $\text{Aut}(\mathcal{A})$. This comprises both of inner and outer automorphisms. Inner automorphisms $\text{Int}(\mathcal{A})$ is a normal subgroup of $\text{Aut}(\mathcal{A})$ defined by

$$\alpha(f) = ufu^*, \quad \forall f \in \mathcal{A}, \quad uu^* = u^*u = 1 \quad (8)$$

The group $\text{Aut}^+(\mathcal{A})$ of automorphisms of the involutive algebra \mathcal{A} are implemented by a unitary operator U in \mathcal{H} commuting with J satisfying

$$\alpha(x) = UxU^{-1} \quad \forall x \in \mathcal{A} \quad (9)$$

For Riemannian manifolds M , this plays the role of the group of diffeomorphisms $\text{Diff}^+(M)$, which preserves the K -homology fundamental class of M . Let \mathcal{E} be a finite projective, hermitian right \mathcal{A} -module, and define the algebra $\mathcal{B} = \text{End}(\mathcal{A})$ as the Morita equivalence of the algebra \mathcal{A} with a hermitian connection ∇ on \mathcal{E} defined as the linear map $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1$ satisfying

$$\begin{aligned} \nabla(\zeta a) &= (\nabla\zeta)a + \zeta \otimes da, & \forall \zeta \in \mathcal{E}, a \in \mathcal{A} \\ d(\zeta, \eta) &= (\zeta, \nabla\eta) - (\nabla\zeta, \eta), & \forall \zeta, \eta \in \mathcal{E} \end{aligned}$$

where $da = [D, a]$ and Ω_D^1 is the bimodule of operators of the form

$$A = \sum_i a_i [D, b_i], \quad a_i, b_i \in \mathcal{A} \quad (10)$$

Since any algebra is Morita equivalent to itself with $\mathcal{E} = \mathcal{A}$, applying the construction given above yields the inner deformation of the spectral geometry. The unitary equivalence is implemented by the representation $u \rightarrow \tilde{U} = u(JuJ^{-1}) = u(u^o)^*$ so that the Dirac operator that includes inner fluctuations

$$D_A = D + A + JAJ^{-1} \quad (11)$$

where $A = A^*$ transforms as $D_A \rightarrow \tilde{U}D_A\tilde{U}^{-1}$ provided that

$$A \rightarrow uAu^* + u[D, u^*] \quad (12)$$

This will ensure that the inner product

$$(\Psi, D_A \Psi) \tag{13}$$

is invariant under the transformation $\Psi \rightarrow \tilde{U}\Psi$. This expression will then take care of all fermionic interactions which, as will be seen in the next section, removes the arbitrariness in specifying the action of the connection on the Hilbert space.

3.2 The spectral action principle

The next breakthrough came a year later in 1996 in the work of Chamseddine and Connes entitled “The spectral action principle” [12]. The basic observation is that for a noncommutative space defined by spectral data, the emphasis is shifted from the coordinates x of a geometric space to the spectrum $\Sigma \sqsubset \mathbb{R}$ of the operator D . We postulate the following hypothesis

$$\text{The physical action depends only on } \Sigma \tag{14}$$

The existence of Riemannian manifolds which are isospectral but not isometric shows that the spectral action principle is stronger than the usual diffeomorphism invariance. In the usual Riemannian case the group $\text{Diff}(M)$ of diffeomorphisms of M is canonically isomorphic to the group $\text{Aut}(\mathcal{A})$ of automorphisms of the algebra $\mathcal{A} = C^\infty(M)$. To each $\varphi \in \text{Diff}(M)$ one associates the algebra preserving map $\alpha_\varphi : \mathcal{A} \rightarrow \mathcal{A}$ given by

$$\alpha_\varphi(f) = f \circ \varphi^{-1} \quad \forall f \in C^\infty(M) = \mathcal{A} \tag{15}$$

The prescription to determine the bosonic action with some cutoff energy scale Λ is to first replace the Hilbert space \mathcal{H} by the subspace \mathcal{H}_Λ defined by

$$\mathcal{H}_\Lambda = \text{range } \chi \left(\frac{D}{\Lambda} \right) \tag{16}$$

where χ is a suitable smooth positive function, restricting both D and \mathcal{A} to this subspace maintaining the commutation relations for the algebra. This procedure is superior to the lattice approximation because it does respect the geometric symmetry group. The *spectral action functional* is then given by the

$$\text{Tr } \chi \left(\frac{D}{\Lambda} \right).$$

For a noncommutative space which is a tensor product of a continuous manifold times a discrete space, the functional $\text{Tr } \chi \left(\frac{D}{\Lambda} \right)$ can be expanded in an

asymptotic series in Λ , rendering the computation amenable to a heat kernel expansion. This procedure will be illustrated in the next section. More general methods to analyze the spectral action have also been developed, see [50] for an early result and also the recent book [48]. An interpretation of the spectral action as the von Neumann entropy of a second-quantized spectral triple has been found recently in [20] (*cf.* [42]).

To summarize, the breakthroughs carried out in the short period 1995-1996, defining the reality operator J and developing the spectral action principle will allow to remove the ambiguities encountered before in the construction of the noncommutative spectral Standard Model.

4 The spectral Standard Model

At the time that the spectral action was formulated, it was clear that this principle forms a unifying framework for gravity and particle physics of the Standard Model. As said, this led to much activity (*cf.* [69]) in the years that followed. Also shortcomings of the approach were pointed out quite quickly, such as the notorious fermion-doubling problem [63, 52]. This doubling—or actually, quadrupling—was due to the incorporation of left-right, particle-anti-particle degrees of freedom both in the continuum spinor space and in the finite noncommutative space. At the technical level this was a crucial starting point, allowing for a product geometry to describe gravity coupled to the Standard Model.

Nevertheless, it was a somewhat disturbing feature which, together with the apparent arbitrariness of the choice of a finite geometry and the absence of neutrino mixing in the model, led Connes to eventually resolve these problems in [31]. At the same time John Barrett [4] arrived at the same conclusion (see also the recent uniqueness result [6]), even though his motivation came from the desire to have noncommutative geometry with a Lorentzian signature.

The crucial insight in both of these works is that one should allow for a KO-dimension for the finite space F which is different from the metric dimension (which is zero). More specifically, the KO-dimension of the finite space should be 6 (modulo 8), so that the product of the continuum M with F is 10 modulo 8. The precise structure of the spectral Standard Model (see Section 4.2) is then best understood using the classification of all irreducible finite noncommutative geometries of KO-dimension 6 which we now briefly recall.

4.1 Classification of irreducible geometries

In [14] Chamseddine and Connes classified *irreducible* finite real spectral triples of KO-dimension 6. This lead to a remarkably concise list of spectral triples, based on the matrix algebras $M_N(\mathbb{C}) \oplus M_N(\mathbb{C})$ for some N . We remark that earlier classification results were obtained [58, 68] which were also exploited in a search Beyond the Standard Model (see Remark 5 below).

Definition 1. *A finite real spectral triple $(A, H, D; J, \gamma)$ is called irreducible if the triple (A, H, J) is irreducible. More precisely, we demand that:*

1. *The representations of A and J in H are irreducible;*
2. *The action of A on H has a separating vector.*

We will prove the main result of [14] using an alternative approach which is based on [75, Sect. 3.4].

Theorem 2. *Let $(A, H, D; J, \gamma)$ be an irreducible finite real spectral triple of KO-dimension 6. Then there exists a positive integer N such that $A \simeq M_N(\mathbb{C}) \oplus M_N(\mathbb{C})$.*

Proof. Let $(A, H, D; J, \gamma)$ be an arbitrary finite real spectral triple. We may then decompose

$$A = \bigoplus_{i=1}^N M_{n_i}(\mathbb{C}), \quad H = \bigoplus_{i,j=1}^N \mathbb{C}^{n_i} \otimes (\mathbb{C}^{n_j})^\circ \otimes V_{ij},$$

with V_{ij} corresponding to the multiplicities as before. Now each $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j}$ is an irreducible representation of A , but in order for H to support a real structure $J : H \rightarrow H$ we need both $\mathbb{C}^{n_i} \otimes (\mathbb{C}^{n_j})^\circ$ and $\mathbb{C}^{n_j} \otimes (\mathbb{C}^{n_i})^\circ$ to be present in H . Moreover, an old result of Wigner [78] for an anti-unitary operator with $J^2 = 1$ assures that already with multiplicities $\dim V_{ij} = 1$ there exists such a real structure. Hence, the irreducibility condition (1) above yields

$$H = \mathbb{C}^{n_i} \otimes (\mathbb{C}^{n_j})^\circ \oplus \mathbb{C}^{n_j} \otimes (\mathbb{C}^{n_i})^\circ,$$

for some $i, j \in \{1, \dots, N\}$. Then, let us consider condition (2) on the existence of a separating vector. Note first that the representation of A in H is faithful only if $A = M_{n_i}(\mathbb{C}) \oplus M_{n_j}(\mathbb{C})$. Second, the stronger condition of a separating vector ξ then implies $n_i = n_j$, as it is equivalent to $A'\xi = H$ for the commutant A' of A in H . Namely, since $A' = M_{n_j}(\mathbb{C}) \oplus M_{n_i}(\mathbb{C})$ with $\dim A' = n_i^2 + n_j^2$, and $\dim H = 2n_i n_j$ we find the desired equality $n_i = n_j$. \square

With the complex finite-dimensional algebras A given as a direct sum $M_N(\mathbb{C}) \oplus M_N(\mathbb{C})$,¹ the additional demand that H carries a symplectic structure $I^2 = -1$ yields real algebras of which A is the complexification. We see that this requires $N = 2k$ so that one naturally considers triples (A, H, J) for which

$$A = M_k(\mathbb{H}) \oplus M_{2k}(\mathbb{C}); \quad H = \mathbb{C}^{2(2k)^2}. \quad (17)$$

4.2 Noncommutative geometry of the Standard Model

The above classification of irreducible finite geometries of KO-dimension 6 forms the starting point for the derivation of the Standard Model from a noncommutative manifold [17]. Hence, it is based on the matrix algebra $M_N(\mathbb{C}) \oplus M_N(\mathbb{C})$ for $N \geq 1$. Let us make the following two additional requirements on the irreducible finite geometry $(A, H_F, D_F; J_F, \gamma_F)$:

1. The finite-dimensional Hilbert space H_F carries a symplectic structure $I^2 = -1$;
2. the grading γ_F induces a non-trivial grading on A , by mapping

$$a \mapsto \gamma_F a \gamma_F,$$

and selects an even subalgebra $A^{\text{ev}} \subset A$ consisting of elements that commute with γ_F .

But the first demand sets $A = M_k(\mathbb{H}) \oplus M_{2k}(\mathbb{C})$, represented on the Hilbert space $\mathbb{C}^{2(2k)^2}$. The second requirement sets $k \geq 2$; we will take the simplest $k = 2$ so that $H_F = \mathbb{C}^{32}$.² Indeed, this allows for a γ_F such that

$$A^{\text{ev}} = \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C}),$$

where \mathbb{H}_R and \mathbb{H}_L are two copies (referred to as *right* and *left*) of the quaternions; they are the diagonal of $M_2(\mathbb{H}) \subset A$. The Hilbert space can then be decomposed according to the defining representations of A^{ev} ,

$$H_F = (\mathbb{C}_R^2 \oplus \mathbb{C}_L^2) \otimes (\mathbb{C}^4)^\circ \oplus \mathbb{C}^4 \otimes ((\mathbb{C}_R^2)^\circ \oplus (\mathbb{C}_L^2)^\circ). \quad (18)$$

¹The case $N = 1$ was exploited successfully in [47] for a noncommutative description of abelian gauge theories.

²Also other algebras that appear in the classification of irreducible geometries of KO-dimension have been considered in the literature: besides the case $N = 4$ that we consider here the simplest case $N = 1$ is relevant for the noncommutative geometric description of quantum electrodynamics [47] and the case $N = 8$ leads to the ‘grand algebra’ of [40, 38].

According to this direct sum decomposition, we write

$$D_F = \begin{pmatrix} S & T^* \\ T & \bar{S} \end{pmatrix} \quad (19)$$

Moreover, J_F is the anti-unitary operator that flips the two 16-dimensional components in Equation (18).

The key result is that if we assume that T is non-trivial, then the first-order condition selects the maximal subalgebra of the Standard Model, that is to say, $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$.

Proposition 3. [17, Prop. 2.11] *Up to $*$ -automorphisms of A^{ev} , there is a unique $*$ -subalgebra $A_F \subset A^{\text{ev}}$ of maximal dimension that allows $T \neq 0$ in (19). It is given by*

$$A_F = \left\{ \left(q_\lambda, q, \begin{pmatrix} q & 0 \\ 0 & m \end{pmatrix} \right) : \lambda \in \mathbb{C}, q \in \mathbb{H}_L, m \in M_3(\mathbb{C}) \right\} \subset \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C}),$$

where $\lambda \mapsto q_\lambda$ is the embedding of \mathbb{C} into \mathbb{H} , with

$$q_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}.$$

Consequently, $A_F \simeq \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$.

The restriction of the representation of A on H_F to the subalgebra A_F gives a decomposition of H_F into irreducible (left and right) representations of \mathbb{C} , \mathbb{H}_L and $M_3(\mathbb{C})$. For instance,

$$(\mathbb{C}_R^2 \oplus \mathbb{C}_L^2) \otimes (\mathbb{C}^4)^\circ \rightsquigarrow (\mathbb{C} \oplus \bar{\mathbb{C}} \oplus \mathbb{C}_L^2) \otimes ((\mathbb{C})^\circ \oplus (\mathbb{C}^3)^\circ). \quad (20)$$

and similarly for $\mathbb{C}^4 \otimes ((\mathbb{C}_R^2)^\circ \oplus (\mathbb{C}_L^2)^\circ)$. In order to connect to the physics of the Standard Model, let us introduce an orthonormal basis for H_F that can be recognized as the fermionic particle content of the Standard Model, and subsequently write the representation of A_F in terms of this basis.

We let the subspace of H_F displayed in Equation (20) be represented by basis vectors $\{\nu_R, e_R, (\nu_L, e_L)\}$ of the so-called *lepton space* H_l and basis vectors $\{u_R, d_R, (u_L, d_L)\}$ of the *quark space* H_q . Their reflections with respect to J_F are the *anti-lepton space* $H_{\bar{l}}$ and the *anti-quark space* $H_{\bar{q}}$, spanned by $\{\bar{\nu}_R, \bar{e}_R, (\bar{\nu}_L, \bar{e}_L)\}$ and $\{\bar{u}_R, \bar{d}_R, (\bar{u}_L, \bar{d}_L)\}$, respectively. The three colors of the quarks are given by a tensor factor \mathbb{C}^3 and when we take into account *three generations* of fermions and anti-fermions by tripling the above finite-dimensional Hilbert space we obtain

$$H_F := (H_l \oplus H_{\bar{l}} \oplus H_q \oplus H_{\bar{q}})^{\oplus 3}.$$

Note that $H_l = \mathbb{C}^4$, $H_q = \mathbb{C}^4 \otimes \mathbb{C}^3$, $H_{\bar{l}} = \mathbb{C}^4$, and $H_{\bar{q}} = \mathbb{C}^4 \otimes \mathbb{C}^3$.

An element $a = (\lambda, q, m) \in A_F$ acts on the space of leptons H_l as $q_\lambda \oplus q$, and acts on the space of quarks H_q as $(q_\lambda \oplus q) \otimes 1_3$. For the action of a on an anti-lepton $\bar{l} \in H_{\bar{l}}$ we have $a\bar{l} = \lambda 1_4 \bar{l}$, and on an anti-quark $\bar{q} \in H_{\bar{q}}$ we have $a\bar{q} = (1_4 \otimes m)\bar{q}$.

The \mathbb{Z}_2 -grading γ_F is such that left-handed particles have eigenvalue $+1$ and right-handed particles have eigenvalue -1 . The anti-linear operator J_F interchanges particles with their anti-particles, so $J_F f = \bar{f}$ and $J_F \bar{f} = f$, with f a lepton or quark.

The first indication that the subalgebra A_F is relevant for the Standard Model—to say the least—comes from the fact that the Standard Model gauge group can be derived from the unitaries in A_F . We restrict to the *unimodular gauge group*,

$$\text{SU}(A_F) = \{u \in A_F : u^* u = u u^* = 1, \det(u) = 1\}$$

where \det is the determinant of the action of u in H_F . It then follows that, up to a finite abelian group we have

$$\text{SU}(A_F) \sim U(1) \times \text{SU}(2) \times \text{SU}(3)$$

and the hypercharges are derived from the unimodularity condition to be the usual ones:

Particle	ν_R	e_R	ν_L	e_L	u_R	d_R	u_L	d_L
Hypercharge	0	-2	-1	-1	$\frac{4}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Let us now turn to the form of the finite Dirac operator, and see what we can say about the components of the matrix D_F as displayed in (19). Recall that we are looking for a self-adjoint operator D_F in H_F that commutes with J_F , anti-commutes with γ_F , and fulfills the first-order conditions with respect to A_F :

$$[[D, a], JbJ^{-1}] = 0; \quad (a, b \in A_F).$$

We also require that D_F commutes with the subalgebra $\mathbb{C}_F = \{(\lambda, \lambda, 0)\} \subset A_F$ which physically speaking corresponds to the fact that the photon remains massless. Then it turns out [31, Theorem 1] (see also [17, Theorem 2.21]) that any D_F that satisfies these assumptions is of the following form:

in terms of the decomposition of H_F in particle ($H_l^{\oplus 3} \oplus H_q^{\oplus 3}$) and anti-particles ($H_{\bar{l}}^{\oplus 3} \oplus H_{\bar{q}}^{\oplus 3}$) the operator S is

$$S_l := S|_{H_l^{\oplus 3}} = \begin{pmatrix} 0 & 0 & Y_\nu^* & 0 \\ 0 & 0 & 0 & Y_e^* \\ Y_\nu & 0 & 0 & 0 \\ 0 & Y_e & 0 & 0 \end{pmatrix}, \quad (21)$$

$$S_q \otimes 1_3 := S|_{H_q^{\oplus 3}} = \begin{pmatrix} 0 & 0 & Y_u^* & 0 \\ 0 & 0 & 0 & Y_d^* \\ Y_u & 0 & 0 & 0 \\ 0 & Y_d & 0 & 0 \end{pmatrix} \otimes 1_3, \quad (22)$$

where Y_ν , Y_e , Y_u and Y_d are some 3×3 matrices acting on the three generations, and 1_3 acting on the three colors of the quarks. The symmetric operator T only acts on the right-handed (anti)neutrinos, so it is given by $T\nu_R = Y_R \bar{\nu}_R$, for a certain 3×3 symmetric matrix Y_R , and $Tf = 0$ for all other fermions $f \neq \nu_R$. Note that ν_R here stands for a vector with 3 components for the number of generations.

The above classification result shows that the Dirac operators D_F give all the required features, such as mixing matrices for quarks and leptons, unbroken color and the see-saw mechanism for right-handed neutrinos. Let us illustrate the latter in some more detail. The mass matrix restricted to the subspace of H_F with basis $\{\nu_L, \nu_R, \bar{\nu}_L, \bar{\nu}_R\}$ is given by

$$\begin{pmatrix} 0 & Y_\nu^* & Y_R^* & 0 \\ Y_\nu & 0 & 0 & 0 \\ Y_R & 0 & 0 & \bar{Y}_\nu^* \\ 0 & 0 & \bar{Y}_\nu & 0 \end{pmatrix}.$$

Suppose we consider only one generation, so that $Y_\mu = m_\nu$ and $Y_R = m_R$ are just scalars. The eigenvalues of the above mass matrix are then given by

$$\pm \frac{1}{2} m_R \pm \frac{1}{2} \sqrt{m_R^2 + 4m_\nu^2}.$$

If we assume that $m_\nu \ll m_R$, then these eigenvalues are approximated by $\pm m_R$ and $\pm \frac{m_\nu^2}{m_R}$. This means that there is a heavy neutrino, for which the Dirac mass m_ν may be neglected, so that its mass is given by the Majorana mass m_R . However, there is also a light neutrino, for which the Dirac and Majorana terms conspire to yield a mass $\frac{m_\nu^2}{m_R}$, which is in fact much smaller

than the Dirac mass m_ν . This is called the *seesaw mechanism*. Thus, even though the observed masses for these neutrinos may be very small, they might still have large Dirac masses (or Yukawa couplings).

Remark 4. *Of course, in the physical applications one chooses Y_ν, Y_e to be the Yukawa mass matrices and Y_R is the Majorana mass matrix. There has been searches for additional conditions to be satisfied by the spectral triple (A_F, H_F, D_F) to further constrain the form of D_F , see for instance [11, 8, 59, 36, 37].*

4.3 The gauge and scalar fields as inner fluctuations

We here derive the precise form of internal fluctuations A_μ for the above spectral triple of the Standard Model (following [17, Sect. 3.5] or [75, Sect. 11.5]).

Take two elements $a = (\lambda, q, m)$ and $b = (\lambda', q', m')$ of the algebra $\mathcal{A} = C^\infty(\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}))$. According to the representation of A_F on H_F , the inner fluctuations $A_\mu = -ia\partial_\mu b$ decompose as

$$\Lambda_\mu := -i\lambda\partial_\mu\lambda'; \quad \Lambda'_\mu := -i\bar{\lambda}\partial_\mu\bar{\lambda}'$$

on ν_R and e_R , respectively, and as

$$Q_\mu := -iq\partial_\mu q'; \quad V'_\mu := -im\partial_\mu m'$$

acting on (ν_L, e_L) and $H_{\bar{q}}$, respectively. On all other components of H_F the gauge field A_μ acts as zero. Imposing the hermiticity $\Lambda_\mu = \Lambda_\mu^*$ implies $\Lambda_\mu \in \mathbb{R}$, and also automatically yields $\Lambda'_\mu = -\Lambda_\mu$. Furthermore, $Q_\mu = Q_\mu^*$ implies that Q_μ is a real-linear combination of the Pauli matrices, which span $isu(2)$. Finally, the condition that V'_μ be hermitian yields $V'_\mu \in iu(3)$, so V'_μ is a $U(3)$ gauge field. As mentioned above, we need to impose the unimodularity condition to obtain an $SU(3)$ gauge field. Hence, we require that the trace of the gauge field A_μ over H_F vanishes, and we obtain

$$\mathrm{Tr}|_{H_{\bar{q}}}(\Lambda_\mu 1_4) + \mathrm{Tr}|_{H_{\bar{q}}}(1_4 \otimes V'_\mu) = 0 \quad \implies \quad \mathrm{Tr}(V'_\mu) = -\Lambda_\mu.$$

Therefore, we can define a traceless $SU(3)$ gauge field V_μ by $\bar{V}_\mu := -V'_\mu - \frac{1}{3}\Lambda_\mu$. The action of the gauge field $B_\mu = A_\mu - J_F A_\mu J_F^{-1}$ on the fermions is

then given by

$$\begin{aligned}
B_\mu|_{H_l} &= \begin{pmatrix} 0 & 0 \\ 0 & -2\Lambda_\mu & Q_\mu - \Lambda_\mu 1_2 \end{pmatrix}, \\
B_\mu|_{H_q} &= \begin{pmatrix} \frac{4}{3}\Lambda_\mu 1_3 + V_\mu & 0 \\ 0 & -\frac{2}{3}\Lambda_\mu 1_3 + V_\mu & (Q_\mu + \frac{1}{3}\Lambda_\mu 1_2) \otimes 1_3 + 1_2 \otimes V_\mu \end{pmatrix}.
\end{aligned} \tag{23}$$

for some $U(1)$ gauge field Λ_μ , an $SU(2)$ gauge field Q_μ and an $SU(3)$ gauge field V_μ .

Note that the coefficients in front of Λ_μ in the above formulas are precisely the aforementioned (and correct!) hypercharges of the corresponding particles.

Next, let us turn to the scalar field ϕ , which is given by

$$\phi|_{H_l} = \begin{pmatrix} 0 & Y^* \\ Y & 0 \end{pmatrix}, \quad \phi|_{H_q} = \begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix} \otimes 1_3, \quad \phi|_{H_{\bar{l}}} = 0, \quad \phi|_{H_{\bar{q}}} = 0, \tag{24}$$

where we now have, for complex fields ϕ_1, ϕ_2 ,

$$Y = \begin{pmatrix} Y_\nu \phi_1 & -Y_e \bar{\phi}_2 \\ Y_\nu \phi_2 & Y_e \bar{\phi}_1 \end{pmatrix}, \quad X = \begin{pmatrix} Y_u \phi_1 & -Y_d \bar{\phi}_2 \\ Y_u \phi_2 & Y_d \bar{\phi}_1 \end{pmatrix}.$$

The scalar field Φ is then given by

$$\Phi = D_F + \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} + J_F \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} J_F^* = \begin{pmatrix} S + \phi & T^* \\ T & (S + \phi) \end{pmatrix}. \tag{25}$$

Finally, one can compute that the action of the gauge group $SU(A_F)$ by conjugation on the fluctuated Dirac operator

$$D_\omega = D \otimes 1 + \gamma^\mu \otimes B_\mu + \gamma_M \otimes \Phi$$

is implemented by

$$\begin{aligned}
\Lambda_\mu &\mapsto \Lambda_\mu - i\lambda \partial_\mu \bar{\lambda}, & Q_\mu &\mapsto q Q_\mu q^* - i q \partial_\mu q^*, & \bar{V}_\mu &\mapsto m \bar{V}_\mu m^* - i m \partial_\mu m^*, \\
H &\mapsto \bar{\lambda} q H,
\end{aligned}$$

for $\lambda \in C^\infty(M, U(1))$, $q \in C^\infty(M, SU(2))$ and $m \in C^\infty(M, SU(3))$ and we have written the *Higgs doublet* as

$$H := \begin{pmatrix} \phi_1 + 1 \\ \phi_2 \end{pmatrix}$$

For the detailed computation we refer to [17, Sect. 3.5] or [75, Prop. 11.5].

Summarizing, the gauge fields derived take values in the Lie algebra $u(1) \oplus su(2) \oplus su(3)$ and transform according to the usual Standard Model gauge transformations. The scalar field ϕ transforms as the Standard Model Higgs field in the defining representation of $SU(2)$, with hypercharge -1 .

4.4 Spectral action

The spectral action for the above spectral Standard Model has been computed in full detail in [17, Section 4.2] and confirmed in *e.g.* [75, Theorem 11.10]. Since it would lie beyond the scope of the present review, we refrain from repeating this computation. Instead, we summarize the main result, which is that the Lagrangian derived from the spectral action is

$$\begin{aligned} S_B = \int & \left(\frac{48\chi_4\Lambda^4}{\pi^2} - \frac{c\chi_2\Lambda^2}{\pi^2} + \frac{d\chi(0)}{4\pi^2} + \left(\frac{c\chi(0)}{24\pi^2} - \frac{4\chi_2\Lambda^2}{\pi^2} \right) s - \frac{3\chi(0)}{10\pi^2} (C_{\mu\nu\rho\sigma})^2 \right. \\ & + \frac{1}{4} Y_{\mu\nu} Y^{\mu\nu} + \frac{1}{4} W_{\mu\nu}^a W^{\mu\nu,a} + \frac{1}{4} G_{\mu\nu}^i G^{\mu\nu,i} + \frac{b\pi^2}{2a^2\chi(0)} |H|^4 \\ & \left. - \frac{2a\chi_2\Lambda^2 - e\chi(0)}{a\chi(0)} |H|^2 + \frac{1}{12} s |H|^2 + \frac{1}{2} |D_\mu H|^2 \right) \sqrt{g} d^4x, \end{aligned}$$

where $\chi_j = \int_0^\infty \chi(v) v^{j-1} dv$ are the moments of the function χ , $j > 0$, $s = -R$ is the scalar curvature, $Y_{\mu\nu}$, $W_{\mu\nu}$ and $G_{\mu\nu}$ are the field strengths of Y_μ , Q_μ and V_μ , respectively and the covariant derivative $D_\mu H$ is given by

$$D_\mu H = \partial_\mu H + \frac{1}{2} i g_2 W_\mu^a \sigma^a H - \frac{1}{2} i g_1 Y_\mu H. \quad (26)$$

Moreover, we have defined the following constants

$$\begin{aligned} a &= \text{Tr} (Y_\nu^* Y_\nu + Y_e^* Y_e + 3Y_u^* Y_u + 3Y_d^* Y_d), \\ b &= \text{Tr} ((Y_\nu^* Y_\nu)^2 + (Y_e^* Y_e)^2 + 3(Y_u^* Y_u)^2 + 3(Y_d^* Y_d)^2), \\ c &= \text{Tr} (Y_R^* Y_R), \\ d &= \text{Tr} ((Y_R^* Y_R)^2), \\ e &= \text{Tr} (Y_R^* Y_R Y_\nu^* Y_\nu). \end{aligned} \quad (27)$$

The normalization of the kinetic terms imposes a relation between the coupling constants g_1, g_2, g_3 and the coefficients χ_0 , of the form

$$\frac{\chi(0)}{2\pi^2} g_3^2 = \frac{\chi(0)}{2\pi^2} g_2^2 = \frac{5\chi(0)}{6\pi^2} g_1^2 = \frac{1}{4}. \quad (28)$$

The coupling constants are then related by

$$g_3^2 = g_2^2 = \frac{5}{3} g_1^2,$$

which is precisely the relation between the coupling constants at unification, common to grand unified theories (GUT). We shall further discuss this in Section 4.6.

4.5 Fermionic action in KO-dimension 6

As already announced above, the shift to KO-dimension 6 for the finite space solved the fermion doubling problem of [63]. Let us briefly explain how this works, following [31].

The crucial observation is that in KO-dimension $2 \equiv 4 + 6 \pmod{8}$ the following pairing

$$(\psi, \psi') \mapsto (J\psi, D_\omega \psi')$$

is a skew-symmetric form on the +1-eigenspace of γ in \mathcal{H} . This skew-symmetry is in concordance with the Grassmann nature of fermionic fields ψ , guaranteeing that the following action functional is in fact non-zero:

$$S_F = \frac{1}{2} \langle J\xi, D_A \xi \rangle$$

for ξ a Grassmann variable in the +1-eigenspace of γ .

This then solves the fermion doubling, or actually quadrupling as follows. First, the restriction to the chiral subspace of γ takes care of a factor of two. Then, the functional integral involving anti-commuting Grassman variables delivers a Pfaffian, which takes care of a square root. That this indeed works has been worked out in full detail for the case of the Standard Model in [17, Section 4.4.1] or [75, Section 11.4].

4.6 Phenomenological consequences

The first phenomenological consequence one can derive from the spectral Standard Model is an upper bound on the mass of the top quark. In fact,

the appearance of the constant a in both the fermionic and the bosonic action allows to derive

$$\text{Tr} (m_\nu^* m_\nu + m_e^* m_e + 3m_u^* m_u + 3m_d^* m_d) = 2g_2^2 v^2 = 8M_W^2. \quad (29)$$

It is natural to assume that the mass m_{top} of the top quark is much larger than all other fermion masses, except possibly a Dirac mass that arises from the seesaw mechanism as was described above. If we write $m_\nu = \rho m_{\text{top}}$ then the above relation would yield the constraint

$$m_{\text{top}} \lesssim \sqrt{\frac{8}{3 + \rho^2}} M_W. \quad (30)$$

The relations (28) between the coupling constants and $\chi(0)$ suggests that we have grand unification of the coupling constants. Moreover, from the action functional we see that the quartic Higgs coupling constant λ is related to $\chi(0)$ as well via

$$\lambda = 24 \frac{b}{a^2} g_2^2.$$

Thus, the spectral Standard Model imposes relations between the coupling constants and bounds on the fermion masses. These relations were used in [17] as input at (or around) grand unification scale Λ_{GUT} , and then run down using one-loop renormalization group equations to 'low energies' where falsifiable predictions were obtained.

In fact, the mass of the top quark can indeed be found to get an acceptable value, however, for the Higgs mass it was found that

$$167 \text{ GeV} \leq m_h \leq 176 \text{ GeV}.$$

Given that there were not much models in particle physics around that could produce falsifiable predictions, it is somewhat ironical that the first exclusion results on the mass of the Higgs that appeared in 2009 from Fermilab hit exactly this region. See Figure 1. And, of course, with the discovery of the Higgs at $m_h \approx 125.5 \text{ GeV}$ in [1, 26] one could say that the spectral Standard Model was not in a particularly good shape at that time.

5 Beyond the Standard Model with noncommutative geometry

Even though the incompatibility between the spectral Standard Model and the experimental discovery of the Higgs with a relatively low mass was not

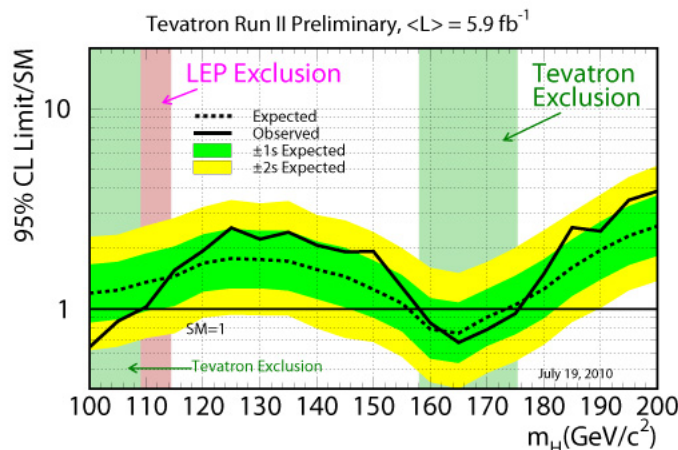


Figure 1: Observed and expected exclusion limits for a Standard Model Higgs boson at the 95-percent confidence level for the combined CDF and DZero analyses. (Fermilab)

an easy stroke at the time, it also led to a period of reflection and reconsideration of the premises of the noncommutative geometric approach. In fact, it was the beginning of yet another exciting chapter in our story on the spectral model of gravity coupled with matter. As we will see in this and the next chapter, once again the input from experiment is taken as a guiding principle in our search for the spectral model that goes Beyond the Standard Model.

Remark 5. *We do not pretend to give a complete overview of the literature here, but only indicate some of the highlights and actively ongoing research areas.*

Other searches beyond the Standard Model with noncommutative geometry include [53, 70, 71, 73, 72, 74], adopting a slightly different approach to almost-commutative manifolds as we do.

There is another aspect that was studied is the connection between supersymmetry and almost-commutative manifolds. It turned out to be very hard —if not impossible— to combine the two. A first approach is [13] and more recently the intersection was studied in [9, 10, 5].

5.1 Resilience of the spectral Standard Model

In 2012 it was realized how a small correction of the spectral Standard Model gives an intriguing possibility to go beyond the Standard Model, solving at the same time a problem with the stability of the Higgs vacuum given the measured low mass m_h . This is based on [16], but for which some of the crucial ingredients surprisingly enough were already present in the 2010 paper [15].

Namely, in the definition of the finite Dirac operator D_F of Equation 19, we can replace Y_R by $Y_R\sigma$, where σ is a real scalar field on M . Strictly speaking, this brings us out of the class of almost-commutative manifolds $M \times F$, since part of D_F now varies over M and this was the main reason why it was disregarded before. However, since from a physical viewpoint there was no reason to assume Y_R to be constant, it was treated as a scalar field already in [15]. This was only fully justified in subsequent papers (as we will see in the next subsections) where the scalar field σ arises as the relic of a spontaneous symmetry breaking mechanism, similar to the Higgs field h in the electroweak sector of the Standard Model. We will discuss a few of the existing approaches in the literature in the next few sections. For now, let us simply focus on the phenomenological consequences of this extra scalar field.

Thus we replace Y_R by $Y_R\sigma$ and analyze the additional terms in the spectral action. The scalar sector becomes

$$S'_H := \int_M \left(\frac{bf(0)}{2\pi^2} |H|^4 - \frac{2af_2\Lambda^2}{\pi^2} |H|^2 + \frac{ef(0)}{\pi^2} \sigma^2 |H|^2 - \frac{cf_2\Lambda^2}{\pi^2} \sigma^2 + \frac{df(0)}{4\pi^2} \sigma^4 + \frac{af(0)}{2\pi^2} |D_\mu H|^2 + \frac{1}{4\pi^2} f(0) c(\partial_\mu \sigma)^2 \right) \sqrt{g} dx,$$

where we ignored the coupling to the scalar curvature.

We exploit the approximation that m_{top} , m_ν and m_R are the dominant mass terms. Moreover, as before we write $m_\nu = \rho m_{top}$. That is, the expressions for a, b, c, d and e in (27) now become

$$\begin{aligned} a &\approx m_{top}^2 (\rho^2 + 3), \\ b &\approx m_{top}^4 (\rho^4 + 3), \\ c &\approx m_R^2, \\ d &\approx m_R^4, \\ e &\approx \rho^2 m_R^2 m_{top}^2. \end{aligned}$$

In a unitary gauge, where $H = \begin{pmatrix} h \\ 0 \end{pmatrix}$, we arrive at the following potential:

$$\mathcal{L}_{pot}(h, \sigma) = \frac{1}{24}\lambda_h h^4 + \frac{1}{2}\lambda_{h\sigma} h^2 \sigma^2 + \frac{1}{4}\lambda_\sigma \sigma^4 - \frac{4g_2^2}{\pi^2} f_2 \Lambda^2 (h^2 + \sigma^2),$$

where we have defined coupling constants

$$\lambda_h = 24 \frac{\rho^4 + 3}{(\rho^2 + 3)^2} g_2^2, \quad \lambda_{h\sigma} = \frac{8\rho^2}{\rho^2 + 3} g_2^2, \quad \lambda_\sigma = 8g_2^2. \quad (31)$$

This potential can be minimized, and if we replace h by $v + h$ and σ by $w + \sigma$, respectively, expanding around a minimum for the terms quadratic in the fields, we obtain:

$$\begin{aligned} \mathcal{L}_{pot}(v + h, w + \sigma)|_{\text{quadratic}} &= \frac{1}{6}v^2 \lambda_h v^2 + 2vw \lambda_{h\sigma} \sigma h + w^2 \lambda_\sigma \sigma^2 \\ &= \frac{1}{2} \begin{pmatrix} h & \sigma \end{pmatrix} M^2 \begin{pmatrix} h \\ \sigma \end{pmatrix}, \end{aligned}$$

where we have defined the mass matrix M by

$$M^2 = 2 \begin{pmatrix} \frac{1}{6}\lambda_h v^2 & \lambda_{h\sigma} v w \\ \lambda_{h\sigma} v w & \lambda_\sigma w^2 \end{pmatrix}.$$

This mass matrix can be easily diagonalized, and if we make the natural assumption that w is of the order of m_R , while v is of the order of M_W , so that $v \ll w$, we find that the two eigenvalues are

$$\begin{aligned} m_+^2 &\sim 2\lambda_\sigma w^2 + 2 \frac{\lambda_{h\sigma}^2}{\lambda_\sigma} v^2, \\ m_-^2 &\sim 2\lambda_h v^2 \left(\frac{1}{6} - \frac{\lambda_{h\sigma}^2}{\lambda_h \lambda_\sigma} \right). \end{aligned}$$

We can now determine the value of these two masses by running the scalar coupling constants λ_h , $\lambda_{h\sigma}$ and λ_σ down to ordinary energy scalar using the renormalization group equations for these couplings that were derived in [51], referring to [16, 75] for full details. The result varies with the chosen value for Λ_{GUT} and the parameter ρ . The mass of σ is essentially given by the largest eigenvalue m_+ which is of the order 10^{12} GeV for all values of Λ_{GUT} and the parameter ρ . The allowed mass range for the Higgs, *i.e.* for m_- , is depicted in Figure 2. The expected value $m_h = 125.5$ GeV is therefore compatible with the above noncommutative model. Moreover, without the σ the λ_h turns negative at energies around 10^{12} GeV. Furthermore, this calculation implies that there is a relation (given by the red line in the Figure) between the ratio m_ν/m_{top} and the unification scale Λ_{GUT} .

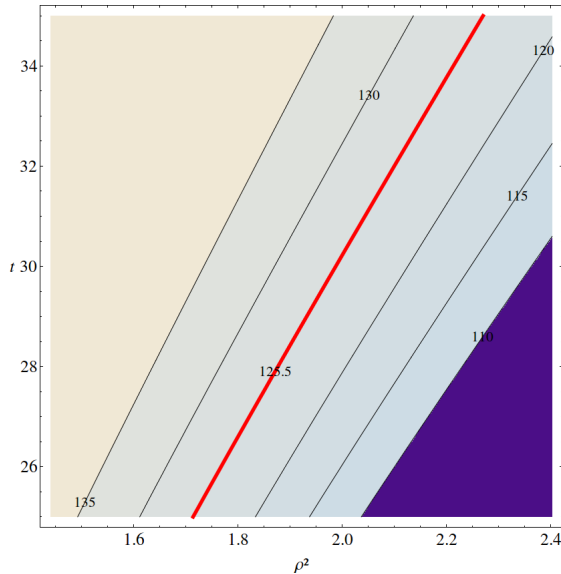


Figure 2: A contour plot of the Higgs mass m_h as a function of ρ^2 and $t = \log(\Lambda_{GUT}/M_Z)$. The red line corresponds to $m_h = 125.5$ GeV.

5.2 Pati–Salam unification and first-order condition

In order to see how we one can use the noncommutative geometric approach to go beyond the Standard Model it is important to trace our steps that led to the spectral Standard Model in the previous Section. The route started with the classification of the algebras of the finite space (*cf.* Equation (17)). The results show that the only algebras which solve the fermion doubling problem are of the form $M_{2a}(\mathbb{C}) \oplus M_{2a}(\mathbb{C})$ where a is an even integer. An arbitrary symplectic constraint is imposed on the first algebra restricting it from $M_{2a}(\mathbb{C})$ to $M_a(\mathbb{H})$. The first non-trivial algebra one can consider is for $a = 2$ with the algebra

$$M_2(\mathbb{H}) \oplus M_4(\mathbb{C}). \quad (32)$$

Coincidentally, and as explained in the introduction, the above algebra comes out as a solution of the two-sided Heisenberg quantization relation between the Dirac operator D and the two maps from the four spin-manifold and the two four spheres $S^4 \times S^4$ [18, 19]. This removes the arbitrary symplectic constraint and replaces it with a relation that quantize the four-volume in terms of two quanta of geometry and have far reaching consequences on the structure of space-time. We will come back to this in the last Section.

The existence of the chirality operator γ that commutes with the algebra breaks the quaternionic matrices $M_2(\mathbb{H})$ to the diagonal subalgebra and leads us to consider the finite algebra

$$\mathcal{A}_F = \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C}). \quad (33)$$

This algebra is the simplest candidate to search for new physics beyond the Standard Model. In fact, the inner automorphism group of $\mathcal{A} = \mathcal{C}^\infty(M) \otimes \mathcal{A}_F$ is recognized as the Pati–Salam gauge group $SU(2)_R \times SU(2)_L \times SU(4)$, and the corresponding gauge bosons appear as inner perturbations of the (spacetime) Dirac operator [21]. Thus, we are considering a spectral Pati–Salam model as a candidate beyond the Standard Model. Let us further analyze this model and its phenomenological consequences.

An element of the Hilbert space $\Psi \in \mathcal{H}$ is represented by

$$\Psi_M = \begin{pmatrix} \psi_A \\ \psi_{A'} \end{pmatrix}, \quad \psi_{A'} = \psi_A^c \quad (34)$$

where ψ_A^c is the conjugate spinor to ψ_A . Thus all primed indices A' correspond to the Hilbert space of conjugate spinors. It is acted on by both the left algebra $M_2(\mathbb{H})$ and the right algebra $M_4(\mathbb{C})$. Therefore the index A can take 16 values and is represented by

$$A = \alpha I \quad (35)$$

where the index α is acted on by quaternionic matrices and the index I by $M_4(\mathbb{C})$ matrices. Moreover, when the grading breaks $M_2(\mathbb{H})$ into $\mathbb{H}_R \oplus \mathbb{H}_L$ the index α is decomposed to $\alpha = \dot{a}, a$ where $\dot{a} = \dot{1}, \dot{2}$ (dotted index) is acted on by the first quaternionic algebra \mathbb{H}_R and $a = 1, 2$ is acted on by the second quaternionic algebra \mathbb{H}_L . When $M_4(\mathbb{C})$ breaks into $\mathbb{C} \oplus M_3(\mathbb{C})$ (due to symmetry breaking or through the use of the order one condition as in [14]) the index I is decomposed into $I = 1, i$ and thus distinguishing leptons and quarks, where the 1 is acted on by the \mathbb{C} and the i by $M_3(\mathbb{C})$. Therefore the various components of the spinor ψ_A are

$$\begin{aligned} \psi_{\alpha I} &= \begin{pmatrix} \nu_R & u_{iR} & \nu_L & u_{iL} \\ e_R & d_{iR} & e_L & d_{iL} \end{pmatrix}, \quad i = 1, 2, 3 \\ &= (\psi_{\dot{a}1}, \psi_{\dot{a}i}, \psi_{a1}, \psi_{ai}), \quad a = 1, 2, \quad \dot{a} = \dot{1}, \dot{2} \end{aligned} \quad (36)$$

This is a general prediction of the spectral construction that there is 16 fundamental Weyl fermions per family, 4 leptons and 12 quarks.

The (finite) Dirac operator can be written in matrix form

$$D_F = \begin{pmatrix} D_A^B & D_{A'}^{B'} \\ D_{A'}^B & D_{A'}^{B'} \end{pmatrix}, \quad (37)$$

and must satisfy the properties

$$\gamma_F D_F = -D_F \gamma_F \quad J_F D_F = D_F J_F \quad (38)$$

where $J_F^2 = 1$. A matrix realization of γ_F and J_F are given by

$$\gamma_F = \begin{pmatrix} G_F & 0 \\ 0 & -\bar{G}_F \end{pmatrix}, \quad G_F = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \quad J_F = \begin{pmatrix} 0_4 & 1_4 \\ 1_4 & 0_4 \end{pmatrix} \circ \text{cc} \quad (39)$$

where cc stands for complex conjugation. These relations, together with the hermiticity of D imply the relations

$$(D_F)_{A'}^{B'} = (\bar{D}_F)_A^B \quad (D_F)_{A'}^B = (\bar{D}_F)_B^{A'} \quad (40)$$

and have the following zero components [15]

$$(D_F)_{aI}^{bJ} = 0 = (D_F)_{\dot{a}I}^{\dot{b}J} \quad (41)$$

$$(D_F)_{aI}^{\dot{b}J'} = 0 = (D_F)_{\dot{a}I}^{b'J'} \quad (42)$$

leaving the components $(D_F)_{aI}^{\dot{b}J}$, $(D_F)_{aI}^{b'J'}$ and $(D_F)_{\dot{a}I}^{\dot{b}J'}$ arbitrary. These restrictions lead to important constraints on the structure of the connection that appears in the inner fluctuations of the Dirac operator. In particular the operator D of the full noncommutative space given by

$$D = D_M \otimes 1 + \gamma_5 \otimes D_F \quad (43)$$

gets modified to

$$D_A = D + A_{(1)} + J A_{(1)} J^{-1} + A_{(2)} \quad (44)$$

where

$$A_{(1)} = \sum a [D, b], \quad A_{(2)} = \sum \hat{a} [A_{(1)}, \hat{b}], \quad \hat{a} = J a J^{-1} \quad (45)$$

We have shown in [21] that components of the connection A which are tensored with the Clifford gamma matrices γ^μ are the gauge fields of the

Pati–Salam model with the symmetry of $SU(2)_R \times SU(2)_L \times SU(4)$. On the other hand, the non-vanishing components of the connection which are tensored with the gamma matrix γ_5 are given by

$$(A)_{aI}^{bJ} \equiv \gamma_5 \Sigma_{aI}^{bJ}, \quad (A)_{aI}^{b'J'} = \gamma_5 H_{aIbJ}, \quad (A)_{\dot{a}I}^{\dot{b}'J'} \equiv \gamma_5 H_{\dot{a}I\dot{b}J} \quad (46)$$

where $H_{aIbJ} = H_{bJaI}$ and $H_{\dot{a}I\dot{b}J} = H_{\dot{b}J\dot{a}I}$, which is the most general Higgs structure possible. These correspond to the representations with respect to $SU(2)_R \times SU(2)_L \times SU(4)$:

$$\Sigma_{aI}^{bJ} = (2_R, 2_L, 1) + (2_R, 2_L, 15) \quad (47)$$

$$H_{aIbJ} = (1_R, 1_L, 6) + (1_R, 3_L, 10) \quad (48)$$

$$H_{\dot{a}I\dot{b}J} = (1_R, 1_L, 6) + (3_R, 1_L, 10) \quad (49)$$

We note, however, that the inner fluctuations form a semi-group and if a component $(D_F)_{aI}^{bJ}$ or $(D_F)_{aI}^{b'J'}$ or $(D_F)_{\dot{a}I}^{\dot{b}'J'}$ vanish, then the corresponding A field will also vanish. We can distinguish three cases: 1) Left-right symmetric Pati–Salam model with fundamental Higgs fields Σ_{aI}^{bJ} , H_{aIbJ} and $H_{\dot{a}I\dot{b}J}$. In this model the field H_{aIbJ} should have a zero vev. 2) A Pati–Salam model where the Higgs field H_{aIbJ} that couples to the left sector is set to zero which is desirable because there is no symmetry between the left and right sectors at low energies. 3) If one starts with $(D_F)_{aI}^{bJ}$ or $(D_F)_{aI}^{b'J'}$ or $(D_F)_{\dot{a}I}^{\dot{b}'J'}$ whose values are given by those that were derived for the Standard Model, then the Higgs fields Σ_{aI}^{bJ} , H_{aIbJ} and $H_{\dot{a}I\dot{b}J}$ will become composite and expressible in terms of more fundamental fields Σ_I^J , $\Delta_{\dot{a}J}$ and $\phi_{\dot{a}}^b$. We refer to this as the composite model. It has the scalar field σ discussed in the previous section as a remnant after spontaneous symmetry breaking [21]. In fact, contrary to some claims in the literature it is possible to perform the potential analysis in this case in unitarity gauge and arrive at the conclusion that the field content contains the scalar field σ (*cf.* Appendix A).

Depending on the precise particle content we may determine the renormalization group equations of the Pati–Salam gauge couplings g_R, g_L, g . In [22] we have run them to look for unification of the coupling $g_R = g_L = g$. The boundary conditions are taken at the intermediate mass scale $\mu = m_R$ to be the usual (e.g. [67, Eq. (5.8.3)])

$$\frac{1}{g_1^2} = \frac{2}{3} \frac{1}{g^2} + \frac{1}{g_R^2}, \quad \frac{1}{g_2^2} = \frac{1}{g^2}, \quad \frac{1}{g_3^2} = \frac{1}{g^2}, \quad (50)$$

in terms of the Standard Model gauge couplings g_1, g_2, g_3 . At the mass scale m_R the Pati–Salam symmetry is broken to that of the Standard Model, and

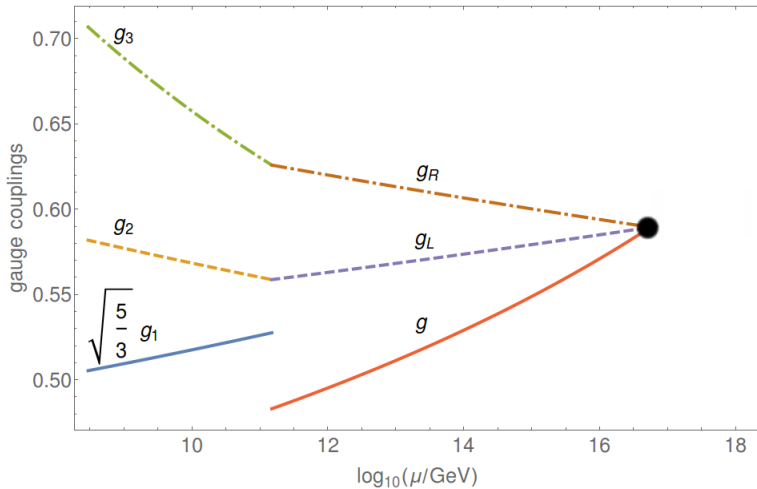


Figure 3: Running of the gauge couplings of the Standard Model gauge couplings (below scale $m_R \approx 10^{11}$ GeV) and the Pati–Salam gauge coupling (above scale m_R) in case 2.

we take it to be the same scale that is present in the see-saw mechanism. It should thus be of the order $10^{11} - 10^{13}$ GeV. What we have found in [22] (and this was confirmed by others in [3]) is that in all three cases it is possible to achieve grand unification of the couplings, while connecting to Standard Model physics in the broken, low-energy phase. An example of a running of the gauge coupling is illustrated in Figure 3.

5.3 Grand symmetry and twisted spectral triples

In [40] the next-to-next case³ in the list of irreducible geometries in Equation (17) was considered: $k = 4$. Thus, one considers

$$A_G = M_4(\mathbb{H}) \oplus M_8(\mathbb{C}); \quad H_F := \mathbb{C}^{128}. \quad (51)$$

where 128 is exactly the number of spinor and internal degrees of freedom combined (including the aforementioned fermion quadruplication). The geometry is then

$$(C^\infty(M, A_G), L^2(M) \otimes H_F, D_M + \gamma_M D_F)$$

³The case $k = 3$ was ruled out by physical considerations [40].

where one has to assume that the spinor bundle on M has been trivialized to gather the spinor and internal fermionic degrees of freedom in a single Hilbert space H_F .

Note that the above geometry is not a direct product of the continuum with a discrete space. In fact, both the algebra and the Dirac operator D_M contain spinor indices. As a consequence the commutator $[D_M, a]$ can become unbounded, thus challenging one of the basic axioms of spectral triples. Instead, it is possible to guarantee that *twisted* commutators are bounded so that this example fits in the general framework of twisted spectral triples developed in [34]. In [41] the authors identify an inner automorphism $\rho = R(\cdot)R$ of A_G such that

$$[D, a]_\rho = Da - \rho(a)D$$

is bounded.

An interesting question that arises at this point is how to generate inner fluctuations of twisted spectral triples. This was analyzed in full detail from a mathematical viewpoint in [61, 62]. One of the intriguing aspects is the self-adjointness of the Dirac operator under fluctuations (even gauge transformations): for this to be respected one has to impose a compatibility between the twist and the fluctuation.

An alternative route was suggested in [39]. Namely, one may drop the above condition of self-adjointness and instead look for operators that are Krein-self-adjoint, using the Krein structure on the Hilbert space that is induced by the operator R (defining the twist ρ). This will have an intriguing appearance of the Lorentzian structure (given by the Krein inner product) from a purely algebraic and Euclidean starting point. Here we also refer to the nice overview given in [64].

5.4 Algebraic constraints on the finite geometry

An interesting question to consider—in particular in light of theories that go Beyond the Standard Model—is whether one can *derive* the restricted form of the Dirac operator D_F in (19). We highlight a few approaches to this question that are present in the literature.

First of all, as mentioned already on page 20, the form of the D_F in terms of the matrices Y_ν, Y_e, Y_u, Y_d and Y_R as in Equations (21) and (22) appears naturally in the study of moduli of finite Dirac operators. The only constraint (in addition to the usual conditions layed out in Section 3.1) there was that the photon remained massless.

An attempt was made to make the latter condition less *ad hoc* is [49, 7, 8]. They proposed to generalize noncommutative geometry to non-associative noncommutative geometry, thus allowing for non-associative algebras. The crucial idea —which goes back to Eilenberg— is to combine the (differential) algebra and (Hilbert space) bimodule into a single algebra, and understand the conditions such as commutant property and first-order conditions as consequences of associativity of the pertinent algebra B . However, this associativity is a strong constraint and accordingly further restrict the geometry described by D_F . Note that non-associative algebras have also been used in the context of noncommutative geometry and particle physics to predict the number of families (to be three) [76]

Another approach to analyzing the form of the Dirac operator D_F by imposing algebraic conditions is taken by [35, 36]. Here the authors adopt the principle that, similar to differential forms in the continuum, the finite Hilbert space should be a Morita equivalence between A and the Clifford algebra generated by A_F and D_F . One finds that the aforementioned form of D_F does not satisfy this condition but additional entries in D_F should be non-zero. This gives rise to a model Beyond the Standard Model: an analysis of the phenomenological consequences is performed in [59, 37]. In [2] it was then found that this model does not exhibit grand unification of the Standard Model couplings.

6 Volume quantization and uniqueness of SM

In the classification of finite noncommutative spaces we arrived at the result that the algebra $\mathcal{A}_F = (\mathbb{H}_R \oplus \mathbb{H}_L) \oplus M_4(\mathbb{C})$ was the first possibility out of many of the form $\mathcal{A}_F = (M_n(\mathbb{H})_R \oplus M_n(\mathbb{H})_L) \oplus M_{4n}(\mathbb{C})$. In addition we made an assumption, that seemed arbitrary, of the existence of anti-linear isometry that reduced the algebra $M_{4n}(\mathbb{C})$ to $(M_n(\mathbb{H})_R \oplus M_n(\mathbb{H})_L)$. It is necessary to have a stronger evidence of the uniqueness of our conclusions that helps us to avoid making the above mentioned assumptions. Surprisingly, the new evidence came in the process of solving a seemingly completely independent problem, encoding low dimensional geometries, and in particular dimension four.

6.1 Higher form of Heisenberg's commutation relations

Starting with the simple example of one dimensional geometries, consider the equation

$$U^* [D, U] = 1, \quad U^* U = 1$$

where D is self-adjoint operator. Assuming that the one dimensional space is a closed curve parameterized by coordinate x and the Dirac operator to be $D = -i\frac{d}{dx} + \alpha$ the above equation simplifies to

$$-iU^*dU = dx$$

Writing $U = e^{in\theta}$ we obtain $dx = nd\theta$. Integrating both sides implies that the length of the one dimensional curve is an integer multiple of 2π , the length of S^1

$$\oint_C dx = n(2\pi)$$

To adopt this construction to higher dimensions, we note that we can characterize the circle S^1 by the equation $Y^A Y^A = 1$, $A = 1, 2$, $Y^{A*} = Y^A$. Assembling the two coordinates Y^1, Y^2 in one matrix, define $Y = Y^A \Gamma_A$, where Γ_A , $A = 1, 2$ are taken to be 2×2 . In addition we identify $\Gamma_1 = \sigma_1$, $\Gamma_2 = \sigma_2$, the Pauli matrices, and define $\Gamma = -i\Gamma_1\Gamma_2 = \sigma_3$ so that $\Gamma_+ = \frac{1}{2}(1 + \Gamma)$ is a projection operator. We notice that we can write

$$Y = \begin{pmatrix} 0 & Y^1 - iY^2 \\ Y^1 + iY^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}$$

where $U = Y^1 - iY^2$ and $U^*U = 1$. The expression

$$\langle \Gamma_+ Y [D, Y] \rangle = 1$$

where $\langle \rangle$ is defined to be the trace over the Clifford algebra defined by Γ_A , gives back the equation $U^* [D, U] = 1$.

For higher dimensional geometries we consider a Riemannian manifold with dimension n and where the algebra \mathcal{A} is taken to be $C^\infty(M)$, the algebra of continuously differentiable functions, while the operator D is identified with the Dirac operator given by

$$D_M = \gamma^\mu \left(\frac{\partial}{\partial x^\mu} + \omega_\mu \right),$$

where $\gamma^\mu = e_a^\mu \gamma^a$ and $\omega_\mu = \frac{1}{4} \omega_{\mu bc} \gamma^{bc}$ is the $SO(n)$ Lie-algebra valued spin-connection with the (inverse) vielbein e_a^μ being the square root of the (inverse) metric $g^{\mu\nu} = e_a^\mu \delta^{ab} e_b^\nu$. The gamma matrices γ^a are anti-hermitian $(\gamma^a)^* = -\gamma^a$ that define the Clifford algebra $\{\gamma^a, \gamma^b\} = -2\delta^{ab}$. The Hilbert space \mathcal{H} is the space of square integrable spinors $L^2(M, S)$. The chirality operator γ in even dimensions is then given by

$$\gamma = (i)^{\frac{n}{2}} \gamma^1 \gamma^2 \dots \gamma^n$$

Starting with manifolds of dimension 2 we first define the two sphere by the equation $Y^A Y^A = 1$, $A = 1, 2, 3$, $Y^{A*} = Y^A$. Assembling the three coordinates Y^1, Y^2, Y^3 in one matrix, defining $Y = Y^A \Gamma_A$, where Γ_A , $A = 1, 2, 3$ are taken to be 2×2 Pauli matrices. Notice that in this case $\Gamma \equiv -i\Gamma_1\Gamma_2\Gamma_3 = 1$ and to generalize equation (6.1) to two dimensions the factor Γ can be dropped, and we write instead

$$\frac{1}{2!} \langle Y [D, Y]^2 \rangle = \gamma$$

The reason we have to include the chirality operator γ on the two dimensional manifold M is that the Dirac operator D appears twice yielding a product of the form $\gamma_1\gamma_2 = -i\gamma$. A simple calculation shows that the above equation in component form is given by

$$\frac{1}{2!} \varepsilon^{\mu\nu} \varepsilon_{ABC} Y^A \partial_\mu Y^B \partial_\nu Y^C = \det(e_\mu^a)$$

which is a constraint on the volume form of M_2 . This implies that the volume of M_2 will be an integer multiple of the area of the unit 2-sphere

$$\begin{aligned} \int_{M_2} d^2x \sqrt{g} &= \int \varepsilon_{ABC} Y^A dY^B dY^C \\ &= n(4\pi) \end{aligned}$$

where n is the winding number. An example of a map Y with winding number n is

$$Y \equiv Y^1 + iY^2 = \frac{2z^n}{|z|^{2n} + 1}, \quad Y^3 = \frac{|z|^{2n} - 1}{|z|^{2n} + 1}, \quad z = x^1 + ix^2$$

From this we deduce that the pullback $Y^*(w_n)$ is a differential form that does not vanish anywhere. This in turn implies that the Jacobian of the map Y does not vanish anywhere, and that Y is a covering of the sphere. The sphere is simply connected, and on each connected component $M_j \subset M_n$, the restriction of the map Y to M_j is a diffeomorphism, implying that the manifold must be disconnected, with each piece having the topology of a sphere. To allow for two dimensional manifolds with arbitrary topology, our first observation is that condition (6.1) involves the commutator of the Dirac operator D and the coordinates Y . In momentum space D is the Feynman-slashed $\gamma^\mu p_\mu$ momentum and Y are the Feynman-slashed coordinates. This

suggests that the quantization condition is a higher form of Heisenberg commutation relation quantizing the phase space formed by coordinates and momenta. We first notice that although the quantization condition is given in terms of the noncommutative data, the operator J is the only one missing. We therefore modify the condition to take J into account. The operator J transforms Y into its commutant $Y' = iJYJ^{-1}$ so that $[Y, Y'] = 0$. Thus let $Y = Y^A \Gamma_A$ and $Y' = iJYJ^{-1}$ and $\Gamma'_A = iJ\Gamma_A J^{-1}$ so that we can write

$$Y = Y^A \Gamma_A, \quad Y' = Y'^A \Gamma'_A,$$

satisfying $Y^2 = 1$ and $Y'^2 = 1$ with the Clifford algebras C_{\pm}

$$\{\Gamma_A, \Gamma_B\} = 2\delta_{AB}, \quad (\Gamma_A)^* = \Gamma_A \quad (52)$$

$$\{\Gamma'_A, \Gamma'_B\} = -2\delta_{AB}, \quad (\Gamma'_A)^* = -\Gamma'_A \quad (53)$$

We immediately see that the Clifford algebra $C_+ = M_2(\mathbb{C})$ and $C_- = \mathbb{H}$. We then define the projection operator $e = \frac{1}{2}(1 + Y)$ satisfying $e^2 = e$ and similarly $e' = \frac{1}{2}(1 + Y')$ satisfying $e'^2 = e'$. From the tensor product of $E = ee'$ satisfying $E^2 = E$, we construct $Z = 2E - 1$ satisfying $Z^2 = 1$ and allowing us to write

$$\frac{1}{2} \langle Z [D, Z]^2 \rangle = \gamma$$

A straightforward calculation reveals that this relation splits as the sum of two non-interfering parts

$$\frac{1}{2} \langle Y [D, Y]^2 \rangle + \frac{1}{2} \langle Y' [D, Y']^2 \rangle = \gamma$$

which in component form reads

$$\frac{1}{2!} \varepsilon^{\mu\nu} \varepsilon_{ABC} \left(Y^A \partial_\mu \bar{Y}^B \partial_\nu Y^C + Y'^A \partial_\mu Y'^B \partial_\nu Y'^C \right) = \det(e_\mu^a)$$

We will show later, when considering the four dimensional case that this modification allows to reconstruct two dimensional manifolds of arbitrary topology from the pullbacks of the maps Y, Y' .

For three dimensional manifolds $\gamma = 1$ and in analogy with the one-dimensional case we write

$$\frac{1}{3!} \langle \Gamma_+ Y [D, Y]^3 \rangle = 1$$

where $Y = Y^A \Gamma_A$, $A = 1, \dots, 4$, $Y^2 = 1$, $Y = Y^*$, Γ_A are 4×4 Clifford algebra matrices C_+ where $\{\Gamma_A, \Gamma_B\} = 2\delta_{AB}$. In this representation of the

Γ matrices we have $\Gamma = \Gamma_5 = \Gamma_1\Gamma_2\Gamma_3\Gamma_4 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}$ so that $\Gamma_+ = \frac{1}{2}(1 + \Gamma)$ is a projection operator. In $d = 3$, we can write

$$Y = Y^A \Gamma_A = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}$$

where U is a unitary 2×2 matrix such that it could be written in the form $U = \exp(i(\alpha_0 1 + \alpha_a \sigma^a))$ so that $U^*U = 1$. It is easy to check that $\langle Y [D, Y]^3 \rangle = 0$ and that the component form of the above relation is

$$\det(e_\mu^a) = \frac{1}{3!} \varepsilon^{\mu\nu\rho} \text{Tr}(U^* \partial_\mu U U^* \partial_\nu U U^* \partial_\rho U)$$

whose integral is the winding number of the $SU(2)$ group manifold. Again, using the reality operator J we act on the Clifford algebra $Y' = iJYJ^{-1}$ so that $[Y, Y'] = 0$, then $\Gamma'_A = iJ\Gamma_A J^{-1}$ satisfies $\{\Gamma'_A, \Gamma'_B\} = -2\delta_{AB}$, $(\Gamma'_A)^* = -\Gamma'_A$. Forming the projection operators $e = \frac{1}{2}(1 + Y)$, $e' = \frac{1}{2}(1 + Y')$, we form the tensor product $E = ee'$ we define the field $Z = 2E - 1$, and thus the two sided relation becomes

$$\frac{1}{3!} \langle \Gamma_+ \Gamma'_+ Z [D, Z]^3 \rangle = 1$$

A lengthy calculation shows that the component form of this relation separates into two parts without interference terms

$$\begin{aligned} \det(e_\mu^a) = \frac{1}{3!} \varepsilon^{\mu\nu\rho} & \left(\text{Tr}(U^* \partial_\mu U U^* \partial_\nu U U^* \partial_\rho U) \right. \\ & \left. + \text{Tr}(U'^* \partial_\mu U' U'^* \partial_\nu U' U'^* \partial_\rho U') \right) \end{aligned}$$

Finally, for four dimensional manifolds the Clifford algebras C_+ and C_- defined as in (52) (53) with Γ_A, Γ'_A , $A = 1, \dots, 5$ are known to be given by $C_+ = M_2(\mathbb{H})$ and $C_- = M_4(\mathbb{C})$. The quantization condition takes the same form as the two dimensional case

$$\frac{1}{4!} \langle Z [D, Z]^4 \rangle = \gamma$$

This relation separates into two non-interfering terms

$$\frac{1}{4!} \langle Y [D, Y]^4 \rangle + \frac{1}{4!} \langle Y' [D, Y']^4 \rangle = \gamma$$

the component form of which is given by

$$\det(e_\mu^a) = \frac{1}{4!} \varepsilon^{\mu\nu\kappa\lambda} \varepsilon_{ABCDE} \left(Y^A \partial_\mu Y^B \partial_\nu Y^C \partial_\kappa Y^D \partial_\lambda Y^E \right. \\ \left. + Y'^A \partial_\mu Y'^B \partial_\nu Y'^C \partial_\kappa Y'^D \partial_\lambda Y'^E \right)$$

One can verify that similar considerations fail when the dimension of the manifold $n > 4$ as there are interference terms between the Y and Y' . Integrating both sides imply

$$\int_{M_4} d^4x \sqrt{g} = \frac{8}{3} \pi^2 (N + N')$$

where N, N' are the winding numbers of the two maps Y, Y' . An example of a map Y with winding number n is given by

$$Y \equiv Y^4 1 + Y^i e_i = \frac{2x^n}{x^n \bar{x}^n + 1}, \\ Y^5 = \frac{x^n \bar{x}^n - 1}{x^n \bar{x}^n + 1},$$

where $x = x^4 1 + x^i e_i$ and $e_i, i = 1, 2, 3$ are the quaternionic complex structures $e_i^2 = -1, e_i e_j = \varepsilon_{ijk} e_k, i \neq j$.

6.2 Volume quantization

Consider the smooth maps $\phi_\pm : M_n \rightarrow S^n$ then their pullbacks ϕ_\pm^* would satisfy

$$\phi_+^*(\alpha) + \phi_-^*(\alpha) = \omega,$$

where α is the volume form on the unit sphere S^n and $\omega(x)$ is an n -form that does not vanish anywhere on M_n . We have shown that for a compact connected smooth oriented manifold with $n < 4$ one can find two maps $\phi_+^*(\alpha)$ and $\phi_-^*(\alpha)$ whose sum does not vanish anywhere, satisfying equation (6.2) such that $\int \omega \in \mathbb{Z}$. The proof for $n = 4$ is more difficult and there is an obstruction unless the second Stiefel–Whitney class w_2 vanishes, which is satisfied if M is required to be a spin-manifold and the volume to be larger than or equal to five units. The key idea in the proof is to note that the kernel of the Jacobian of the map Y is a hypersurface Σ of co-dimension 2 and therefore

$$\dim \Sigma = n - 2.$$

We can then construct a map $Y' = Y \circ \psi$ where ψ is a diffeomorphism on M such that the sum of the pullbacks of Y and Y' does not vanish anywhere. The coordinates Y are defined over a Clifford algebra C_+ spanned by $\{\Gamma_A, \Gamma_B\} = 2\delta_{AB}$. For $n = 2$, $C_+ = M_2(\mathbb{C})$ while for $n = 4$, $C_+ = M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$ where \mathbb{H} is the field of quaternions. However, for $n = 4$, since we will be dealing with irreducible representations we take $C_+ = M_2(\mathbb{H})$. Similarly the coordinates Y' are defined over the Clifford algebra C_- spanned by $\{\Gamma'_A, \Gamma'_B\} = -2\delta_{AB}$ and for $n = 2$, $C_- = \mathbb{H} \oplus \mathbb{H}$ and for $n = 4$, $C_- = M_4(\mathbb{C})$. The operator J acts on the two algebras $C_+ \oplus C_-$ in the form $J(x, y) = (y^*, x^*)$ (i.e. it exchanges the two algebras and takes the Hermitian conjugate). The coordinates $Z = \frac{1}{2}(Y + 1)(Y' + 1) - 1$, then define the matrix algebras [18]

$$\begin{aligned}\mathcal{A}_F &= M_2(\mathbb{C}) \oplus \mathbb{H}, & n &= 2 \\ \mathcal{A}_F &= M_2(\mathbb{H}) \oplus M_4(\mathbb{C}), & n &= 4.\end{aligned}$$

One, however, must remember that the maps Y and Y' are functions of the coordinates of the manifold M and therefore the algebra associated with this space must be

$$\begin{aligned}\mathcal{A} &= C^\infty(M, \mathcal{A}_F) \\ &= C^\infty(M) \otimes \mathcal{A}_F.\end{aligned}$$

To see this consider, for simplicity, the $n = 2$ case with only the map Y . The Clifford algebra $C_- = \mathbb{H}$ is spanned by the set $\{1, \Gamma^A\}$, $A = 1, 2, 3$, where $\{\Gamma^A, \Gamma^B\} = -2\delta^{AB}$. We then consider functions which are made out of words of the variable Y formed with the use of constant elements of the algebra [32]

$$\sum_{i=1}^{\infty} a_1 Y a_2 Y \cdots a_i Y, \quad a_i \in \mathbb{H},$$

which will generate arbitrary functions over the manifold which is the most general form since $Y^2 = 1$. One can easily see that these combinations generate all the spherical harmonics. This result could be easily generalized by considering functions of the fields

$$Z = \frac{1}{2}(Y + 1)(Y' + 1) - 1, \quad Y \in \mathbb{H}, \quad Y' \in M_2(\mathbb{C}),$$

showing that the noncommutative algebra generated by the constant matrices and the Feynman slash coordinates Z is given by [32]

$$\mathcal{A} = C^\infty(M_2) \otimes (\mathbb{H} + M_2(\mathbb{C})).$$

We now restrict ourselves to the physical case of $n = 4$. Here the algebra is given by

$$\mathcal{A} = C^\infty(M_4) \otimes (M_2(\mathbb{H}) + M_4(\mathbb{C})).$$

The associated Hilbert space is

$$\mathcal{H} = L^2(M_4, S) \otimes \mathcal{H}_F.$$

The Dirac operator mixes the finite space and the continuous manifold non-trivially

$$D = D_M \otimes 1 + \gamma_5 \otimes D_F,$$

where D_F is a self adjoint operator in the finite space. The chirality operator is

$$\gamma = \gamma_5 \otimes \gamma_F,$$

and the anti-unitary operator J is given by

$$J = J_M \gamma_5 \otimes J_F,$$

where J_M is the charge-conjugation operator C on M and J_F the anti-unitary operator for the finite space. Thus an element $\Psi \in \mathcal{H}$ is of the form $\Psi = \begin{pmatrix} \psi_A \\ \psi_{A'} \end{pmatrix}$ where ψ_A is a 16 component $L^2(M, S)$ spinor in the fundamental representation of \mathcal{A}_F of the form $\psi_A = \psi_{\alpha I}$ where $\alpha = 1, \dots, 4$ with respect to $M_2(\mathbb{H})$ and $I = 1, \dots, 4$ with respect to $M_4(\mathbb{C})$ and where $\psi_{A'} = C\psi_A^*$ is the charge conjugate spinor to ψ_A [15]. The chirality operator γ must commute with elements of \mathcal{A} which implies that γ_F must commute with elements in \mathcal{A}_F . Commutativity of the chirality operator γ_F with the algebra \mathcal{A}_F and that this $\mathbb{Z}/2$ grading acts non-trivially reduces the algebra $M_2(\mathbb{H})$ to $\mathbb{H}_R \oplus \mathbb{H}_L$ [18]. Thus the γ_F is identified with $\gamma_F = \Gamma^5 = \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4$ and the finite space algebra reduces to

$$\mathcal{A}_F = \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C}).$$

This can be easily seen by noting that an element of $M_2(\mathbb{H})$ takes the form $\begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}$ where each q_i , $i = 1, \dots, 4$, is a 2×2 matrix representing a quaternion. Taking the representation of $\Gamma^5 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}$ to commute with $M_2(\mathbb{H})$ implies that $q_2 = 0 = q_3$, thus reducing the algebra to $\mathbb{H}_R \oplus \mathbb{H}_L$. Therefore the index $\alpha = 1, \dots, 4$ splits into two parts, $\hat{a} = \hat{1}, \hat{2}$ which is a doublet under \mathbb{H}_R and $a = 1, 2$ which is a doublet under \mathbb{H}_L . The spinor

Ψ further satisfies the chirality condition $\gamma\Psi = \Psi$ which implies that the spinors $\psi_{\hat{a}I}$ are in the $(2_R, 1_L, 4)$ with respect to the algebra $\mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$ while ψ_{aI} are in the $(1_R, 2_L, 4)$ representation. The finite space Dirac operator D_F is then a 32×32 Hermitian matrix acting on the 32 component spinors Ψ . In addition we take three copies of each spinor to account for the three families, but will omit writing an index for the families. At present we have no explanation for why the number of generations should be three. The Dirac operator for the finite space is then a 96×96 Hermitian matrix. The Dirac action is then given by [17]

$$(J\Psi, D\Psi).$$

We note that we are considering compact spaces with Euclidean signature and thus the condition $J\Psi = \Psi$ could not be imposed. It could, however, be imposed if the four dimensional space is Lorentzian [4]. The reason is that the KO dimension of the finite space is 6 because the operators D_F , γ_F and J_F satisfy

$$J_F^2 = 1, \quad J_F D_F = D_F J_F, \quad J_F \gamma_F = -\gamma_F J_F.$$

The operators D_M , $\gamma_M = \gamma_5$, and $J_M = C$ for a compact manifold of dimension 4 satisfy

$$J_M^2 = -1, \quad J_M D_M = D_M J_M, \quad J_M \gamma_5 = \gamma_5 J_M.$$

Thus the KO dimension of the full noncommutative space $(\mathcal{A}, \mathcal{H}, D)$ with the decorations J and γ included is 10 and satisfies

$$J^2 = -1, \quad J D = D J, \quad J \gamma = -\gamma J.$$

We have shown in [17] that the path integral of the Dirac action, thanks to the relations $J^2 = -1$ and $J\gamma = -\gamma J$, yields a Pfaffian of the operator D instead of its determinant and thus eliminates half the degrees of freedom of Ψ and have the same effect as imposing the condition $J\Psi = \Psi$.

We have also seen that the operator J sends the algebra \mathcal{A} to its commutant, and thus the full algebra acting on the Hilbert space \mathcal{H} is $\mathcal{A} \otimes \mathcal{A}^\circ$. Under automorphisms of the algebra

$$\Psi \rightarrow U\Psi,$$

where $U = u\hat{u}$ with $u \in \mathcal{A}$, $\hat{u} \in \mathcal{A}^\circ$ with $[u, \hat{u}] = 0$, it is clear that Dirac action is not invariant.

At this point it is clear that we have retrieved all our conclusions we have before arriving at a unique possibility, which is to have a noncommutative space corresponding to the Pati–Salam Model we considered before, and in the special case where the Dirac operator and algebra satisfy the order one condition, the result is the noncommutative space of the Standard Model. We have thus succeeded in obtaining the Pati–Salam Model and Standard Model as unique possibilities starting with the two sided Heisenberg like equation (6.1) thus eliminating all other possibilities obtained in classifying finite noncommutative spaces of KO dimension 6. There is no need to assume the existence of an isometry that reduces the first algebra from $M_4(\mathbb{C})$ to $M_2(\mathbb{H})$, and no need to assume that the KO dimension of the finite space to be 6. These results are very satisfactory and serve to enhance our confidence of the fine structure of space time as given by the above derived noncommutative space.

7 Outlook: towards quantization

Starting with the simple observation that the Higgs field could be interpreted as the link between two parallel sheets separated by a distance of the order of 10^{-16} cm it took enormous effort to identify a noncommutative space where the spectrum of the Standard Model could fit. Small deviations from the model, such as the need for a real structure and a KO dimension 6, were taken as input to fine tune and determine precisely the noncommutative space. The spectral action principle proved to be very efficient way in evaluating the bosonic sector of the theory. Having identified the noncommutative space, the next target was to understand why nature would chose the Standard Model and not any other possibility. A classification of finite spaces revealed the special nature of the the finite part of the noncommutative space identified. Work on encoding manifolds with dimensions equal to four satisfying a higher form of Heisenberg type equation showed that the most general solution of this equation is that of a noncommutative space which is a product of a four-dimensional Riemannian spin-manifold times the finite space corresponding to a Pati–Salam unification model. The Standard Model is a special case of this space where a first order differential condition is satisfied. After a long journey the reasons why nature chose the Standard Model is now reduced to determining solutions of a higher form of Heisenberg equation. With such little input, it is quite satisfying to learn that it is possible to answer many of the questions which puzzled theorists for a long time. We now know why there are 16 fermions per generation,

why the gauge group is $SU(3) \times SU(2) \times U(1)$, an explanation of the Higgs field and origin of spontaneous symmetry breaking. The Spectral model also predicts a Majorana mass for the right-handed neutrinos and explains the see-saw mechanism. We thus understand unification of all fundamental forces as a geometrical theory based on the spectral action principle of a noncommutative space.

Naturally, there are many questions that are still unanswered, and this motivates the need for further research to address these problems using noncommutative geometry considerations. To conclude, we mention few of the possible directions of future research. One important aspect to consider is the renormalizability properties of the spectral model. Another problem is to study the quantum properties of the Dirac operator and whether it could be related to the pullbacks of the maps used in determining the quanta of geometry. The future of noncommutative geometry in the program of unification of all fundamental interactions looks now to be very promising.

A Pati–Salam model: potential analysis

We here include the scalar potential analysis for the composite Pati–Salam model, as described in Section 5.2 above.

If there is unification of lepton and quark couplings, then $\rho = 1$ so that the Σ_J^I -field decouples. In that case we have

$$\begin{aligned} \mathcal{L}_{pot}(\phi_a^b, \Delta_{\dot{a}I}) &= -\mu^2 \phi_a^c \phi_c^{\dot{a}} - \nu^2 \left(\Delta_{\dot{a}K} \bar{\Delta}^{\dot{a}K} \right)^2 + \lambda_\Sigma \phi_a^c \phi_c^b \phi_b^{\dot{d}} \phi_d^{\dot{a}} \\ &+ \lambda_H \left(\Delta_{\dot{a}K} \bar{\Delta}^{\dot{a}L} \Delta_{\dot{b}L} \bar{\Delta}^{\dot{b}K} \right)^2 + \lambda_{H\Sigma} \left(\Delta_{\dot{a}J} \bar{\Delta}^{\dot{a}J} \Delta_{\dot{c}I} \bar{\Delta}^{\dot{c}I} \right) \phi_b^{\dot{c}} \phi_d^{\dot{b}} \end{aligned}$$

where we have absorbed some constant factors by redefining the couplings λ_H , $\lambda_{H\Sigma}$ and λ_Σ .

We choose unitarity gauge for the Δ and ϕ -fields, in the following precise sense.

Lemma 6. *For each value of the fields $\{\phi_a^b, \Delta_{\dot{a}I}\}$ there is an element $(u_R, u_L, u) \in SU(2)_R \times SU(2)_L \times SU(4)$ such that*

$$u_R \begin{pmatrix} \phi_1^1 & \phi_1^2 \\ \phi_2^1 & \phi_2^2 \end{pmatrix} u_L^* = \begin{pmatrix} h & 0 \\ 0 & \chi \end{pmatrix}$$

and

$$u_R \begin{pmatrix} \Delta_{i1} & \Delta_{i2} & \Delta_{i3} & \Delta_{i4} \\ \Delta_{\dot{2}1} & \Delta_{\dot{2}2} & \Delta_{\dot{2}3} & \Delta_{\dot{2}4} \end{pmatrix} u^t = \begin{pmatrix} 1 + \delta_0 & 0 & 0 & 0 \\ \delta_1 & \eta_1 & 0 & 0 \end{pmatrix}$$

where $h, \delta_0, \delta_1, \eta_1$ are real fields and χ is a complex field.

Proof. Consider the singular value decomposition of the 2×2 matrix (ϕ_a^b) :

$$\begin{pmatrix} \phi_1^1 & \phi_1^2 \\ \phi_2^1 & \phi_2^2 \end{pmatrix} = U \begin{pmatrix} h & 0 \\ 0 & k \end{pmatrix} V^*$$

for unitary 2×2 matrices U, V and real coefficients h, k . If we define

$$u_R = \begin{pmatrix} 1 & 0 \\ 0 & \det U \end{pmatrix} U^* \in SU(2)_R$$

$$u_L = \begin{pmatrix} 1 & 0 \\ 0 & \det V \end{pmatrix} V^* \in SU(2)_L$$

it follows that

$$u_R \begin{pmatrix} \phi_1^1 & \phi_1^2 \\ \phi_2^1 & \phi_2^2 \end{pmatrix} u_L^* = \begin{pmatrix} h & 0 \\ 0 & k \det UV^* \end{pmatrix} =: \begin{pmatrix} h & 0 \\ 0 & \chi \end{pmatrix}.$$

Next, we consider $\Delta_{\dot{a}I}$ and write

$$(\Delta_{\dot{a}I}) = \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix}, \quad \text{with } u_a^* = (\Delta_{\dot{a}1} \quad \Delta_{\dot{a}2} \quad \Delta_{\dot{a}3} \quad \Delta_{\dot{a}4})$$

for $a = 1, 2$. We may suppose that the vectors u_1, u_2 are such that their inner product $u_1^* u_2$ is a real number. Indeed, if this is not the case, then multiply $\Delta_{\dot{a}I}$ by a matrix in $SU(2)_R$ as follows:

$$\begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix} \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} = \begin{pmatrix} \alpha u_1^* \\ \alpha^* u_2^* \end{pmatrix}.$$

Now the inner product is $(\alpha^* u_1)^* \alpha u_2 = (\alpha)^2 u_1^* u_2$ and we may choose α so as to cancel the phase of $u_1^* u_2$. Moreover, this transformation respects the above form of ϕ_a^b after a $SU(2)_L$ -transformation of exactly the same form:

$$\begin{pmatrix} h & 0 \\ 0 & \chi \end{pmatrix} \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & \chi \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix}^* = \begin{pmatrix} h & 0 \\ 0 & \chi \end{pmatrix}.$$

Thus let us continue with the vectors u_1, u_2 satisfying $u_1^* u_2 \in \mathbb{R}$. We apply Gram-Schmidt orthonormalization to u_1 and u_2 , to arrive at the following orthonormal set of vectors $\{e_1, e_2\}$ in \mathbb{C}^4 :

$$e_1 = \frac{u_1}{\|u_1\|}; \quad e_2 = \frac{u_2 - \frac{u_1^* u_2}{\|u_1\|} u_1}{\|u_2 - \frac{u_1^* u_2}{\|u_1\|} u_1\|}.$$

We complete this set by choosing two additional orthonormal vectors e_3 and e_4 and write a unitary 4×4 matrix:

$$U = (e_1 \ e_2 \ e_3 \ e_4)$$

The sought-for matrix $u \in SU(4)$ is determined by

$$u^t = U \begin{pmatrix} 1_3 & 0 \\ 0 & \det U^* \end{pmatrix}$$

so as to give

$$(\Delta_{\dot{a}I}) u^t = \begin{pmatrix} u_1^* e_1 & 0 & 0 & 0 \\ u_2^* e_1 & u_2^* e_2 & 0 & 0 \end{pmatrix} =: \begin{pmatrix} 1 + \delta_0 & 0 & 0 & 0 \\ \delta_1 & \eta_1 & 0 & 0 \end{pmatrix} \quad \square$$

Remark 7. *Note that this is compatible with the dimension of the quotient of the space of field values by the group. Indeed, the fields ϕ_a^b and $\Delta_{\dot{a}I}$ span a real 24-dimensional space (at each manifold point). The dimension of the orbit space is then $24 - \dim P$ with P a principal orbit of the action of $SU(2)_R \times SU(2)_L \times SU(4)$ on the space of field values. This dimension $\dim P$ is determined by the dimension of the group and of a principal isotropy group.*

First, we see that up to conjugation there is always a $SU(2)$ -subgroup of $SU(4)$ leaving $\Delta_{\dot{a}I}$ invariant: it corresponds to $SU(2)$ -transformations in the space orthogonal to the vectors Δ_{1I} and Δ_{2I} in \mathbb{C}^4 . Moreover, one can compute that the isotropy subgroup of the field values

$$(\phi_a^b) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad (\Delta_{\dot{a}I}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

is given by $\mathbb{Z}_2 \times SU(2)$. Hence, the dimension of the principal orbit is $21 - 3 = 18$ so that the orbit space is 6-dimensional. This corresponds to the 4 real fields $h, \delta_0, \delta_1, \eta_1$ and the complex field χ .

We allow for the colour $SU(3)$ -symmetry not to be broken spontaneously, hence we only choose unitarity gauge in the $SU(2)_R \times SU(2)_L \times U(1)$ -representations. That is, we retain the row vector $\Delta_{\dot{2}I}$ for $I = 1, \dots, 4$ as a variable and write

$$(\Delta_{\dot{a}I}) = \begin{pmatrix} \sqrt{w} + \delta_0/\sqrt{w} & 0 & 0 & 0 \\ \delta_1/\sqrt{w} & \eta_1/\sqrt{w} & \eta_2/\sqrt{w} & \eta_3/\sqrt{w} \end{pmatrix}$$

so that (η_i) forms a scalar $SU(3)$ -triplet field (so-called *scalar leptiquarks*). The reason for the rescaling with \sqrt{w} is that it yields the right kinetic terms for δ_0, δ_1 and η . Indeed, from the spectral action we then have

$$\begin{aligned} \frac{1}{2} \partial_\mu H_{\dot{a}I\dot{b}J} \partial^\mu H^{\dot{a}I\dot{b}J} &= \frac{1}{2} \partial_\mu \left(\Delta_{\dot{a}J} \Delta_{\dot{b}I} \right) \partial^\mu \left(\Delta^{\dot{a}J} \Delta^{\dot{b}I} \right) \\ &\sim \sum_{a=0}^1 \partial_\mu \delta_a \partial^\mu \delta^a + \partial_\mu \eta \partial^\mu \eta^* + \text{higher order} \end{aligned}$$

The scalar potential becomes in terms of the fields $h, \chi, \delta_0, \delta_1, \eta_i$:

$$\begin{aligned} \mathcal{L}_{pot}(h, \chi, \delta_0, \delta_1, \eta) &= -\mu^2 (h^2 + |\chi|^2) - \nu^2 \left((w + \delta_0)^2 + \delta_1^2 + |\eta|^2 \right)^2 / w^2 \\ &\quad + \lambda_{H\Sigma} \left((w + \delta_0)^2 h^2 + (\delta_1^2 + |\eta|^2) |\chi|^2 \right) \left((w + \delta_0)^2 + \delta_1^2 + |\eta|^2 \right) / w^2 \\ &\quad + \lambda_H \left((w + \delta_0)^4 + 2(w + \delta_0)^2 \delta_1^2 + (\delta_1^2 + |\eta|^2)^2 \right) / w^4 + \lambda_\Sigma (h^4 + |\chi|^4) \end{aligned}$$

As we are interested in the truncation to the Standard Model, we look for extrema with $\langle \delta_1 \rangle = \langle \eta_i \rangle = 0$, whilst setting $\langle h \rangle = v, \langle \delta_0 \rangle = 0, \langle \chi \rangle = x$. Note that the symmetry of these vevs is

$$\begin{aligned} &\left\{ \left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^* \end{pmatrix}, \begin{pmatrix} \lambda^* & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda^* & 0 \\ 0 & m \end{pmatrix} \right) : \lambda \in U(1), m \in SU(3) \right\} \\ &\quad \subset SU(2)_R \times SU(2)_L \times SU(4) \end{aligned}$$

In other words, $SU(2)_R \times SU(2)_L \times SU(4)$ is broken by the above vevs to $U(1) \times SU(3)$.

The first derivative of V vanishes for these vevs precisely if

$$\begin{aligned} 2v(w^2 \lambda_{H\Sigma} + 2v^2 \lambda_\Sigma - \mu^2) &= 0, \\ 4x^3 \lambda_\Sigma - 2x\mu^2 &= 0, \\ 4w(2w^2 \lambda_H + v^2 \lambda_{H\Sigma} - \nu^2) &= 0. \end{aligned}$$

This gives rise to the fine-tuning of v, w as in [16]:

$$w^2 \lambda_{H\Sigma} + 2v^2 \lambda_\Sigma - \mu^2, \quad 2w^2 \lambda_H + v^2 \lambda_{H\Sigma} - \nu^2$$

choosing μ and ν such that the solutions v, w are of the desired orders. Moreover, we find that the vev for χ either vanishes or is equal to $x = \sqrt{\mu^2/2\lambda_\Sigma}$. Note that this latter vev appears precisely at the entry $k^d h$ (or $k^e h$) of the finite Dirac operator, which we have disregarded by setting $\rho = 1$.

If $\langle \chi \rangle = x = 0$ then the Hessian is (derivatives with respect to $h, \chi, \delta_0, \delta_1, \eta$):

$$\begin{pmatrix} 8v^2\lambda_\Sigma & 0 & 8vw\lambda_{H\Sigma} & 0 & 0 \\ 0 & -2w^2\lambda_{H\Sigma}-4v^2\lambda_\Sigma & 0 & 0 & 0 \\ 8vw\lambda_{H\Sigma} & 0 & 32w^2\lambda_H & 0 & 0 \\ 0 & 0 & 0 & -2v^2\lambda_{H\Sigma} & 0 \\ 0 & 0 & 0 & 0 & -8\lambda_H w^2 - 2v^2\lambda_{H\Sigma} w^2 \mathbf{1}_3 \end{pmatrix}$$

where the $\mathbf{1}_3$ is the identity matrix in colour space, corresponding to the η -field. This Hessian is not positive definite so we disregard the possibility that $\langle \chi \rangle = 0$.

If $x = \sqrt{\mu^2/2\lambda_\Sigma}$ then the Hessian is

$$\begin{pmatrix} 8v^2\lambda_\Sigma & 0 & 8vw\lambda_{H\Sigma} & 0 & 0 \\ 0 & 4w^2\lambda_{H\Sigma}+8v^2\lambda_\Sigma & 0 & 0 & 0 \\ 8vw\lambda_{H\Sigma} & 0 & 32w^2\lambda_H & 0 & 0 \\ 0 & 0 & 0 & w^2\frac{\lambda_{H\Sigma}^2}{\lambda_\Sigma} & 0 \\ 0 & 0 & 0 & 0 & w^2\frac{\lambda_{H\Sigma}^2-8\lambda_H\lambda_\Sigma}{\lambda_\Sigma}\mathbf{1}_3 \end{pmatrix}$$

which is positive-definite if

$$\lambda_{H\Sigma}^2 \geq 8\lambda_H\lambda_\Sigma. \quad (54)$$

Note that this relation may hold only at high-energies. The masses for χ, δ_1 and η are then readily found to be:

$$\begin{aligned} m_\chi^2 &= 4w^2\lambda_{H\Sigma} + 8v^2\lambda_\Sigma, \\ m_{\delta_1}^2 &= w^2\frac{\lambda_{H\Sigma}^2}{\lambda_\Sigma}, \\ m_\eta^2 &= w^2\frac{\lambda_{H\Sigma}^2 - 8\lambda_H\lambda_\Sigma}{\lambda_\Sigma}. \end{aligned}$$

Under the assumption that $v^2 \approx 10^2 \text{ GeV}$, $w^2 \approx 10^{11} \text{ GeV}$ we have $m_\chi^2 \approx 10^{11} \text{ GeV}$ and $m_{\delta_1}^2, m_\eta \approx 10^{11} \text{ GeV}$.

The (non-diagonal) h and δ_0 sector has mass eigenstates as in [16]:

$$\begin{aligned} m_\pm^2 &= 16w^2\lambda_H + 4v^2\lambda_\Sigma \\ &\pm 4\sqrt{16w^4\lambda_H^2 + v^4\lambda_\Sigma^2 + 4v^2w^2(\lambda_{H\Sigma}^2 - 2\lambda_H\lambda_\Sigma)} \end{aligned}$$

Under the assumption that $v^2 \ll w^2$ we can expand the square root:

$$\begin{aligned}
& 4\sqrt{16\lambda_H^2 w^4 \left(1 + \frac{\lambda_\Sigma^2 v^4}{\lambda_H^2 w^4} + \frac{\lambda_{H\Sigma}^2 - 2\lambda_H \lambda_\Sigma}{4\lambda_H^2} \frac{v^2}{w^2}\right)} \\
& \approx 16\lambda_H w^2 \left(1 + \frac{\lambda_{H\Sigma}^2 - 2\lambda_H \lambda_\Sigma}{8\lambda_H^2} \frac{v^2}{w^2}\right) \\
& = 16\lambda_H w^2 + \frac{2\lambda_{H\Sigma}^2}{\lambda_H} v^2 - 4\lambda_\Sigma v^2.
\end{aligned}$$

Consequently,

$$\begin{aligned}
m_+ & \approx 32\lambda_H w^2 + 2\frac{2\lambda_{H\Sigma}^2}{\lambda_H} v^2, \\
m_- & \approx 8\lambda_\Sigma v^2 \left(1 - \frac{\lambda_{H\Sigma}^2}{4\lambda_H \lambda_\Sigma}\right).
\end{aligned}$$

which are of the order of 10^{11} and 10^2 GeV, respectively. This requires that we have at low energies

$$4\lambda_H \lambda_\Sigma \geq \lambda_{H\Sigma}^2, \quad (55)$$

which fully agrees with [16] when we identify $\delta_0 \equiv \sigma$ and with the couplings related via

$$\lambda_H = \frac{1}{4}\lambda_\sigma, \quad \lambda_{H\Sigma} = \frac{1}{2}\lambda_{h\sigma}, \quad \lambda_\Sigma = \frac{1}{4}\lambda_h$$

Note the tension between Equations (55) and (54), calling for a careful study of the running of the couplings in order to guarantee positive mass eigenstates at their respective energies.

We have summarized the scalar particle content of the above model in Table 1. In terms of the original scalar fields ϕ_a^b and Δ_{aI} the vevs are of the following form:

$$\begin{aligned}
(\phi_a^b) &= \begin{pmatrix} v & 0 \\ 0 & \sqrt{\mu^2/2\Lambda_\Sigma} \end{pmatrix} \\
(\Delta_{aI}) &= \begin{pmatrix} w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

This shows that there are two scales of spontaneous symmetry breaking: at $10^{11} - 10^{12}$ GeV we have

$$SU(2)_R \times SU(2)_L \times SU(4) \rightarrow U(1)_Y \times SU(2)_L \times SU(3)$$

and then at electroweak scale (both v and μ) we have

$$U(1)_Y \times SU(2)_L \times SU(3) \rightarrow U(1)_Q \times SU(3)$$

	$U(1)_Y$	$SU(2)_L$	$SU(3)$
$\begin{pmatrix} \phi_1^0 \\ \phi_1^+ \end{pmatrix} = \begin{pmatrix} \phi_1^1 \\ \phi_1^2 \end{pmatrix}$	1	2	1
$\begin{pmatrix} \phi_2^- \\ \phi_2^0 \end{pmatrix} = \begin{pmatrix} \phi_2^1 \\ \phi_2^2 \end{pmatrix}$	-1	2	1
δ_0	0	1	1
δ_1	-2	1	1
η	$-\frac{2}{3}$	1	3

Table 1: Scalar particle content with SM-representations

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