

DESCENT OF HILBERT C^* -MODULES

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ABSTRACT. Let F be a right Hilbert C^* -module over a C^* -algebra B , and suppose that F is equipped with a left action, by compact operators, of a second C^* -algebra A . Tensor product with F gives a functor from Hilbert C^* -modules over A to Hilbert C^* -modules over B . We prove that under certain conditions (which are always satisfied if, for instance, A is nuclear), the image of this functor can be described in terms of coactions of a certain coalgebra canonically associated to F . We then discuss several examples that fit into this framework: parabolic induction of tempered group representations; Hermitian connections on Hilbert C^* -modules; Fourier (co)algebras of compact groups; and the maximal C^* -dilation of operator modules over non-self-adjoint operator algebras.

1. INTRODUCTION

The notion of *descent* originated in algebraic geometry as a technique for relating the geometry of a space M to that of a covering space $N \rightarrow M$ ([Gro95]; cf. [BLR90, Chapter 6] for an exposition). The idea is to characterise those structures—vector bundles, or sheaves, for example—defined on N that are pulled back from, and thus ‘descend’ back down to, the base M . Translating geometry into algebra, one is led to consider questions such as that of characterising those B -modules that have the form $X \otimes_A B$ for some A -module X , given an inclusion of rings $A \hookrightarrow B$. In this paper we shall consider an analogue of this question for modules over algebras of Hilbert-space operators, and show that for a large class of algebras (including for instance all nuclear C^* -algebras), the descent problem has a particularly satisfactory solution: in the language of [Gro95], every injective $*$ -homomorphism satisfies strict descent for Hilbert C^* -modules.

The problem that we shall study is closely related to—indeed, as we shall presently explain, it is dual to—the one addressed by Rieffel in his C^* -algebraic generalisation of Mackey’s imprimitivity theorem [Rie74, Theorem 6.29]. Let us recall that Rieffel’s theorem describes the image of the functor

$$\mathcal{F}_1 : \text{Rep}(B) \rightarrow \text{Rep}(A) \quad Y \mapsto F \otimes_B Y,$$

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from the $*$ -representations of a C^* -algebra B to those of a second C^* -algebra A , given by tensor product with a C^* -correspondence (or in the terminology of [Rie74], a Hermitian B -rigged A -module) ${}_A F_B$. Rieffel observed that the A -representations $F \otimes_B Y$ all carry a compatible representation of the C^* -algebra $K = K_B(F)$ of B -compact operators on F , and he proved that the existence of such a representation of K precisely characterises the image of \mathcal{F}_1 .

Alongside \mathcal{F}_1 , each C^* -correspondence ${}_A F_B$ naturally determines a second functor

$$\mathcal{F}_2 : \text{CM}(A) \rightarrow \text{CM}(B) \quad X \mapsto X \otimes_A F,$$

from the category of right Hilbert C^* -modules over A to those over B . Functors of this sort play a central role in operator K -theory (see [Kas84] for a survey). An important instance of this is when the algebra A acts on the correspondence F by B -compact operators, in which case the functor \mathcal{F}_2 induces a map in K -theory $K_*(A) \rightarrow K_*(B)$. For example, if $N \rightarrow M$ is a surjection of compact Hausdorff spaces, then the corresponding inclusion of algebras of continuous functions $C(M) \hookrightarrow C(N)$ yields a C^* -correspondence ${}_{C(M)} C(N)_{C(N)}$ for which the functor \mathcal{F}_2 (and the induced map $K^*(M) \rightarrow K^*(N)$) is given by pull-back of continuous fields of Hilbert spaces from M to N . The problem of describing the image of functors of the form \mathcal{F}_2 is thus an interesting one from the point of view of K -theory.

This problem is also relevant to representation theory, specifically to the tempered representation theory of reductive Lie groups. Given such a group G , and a Levi subgroup $L \subseteq G$, Clare showed in [Cla13] how to construct a C^* -correspondence ${}_{C_r^*(G)} F_{C_r^*(L)}$ whose associated tensor-product functor \mathcal{F}_1 is the well-known functor of *parabolic induction* of tempered unitary representations from L to G . In [CCH16] and [CCH18] we showed that the companion functor \mathcal{F}_2 of *parabolic restriction* also plays an important role in representation theory. Detailed information about the image of this functor can be obtained, via the methods of [CCH16] and [CCH18], from deep representation-theoretic results due to Harish-Chandra, Langlands, and others. It would be of great interest to find an alternative, more geometric description of the image of parabolic restriction, since doing so might yield a new perspective on aspects of tempered representation theory.

Motivated by these K -theoretic and representation-theoretic considerations, in this paper we shall prove (Theorem 5.6) that the image of the functor \mathcal{F}_2 can, under certain conditions, be characterised in terms of *coactions* of an operator coalgebra associated to the correspondence F . Our approach relies on relating the functors \mathcal{F}_1 and \mathcal{F}_2 to one another in categorical terms, which was achieved in [CH16] by embedding both categories Rep and CM into the category OM of *operator modules* (cf. [BLM04]). The functors \mathcal{F}_1 and \mathcal{F}_2 extend, using the *Haagerup tensor product*, to functors between $\text{OM}(A)$ and $\text{OM}(B)$, and in [CH16] we noted that if A acts on F by B -compact operators then these extended functors are adjoint to one another.

This adjunction allows us to bring a standard piece of categorical algebra—namely, the notion of (co)monads and their categories of (co)modules—to bear on the problem of characterising the images of \mathcal{F}_1 and \mathcal{F}_2 . (See e.g., [ML98, Chapter VI], [BW05, Chapter 3], or [Bor94, Chapter 4], for the general theory.) Our description of the image of \mathcal{F}_2 is obtained through a study of the comonad associated to \mathcal{F}_1 and \mathcal{F}_2 . Letting F^* denote the operator B - A bimodule adjoint to F , we find that the Haagerup tensor product $C = F^* \otimes_A^h F$ carries a coalgebra structure; that the Hilbert C^* - B -modules $Y = X \otimes_A F$ all carry a compatible coaction $\delta_Y : Y \rightarrow Y \otimes_B^h C$; and that under certain circumstances the existence of such a coaction precisely characterises the image of \mathcal{F}_2 . (Dually, Rieffel’s imprimitivity theorem can be recovered through a study of the monad associated to \mathcal{F}_1 and \mathcal{F}_2 .)

The appearance of coalgebras and comodules in connection with \mathcal{F}_2 , instead of the algebras and modules entering into Rieffel’s imprimitivity theorem for \mathcal{F}_1 , is easily explained from a category-theoretic perspective: the functor \mathcal{F}_2 is *left*-adjoint to \mathcal{F}_1 (upon extension to operator modules), and left adjoints always produce comodules over comonads rather than modules over monads. The appearance of comodules can also be explained geometrically, by way of a simple example. Consider as above a continuous surjection $N \rightarrow M$ of compact Hausdorff spaces, and let U be an open subset of N . Under what circumstances is U the preimage of an open subset $V \subseteq M$, so that $C_0(U) \cong C_0(V) \otimes_{C(M)} C(N)$ as Hilbert C^* -modules over $C(N)$? An obvious criterion is that the map $U \times_M N \rightarrow N$, $(u, n) \mapsto n$ should have image contained in U , and so give rise (by pullback) to a map of $C(N)$ -modules $C_0(U) \rightarrow C_0(U \times_M N)$. The latter map factors through a map $C_0(U) \rightarrow C_0(U) \otimes_{C(M)}^h C(N)$, which is a coaction of the kind appearing in our Theorem 5.6.

Coalgebras and comodules related to the Haagerup tensor product have previously been studied from (at least) two other, rather different directions. In [ER03], Effros and Ruan showed that the multiplication on a von Neumann algebra M dualises to a coproduct $M_\vee \rightarrow M_\vee \otimes_{\mathbb{C}}^{\text{eh}} M_\vee$ on the predual M_\vee , where \otimes^{eh} is the *extended* Haagerup tensor product. An important motivating example is that of $M = \text{vN}(G)$, the von Neumann algebra of a locally compact group G , whose predual is the Fourier algebra $A(G)$ [Eym64]. In certain circumstances—e.g., if $M = \text{vN}(G)$ for a compact group G (or more generally, a compact quantum group: cf. [Daw10, Chapter 9])—Effros and Ruan’s coproduct takes values in the ordinary Haagerup tensor product, and can thus be compared to the coalgebras studied in this paper. In Section 6.2 we show that in such cases Effros and Ruan’s coalgebras are isomorphic to coalgebras associated as above to C^* -correspondences (actually, to $*$ -representations: $B = \mathbb{C}$). We conclude for instance that the tensor product with the regular representation provides an equivalence between the category $\text{OM}(C^*(G))$ of operator modules over the C^* -algebra of

a compact group G , and the category $\text{OC}(A(G))$ of operator comodules over the Fourier (co)algebra of G .

Comodules involving the Haagerup tensor product have also previously appeared in operator K -theory, under the guise of *connections* on operator modules. Following analogous constructions in algebra (cf. [Con94, CQ95]), Mesland in [Mes14] associated to each operator algebra B the B -bimodule

$$\Omega(B) := \ker \left(B \otimes_{\mathbb{C}}^{\text{h}} B \xrightarrow{\text{multiplication}} B \right),$$

and he studied connections $\nabla : Z \rightarrow Z \otimes_B^{\text{h}} \Omega(B)$ on operator B -modules. More generally, one can replace \mathbb{C} by any closed nondegenerate subalgebra $A \subseteq B$ to obtain a bimodule $\Omega(B, A)$, and a corresponding notion of connection. In the parallel purely algebraic setting, given a connection $\nabla : Z \rightarrow Z \otimes_B \Omega(B, A)$ on a module over a ring B , one finds (cf. [Nus97, Brz03]) that the map

$$\delta : Z \xrightarrow{z \mapsto \nabla(z) + z \otimes 1 \otimes 1} Z \otimes_B B \otimes_A B$$

defines a coaction of the (Sweedler) coring $C = B \otimes_A B$ if and only if the connection ∇ is *flat*; and one obtains in this way an equivalence of categories between C -comodules and flat $\Omega(B, A)$ -connections on B -modules. With a view to applications in K -theory, in Section 6.3 we extend these observations to the C^* -algebraic setting, and we deduce from Theorem 5.6 that for a nondegenerate inclusion of C^* -algebras $A \hookrightarrow B$, where A is (for instance) nuclear, the image of the functor $\text{CM}(A) \rightarrow \text{CM}(B)$, $X \mapsto X \otimes_A B$ —and hence, the image of the induced map in K -theory $K_*(A) \rightarrow K_*(B)$ —can be described in terms of *Hermitian connections* $Z \mapsto Z \otimes_B^{\text{h}} \Omega(B, A)$ on Hilbert C^* - B -modules.

The organisation of the paper is as follows. In Section 2 we recall some background and establish our notation regarding operator algebras and their modules. In Section 3 we introduce the coalgebras and comodules that appear in our main results, which are proven in Sections 4 (on operator modules) and 5 (on Hilbert C^* -modules). In Section 6 we discuss several examples: parabolic induction of representations of reductive groups, following [CCH16, CCH18, CH16] (Section 6.1); the coalgebras of Effros and Ruan (Section 6.2); and Hermitian connections on Hilbert C^* -modules (Section 6.3). Finally, in Section 6.4, we contrast these C^* -algebraic examples with a non-self-adjoint example, namely the inclusion of a non-self-adjoint operator algebra into its maximal C^* -algebra (cf. [Ble99]).

Let us conclude this introduction with a few words on our proofs. We indicated above that the the formulation of our main result is obtained by ‘reversing the arrows’ in (an appropriate formulation of) Rieffel’s imprimitivity theorem. The same is not true of the proofs, however, due to an inherent asymmetry of the Haagerup tensor product. A theorem of Beck (cf. [BW05, Theorem 3.3.14]) shows that a necessary condition for Rieffel’s theorem to hold is that the functor \mathcal{F}_1 (extended to operator modules)

should preserve certain cokernels; while a necessary condition for our results to hold is that (the extension of) the functor \mathcal{F}_2 should preserve certain kernels. The Haagerup tensor product over a C^* -algebra in fact preserves all cokernels, but it does not preserve all kernels (see Section 2.3). Our verification that the functor \mathcal{F}_2 preserves the appropriate class of kernels (Lemma 4.9) adapts techniques from [Mes00, JT04, BP88, Mes18]—all based on some form of duality—to the context of operator modules. (Some extra care is required, owing to the fact that the operator-space dual of an operator module is not, in general, an operator module.) We also rely heavily on a theorem of Anantharaman-Delaroche and Pop on the exactness of the Haagerup tensor product ([ADP02], cf. Theorem 2.5).

2. PRELIMINARIES

2.1. Operator algebras and their modules. We mostly adhere to the terminology and notation of [BLM04] with regard to operator algebras and their modules. Let us briefly recall the definitions.

An *operator space* is a complex vector space X equipped for each $n \geq 1$ with a Banach space norm on the matrix space $M_n(X)$, for which there exist a Hilbert space H and an embedding $X \rightarrow B(H)$ such that the induced embeddings $M_n(X) \rightarrow M_n(B(H)) \cong B(H^{\oplus n})$ are isometric for every n , where $B(H^{\oplus n})$ is given the operator norm.

A linear map of operator spaces $f : X \rightarrow Y$ is *completely bounded* if the quantity

$$\|f\|_{\text{cb}} := \sup_n \|M_n(f) : M_n(X) \rightarrow M_n(Y)\|$$

is finite. The notions of *completely contractive* and *completely isometric* maps are defined in a similar way. A *completely bounded isomorphism* is a completely bounded linear bijection whose inverse is also completely bounded. A *complete embedding* is a map which has closed range and which is a completely bounded isomorphism onto its image.

An *operator algebra* is an operator space A equipped with an associative bilinear product satisfying $\|aa'\| \leq \|a\| \cdot \|a'\|$ for every $n \geq 1$ and every $a, a' \in M_n(A)$. We shall always assume that our operator algebras are *approximately unital*, meaning that they possess a contractive approximate unit. This latter assumption excludes many interesting examples of non-self-adjoint operator algebras; on the other hand, every C^* -algebra is an approximately unital operator algebra.

A (right) *operator module* over an operator algebra A is an operator space X which is also a right A -module and whose norms satisfy $\|xa\| \leq \|x\| \cdot \|a\|$ for every $n \geq 1$ and every $a \in M_n(A)$ and $x \in M_n(X)$. In this case A and X can always be represented completely isometrically as spaces of Hilbert space operators, in such a way that the product in A and the module action on X become composition of operators. Important examples include Hilbert C^* -modules, which can be represented as Hilbert space operators via the ‘linking algebra’ construction; and Hilbert-space representations $A \rightarrow B(H)$, for

which the Hilbert space H can be represented as rank-one operators on the conjugate Hilbert space H^* . Left operator modules, and operator bimodules, are defined in a similar way to right modules. In this paper ‘module’ means ‘right module’ unless otherwise specified. An operator A -module X is *nondegenerate* if every element of X has the form xa for some $a \in A$ and $x \in X$. Equivalently—by the Cohen factorisation theorem and our assumption that A is approximately unital— X is nondegenerate if the elements of the form xa span a dense subspace of X . We shall work exclusively with nondegenerate modules.

If E is an operator A - B bimodule, where A and B are approximately unital operator algebras, and X is a right operator A -module, then the *Haagerup tensor product* $X \otimes_A^h E$ is a right operator B -module. See [BLM04, 1.5 & 3.4] for the definition and basic properties of the Haagerup tensor product. If X is a nondegenerate right (respectively, left) operator A -module, then the multiplication maps $X \otimes_A^h A \rightarrow X$ (respectively, $A \otimes_A^h X \rightarrow X$) are completely isometric isomorphisms, a fact that we shall frequently use without comment.

On operator modules over C^* -algebras there is a formal adjoint operation, which allows one to exchange left and right modules. If ${}_A X_B$ is an operator bimodule over C^* -algebras A and B , then the adjoint X^* is an operator B - A bimodule: as a vector space X^* is the complex-conjugate of X ; the bimodule structure is defined by $b \cdot x^* \cdot a := (a^* x b^*)^*$ (where $x \mapsto x^*$ is the canonical conjugate-linear isomorphism $X \rightarrow X^*$); and the operator-space structure is given by the norms $\|[x_{ij}^*]\|_{M_n(X^*)} := \|[x_{ji}]\|_{M_n(X)}$. The assignment $X \mapsto X^*$ is a functor, from operator A - B bimodules to operator B - A bimodules: each completely bounded bimodule map $t : X \rightarrow Y$ induces a completely bounded (with the same norm) bimodule map $t^* : X^* \rightarrow Y^*$ via $t^*(x^*) := t(x)^*$. If X is a right operator A -module, and Y is a left operator A -module, then the map

$$(2.1) \quad (X \otimes_A^h Y)^* \rightarrow Y^* \otimes_A^h X^*, \quad (x \otimes y)^* \mapsto y^* \otimes x^*$$

is a completely isometric isomorphism.

2.2. The OS_1 -category of operator modules. For an approximately unital operator algebra A we denote by $\text{OM}(A)$ the category whose objects are nondegenerate right operator A -modules, and whose morphisms are the completely bounded A -module maps. The space of such maps from X to Y will be denoted $\text{CB}_A(X, Y)$.

It will be important in what follows to note that $\text{OM}(A)$ carries extra structure: its morphism sets $\text{CB}_A(X, Y)$ are operator spaces, and composition of morphisms gives rise to completely contractive maps

$$\text{CB}_A(Y, Z) \widehat{\otimes} \text{CB}_A(X, Y) \rightarrow \text{CB}_A(X, Z),$$

where $\widehat{\otimes}$ is the projective tensor product of operator spaces (cf. [BLM04, 1.2.19 & 1.5.11]). Let us refer to a category whose morphism sets and composition law satisfy the above conditions as an *OS_1 -category*. (This is

an example of an *enriched category*.) A functor $\mathcal{F} : \mathbf{A} \rightarrow \mathbf{B}$ of OS_1 -categories will be called an OS_1 -*functor*, or a *completely contractive functor*, if the induced maps on morphism sets are completely contractive linear maps of operator spaces. An example of an OS_1 -functor is the functor

$$\text{OM}(A) \rightarrow \text{OM}(B), \quad X \mapsto X \otimes_A^{\text{h}} E$$

of Haagerup tensor product with an operator bimodule; see [Cri18, Appendix] for a proof of this well-known fact.

Blecher has shown that the OS_1 -category $\text{OM}(A)$ is a complete Morita invariant of A . To state this precisely, let us introduce the following terminology.

Definition 2.2. Let \mathbf{A} and \mathbf{B} be OS_1 -categories. A *complete contraction* in \mathbf{A} is a morphism t with $\|t\| \leq 1$. We write \mathbf{A}_1 for the subcategory of \mathbf{A} having the same objects as \mathbf{A} , and having as morphisms the complete contractions in \mathbf{A} . A *completely isometric isomorphism* in \mathbf{A} is an isomorphism t such that both t and t^{-1} are complete contractions: i.e., an isomorphism in \mathbf{A}_1 . A *completely isometric natural isomorphism* of functors $\mathcal{F}, \mathcal{G} : \mathbf{A} \rightarrow \mathbf{B}$ is a natural isomorphism $\xi : \mathcal{F} \rightarrow \mathcal{G}$ such that for each $X \in \mathbf{A}$ the morphism $\xi_X : \mathcal{F}(X) \rightarrow \mathcal{G}(X)$ is a completely isometric isomorphism. A *completely isometric equivalence* between \mathbf{A} and \mathbf{B} is an OS_1 -functor $\mathcal{F} : \mathbf{A} \rightarrow \mathbf{B}$ for which there exists an OS_1 -functor $\mathcal{G} : \mathbf{B} \rightarrow \mathbf{A}$ and completely isometric natural isomorphisms $\mathcal{F} \circ \mathcal{G} \cong \text{id}_{\mathbf{B}}$ and $\mathcal{G} \circ \mathcal{F} \cong \text{id}_{\mathbf{A}}$.

We note that if $\mathcal{F} : \mathbf{A} \rightarrow \mathbf{B}$ is an OS_1 -functor, then \mathcal{F} restricts to a functor $\mathcal{F}_1 : \mathbf{A}_1 \rightarrow \mathbf{B}_1$. If \mathcal{F} is a completely isometric equivalence then \mathcal{F}_1 is an equivalence.

The following result is due to Blecher [Ble01b, Theorem 1.2].

Theorem 2.3. *If A and B are C^* -algebras and $\mathcal{F} : \text{OM}(A) \rightarrow \text{OM}(B)$ is a completely isometric equivalence, then \mathcal{F} is completely isometrically isomorphic to the functor of Haagerup tensor product with a Morita equivalence bimodule.*

In Section 6.4 we shall make use of the categorical notion of *kernels* and *cokernels*. Recall (e.g., from [ML98, VIII.1]) that a *kernel* of a morphism $t : X \rightarrow Y$ in a category \mathbf{A} with a zero object is a morphism $i : I \rightarrow X$ such that $t \circ i = 0$, and such that any other morphism $j : J \rightarrow X$ having $t \circ j = 0$ factors uniquely as a composition $J \rightarrow I \xrightarrow{i} X$. The notion of a *cokernel* of t is defined analogously as a map $q : Y \rightarrow Q$ which satisfies $q \circ t = 0$ and which is universal for this property. Kernels and cokernels, when they exist, are unique up to isomorphism.

Lemma 2.4. *Let A be an operator algebra and let $t : X \rightarrow Y$ be a morphism in $\text{OM}(A)$ (respectively, in $\text{OM}(A)_1$). The inclusion $i : \ker(t) \rightarrow X$ is a kernel of t , and the quotient mapping $q : Y \rightarrow Y/\overline{\text{image}(t)}$ is a cokernel of t in $\text{OM}(A)$ (respectively, in $\text{OM}(A)_1$). In particular, the kernels in*

$\text{OM}(A)$ are precisely the complete embeddings, while the kernels in $\text{OM}(A)_1$ are precisely the complete isometries.

Proof. The maps i and q are easily seen to possess the necessary universal properties (cf. [BLM04, 1.2.15]). The last assertion follows from the uniqueness of kernels: every kernel of t in $\text{OM}(A)$ (respectively, $\text{OM}(A)_1$) is conjugate, by a completely bounded (respectively, completely isometric) isomorphism, to the complete isometry $i : \ker(t) \rightarrow X$. \square

2.3. Exactness of the Haagerup tensor product. The simplicity of the descent theorem for operator modules over C^* -algebras, compared to other kinds of algebras, is largely due to the following exactness property of the Haagerup tensor product, established by Anantharaman-Delaroche and Pop [ADP02, Corollary, p.411]

Theorem 2.5. *Let A be a C^* -algebra, let X be a right operator A -module, and let Y be a left operator A -module. Let I be a closed submodule of X , and let $i : I \rightarrow X$ and $q : X \rightarrow X/I$ denote the inclusion and the quotient mappings, respectively. Consider the maps*

$$I \otimes_A^h Y \xrightarrow{i \otimes \text{id}_Y} X \otimes_A^h Y \xrightarrow{q \otimes \text{id}_Y} (X/I) \otimes_A^h Y.$$

- (a) *The map $i \otimes \text{id}_Y$ is a complete isometry.*
- (b) *The map $q \otimes \text{id}_Y$ is a complete quotient mapping; that is, the induced map $(X \otimes_A^h Y)/\ker(q \otimes \text{id}_Y) \rightarrow (X/I) \otimes_A^h Y$ is a completely isometric isomorphism.*
- (c) *The image of $i \otimes \text{id}_Y$ equals the kernel of $q \otimes \text{id}_Y$.*

Proof. The assertions (a) and (b) are proved in [ADP02] (cf. [BLM04, 1.5.5 & 3.6.5]). Part (c) is easier, and presumably well known: we clearly have an inclusion $\text{image}(i \otimes \text{id}_Y) \subseteq \ker(q \otimes \text{id}_Y)$, whence a mapping

$$(2.6) \quad (X \otimes_A^h Y)/\text{image}(i \otimes \text{id}_Y) \rightarrow (X/I) \otimes_A^h Y.$$

It is a straightforward matter to check that the formula

$$(X/I) \times Y \rightarrow (X \otimes_A^h Y)/\text{image}(i \otimes \text{id}_Y), \quad (x+I, y) \mapsto (x \otimes y) + \text{image}(i \otimes \text{id}_Y)$$

defines a completely contractive (in the sense of [BLM04, 1.5.4]) A -balanced bilinear map, and hence induces a map on $(X/I) \otimes_A^h Y$ that is inverse to (2.6). \square

Remark 2.7. Note that the equality $\ker(t \otimes \text{id}_Y) = \ker(t) \otimes_A^h Y$ is not generally valid if t is not a quotient map. For example, let B be a C^* -algebra containing A as a subalgebra, and let $t : A \rightarrow A$ be given by $a \mapsto a_0 a$, where a_0 is an element of A which is not a zero-divisor in A , but which is a zero-divisor in B . Then the kernel of t is trivial, whereas the kernel of $t \otimes \text{id}_B : A \otimes_A^h B \rightarrow A \otimes_A^h B$ is not trivial. Examples of this kind clearly do not arise when $A = \mathbb{C}$, and indeed Blecher and Smith have shown that in this case one does have $\ker(t \otimes \text{id}_Y) = \ker(t) \otimes^h Y$ for every completely bounded map t : see [BS92, Remark, p.137] and [ASS93, Theorem 2.4]. We

thank David Blecher for helpful discussions of this and other aspects of the exactness of \otimes^h .

2.4. Weak expectations. As a final terminological preliminary, we recall (e.g., from [BO08, Section 3.6]; cf. [Lan95]) that a *weak expectation* for an inclusion of C^* -algebras $A \hookrightarrow D$ is a contractive, completely positive map $\iota : D \rightarrow A^{\vee\vee}$ (where \vee indicates the dual Banach space) extending the canonical embedding $A \hookrightarrow A^{\vee\vee}$. Let us remark that such an ι is automatically a completely contractive A -bimodule map (see [BO08, Section 1.5]). A C^* -algebra A has Lance's *weak expectation property* if every inclusion $A \hookrightarrow D$ admits a weak expectation. Every nuclear C^* -algebra has the weak expectation property, as do many non-nuclear algebras. It is an open problem—equivalent, as shown by Kirchberg [Kir93], to Connes's embedding problem—to determine whether the C^* -algebras of nonabelian free groups have the weak expectation property. Note that non-self-adjoint operator algebras never have (the appropriate non-self-adjoint analogue of) this property: see [BLM04, Theorem 7.1.7].

3. THE COALGEBRA OF AN ADJOINT PAIR OF BIMODULES

This section introduces the basic operator-algebra structures—operator coalgebras and their comodules—that appear in our descent theorems for Hilbert C^* -modules and operator modules. While we are mostly interested in coalgebras associated to C^* -correspondences, interesting examples do arise in other contexts (see, e.g., Section 6.4), and in order to cover such examples we work throughout Sections 3 and 4 in the more general setting described below.

3.1. Adjoint pairs of operator bimodules.

Definition 3.1. Let A and B be approximately unital operator algebras, and let ${}_A L_B$ and ${}_B R_A$ be a pair of nondegenerate operator bimodules. We shall call (L, R) an *adjoint pair of bimodules* if we are given completely contractive bimodule maps

$$\eta : A \rightarrow L \otimes_B^h R \quad \text{and} \quad \varepsilon : R \otimes_A^h L \rightarrow B$$

(referred to, respectively, as the *unit* and *counit*) for which the diagrams (3.2)

$$\begin{array}{ccc} R \otimes_A^h A & \xrightarrow{\text{id}_R \otimes \eta} & R \otimes_A^h L \otimes_B^h R & & A \otimes_A^h L & \xrightarrow{\eta \otimes \text{id}_L} & L \otimes_B^h R \otimes_A^h L \\ \cong \downarrow & & \downarrow \varepsilon \otimes \text{id}_R & & \cong \downarrow & & \downarrow \text{id}_L \otimes \varepsilon \\ R & \xleftarrow{\cong} & B \otimes_B^h R & & L & \xleftarrow{\cong} & L \otimes_B^h B \end{array}$$

commute. In this situation we will use the notation $K := L \otimes_A^h R$ and $C := R \otimes_A^h L$.

To explain the choice of terminology, let us consider for each $X \in \text{OM}(A)$ the map $\eta_X : X \rightarrow X \otimes_A^{\text{h}} K$ defined as the composition

$$\eta_X : X \xrightarrow{\cong} X \otimes_A^{\text{h}} A \xrightarrow{\text{id}_X \otimes \eta} X \otimes_A^{\text{h}} K,$$

and for each $Y \in \text{OM}(B)$ the map $\varepsilon_Y : Y \otimes_B^{\text{h}} C \rightarrow Y$ defined as the composition

$$\varepsilon_Y : Y \otimes_B^{\text{h}} C \xrightarrow{\text{id}_Y \otimes \varepsilon} Y \otimes_B^{\text{h}} B \xrightarrow{\cong} Y.$$

The maps η_X and ε_Y are natural in X and Y (respectively), meaning that for all morphisms $t \in \text{CB}_A(X, X')$ and $s \in \text{CB}_B(Y, Y')$ the diagrams

$$(3.3) \quad \begin{array}{ccc} X & \xrightarrow{t} & X' \\ \eta_X \downarrow & & \downarrow \eta_{X'} \\ X \otimes_A^{\text{h}} K & \xrightarrow{t \otimes \text{id}_K} & X' \otimes_A^{\text{h}} K \end{array} \quad \begin{array}{ccc} Y & \xrightarrow{s} & Y' \\ \varepsilon_Y \uparrow & & \uparrow \varepsilon_{Y'} \\ Y \otimes_B^{\text{h}} C & \xrightarrow{s \otimes \text{id}_C} & Y' \otimes_B^{\text{h}} C \end{array}$$

commute. These natural maps are the unit and counit of a completely isometric adjunction between the Haagerup tensor product functors

$$\mathcal{L} : \text{OM}(A) \xrightarrow{X \mapsto X \otimes_A^{\text{h}} L} \text{OM}(B) \quad \text{and} \quad \mathcal{R} : \text{OM}(B) \xrightarrow{Y \mapsto Y \otimes_B^{\text{h}} R} \text{OM}(A).$$

That is to say, the map

$$\text{CB}_B(X \otimes_A^{\text{h}} L, Y) \xrightarrow{t \mapsto (t \otimes \text{id}_R) \circ \eta_X} \text{CB}_A(X, Y \otimes_B^{\text{h}} R)$$

is a completely isometric natural isomorphism, whose inverse is given by $s \mapsto \varepsilon_Y \circ (s \otimes \text{id}_L)$. See [ML98, Chapter IV] for the general notion of adjoint functors.

Example 3.4 (C^* -correspondences). Let A and B be C^* -algebras, and let ${}_A F_B$ be a C^* -correspondence from A to B : that is, a right Hilbert C^* -module over B equipped with a $*$ -homomorphism from A into the C^* -algebra of adjointable operators on F . (See [Lan95] or [BLM04, Chapter 8] for the relevant background on Hilbert C^* -modules.) Equip F with its canonical operator space structure (cf. [BLM04, 8.2]). Blecher has shown that the tensor product functor $\otimes_A^{\text{h}} F : \text{OM}(A) \rightarrow \text{OM}(B)$ coincides, on the subcategory of Hilbert C^* -modules, with the functor \mathcal{F}_2 considered in the introduction; see [BLM04, Theorem 8.2.11].

Suppose now that the image of A under the action homomorphism lies in the ideal $\text{K}_B(F)$ of compact operators on F . Let F^* denote the operator B - A bimodule adjoint to F . Another theorem of Blecher [BLM04, Corollary 8.2.15] identifies $\text{K}_B(F)$ with the Haagerup tensor product $K = F \otimes_B^{\text{h}} F^*$, and in [CH16] we observed that the unit

$$\eta : A \xrightarrow{\text{action}} \text{K}_B(F) \cong F \otimes_B^{\text{h}} F^*$$

and the counit

$$\varepsilon : F^* \otimes_B^{\text{h}} F \xrightarrow{x^* \otimes y \mapsto \langle x|y \rangle} B$$

make (F, F^*) into an adjoint pair of operator bimodules. See [CH16] and [Cri18] (where the opposite convention regarding left versus right modules is used) for more on this example. See also [BKM18] for an analogue of C^* -correspondences, and ‘compact’ operators thereon, for non-selfadjoint operator $*$ -algebras.

Example 3.5 (Subalgebras). Let B be an operator algebra and let $A \subseteq B$ be a norm-closed subalgebra, which is nondegenerate in the sense that $AB = B = BA$. Then the unit

$$\eta : A \xrightarrow{\text{inclusion}} B \cong B \otimes_B^h B$$

and the counit

$$\varepsilon : B \otimes_A^h B \xrightarrow{\text{multiplication}} B$$

make $({}_A B_B, B B_A)$ into an adjoint pair of operator bimodules. The corresponding functors are the restriction functor $\text{OM}(B) \rightarrow \text{OM}(A)$, and its left-adjoint induction functor $\text{OM}(A) \xrightarrow{X \mapsto X \otimes_A^h B} \text{OM}(B)$. Note that if A and B are C^* -algebras then this example is an instance of the previous one: B is a Hilbert C^* -module over itself, with inner product $\langle b_1 | b_2 \rangle = b_1^* b_2$, and we have $K_B(B) = B$ (acting by left multiplication). Note further that the operator B -bimodule $C = B \otimes_A^h B$ can in this case be identified with the closed linear span of the products $b_1 * b_2$ in the amalgamated free product C^* -algebra $B *_A B$; see [CES87, Theorem 3.1], [Pis96, Lemma 1.14], [Oza04, p.515].

Remark 3.6. We insist upon η and ε being completely contractive, rather than merely completely bounded, in order to ensure that we eventually obtain completely isometric equivalences of OS_1 -categories (cf. Theorem 2.3). As is often the case in operator algebra theory (see [BLM04, Chapter 5], for example), much of what we do below admits a parallel ‘completely bounded’ version. We shall not say any more about that here, except to point out an example: the unit of the adjunction in [CH16, Theorem 4.15] is not completely contractive. (The unit can be made completely contractive by a simple rescaling, but then the counit would cease to be completely contractive.)

3.2. Operator coalgebras and comodules. We temporarily suspend the standing notation from the previous section to make a general definition.

Definition 3.7. Let B be an approximately unital operator algebra. An *operator B -coalgebra* is a nondegenerate operator B -bimodule C equipped with completely contractive B -bimodule maps

$$\delta : C \rightarrow C \otimes_B^h C \quad \text{and} \quad \varepsilon : C \rightarrow B$$

making the diagrams

$$\begin{array}{ccc}
C & \xrightarrow{\delta} & C \otimes_B^h C \\
\delta \downarrow & & \downarrow \delta \otimes \text{id}_C \\
C \otimes_B^h C & \xrightarrow{\text{id}_C \otimes \delta} & C \otimes_B^h C \otimes_B^h C
\end{array}
\qquad
\begin{array}{ccccc}
C \otimes_B^h C & \xleftarrow{\delta} & C & \xrightarrow{\delta} & C \otimes_B^h C \\
\text{id}_C \otimes \varepsilon \downarrow & & \downarrow \text{id}_C & & \downarrow \varepsilon \otimes \text{id}_C \\
C \otimes_B^h B & \xrightarrow{\cong} & C & \xleftarrow{\cong} & B \otimes_B^h C
\end{array}$$

commute.

A *right operator C -comodule* is a pair (Z, δ_Z) consisting of a nondegenerate right operator B -module Z and a completely contractive B -module map $\delta_Z : Z \rightarrow Z \otimes_B^h C$ making the diagrams

(3.8)

$$\begin{array}{ccc}
Z & \xrightarrow{\delta_Z} & Z \otimes_B^h C \\
\delta_Z \downarrow & & \downarrow \text{id}_Z \otimes \delta \\
Z \otimes_B^h C & \xrightarrow{\delta_Z \otimes \text{id}_C} & Z \otimes_B^h C \otimes_B^h C
\end{array}
\qquad
\text{and}
\qquad
\begin{array}{ccc}
Z & \xrightarrow{\delta_Z} & Z \otimes_B^h C \\
\text{id}_Z \downarrow & \swarrow \varepsilon_Z & \\
Z & &
\end{array}$$

commute. The space of morphisms $\text{CB}_C(Z, W)$ between operator C -comodules (Z, δ_Z) and (W, δ_W) is defined to be the set of maps $t \in \text{CB}_B(Z, W)$ for which the diagram

(3.9)

$$\begin{array}{ccc}
Z & \xrightarrow{\delta_Z} & Z \otimes_B^h C \\
t \downarrow & & \downarrow t \otimes \text{id}_C \\
W & \xrightarrow{\delta_W} & W \otimes_B^h C
\end{array}$$

commutes. Let $\text{OC}(C)$ denote the category of right operator C -comodules, with morphisms $\text{CB}_C(Z, W)$ as above. We denote by $\text{OC}^{\text{cb}}(C)$ the category whose objects are pairs (Z, δ_Z) , where Z is an operator B -module, and $\delta_Z : Z \rightarrow Z \otimes_B^h C$ is a completely *bounded* B -module map making the diagrams (3.8) commute. The morphisms in $\text{OC}^{\text{cb}}(C)$ are the completely bounded B -module maps making the diagram (3.9) commute.

Remarks 3.10. (a) The morphism spaces $\text{CB}_C(Z, W)$ in $\text{OC}(C)$ and $\text{OC}^{\text{cb}}(C)$ are closed subspaces of the operator spaces $\text{CB}_B(Z, W)$; hence $\text{OC}(C)$ and $\text{OC}^{\text{cb}}(C)$ are an OS_1 -categories.

- (b) The coproduct δ for an operator coalgebra is a complete isometry: indeed, δ is a complete contraction, and is split by the complete contraction $\text{id}_C \otimes \varepsilon$. Similarly, the coaction δ_Z for an object in $\text{OC}(C)$ (respectively, in $\text{OC}^{\text{cb}}(C)$) is a complete isometry (respectively, a complete embedding).
- (c) The notions of left comodules and bi-comodules, and of morphisms between them, are defined via the obvious modifications of the above definition. Following our convention for modules, ‘comodule’ means ‘right comodule’ in the absence of further specification.

- (d) The category $\text{OC}^{\text{cb}}(C)$ is the Eilenberg-Moore category for the comonad $\mathcal{C} := \otimes_B^{\text{h}} C$ on $\text{OM}(B)$. (See [ML98, Chapter VI] for the terminology.) Since the functor \mathcal{C} and the structure maps δ and ε are completely contractive, \mathcal{C} restricts to a comonad \mathcal{C}_1 on $\text{OM}(B)_1$, whose Eilenberg-Moore category is $\text{OC}(C)_1$. The category $\text{OC}(C)$ thus sits between the Eilenberg-Moore categories for \mathcal{C}_1 and \mathcal{C} : it has the same objects as the former, and the same morphisms as the latter.

3.3. The coalgebra of an adjoint pair. We now return to the setting of Section 3.1: A and B are approximately unital operator algebras, and $({}_A L_B, {}_B R_A)$ is an adjoint pair of operator bimodules with unit $\eta : A \rightarrow K := L \otimes_B^{\text{h}} R$ and counit $\varepsilon : C := R \otimes_A^{\text{h}} L \rightarrow B$.

Definition 3.11. On the operator B -bimodule $C = R \otimes_A^{\text{h}} L$ we consider the coproduct $\delta : C \rightarrow C \otimes_B^{\text{h}} C$ defined by

$$\delta := \eta_R \otimes \text{id}_L : R \otimes_A^{\text{h}} L \rightarrow R \otimes_A^{\text{h}} L \otimes_B^{\text{h}} R \otimes_A^{\text{h}} L.$$

The associativity of the Haagerup tensor product ensures that δ is coassociative, while the commutativity of the diagrams (3.2) ensures that

$$(\varepsilon \otimes \text{id}_C) \circ \delta = (\text{id}_C \otimes \varepsilon) \circ \delta = \text{id}_C.$$

Thus the counit ε and the coproduct δ furnish C with the structure of an operator B -coalgebra.

Examples 3.12. (a) The map $\delta_L : L \rightarrow L \otimes_A^{\text{h}} C$ defined as the composition

$$L \xrightarrow{\cong} A \otimes_A^{\text{h}} L \xrightarrow{\eta \otimes \text{id}_L} K \otimes_A^{\text{h}} L = L \otimes_B^{\text{h}} C$$

furnishes L with the structure of a right operator C -comodule. Similarly the map $\delta_R := \eta_R$ makes R a left operator C -comodule.

(b) If X is any operator A -module, then setting

$$\delta_{X \otimes_A^{\text{h}} L} := \eta_X \otimes \text{id}_L = \text{id}_X \otimes \delta_L$$

yields an operator C -comodule $(X \otimes_A^{\text{h}} L, \delta_{X \otimes_A^{\text{h}} L})$.

Definition 3.13. We consider the *comparison functor* $\mathcal{L}_C : \text{OM}(A) \rightarrow \text{OC}(C)$ defined on objects by $X \mapsto (X \otimes_A^{\text{h}} L, \delta_{X \otimes_A^{\text{h}} L})$ (see Example 3.12), and on morphisms by $t \mapsto t \otimes \text{id}_L$.

The comparison functor \mathcal{L}_C is given on morphisms by a Haagerup tensor product, so it is an OS_1 -functor. Note that there is a forgetful functor $\text{OC}(C) \rightarrow \text{OM}(B)$, $(Z, \delta_Z) \mapsto Z$, making the diagram

$$(3.14) \quad \begin{array}{ccc} & & \text{OC}(C) \\ & \nearrow \mathcal{L}_C & \downarrow \text{forget} \\ \text{OM}(A) & \xrightarrow{\mathcal{L}} & \text{OM}(B) \end{array}$$

commute.

4. DESCENT OF OPERATOR MODULES

The notation in this section is the same as in the previous one: A and B are approximately unital operator algebras, and $({}_A L_B, {}_B R_A)$ is an adjoint pair of operator bimodules with unit $\eta : A \rightarrow K := L \otimes_B^h R$ and counit $\varepsilon : C := R \otimes_A^h L \rightarrow B$.

Theorem 4.1. *Suppose that A is a C^* -algebra, and that there is a completely contractive A -bimodule map $\iota : K = L \otimes_B^h R \rightarrow A^{\vee\vee}$ such that $\iota \circ \eta : A \rightarrow A^{\vee\vee}$ is the canonical embedding. Then the comparison functor $\mathcal{L}_C : \text{OM}(A) \rightarrow \text{OC}(C)$ is a completely isometric equivalence.*

- Remarks 4.2.** (a) A more explicit version of the theorem, including an identification of an inverse $\text{OC}(C) \rightarrow \text{OM}(A)$, is stated below as Proposition 4.7.
- (b) If A has the weak expectation property (e.g., if A is nuclear) then the hypothesis involving ι can be replaced by the simpler hypothesis that L is faithful as a left A -module: see Lemma 4.5
- (c) In view of (3.14), Theorem 4.1 gives the following characterisation of the image of \mathcal{L} : an operator B -module Y is completely isometrically isomorphic to one of the form $X \otimes_A^h L$ if and only if Y admits a C -comodule structure; and a B -linear map $t \in \text{CB}_B(Y_1, Y_2)$ between two such modules is of the form $s \otimes \text{id}_L$ if and only if t is a map of C -comodules.
- (d) The significance of \mathcal{L}_C being a *completely isometric* equivalence is that, in view of Theorem 2.3, the OS_1 -category $\text{OC}(C)$ becomes a complete Morita invariant of the C^* -algebra A .

We shall prove Theorem 4.1 as a consequence of a theorem of Beck (cf. [ML98, Section VI.7], [BW05, Theorem 3.3.14], or [Bor94, Theorem 4.4.4]). In the following sequence of lemmas we verify that the hypotheses of Beck's theorem are satisfied.

Lemma 4.3. *The operator space K is completely isometrically isomorphic to an approximately unital operator algebra in such a way that $\eta : A \rightarrow K$ is an algebra homomorphism.*

Proof. The completely contractive map $\mu : K \times K \rightarrow K$ defined by

$$(l_1 \otimes r_1, l_2 \otimes r_2) \mapsto l_1 \varepsilon(r_1 \otimes l_2) \otimes r_2$$

gives K the structure of an associative algebra, and the commutativity of (3.2) ensures that the map $\eta : A \rightarrow K$ is an algebra homomorphism. The quotient map $K \otimes^h K \rightarrow K \otimes_A^h K$ and the counit map $R \otimes_A^h L \rightarrow B$ are both completely contractive, and so μ extends to a completely contractive map $K \otimes^h K \rightarrow K$. Since K is nondegenerate as an A -bimodule, the image under the contraction η of a contractive approximate unit for A will be a contractive approximate unit for K . A theorem of Blecher, Ruan and Sinclair [BLM04, Theorem 2.3.2] then implies that K is completely isometrically isomorphic to an operator algebra. \square

Lemma 4.4. *Suppose that A is a C^* -algebra and that $\eta : A \rightarrow K$ is injective. Then for each $X \in \text{OM}(A)$ the natural transformation $\eta_X : X \rightarrow X \otimes_A^h K$ is a complete isometry.*

Proof. Lemma 4.3 implies that there exists a completely isometric algebra embedding of K into a C^* -algebra D ; let us fix such an embedding. The map $\eta : A \rightarrow K \hookrightarrow D$ is a contractive, injective algebra homomorphism of C^* -algebras, and hence it is a complete isometry (see e.g. [BLM04, A.5.8]). Now η_X is the composition of the completely isometric isomorphism $X \cong X \otimes_A^h A$ with the map $\text{id}_X \otimes \eta$, which is a complete isometry by Theorem 2.5. \square

The following Lemma shows that the hypotheses of Theorem 4.1 can be simplified if A has the weak expectation property:

Lemma 4.5. *Let $({}_A L_B, {}_B R_A)$ be an adjoint pair of operator bimodules, where A is a C^* -algebra with the weak expectation property. The following are equivalent:*

- (a) L is faithful as a left A -module.
- (b) R is faithful as a right A -module.
- (c) The unit map $\eta : A \rightarrow K$ is injective.
- (d) There is a completely contractive A -bimodule map $\iota : K \rightarrow A^{\vee\vee}$ such that $\iota \circ \eta : A \rightarrow A^{\vee\vee}$ is the canonical embedding.

Proof. The commutative diagrams (3.2) show that the kernel of η annihilates the modules L and R , and so each of (a) and (b) implies (c). On the other hand, if $a \in A$ annihilates the left A -module L , then a also annihilates $K = L \otimes_B^h R$ as a left A -module, and in particular we have $a\eta(a') = \eta(a)a' = 0$ for all $a' \in A$. Since K is nondegenerate as a right A -module, we conclude from this last equality that $\eta(a) = 0$; thus (c) implies (a). Switching left and right in the preceding argument shows that (c) implies (b) too. Clearly (d) implies (c). To see that (c) implies (d), choose a completely isometric embedding of K into a C^* -algebra D (this is possible by Lemma 4.3). Then the composition $A \xrightarrow{\eta} K \rightarrow D$ is a completely isometric algebra homomorphism (Lemmas 4.3 and 4.4), and is therefore an injective $*$ -homomorphism ([BLM04, A.5.8] again). Now, since A has the weak expectation property, there is a completely contractive A -bimodule map $\iota : D \rightarrow A^{\vee\vee}$ whose composition with $\eta : A \rightarrow K$ is the canonical embedding. \square

To prove Theorem 4.1 we shall exhibit an explicit inverse to \mathcal{L}_C . The construction is dictated by abstract categorical considerations; see e.g. [BW05, 3.2.4] for the general construction.

Definition/Lemma 4.6. *There is a completely contractive functor*

$$\mathcal{R}_C : \text{OC}^{\text{cb}}(C) \rightarrow \text{OM}(A)$$

given on objects by

$$\mathcal{R}_C(Z, \delta_Z) = Z \boxtimes_C R := \left\{ \xi \in Z \otimes_B^h R \mid (\text{id}_Z \otimes \eta_R)(\xi) = (\delta_Z \otimes \text{id}_R)(\xi) \right\}$$

and on morphisms by $\mathcal{R}_C(t) := (t \otimes \text{id}_R)|_{Z \boxtimes_C R}$.

Proof. Note firstly that for each $Z \in \text{OC}(C)$ the space $Z \boxtimes_C R$ is a closed A -submodule of $Z \otimes_B^h R$, and is therefore an operator A -module. The fact that \mathcal{R}_C is a functor—that is, that $t \otimes \text{id}_R$ maps $Z \boxtimes_C R$ into $W \boxtimes_C R$ for each morphism $t \in \text{CB}_C(Z, W)$ —is easily checked as in the algebraic case (see e.g. [BW03, 1.10]). The functor \mathcal{R}_C is completely contractive because \mathcal{R} is completely contractive, and because restriction of completely bounded maps to a closed subspace is likewise completely contractive. \square

We shall prove the following more precise version of Theorem 4.1. To understand the statement, note that for each $X \in \text{OM}(A)$ the A -module $\mathcal{R}_C \mathcal{L}_C(X) = (X \otimes_A^h L) \boxtimes_C R$ is a closed submodule of $X \otimes_A^h K = X \otimes_A^h L \otimes_A^h R$; while for each $(Z, \delta_Z) \in \text{OC}^{\text{cb}}(C)$ the C -comodule $\mathcal{L}_C \mathcal{R}_C(Z) = (Z \boxtimes_C R) \otimes_A^h L$ can be identified with a closed B -submodule of $Z \otimes_B^h C = Z \otimes_B^h R \otimes_A^h L$, via the completely isometric embedding

$$(Z \boxtimes_C R) \otimes_A^h L \xrightarrow{\text{inclusion} \otimes \text{id}_L} Z \otimes_B^h R \otimes_A^h L.$$

- Proposition 4.7.** (a) *Suppose that A is a C^* -algebra, and that $\eta : A \rightarrow K$ is injective. For each $X \in \text{OM}(A)$ the map $\eta_X : X \rightarrow X \otimes_A^h K$ restricts to a completely isometric isomorphism of operator A -modules $\eta : X \rightarrow \mathcal{R}_C \mathcal{L}_C(X)$.*
- (b) *Suppose that A is a C^* -algebra and that there exists a completely contractive A -bimodule map $\iota : K \rightarrow A^{\vee\vee}$ such that $\iota \circ \eta : A \rightarrow A^{\vee\vee}$ is the canonical embedding. Then for each $(Z, \delta_Z) \in \text{OC}^{\text{cb}}(C)$ the map $\delta_Z : Z \rightarrow Z \otimes_B^h C$ restricts to a completely bounded natural isomorphism of operator C -comodules $\delta_Z : Z \rightarrow \mathcal{L}_C \mathcal{R}_C(Z)$, which is completely isometric if $(Z, \delta_Z) \in \text{OC}(C)$. In particular, $\mathcal{L}_C : \text{OM}(A) \rightarrow \text{OC}(C)$ and $\mathcal{R}_C : \text{OC}(C) \rightarrow \text{OM}(A)$ are mutually inverse completely isometric equivalences.*

The technical heart of the proof of Proposition 4.7 is contained in the following two lemmas.

Lemma 4.8. *Assume the hypotheses of Theorem 4.1. For each $X \in \text{OM}(A)$ there is a completely bounded map $\sigma_X : X^\vee \rightarrow (X \otimes_A^h K)^\vee$ which satisfies $\eta_X^\vee \circ \sigma_X = \text{id}_{X^\vee}$, and which is natural in the sense that for every morphism $t \in \text{CB}_A(X, Y)$ in $\text{OM}(A)$ one has $\sigma_X \circ t^\vee = (t \otimes \text{id}_K)^\vee \circ \sigma_Y$.*

Proof. As explained in [BLM04, 3.8.9], the operator A -module structure on X extends to a compatible operator $A^{\vee\vee}$ -module structure on the bidual operator space $X^{\vee\vee}$. Consider the completely contractive, right- A -linear map $\tau_X : X \otimes_A^h K \rightarrow X^{\vee\vee}$ defined as the composition

$$X \otimes_A^h K \xrightarrow{[X \hookrightarrow X^{\vee\vee}] \otimes \iota} X^{\vee\vee} \otimes_A^h A^{\vee\vee} \xrightarrow{\text{act}} X^{\vee\vee}.$$

Note that for $x \in X$ and $a \in A$ we have

$$\tau_X \circ \eta_X(xa) = \tau_X(x \otimes \eta(a)) = x \cdot \iota \circ \eta(a) = xa,$$

showing that $\tau_X \circ \eta_X$ is the canonical embedding $X \hookrightarrow X^{\vee\vee}$. Dualising τ_X , and restricting to the image of the natural embedding $X^\vee \hookrightarrow X^{\vee\vee\vee}$, we obtain a completely contractive map

$$\sigma_X := \tau_X^\vee|_{X^\vee} : X^\vee \rightarrow (X \otimes_A^h K)^\vee.$$

Since $\tau_X \circ \eta_X$ is the embedding $X \hookrightarrow X^{\vee\vee}$, it follows by duality that $\eta_X^\vee \circ \sigma_X = \text{id}_{X^\vee}$. Since σ_X and η_X^\vee are both completely contractive, this last equality implies that σ_X is a complete isometry.

To verify the asserted naturality, note that if $t \in \text{CB}_A(X, Y)$ then the bidual map $t^{\vee\vee} : X^{\vee\vee} \rightarrow Y^{\vee\vee}$ is $A^{\vee\vee}$ -linear: this follows from the separate weak- $*$ continuity of the bidual module action (see [BLM04, 3.8.9]). We thus have for every $k \in K$ and $x \in X$ that

$$t^{\vee\vee} \circ \tau_X(x \otimes k) = t^{\vee\vee}(x\iota(k)) = t(x)\iota(k) = \tau_Y \circ (t \otimes \text{id}_K)(x \otimes k),$$

from which it follows by duality that $\sigma_X \circ t^\vee = (t \otimes \text{id}_K)^\vee \circ \sigma_Y$ as required. \square

Lemma 4.9. *Assume the hypotheses of Theorem 4.1, and suppose that $f, g : X \rightarrow Y$ is a pair of morphisms in $\text{OM}(A)$ such that there exists a morphism $\lambda : Y \otimes_A^h L \rightarrow X \otimes_A^h L$ in $\text{OM}(B)$ satisfying $\lambda \circ (f \otimes \text{id}_L) = \text{id}_{X \otimes_A^h L}$ and $(h \otimes \text{id}_L) \circ \lambda \circ (g \otimes \text{id}_L) = 0$, where $h := f - g$. Then the map $h : X \rightarrow Y$ has closed range, and induces a completely bounded isomorphism $X/\ker h \rightarrow \text{image}(h)$.*

Proof. Consider the map $s : X^\vee \rightarrow Y^\vee$ defined as the composition $s := \eta_Y^\vee \circ (\lambda \otimes \text{id}_R)^\vee \circ \sigma_X$. Noting that the given relations between f, g, h , and λ immediately give

$$(h \otimes \text{id}_K) \circ (\lambda \otimes \text{id}_R) \circ (h \otimes \text{id}_K) = h \otimes \text{id}_K,$$

we compute using the naturality properties of η and σ and find that

$$\begin{aligned} h^\vee \circ s \circ h^\vee &= h^\vee \circ \eta_Y^\vee \circ (\lambda \otimes \text{id}_R)^\vee \circ \sigma_X \circ h^\vee \\ &= \eta_X^\vee \circ (h \otimes \text{id}_K)^\vee \circ (\lambda \otimes \text{id}_R)^\vee \circ (h \otimes \text{id}_K)^\vee \otimes \sigma_Y \\ &= \eta_X^\vee \circ \sigma_X \circ h^\vee = h^\vee. \end{aligned}$$

The rest of the proof is an application of basic duality theory. The claim is that if X and Y are operator spaces, and if $h : X \rightarrow Y$ and $s : X^\vee \rightarrow Y^\vee$ are completely bounded maps satisfying $h^\vee s h^\vee = h^\vee$, then h has closed range and the induced map

$$\tilde{h} : X/\ker(h) \rightarrow \text{image}(h), \quad x + \ker(h) \mapsto h(x)$$

is an isomorphism. The equality $h^\vee s h^\vee = h^\vee$ implies that $h^\vee s$ is an idempotent on X^\vee with the same image as h^\vee , and so h^\vee has closed range. It now follows from standard Banach space theory that h also has closed range. Moreover, the map

$$\widetilde{h}^\vee : Y^\vee/\ker(h^\vee) \rightarrow \text{image}(h^\vee), \quad y^\vee + \ker(h^\vee) \mapsto h^\vee(y^\vee)$$

is a completely bounded isomorphism, with inverse $x^\vee \mapsto \sigma(x^\vee) + \ker(h^\vee)$.

The standard duality relations between subspaces, quotients, images and kernels (cf. [BLM04, 1.4.4]) give a commuting diagram

$$\begin{array}{ccc} \text{image}(h)^\vee & \xrightarrow{(\tilde{h})^\vee} & (X/\ker(h))^\vee \\ \cong \downarrow & & \downarrow \cong \\ Y^\vee/\ker(h^\vee) & \xrightarrow{\tilde{h}^\vee} & \text{image}(h^\vee) \end{array}$$

from which we conclude that $(\tilde{h})^\vee$ is, like \tilde{h}^\vee , an isomorphism. Thus \tilde{h} is an isomorphism as well (cf. [BLM04, 1.4.3]). \square

Corollary 4.10. *Assume the hypotheses of Theorem 4.1. If f , g and $h = f - g$ are as in Lemma 4.9, then the map*

$$\ker(h) \otimes_A^h L \xrightarrow{\text{inclusion} \otimes \text{id}_L} X \otimes_A^h L$$

is a complete isometry onto $\ker(h \otimes \text{id}_L)$.

Proof. Lemma 4.9 implies that the restricted map $h : X \rightarrow \text{image}(h)$ is conjugate, by a completely bounded isomorphism, to the quotient map $X \rightarrow X/\ker h$, and so the corollary follows from Theorem 2.5. \square

The proof of Proposition 4.7 follows from Lemma 4.4 and Corollary 4.10 by an argument due to Beck (cf. [ML98, Section VI.7], [BW05, Theorem 3.3.14], or [Bor94, Theorem 4.4.4]), which applies in a very general context and whose statement and proof are typically couched in correspondingly general categorical language. For the benefit of the reader having only a slight acquaintance with (or tolerance for) category theory, let us give a self-contained presentation of Beck's argument in the present setting.

Proof of Proposition 4.7(a). The map $\eta_X : X \rightarrow X \otimes_A^h K$ is a complete isometry by Lemma 4.4, and its image is contained in $\mathcal{R}_C \mathcal{L}_C(X)$ by the naturality property (3.3) of η :

$$\begin{aligned} (\delta_{\mathcal{L}_C(X)} \otimes \text{id}_R) \circ \eta_X &= (\eta_X \otimes \text{id}_K) \circ \eta_X = \eta_{X \otimes_A^h K} \circ \eta_X \\ &= (\text{id}_X \otimes \eta_K) \circ \eta_X = (\text{id}_{\mathcal{L}_C(X)} \otimes \eta_R) \circ \eta_X. \end{aligned}$$

To prove the reverse inclusion $\mathcal{R}_C \mathcal{L}_C(X) \subseteq \text{image}(\eta_X)$, we consider the quotient map $q : K \rightarrow K/A$; this is a map of operator A -bimodules. Theorem 2.5, modified in the obvious way so that the roles of left and right modules are reversed, implies that the image of $\eta_X = \text{id}_X \otimes \eta$ is equal to the kernel of $\text{id}_X \otimes q$. We have

$$\begin{aligned} (\text{id}_X \otimes \eta_{K/A}) \circ (\text{id}_X \otimes q) \Big|_{\mathcal{R}_C \mathcal{L}_C(X)} &= (\text{id}_X \otimes q \otimes \text{id}_K) \circ (\text{id}_X \otimes \eta_K) \Big|_{\mathcal{R}_C \mathcal{L}_C(X)} \\ &= (\text{id}_X \otimes q \otimes \text{id}_K) \circ (\eta_X \otimes \text{id}_K) \Big|_{\mathcal{R}_C \mathcal{L}_C(X)} \\ &= 0, \end{aligned}$$

where the first equality holds by the naturality of η , the second holds by the definition of $\mathcal{R}_C \mathcal{L}_C(X)$, and the third holds because $q \circ \eta = 0$. Since

$\text{id}_X \otimes \eta_{K/A}$ is a complete isometry, by Lemma 4.4, we conclude from the above equality that $\mathcal{R}_C \mathcal{L}_C(X) \subseteq \ker(\text{id}_X \otimes q) = \text{image}(\eta_X)$ as required. \square

Proof of Proposition 4.7(b). Consider a comodule $(Z, \delta_Z) \in \text{OC}^{\text{cb}}(C)$. The map δ_Z is natural in Z , by virtue of the defining property (3.9) of the morphisms in $\text{OC}(C)$. We observed in Remarks 3.10 that δ_Z is a complete embedding, and a complete isometry if $(C, \delta_C) \in \text{OC}(C)$.

We are going to apply Lemma 4.9 to the A -modules $X = Z \otimes_B^h R$ and $Y = Z \otimes_B^h C \otimes_B^h R$, and the maps

$$Z \otimes_B^h R \xrightleftharpoons[g = \delta_Z \otimes \text{id}_R]{f = \text{id}_Z \otimes \eta_R} Z \otimes_B^h C \otimes_B^h R, \quad Z \otimes_B^h C \otimes_B^h C \xrightarrow{\lambda = \text{id}_Z \otimes \varepsilon_C} Z \otimes_B^h C$$

and $h = f - g$. We have

$$\lambda \circ (f \otimes \text{id}_L) = (\text{id}_Z \otimes \varepsilon_C) \circ (\text{id}_Z \otimes \delta) = \text{id}_{Z \otimes_B^h C}$$

by the coalgebra axioms (Definition 3.7), while

$$\begin{aligned} (h \otimes \text{id}_L) \circ \lambda \circ (g \otimes \text{id}_L) &= (\text{id}_Z \otimes \delta - \delta_Z \otimes \text{id}_C) \circ (\text{id}_Z \otimes \varepsilon_C) \circ (\delta_Z \otimes \text{id}_C) \\ &= (\text{id}_Z \otimes \delta - \delta_Z \otimes \text{id}_C) \circ \delta_Z \circ \varepsilon_Z = 0, \end{aligned}$$

where the second equality holds by virtue of the naturality (3.3) of ε , and the third holds by (3.8). Thus Lemma 4.9 and Corollary 4.10 apply, giving an equality of B -modules

$$\mathcal{L}_C \mathcal{R}_C(Z) := \ker(h) \otimes_A^h L = \ker(h \otimes \text{id}_L).$$

Now the coassociativity property (3.8) shows as above that $\text{image}(\delta_Z) \subseteq \ker(h \otimes \text{id}_L) = \mathcal{L}_C \mathcal{R}_C(Z)$. The reverse inclusion $\ker(h \otimes \text{id}_L) \subseteq \text{image}(\delta_Z)$ follows from the counitality of δ and the naturality of ε , thus:

$$\begin{aligned} \text{id}_{Z \otimes_B^h C} - \delta_Z \circ \varepsilon_Z &= (\text{id}_Z \otimes \varepsilon_C) \circ (\text{id}_Z \otimes \delta) - (\text{id}_Z \otimes \varepsilon_C) \circ (\delta_Z \otimes \text{id}_C) \\ &= (\text{id}_Z \otimes \varepsilon_C) \circ (h \otimes \text{id}_L). \end{aligned}$$

We conclude that δ_Z is a completely bounded (or if $(Z, \delta_Z) \in \text{OC}(C)$, completely isometric) natural isomorphism of B -modules, from Z to $\mathcal{L}_C \mathcal{R}_C(Z)$.

It remains to show that δ_Z is an isomorphism of C -comodules, which is easy: (3.8) ensures that δ_Z is a map of comodules, and then the usual algebraic argument shows that the inverse δ_Z^{-1} of the B -module isomorphism δ_Z is a map of comodules too. \square

Remark 4.11. Fix a third operator algebra D , and consider the OS_1 -categories $\text{OM}(D, A)$ and $\text{OM}(D, B)$ of operator D - A bimodules and operator D - B bimodules, with morphisms the completely bounded bimodule maps. If $({}_A L_B, {}_B R_A)$ is an adjoint pair of operator bimodules, and if $C = R \otimes_A^h L$ is the associated coalgebra, then we consider the OS_1 -category $\text{OC}(D, C)$ whose objects are pairs (Z, δ_Z) consisting of an operator D - B bimodule Z and a completely contractive D - B -bimodule map $\delta_Z : Z \rightarrow Z \otimes_B^h C$ making the diagrams (3.8) commute; the morphisms in $\text{OC}(D, C)$ are the

completely bounded D - B -bimodule maps making (3.9) commute. Just as in the case $D = \mathbb{C}$ we have a comparison functor

$$(4.12) \quad \text{OM}(D, A) \rightarrow \text{OC}(D, C), \quad X \mapsto (X \otimes_A^{\text{h}} L, \eta_X \otimes \text{id}_L),$$

and the proof of Theorem 4.1 carries over verbatim to this setting: if A is a C^* -algebra and there is a map $\iota : K \rightarrow A^{\vee\vee}$ as in Theorem 4.1, then the functor (4.12) is a completely isometric equivalence.

5. DESCENT OF HILBERT C^* -MODULES

In this section we return to the setting of Example 3.4: we consider the adjoint pair $({}_A F_B, {}_B F_A^*)$ associated to a C^* -correspondence ${}_A F_B$ between C^* -algebras A and B , where A acts on F through a nondegenerate $*$ -homomorphism into the C^* -algebra $K_B(F) \cong K$ of B -compact operators on F . We are going to prove a C^* -module version of Theorem 4.1 for this adjoint pair.

We thus consider for each C^* -algebra A the category $\text{CM}(A)$ of right Hilbert C^* -modules over A , with adjointable A -module maps (notation: $\text{CB}_A^*(X, Y)$) as morphisms. We refer to [Lan95] or [BLM04, Chapter 8] for background on Hilbert C^* -modules; see in particular [BLM04, 8.2.1] for the canonical operator-module structure on a Hilbert C^* -module, and [BLM04, 8.2.11] for the completely isometric identification between the Haagerup tensor product and the Hilbert C^* -module tensor product.

The operator B -coalgebra $C = F^* \otimes_A^{\text{h}} F$ associated to the adjoint pair (F, F^*) carries a completely isometric, conjugate-linear involution

$$* : C \rightarrow C, \quad (f_1^* \otimes f_2)^* := f_2^* \otimes f_1.$$

This involution satisfies $(b_1 c b_2)^* = b_2^* c^* b_1^*$ for all $b_1, b_2 \in B$ and $c \in C$. The involution is also compatible with the coalgebra structure: indeed, we have a second completely isometric, conjugate-linear involution

$$* : C \otimes_B^{\text{h}} C \rightarrow C \otimes_B^{\text{h}} C, \quad (c_1 \otimes c_2)^* := c_2^* \otimes c_1^*,$$

and one has $\delta(c)^* = \delta(c^*)$ and $\varepsilon(c)^* = \varepsilon(c^*)$ for all $c \in C$.

Definition 5.1. A *Hilbert C^* -comodule* over C is a right Hilbert C^* -module Z over B , equipped with a completely contractive B -linear map $\delta_Z : Z \rightarrow Z \otimes_B^{\text{h}} C$ making (3.8) commute, such that

$$(5.2) \quad \langle z_1 | \delta_Z(z_2) \rangle_C = \langle z_2 | \delta_Z(z_1) \rangle_C^*$$

for all $z_1, z_2 \in Z$. Here we are applying the pairing

$$\langle \cdot | \cdot \rangle_C : Z \times (Z \otimes_B^{\text{h}} C) \rightarrow C, \quad \langle z_1 | z_2 \otimes c \rangle_C := \langle z_1 | z_2 \rangle c.$$

A morphism of Hilbert C^* -comodules $(Z, \delta_Z) \rightarrow (W, \delta_W)$ is an adjointable map of Hilbert C^* -modules $t : Z \rightarrow W$ making (3.9) commute. We denote by $\text{CC}(C)$ the category of right Hilbert C^* -comodules over C , and by $\text{CB}_C^*(Z, W)$ the space of morphisms in $\text{CC}(C)$.

We shall also consider the category $\text{CC}^{\text{cb}}(C)$ of *completely bounded* Hilbert C^* -comodules: that is, pairs (Z, δ_Z) which are as in Definition 5.1, except that the coaction δ_Z is allowed to be merely completely bounded, rather than completely contractive. We will later show that in the situations of interest to us, δ_Z is in fact automatically completely contractive: see Corollary 5.16.

Lemma 5.3. *If $t : (Z, \delta_Z) \rightarrow (W, \delta_W)$ is a morphism in $\text{CC}^{\text{cb}}(C)$, then the adjoint map $t^* : W \rightarrow Z$ is also a morphism in $\text{CC}^{\text{cb}}(C)$.*

Proof. For each $z \in Z$ and $w \in W$ we have

$$\begin{aligned} \langle z | \delta_Z(t^*w) \rangle_C &= \langle t^*w | \delta_Z(z) \rangle_C^* = \langle w | (t \otimes \text{id}_C) \delta_Z(z) \rangle_C^* \\ &= \langle w | \delta_W(tz) \rangle_C^* = \langle tz | \delta_W(w) \rangle_C = \langle z | (t^* \otimes \text{id}_C) \delta_W(w) \rangle_C, \end{aligned}$$

where the first and the fourth equalities hold because of the Hermitian property (5.2) of δ_Z and δ_W ; the second and the fifth equalities hold because t and t^* are adjoints of one another; and the third equality holds because t was assumed to be a comodule map.

Now, Blecher has shown ([Ble97, Theorem 3.10], cf. [BLM04, Corollary 8.2.15]) that as z ranges over Z , the maps

$$\langle z | : Z \otimes_B^{\text{h}} C \rightarrow C, \quad \xi \mapsto \langle z | \xi \rangle_C$$

separate the points of $Z \otimes_B^{\text{h}} C$. It follows from this, and from the above computation, that $\delta_Z(t^*w) = (t^* \otimes \text{id}_C) \delta_W(w)$ for all $w \in W$, and so t^* is indeed a map of comodules. \square

The category $\text{CM}(B)$ of Hilbert C^* -modules over B is a C^* -category: each of its morphism spaces $\text{CB}_B^*(X, Y)$ carries a Banach-space norm and a conjugate-linear map $*$: $\text{CB}_B^*(X, Y) \rightarrow \text{CB}_B^*(Y, X)$ satisfying the natural ‘multi-object’ analogues of the axioms of a C^* -algebra; and for each $t \in \text{CB}_B^*(X, Y)$ we have $t^*t \geq 0$ in the C^* -algebra $\text{CB}_B^*(X, X)$. See [GLR85] for more on the notion of a C^* -category. The morphism spaces $\text{CB}_C^*(X, Y)$ in $\text{CC}^{\text{cb}}(C)$ are obviously closed subspaces of the morphism spaces $\text{CB}_B^*(X, Y)$ in $\text{CM}(B)$, and Lemma 5.3 shows that the involution $*$ on $\text{CB}_B^*(_, _)$ restricts to an involution on $\text{CB}_C^*(_, _)$. Thus:

Corollary 5.4. *The categories $\text{CC}(C)$ and $\text{CC}^{\text{cb}}(C)$ are C^* -categories. \square*

For each right Hilbert C^* -module X over A , the Haagerup tensor product $\mathcal{F}(X) := X \otimes_A^{\text{h}} F$ is a right Hilbert C^* -module over B , with inner product

$$\langle x_1 \otimes f_1 | x_2 \otimes f_2 \rangle = \langle f_1 | \langle x_1 | x_2 \rangle f_2 \rangle.$$

We consider, as in Example 3.12, the completely contractive coaction

$$\delta_{\mathcal{F}(X)} := \eta_X \otimes \text{id}_F : X \otimes_A^{\text{h}} F \rightarrow X \otimes_A^{\text{h}} F \otimes_B^{\text{h}} C.$$

Lemma 5.5. *For each Hilbert C^* -module X over A , the pair $(\mathcal{F}(X), \delta_{\mathcal{F}(X)})$ is a Hilbert C^* -comodule over C .*

Proof. We must show that the coaction $\delta_{\mathcal{F}(X)}$ satisfies the Hermitian property (5.2); i.e., that

$$\langle x_1 \otimes a_1 f_1 | x_2 \otimes \eta(a_2) \otimes f_2 \rangle_C = \langle x_2 \otimes a_2 f_2 | x_1 \otimes \eta(a_1) \otimes f_1 \rangle_C^*$$

for all $f_1, f_2 \in F$, $a_1, a_2 \in A$ and $x_1, x_2 \in X$.

We claim that for each $k \in K_B(F) \cong F \otimes_B^h F^*$ one has

$$\langle x_1 \otimes f_1 | x_2 \otimes k \otimes f_2 \rangle_C = (k^* \langle x_2 | x_1 \rangle f_1)^* \otimes f_2.$$

Indeed, it suffices to check this for a ‘rank-one’ operator $k = f_2 \otimes f_3^*$, which is done by a straightforward computation. It follows from this that

$$\begin{aligned} \langle x_1 \otimes a_1 f_1 | x_2 \otimes \eta(a_2) \otimes f_2 \rangle_C &= (\eta(a_2^*) \langle x_2 | x_1 \rangle a_1 f_1)^* \otimes f_2 \\ &= (\langle x_2 a_2 | x_1 a_1 \rangle f_1)^* \otimes f_2 = f_1^* \otimes \langle x_1 a_1 | x_2 a_2 \rangle f_2 \\ &= ((\eta(a_1^*) \langle x_1 | x_2 \rangle a_2 f_2)^* \otimes f_1)^* = \langle x_2 \otimes a_2 f_2 | x_1 \otimes \eta(a_1) \otimes f_1 \rangle_C^* \end{aligned}$$

as required. \square

Setting $\mathcal{F}_C(t) := t \otimes \text{id}_C$ for each morphism $t \in \text{CM}(A)$, we obtain a $*$ -functor

$$\mathcal{F}_C : \text{CM}(A) \rightarrow \text{CC}(C).$$

We shall prove the following as a corollary to Theorem 4.1:

Theorem 5.6. *Let ${}_A F_B$ be a C^* -correspondence between C^* -algebras A and B . Assume that A acts faithfully by B -compact operators on F , and that the inclusion $\eta : A \rightarrow K_B(F)$ admits a weak expectation $K_B(F) \rightarrow A^{\vee\vee}$. Then the functor*

$$\mathcal{F}_C : \text{CM}(A) \rightarrow \text{CC}(C), \quad X \mapsto (X \otimes_A^h F, \eta_X \otimes \text{id}_F)$$

is a unitary equivalence of C^ -categories, from the category of right Hilbert C^* -modules over A to the category of right Hilbert C^* -comodules over C .*

Remark 5.7. We note, as in Remark 4.2(c), that Theorem 5.6 gives a description of the image of the functor \mathcal{F} : a Hilbert C^* -module Y over B is unitarily isomorphic to one of the form $X \otimes_A F$ for some Hilbert C^* - A -module X , if and only if Y can be equipped with the structure of a Hilbert C^* -comodule over C . Similar considerations apply to morphisms.

To construct an inverse to \mathcal{F}_C we consider, for each (completely bounded) Hilbert C^* -comodule $(Z, \delta_Z) \in \text{CC}^{\text{cb}}(C)$, the right operator A -module $Z \boxtimes_C F^*$ defined as in Definition/Lemma 4.6:

$$Z \boxtimes_C F^* := \left\{ \xi \in Z \otimes_B^h F^* \mid (\text{id}_Z \otimes \eta_{F^*})(\xi) = (\delta_Z \otimes \text{id}_{F^*})(\xi) \right\}.$$

The module F^* is right Hilbert C^* -module over the C^* -algebra $K = K_B(F) \cong F \otimes_B^h F^*$ of B -compact operators on F : the K -valued inner product is given by $\langle f_1^* | f_2^* \rangle := f_1 \otimes f_2^*$. The left action of B on F^* is via a $*$ -homomorphism into the C^* -algebra of K -adjointable operators, and it follows from this that

the Haagerup tensor product $Z \otimes_B^h F^*$ is a right Hilbert C^* -module over K , with inner product

$$(5.8) \quad \langle z_1 \otimes f_1^* | z_2 \otimes f_2^* \rangle := f_1 \langle z_1 | z_2 \rangle \otimes f_2^*.$$

Lemma 5.9. *Suppose that the action homomorphism $\eta : A \rightarrow K$ is injective. For each Hilbert C^* -comodule $(Z, \delta_Z) \in \text{CC}^{\text{cb}}(C)$ one has $\langle \xi | \zeta \rangle \in \eta(A) \subseteq K$ for all $\xi, \zeta \in Z \boxtimes_C F^*$. Consequently, $Z \boxtimes_C F^*$ is a Hilbert C^* -module over A , with inner product $\langle \xi | \zeta \rangle_A := \eta^{-1}(\langle \xi | \zeta \rangle_K)$.*

Proof. We claim that

$$(5.10) \quad \eta(A) = \{k \in K \mid f_1^* \otimes k(f_2) = (k^*(f_1))^* \otimes f_2 \text{ in } C, \text{ for all } f_1, f_2 \in F\}.$$

To prove this we first note that Proposition 4.7(a), applied to $X = A$, shows that

$$(5.11) \quad \eta(A) = \left\{ a_1 k a_2 \in K \mid \eta(a_1) \otimes k a_2 = a_1 k \otimes \eta(a_2) \text{ in } K \otimes_A^h K \right\}.$$

Now [Ble97, Theorem 3.10] implies that as f_1 and f_2 range over F , the maps

$$K \otimes_A^h K \rightarrow C, \quad k_1 \otimes k_2 \mapsto (k_1^*(f_1))^* \otimes k_2(f_2)$$

separate the points of $K \otimes_A^h K$, and the identity (5.10) follows from this fact combined with (5.11).

We shall use the pairings

$$\begin{aligned} \langle \cdot | \cdot \rangle_F : Z \times Z \otimes_B^h F^* &\rightarrow F, & \langle z_1 | z_2 \otimes f^* \rangle_F &:= f \langle z_2 | z_1 \rangle & \text{and} \\ \langle \cdot | \cdot \rangle_Z : Z \otimes_B^h F^* \times F &\rightarrow Z, & \langle z \otimes f_1^* | f_2 \rangle_Z &:= z \langle f_1 | f_2 \rangle, \end{aligned}$$

which are related to the K -valued inner product (5.8) by the equality

$$(5.12) \quad \langle \xi | \zeta \rangle(f) = \langle \langle \zeta | f \rangle_Z | \xi \rangle_F$$

for $\xi, \zeta \in Z \otimes_B^h F^*$ and $f \in F$.

For each $f \in F$ and $z \in Z$ consider the map

$$(5.13) \quad \langle z | \cdot | f \rangle : Z \otimes_B^h C \otimes_B^h F^* \rightarrow C, \quad z_1 \otimes c \otimes f_1^* \mapsto \langle z | z_1 \rangle c \langle f_1 | f_2 \rangle.$$

It is easily checked that the composition

$$Z \otimes_B^h F^* \xrightarrow{\text{id}_Z \otimes \eta_{F^*}} Z \otimes_B^h C \otimes_B^h F^* \xrightarrow{\langle z | \cdot | f \rangle} C$$

is given by $\xi \mapsto \langle z | \xi \rangle_F^* \otimes f$, while the composition

$$Z \otimes_B^h F^* \xrightarrow{\delta_Z \otimes \text{id}_{F^*}} Z \otimes_B^h C \otimes_B^h F^* \xrightarrow{\langle z | \cdot | f \rangle} C$$

is given by $\xi \mapsto \langle z \mid \delta_Z(\langle \xi | f \rangle_Z) \rangle_C$. Consulting the definition of $Z \boxtimes_C F^*$, we find that

$$(5.14) \quad \langle z | \xi \rangle_F^* \otimes f = \langle z \mid \delta_Z(\langle \xi | f \rangle_Z) \rangle_C \quad \text{for all } f \in F, z \in Z, \xi \in Z \boxtimes_C F^*.$$

Now applying (5.12) and (5.14), we find for $\xi, \zeta \in Z \boxtimes_C F^*$ and $f_1, f_2 \in F$ that

$$\begin{aligned} (\langle \xi | \zeta \rangle^* (f_1))^* \otimes f_2 &= (f_2^* \otimes \langle \zeta | \xi \rangle f_1)^* = (f_2^* \otimes \langle \langle \xi | f_1 \rangle_Z | \zeta \rangle)^* \\ &= \langle \langle \xi | f_1 \rangle_Z | \zeta \rangle_F^* \otimes f_2 = \langle \langle \xi | f_1 \rangle_Z | \delta_Z (\langle \zeta | f_2 \rangle_Z) \rangle_C. \end{aligned}$$

The Hermitian property (5.2) of δ_Z translates, through the above chain of equalities, to the equality

$$(\langle \xi | \zeta \rangle^* f_1)^* \otimes f_2 = ((\langle \zeta | \xi \rangle^* f_2)^* \otimes f_1)^* = f_1^* \otimes \langle \xi | \zeta \rangle f_2.$$

In view of (5.10), this computation shows that that $\langle \xi | \zeta \rangle \in \eta(A)$ for all $\xi, \zeta \in Z \boxtimes_C F^*$, and so the latter module is indeed a Hilbert C^* -module over A . \square

Lemma 5.15. *Suppose that $\eta : A \rightarrow K$ is injective. The assignment $(Z, \delta_Z) \mapsto Z \boxtimes_C F^*$ extends to a $*$ -functor $\mathcal{F}_C^* : \text{CC}^{\text{cb}}(C) \rightarrow \text{CM}(A)$, defined on morphisms by $\mathcal{F}_C^*(t) := (t \otimes \text{id}_{F^*})|_{Z \boxtimes_C F^*}$.*

Proof. If $t : (Z, \delta_Z) \rightarrow (W, \delta_W)$ is an adjointable map of C^* -comodules, then the maps

$$t \otimes \text{id}_{F^*} : Z \otimes_B^{\text{h}} F^* \rightarrow W \otimes_B^{\text{h}} F^* \quad \text{and} \quad t^* \otimes \text{id}_{F^*} : W \otimes_B^{\text{h}} F^* \rightarrow Z \otimes_B^{\text{h}} F^*$$

are mutually adjoint maps of Hilbert C^* -modules over K . Since both t and t^* are comodule maps (Lemma 5.3), $t \otimes \text{id}_{F^*}$ and $t^* \otimes \text{id}_{F^*}$ restrict to maps between the closed A -submodules $Z \boxtimes_C F^*$ and $W \boxtimes_C F^*$, and we have by the definition of the A -valued inner products on $Z \boxtimes_C F^*$ and $W \boxtimes_C F^*$ that

$$\langle \mathcal{F}_C^*(t) \xi | \zeta \rangle = \eta^{-1} (\langle (t \otimes \text{id}_{F^*}) \xi | \zeta \rangle) = \eta^{-1} (\langle \xi | (t^* \otimes \text{id}_{F^*}) \zeta \rangle) = \langle \xi | \mathcal{F}_C(t^*) \zeta \rangle.$$

Thus $\mathcal{F}_C(t)^* = \mathcal{F}_C(t^*)$ as maps of Hilbert C^* -modules over A , and this shows that \mathcal{F}_C is a $*$ -functor from $\text{CC}^{\text{cb}}(C)$ to $\text{CM}(A)$. \square

Proof of Theorem 5.6. Proposition 4.7, with $L = F$ and $R = F^*$, implies that for each $X \in \text{CM}(A)$ there is a completely isometric natural isomorphism of right operator A -modules $X \xrightarrow{\cong} \mathcal{F}_C^* \mathcal{F}_C(X)$; and that for each $(Z, \delta_Z) \in \text{CC}(C)$ there is a completely isometric natural isomorphism of right operator C -comodules $(Z, \delta_Z) \xrightarrow{\cong} \mathcal{F}_C \mathcal{F}_C^*(Z, \delta_Z)$. These are isometric module isomorphisms between Hilbert C^* -modules, and so by a theorem of Lance (cf. [BLM04, 8.1.8]) they are in fact unitary isomorphisms. \square

Corollary 5.16. *Suppose that A acts faithfully by B -compact operators on a C^* -correspondence ${}_A F_B$, and that the inclusion $\eta : A \rightarrow \text{K}_B(F)$ admits a weak expectation $\text{K}_B(F) \rightarrow A^{\vee\vee}$. Then for the coalgebra $C = F^* \otimes_B^{\text{h}} F$ one has $\text{CC}^{\text{cb}}(C) = \text{CC}(C)$: every completely bounded Hilbert C^* -comodule over C is in fact completely contractive.*

Proof. Let $(Z, \delta_Z) \in \text{CC}^{\text{cb}}(C)$ be a completely bounded comodule. Proposition 4.7(b) shows that the image of the coaction $\delta_Z : Z \rightarrow Z \otimes_B^{\text{h}} C$ is equal to the Hilbert C^* -module $(Z \boxtimes_C F^*) \otimes_A^{\text{h}} F$. The A -valued inner product on

$Z \boxtimes_C F^*$ was defined (in Lemma 5.9) by restriction from the K -valued inner product on $Z \otimes_B^h F^*$, and it follows from this that the map

$$(Z \boxtimes_C F^*) \otimes_A^h F \xrightarrow{\xi \otimes f \mapsto \xi \otimes f} Z \otimes_B^h F^* \otimes_K^h F$$

is completely isometric. Now the map

$$F^* \otimes_K^h F \rightarrow B, \quad f_1^* \otimes f_2 \mapsto \langle f_1 | f_2 \rangle = \varepsilon(f_1^* \otimes_A f_2)$$

is also completely isometric: indeed, it is an isomorphism of C^* -algebras from the algebra of K -compact operators on F^* to the closed ideal $\text{image}(\varepsilon) \subseteq B$. It follows from these two observations that the map $\varepsilon_Z : Z \otimes_B^h C \rightarrow Z$ is completely isometric on the image of δ_Z . Since $\varepsilon_Z \circ \delta_Z = \text{id}_Z$, we conclude that δ_Z is also a complete isometry. \square

Remark 5.17. One can adapt Theorem 5.6 to categories of bimodules, similarly to what was explained in Remark 4.11. So consider, for C^* -algebras A , B and D , the categories $\text{CM}(D, A)$ and $\text{CM}(D, B)$ of C^* -correspondences from D to A , and C^* -correspondences from D to B . Let ${}_A F_B$ be a C^* -correspondence on which A acts by B -compact operators, and consider the associated coalgebra $C = F^* \otimes_A^h F$. The proof of Theorem 5.6 carries over verbatim to this setting, and gives for instance the following criterion for factoring operators on C^* -correspondences: with the same assumptions as Theorem 5.6, if $Z \in \text{CM}(D, B)$ is a C^* -correspondence and $t : Z \rightarrow Z$ is an adjointable bimodule map, then there exists a correspondence $X \in \text{CM}(D, A)$, an adjointable bimodule map $s : X \rightarrow X$, and a unitary bimodule isomorphism $u : Z \rightarrow X \otimes_A^h F$ with $t = u^*(s \otimes \text{id}_F)u$, if and only if there exists a completely bounded D - B -linear coaction $\delta_Z : Z \rightarrow Z \otimes_B^h C$ satisfying (5.2), and having $\delta_Z \circ t = (t \otimes \text{id}_C) \circ \delta_Z$.

6. EXAMPLES

6.1. Parabolic induction. Let G be a real reductive group, and let P be a parabolic subgroup of G with unipotent radical N . The quotient $L := P/N$ is another real reductive group. An example is $G = \text{GL}_n(\mathbb{R})$ with (for $n = a + b$)

$$P = \begin{bmatrix} \text{GL}_a(\mathbb{R}) & M_{a \times b}(\mathbb{R}) \\ 0 & \text{GL}_b(\mathbb{R}) \end{bmatrix}, \quad N = \begin{bmatrix} 1_a & M_{a \times b}(\mathbb{R}) \\ 0 & 1_b \end{bmatrix}, \quad L \cong \text{GL}_a(\mathbb{R}) \times \text{GL}_b(\mathbb{R}).$$

Clare [Cla13] showed how to construct a C^* -correspondence, which we shall denote here by ${}_{C_r^*(G)} C_r^*(G/N) {}_{C_r^*(L)}$, whose associated tensor-product functor

$$\text{Ind}_P : \text{Rep}(C_r^*(L)) \rightarrow \text{Rep}(C_r^*(G)), \quad H \mapsto C_r^*(G/N) \otimes_{C_r^*(L)}^h H$$

is isomorphic to the well-known functor of *parabolic induction* of tempered unitary representations. This correspondence was further analysed in [CCH16] and [CCH18], where we showed that the adjoint operator bimodule

$$C_r^*(N \backslash G) := {}_{C_r^*(L)} C_r^*(G/N) {}_{C_r^*(G)}^*$$

is completely boundedly isomorphic to a C^* -correspondence. This implies that the *parabolic restriction* functor

$$\text{Res}_P : \text{OM}(C_r^*(G)) \rightarrow \text{OM}(C_r^*(L)), \quad X \mapsto X \otimes_{C_r^*(G)}^{\text{h}} C_r^*(G/N)$$

sends Hilbert-space representations of $C_r^*(G)$ to Hilbert-space representations of $C_r^*(L)$, and hence furnishes a two-sided adjoint to Ind_P on tempered unitary representations. The arguments in [CCH16] relied on deep theorems from representation theory, and it would be of great interest to find an alternative, more geometric route to this adjunction theorem.

The operator $C_r^*(L)$ -bimodule

$$C_P := C_r^*(N \backslash G) \otimes_{C_r^*(G)}^{\text{h}} C_r^*(G/N)$$

is of great interest from a representation-theoretic standpoint: indeed, it follows from [BLM04, 1.5.14 & 3.5.10] that for every pair of tempered unitary representations H_1, H_2 of L , the space of intertwining operators between the parabolically induced representations $\text{Ind}_P H_1$ and $\text{Ind}_P H_2$ can be recovered as

$$\text{B}_G(\text{Ind}_P H_1, \text{Ind}_P H_2) \cong \left(H_2^* \otimes_{C_r^*(L)}^{\text{h}} C_P \otimes_{C_r^*(L)}^{\text{h}} H_1 \right)^\vee.$$

The results of this paper show that C_P carries extra algebraic structure that is very relevant to studying the representations of G .

Corollary 6.1. *The operator bimodule C_P is an operator $C_r^*(L)$ -coalgebra, and the parabolic restriction functor $\text{Res}_P : X \rightarrow X \otimes_{C_r^*(G)}^{\text{h}} C_r^*(G/N)$ induces a completely isometric equivalence of OS_1 -categories*

$$\text{Res}_P : \text{OM}(C_r^*(G)_P) \xrightarrow{\cong} \text{OC}(C_P),$$

and a unitary equivalence of C^* -categories

$$\text{Res}_P : \text{CM}(C_r^*(G)_P) \xrightarrow{\cong} \text{CC}(C_P),$$

where $C_r^*(G)_P$ denotes the image of $C_r^*(G)$ under the action homomorphism $C_r^*(G) \rightarrow \text{K}_{C_r^*(L)}(C_r^*(G/N))$.

Proof. It was observed by Clare that simple geometric considerations (the compactness of G/P) ensure that $C_r^*(G)$ acts on $C_r^*(G/N)$ through $C_r^*(L)$ -compact operators. It follows that $(C_r^*(G/N), C_r^*(N \backslash G))$ is an adjoint pair of bimodules in the sense of Definition 3.1, and so the bimodule C_P is a $C_r^*(L)$ -coalgebra as in Section 3.3. The C^* -algebra $C_r^*(G)$ is nuclear—indeed, by a theorem of Harish-Chandra [HC53, Theorem 7], it is of type I—and hence its quotient $C_r^*(G)_P$ is also nuclear and thus has the weak expectation property. Since $C_r^*(G/N)$ is faithful as a $C_r^*(G)_P$ -module by definition, the asserted equivalences follow from Lemma 4.5 and Theorems 4.1 and 5.6. \square

Remark 6.2. The results of [CCH16] and [CCH18] allow us to say more about the coalgebra C_P : the $C_r^*(G)$ -valued inner product on $C_r^*(N \backslash G)$ constructed in [CCH16] furnishes C_P with a completely bounded multiplication $C_P \otimes_{C_r^*(L)}^{\text{h}} C_P \rightarrow C_P$, and as an algebra C_P is completely boundedly

isomorphic to the C^* -algebra of $C_r^*(G)$ -compact operators on $C_r^*(N \setminus G)$. The algebra and coalgebra structures on C_P are compatible with one another: the coproduct is a map of C_P -bimodules, and the product is a map of C_P -bi-comodules; in other words C_P is a (non-unital) *Frobenius algebra* in the category of operator B -bimodules. Just as for Frobenius algebras in the purely algebraic setting (cf. [Abr99]), one has an equivalence $\text{OC}^{\text{cb}}(C_P) \cong \text{OM}^{\text{cb}}(C_P)$ between the categories of (completely bounded) comodules and modules over C_P . Similar considerations apply to any locally adjoint pair of C^* -correspondences admitting a unit, in the terminology of [CCH18].

6.2. The Fourier coalgebra of a compact group. To begin with we let $A \subseteq \mathbb{K}(H)$ be a C^* -algebra of compact operators, which is nondegenerate in the sense that $H = AH$. Regarding H as a C^* -correspondence from A to \mathbb{C} puts us in the setting of Example 3.4: the pair $({}_A H_{\mathbb{C}}, {}_{\mathbb{C}} H_A^*)$ is an adjoint pair of operator bimodules, with unit the inclusion map $A \rightarrow \mathbb{K}(H) \cong H \otimes_{\mathbb{C}}^{\text{h}} H^*$ and counit the inner product $H^* \otimes_A^{\text{h}} H \rightarrow \mathbb{C}$.

The operator \mathbb{C} -coalgebra $C = H^* \otimes_A^{\text{h}} H$ associated to this adjoint pair is isomorphic to one previously studied in [ER03], as we shall now explain. We note firstly that if $A \subseteq \mathbb{B}(H)$ is any nondegenerate C^* -algebra of operators, then the pairing

$$(H^* \otimes_A^{\text{h}} H) \times A' \rightarrow \mathbb{C}, \quad \langle\langle \xi^* \otimes \zeta | t \rangle\rangle := \langle \xi | t \zeta \rangle \quad (\xi, \zeta \in H, t \in A')$$

induces a completely isometric isomorphism of operator spaces

$$(6.3) \quad H^* \otimes_A^{\text{h}} H \xrightarrow{\cong} A'_V$$

between the Haagerup tensor product $H^* \otimes_A^{\text{h}} H$ and the predual of the commutant $A' \subseteq \mathbb{B}(H)$. This is a special case of [BLM04, Corollary 3.5.10].

In [ER03], Effros and Ruan showed that the predual M_V of any von Neumann algebra M carries a coalgebra structure dual to the algebra structure of M : the inclusion-of-the-unit map $u : \mathbb{C} \rightarrow M$ dualises to give a counit $u_V : M_V \rightarrow \mathbb{C}$, while the multiplication map $\mu : M \otimes^{\sigma\text{h}} M \rightarrow M$ dualises to give a coproduct $\mu_V : M_V \rightarrow M_V \otimes^{\text{eh}} M_V$. Here $\otimes^{\sigma\text{h}}$ and \otimes^{eh} denote the *normal* and the *extended* Haagerup tensor products, respectively. We refer the reader to [ER03] for details.

Lemma 6.4. *If $A \subseteq \mathbb{K}(H)$ is a nondegenerate C^* -algebra of compact operators, then Effros and Ruan's coproduct $\mu_V : A'_V \rightarrow A'_V \otimes^{\text{eh}} A'_V$ has image contained in $A'_V \otimes^{\text{h}} A'_V$, and the map (6.3) is a completely isometric isomorphism of operator \mathbb{C} -coalgebras.*

Proof. Let us denote the isomorphism (6.3) by $\varphi : C \rightarrow A'_\vee$. We are claiming that the diagrams

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & A'_\vee \\ \delta \downarrow & & \downarrow \mu_\vee \\ C \otimes^{\text{h}} C & \xrightarrow{i \circ (\varphi \otimes \varphi)} & A'_\vee \otimes^{\text{eh}} A'_\vee \end{array} \quad \text{and} \quad \begin{array}{ccc} C & \xrightarrow{\varphi} & A'_\vee \\ \varepsilon \downarrow & \swarrow u_\vee & \\ \mathbb{C} & & \end{array}$$

commute, where $i : A'_\vee \otimes^{\text{h}} A'_\vee \rightarrow A'_\vee \otimes^{\text{eh}} A'_\vee$ is the canonical inclusion.

By the definition of the coproduct μ_\vee , and by the fact that the image of the algebraic tensor product $A' \otimes A'$ is weak*-dense in $A' \otimes^{\text{sh}} A'$ (see [ER03, Lemma 5.8]), to show that the first diagram commutes it will suffice to prove that

$$\langle\langle \varphi(\xi^* \otimes a\zeta) | t_1 t_2 \rangle\rangle = \langle\langle i \circ (\varphi \otimes \varphi) \circ \delta(\xi^* \otimes a\zeta) | t_1 \otimes t_2 \rangle\rangle$$

for all $\xi, \zeta \in H$, all $a \in A$ and all $t_1, t_2 \in A'$ (where $\langle\langle \cdot, \cdot \rangle\rangle$ denote the duality pairings). Approximating a in norm by finite-rank operators

$$a = \lim_{n \rightarrow \infty} \sum_{m=1}^{r_n} \alpha_{n,m} \otimes \beta_{n,m}^* \quad (\alpha_{n,m}, \beta_{n,m} \in H)$$

we have

$$\delta(\xi^* \otimes a\zeta) = \lim_n \sum_m (\xi^* \otimes \alpha_{n,m}) \otimes (\beta_{n,m}^* \otimes \zeta),$$

and so

$$\begin{aligned} & \langle\langle i \circ (\varphi \otimes \varphi) \circ \delta(\xi^* \otimes a\zeta) | t_1 \otimes t_2 \rangle\rangle \\ &= \lim_n \sum_m \langle\langle \varphi(\xi^* \otimes \alpha_{n,m}) | t_1 \rangle\rangle \langle\langle \varphi(\beta_{n,m}^* \otimes \zeta), t_2 \rangle\rangle \\ &= \lim_n \sum_m \langle \xi | t_1 \alpha_{n,m} \rangle \langle \beta_{n,m} | t_2 \zeta \rangle \\ &= \langle \xi | t_1 a t_2 \zeta \rangle = \langle \xi | t_1 t_2 a \zeta \rangle = \langle\langle \varphi(\xi^* \otimes a\zeta) | t_1 t_2 \rangle\rangle. \end{aligned}$$

Thus the first diagram commutes.

The commutativity of the second diagram is easily seen by noting that

$$\varepsilon(\xi^* \otimes \zeta) = \langle \xi | \zeta \rangle = \langle\langle \varphi(\xi^* \otimes \zeta) | 1 \rangle\rangle = u_\vee(\varphi(\xi^* \otimes \zeta)). \quad \square$$

Since A has the weak expectation property (indeed, A is nuclear), Theorem 4.1 gives the following identification of A -modules and A'_\vee -comodules:

Corollary 6.5. *If $A \subseteq K(H)$ is a nondegenerate C^* -algebra of compact operators then the comparison functor*

$$\text{OM}(A) \rightarrow \text{OC}(A'_\vee), \quad X \mapsto (X \otimes_A^{\text{h}} H, \eta_X \otimes \text{id}_H)$$

is a completely isometric equivalence. □

Now let G be a compact group, and consider the right-regular representation ρ of G on $L^2(G)$. As is well known, the commutant $\rho(G)'$ is isomorphic to the group von Neumann algebra of G , whose predual $\rho(G)'_\vee$ is the *Fourier*

algebra $A(G)$ of continuous functions $G \rightarrow \mathbb{C}$ which are matrix coefficients of the left-regular representation λ : i.e., those functions of the form

$$c_{\xi^* \otimes \zeta} : g \mapsto \langle \xi | \lambda(g) \zeta \rangle_{L^2(G)}$$

for some $\xi, \zeta \in L^2(G)$. The duality pairing between an operator $t \in \rho(G)'$ and a function $c_{\xi^* \otimes \zeta} \in A(G)$ is given explicitly by

$$\langle c_{\xi^* \otimes \zeta} | t \rangle = \langle \xi | t \zeta \rangle_{L^2(G)}.$$

See [Eym64].

Putting $A = C^*(G)$ and $H = L^2(G)$ (on which $C^*(G)$ acts via ρ) in the discussion above, we find that the map

$$(6.6) \quad L^2(G)^* \otimes_{C^*(G)}^h L^2(G) \rightarrow A(G), \quad \xi^* \otimes \zeta \mapsto c_{\xi^* \otimes \zeta}$$

is a completely isometric isomorphism of operator \mathbb{C} -coalgebras, where the coproduct on $A(G)$ is dual to the multiplication in G , and the counit is evaluation at the identity. Corollary 6.5 then gives the following identification between operator $C^*(G)$ -modules and operator $A(G)$ -comodules:

Corollary 6.7. *For each compact group G the functor*

$$\text{OM}(C^*(G)) \rightarrow \text{OC}(A(G)), \quad X \mapsto X \otimes_{C^*(G)}^h L^2(G)$$

is a completely isometric equivalence. □

Remark 6.8. It would certainly be interesting to extend Corollary 6.7 beyond the setting of compact groups. The identification (6.6) is valid for every locally compact group G ; but when G is not compact the coproduct on $A(G)$ takes values in the *extended* Haagerup tensor product, and so this case lies beyond the scope of Theorem 4.1. It may be possible to remedy this by extending our framework to include coproducts and coactions taking values in spaces of multipliers, as is frequently done in the setting of C^* -algebraic quantum groups, and as is suggested in the present context by Daws in [Daw10, Section 9.3].

6.3. Subalgebras and flat connections. In this section we study non-degenerate inclusions of C^* -algebras $A \hookrightarrow B$ (nondegeneracy meaning that $B = AB = BA$). We consider the B -coalgebra $C = B \otimes_A^h B$ associated to the adjoint pair of operator bimodules $({}_A B_B, {}_B B_A)$, with unit $\eta : A \rightarrow B \cong B \otimes_B^h B$ the inclusion, and counit $B \otimes_A^h B \rightarrow B$ the product. Considering B as a Hilbert C^* -module over itself, with inner product $\langle b_1 | b_2 \rangle = b_1^* b_2$, puts us in the context of Section 5. In this section we shall adapt some algebraic observations of Nuss [Nus97] and Brzezinski [Brz03] to this C^* -algebraic setting to give a description of the C^* -comodule category $\text{CC}(C)$ in terms of flat connections on Hilbert C^* -modules over B . (A similar description applies to the category $\text{OC}(C)$ of operator comodules. These categories may likewise be described in terms of *descent data* and *involutive twists*, as is done in the algebraic setting in [Cip76] and [Nus97].)

Let $\Omega = \Omega(B, A) := \ker \varepsilon \subseteq C$ denote the kernel of the multiplication map. This is the analogue in the operator-algebraic setting of the module of relative one-forms of [CQ95]. The case where $A = \mathbb{C}$ has previously been considered by Mesland in [Mes14]. Note that Ω is stable under the involution $(b_1 \otimes b_2)^* := b_2^* \otimes b_1^*$ on C .

We do not assume that the algebra B has a unit; but of course B can be embedded completely isometrically in a unital C^* -algebra B^+ , and Theorem 2.5 ensures that this embedding induces completely isometric embeddings of Haagerup tensor products $B \otimes_A^h B \hookrightarrow B^+ \otimes_A^h B^+$ (and likewise for higher tensor powers). Note that the nondegeneracy of B over A ensures that elements of the form $b \otimes 1, 1 \otimes b \in B^+ \otimes_A^h B^+$ belong to the submodule $B \otimes_A^h B$.

We consider the completely bounded A -bimodule maps

$$\begin{aligned} d : B &\rightarrow \Omega, & b &\mapsto 1 \otimes b - b \otimes 1 \quad \text{and} \\ d^1 : \Omega &\rightarrow \Omega \otimes_B^h \Omega, & \omega &\mapsto q \circ (d \otimes d)|_{\Omega}(\omega), \end{aligned}$$

where $q : \Omega \otimes_A^h \Omega \rightarrow \Omega \otimes_B^h \Omega$ is the quotient mapping. These maps satisfy the Leibniz rules

$$\begin{aligned} d(bb') &= d(b)b' + bd(b'), \\ d^1(b\omega) &= d(b) \otimes \omega + bd^1(\omega), \\ d^1(\omega b) &= d^1(\omega)b - \omega \otimes d(b) \end{aligned}$$

for all $b, b' \in B$ and all $\omega \in \Omega$, and one has $d^1 \circ d = 0$. Moreover, $d(b^*) = -d(b)^*$ and $d^1(\omega^*) = d^1(\omega)^*$ for all $b \in B$ and $\omega \in \Omega$. (This structure may be extended to a differential graded $*$ -algebra, but we won't need to consider that here.)

Definition 6.9. (cf. [CQ95, Section 8], [Mes14, Section 5]) An Ω -connection on a right operator B -module Z is a completely bounded A -linear map $\nabla : Z \rightarrow Z \otimes_B^h \Omega$ satisfying

$$\nabla(zb) = \nabla(z)b + z \otimes d(b)$$

for all $b \in B$ and $z \in Z$. The *curvature* of a connection ∇ is the composite map

$$Z \xrightarrow{\nabla} Z \otimes_B^h \Omega \xrightarrow{\text{id}_Z \otimes d^1 + \nabla \otimes \text{id}_{\Omega}} Z \otimes_B^h \Omega \otimes_B^h \Omega,$$

and a *flat* connection is one whose curvature is zero. If Z is a right Hilbert C^* -module over B , then a connection $\nabla : Z \rightarrow Z \otimes_B^h \Omega$ is called *Hermitian* if it satisfies

$$(6.10) \quad \langle z_1 | \nabla(z_2) \rangle_{\Omega} - \langle z_2 | \nabla(z_1) \rangle_{\Omega}^* = d(\langle z_1 | z_2 \rangle)$$

for all $z_1, z_2 \in Z$, where we are using the pairing

$$\langle \cdot | \cdot \rangle_{\Omega} : Z \times (Z \otimes_B^h \Omega) \rightarrow \Omega, \quad \langle z_1 | z_2 \otimes \omega \rangle_{\Omega} := \langle z_1 | z_2 \rangle \omega.$$

Let $\text{Con}^f(B, A)$ denote the category whose objects (Z, ∇_Z) are right Hilbert C^* -modules over B equipped with flat, Hermitian $\Omega(B, A)$ -connections, with

morphisms $t : (W, \nabla_W) \rightarrow (Z, \nabla_Z)$ the adjointable maps of Hilbert C^* -modules satisfying $\nabla_Z \circ t = (t \otimes \text{id}_\Omega) \circ \nabla_W$.

Example 6.11. Every countably generated Hilbert C^* -module Z over B may be equipped with a Hermitian $\Omega(B, A)$ -connection: the construction of Grassmann connections in [CQ95, Proposition 8.1] and [Mes14, Corollary 5.15] carries over verbatim to this setting.

The category $\text{Con}^f(B, A)$ is a C^* -category: this can be shown by a direct argument as in Lemma 5.3, but it also follows easily from the following equivalence between flat connections and comodules over $C = B \otimes_A^h B$, which is proved just as in the algebraic case (cf. [Brz03, Theorem 4.4]).

Lemma 6.12. *Let (Z, δ_Z) be a completely bounded Hilbert C^* -comodule over C . The map ∇_{δ_Z} defined as the composition*

$$Z \xrightarrow{\delta_Z} Z \otimes_B^h C \xrightarrow{\text{id}_Z \otimes \theta} Z \otimes_B^h \Omega,$$

where $\theta(c) := c - \varepsilon(c) \otimes 1$, is a flat Hermitian connection. Conversely, if ∇_Z is a flat Hermitian $\Omega(B, A)$ -connection on a right Hilbert C^* -module Z over B , then the map δ_{∇_Z} defined by

$$\delta_{\nabla_Z} : Z \rightarrow Z \otimes_B^h C, \quad z \mapsto \nabla_Z(z) + z \otimes 1 \otimes 1$$

gives Z the structure of a completely bounded Hilbert C^* -comodule over C . The functors

$$\text{CC}^{\text{cb}}(C) \xrightarrow{\mathcal{E}} \text{Con}^f(B, A), \quad \mathcal{E}(Z, \delta_Z) = (Z, \nabla_{\delta_Z}), \quad \mathcal{E} = \text{id on morphisms}$$

$$\text{Con}^f(B, A) \xrightarrow{\mathcal{D}} \text{CC}^{\text{cb}}(C), \quad \mathcal{D}(Z, \nabla_Z) = (Z, \delta_{\nabla_Z}), \quad \mathcal{D} = \text{id on morphisms}$$

are mutually inverse isomorphisms of categories. \square

Corollary 6.13. *Let A be a C^* -algebra, embedded as a nondegenerate sub-algebra of a C^* -algebra B , and suppose that there is a weak expectation $B \rightarrow A^{\vee\vee}$. Then the functors*

$$\text{CM}(A) \rightarrow \text{Con}^f(B, A), \quad X \mapsto (X \otimes_A^h B, \nabla(x \otimes b) := x \otimes 1 \otimes d(b))$$

and

$$\text{Con}^f(B, A) \rightarrow \text{CM}(A), \quad (Z, \nabla_Z) \mapsto \ker(\nabla_Z)$$

are mutually inverse unitary equivalences of C^* -categories.

Proof. First note that Corollary 5.16 gives an equality $\text{CC}^{\text{cb}}(C) = \text{CC}(C)$. Now the functors in question are equal, respectively, to the compositions

$$\text{CM}(A) \xrightarrow{\mathcal{F}_C} \text{CC}(C) \xrightarrow{\mathcal{E}} \text{Con}^f(B, A) \quad \text{and} \quad \text{Con}^f(B, A) \xrightarrow{\mathcal{D}} \text{CC}(C) \xrightarrow{\mathcal{F}_C^*} \text{CM}(A)$$

and so the corollary follows from Theorem 5.6 and Lemma 6.12. \square

Example 6.14. Setting $A = \mathbb{C}$ we find that for a unital C^* -algebra B , the Hilbert C^* -modules over B admitting a flat, Hermitian $\Omega(B, \mathbb{C})$ -connection are precisely the ‘free modules’ $B^{\oplus n}$ (where n may be infinite).

- Remarks 6.15.** (a) Once again (cf. Remark 5.17) there is a bimodule version of Corollary 6.13, relating C^* -correspondences ${}_D E_A$ to flat, D -linear, Hermitian connections on C^* -correspondences ${}_D F_B$, for each C^* -algebra D .
- (b) It follows from Corollary 6.13 that the K -theory of A can be described in terms of the category $\text{Con}^f(B, A)$, in such a way that the map $K_*(A) \rightarrow K_*(B)$ induced by the inclusion $A \rightarrow B$ corresponds to the one induced by the forgetful functor $\text{Con}^f(B, A) \rightarrow \text{CM}(B)$.

6.4. The maximal C^* -dilation. For our final example we consider the descent problem for the inclusion $A \hookrightarrow C^*A$ of a not-necessarily-self-adjoint operator algebra A into its *maximal C^* -algebra*. This embedding is characterised by the property that if D is a C^* -algebra generated by a completely isometrically embedded copy of A , then the identity map on A extends uniquely to a $*$ -homomorphism $C^*A \rightarrow D$. If A is a C^* -algebra then $C^*A = A$ and the inclusion is the identity. See [Ble99].

The passage from (modules over) A to (modules over) C^*A was used in [Ble01a] and [BS04] as a way to relate the representation theory of non-self-adjoint operator algebras to the better understood C^* -algebra theory. Given an operator algebra A , one considers the *maximal C^* -dilation* functor

$$\mathcal{L} : \text{OM}(A) \rightarrow \text{OM}(C^*A), \quad X \mapsto X \otimes_A^{\text{h}} C^*A,$$

which is left-adjoint to the forgetful functor $\mathcal{R} : \text{OM}(C^*A) \rightarrow \text{OM}(A)$. This is the pair of functors corresponding to the adjoint pair of operator bimodules $({}_A C^*A {}_{C^*A}, {}_{C^*A} C^*A {}_A)$ (cf. Example 3.5). Letting C denote the associated operator C^*A -coalgebra $C^*A \otimes_A^{\text{h}} C^*A$, we have a comparison functor

$$\mathcal{L}_C : \text{OM}(A) \rightarrow \text{OC}(C), \quad X \mapsto (X \otimes_A^{\text{h}} C^*A, \eta_X \otimes \text{id}_{C^*A}).$$

Theorems of Beck and of Blecher imply that \mathcal{L}_C is completely isometrically fully faithful:

Proposition 6.16. *For each pair of operator A -modules $X, Y \in \text{OM}(A)$ the natural map*

$$\mathcal{L}_C : \text{CB}_A(X, Y) \rightarrow \text{CB}_C(\mathcal{L}_C(X), \mathcal{L}_C(Y))$$

is a completely isometric isomorphism.

Proof. The functor \mathcal{L}_C is given on morphisms by a Haagerup tensor product, so it is completely contractive. Blecher showed in [Ble99, Corollary 3.11] that for each $X \in \text{OM}(A)$ the natural map

$$\eta_X : X \rightarrow X \otimes_A^{\text{h}} C^*A, \quad xa \mapsto x \otimes a$$

is a completely isometric embedding, and since for each $t \in \text{CB}_A(X, Y)$ the diagram

$$\begin{array}{ccc} X & \xrightarrow{t} & Y \\ \eta_X \downarrow & & \downarrow \eta_Y \\ X \otimes_A^{\text{h}} C^*A & \xrightarrow{\mathcal{L}_C(t) = t \otimes \text{id}_{C^*A}} & Y \otimes_A^{\text{h}} C^*A \end{array}$$

commutes, this implies that the map $t \mapsto \mathcal{L}_C(t)$ is in fact a complete isometry.

The proof that the map \mathcal{L}_C is surjective is a special case of an argument due to Beck (cf. [BW05, Theorem 3.13]). Using [Ble99, Corollary 3.11] once again, one shows as in the proof of Proposition 4.7(a) that

$$\eta_X(X) = \left\{ \xi \in X \otimes_A^{\text{h}} C^*A \mid (\text{id}_X \otimes \eta_{C^*A})(\xi) = (\eta_X \otimes \text{id}_{C^*A})(\xi) \right\}$$

for every $X \in \text{OM}(A)$. Using this, and the naturality of η , one shows that each comodule map $s \in \text{CB}_C(\mathcal{L}_C(X), \mathcal{L}_C(Y))$ restricts to a map $\eta_X(X) \rightarrow \eta_Y(Y)$. A simple computation then shows that $s = t \otimes \text{id}_{C^*A}$ for the map $t := \eta_Y^{-1} \circ s \circ \eta_X \in \text{CB}_A(X, Y)$. \square

Proposition 6.16 implies that the category $\text{OM}(A)$ can be identified, completely isometrically, with a full subcategory of $\text{OC}(C)$. In the interests of clarifying the relationship between the representation theories of A and of C^*A , it would be useful to characterise this subcategory. In this section we shall prove a first result in this direction: if A is not a C^* -algebra then the subcategory in question is a *proper* subcategory of $\text{OC}(C)$.

Theorem 6.17. *Let A be a locally unital operator algebra. Then A is a C^* -algebra if and only if the comparison functor $\mathcal{L}_C : \text{OM}(A) \rightarrow \text{OC}(C)$ for the operator C^*A -coalgebra $C = C^*A \otimes_A^{\text{h}} C^*A$ is a completely isometric equivalence.*

We shall prove Theorem 6.17 as a corollary of the following general property of operator coalgebras over C^* -algebras. Recall from Section 2.2 that for an operator B -coalgebra C , we denote by $\text{OC}(C)_1$ and $\text{OM}(B)_1$ the subcategories of completely contractive maps in $\text{OC}(C)$ and $\text{OM}(B)$ (respectively). Let us say that a pair of morphisms (i, q) is a *kernel-cokernel pair* if i is a kernel of q and q is a cokernel of i .

Lemma 6.18. *Let B be a C^* -algebra and let C be an operator B -coalgebra. Then the forgetful functors $\mathcal{F} : \text{OC}(C) \rightarrow \text{OM}(B)$ and $\mathcal{F}_1 : \text{OC}(C)_1 \rightarrow \text{OM}(B)_1$ preserve kernel-cokernel pairs.*

Proof. We shall present the proof for \mathcal{F}_1 , from which the proof for \mathcal{F} differs only in notation. Let $i : (I, \delta_I) \rightarrow (Z, \delta_Z)$ and $q : (Z, \delta_Z) \rightarrow (Q, \delta_Q)$ be a kernel-cokernel pair in $\text{OC}(C)_1$. The functor \mathcal{F}_1 has a right adjoint—namely, the ‘free comodule’ functor $Y \mapsto (Y \otimes_B^{\text{h}} C, \text{id}_Y \otimes \delta)$ —and so \mathcal{F}_1 preserves cokernels [ML98, V.5]. Therefore $\mathcal{F}_1(q)$ is a cokernel of $\mathcal{F}_1(i)$ in $\text{OM}(B)_1$.

To prove that $\mathcal{F}_1(i)$ is a kernel of $\mathcal{F}_1(q)$ in $\text{OM}(B)_1$, we will show that the map q has a kernel $j : (J, \delta_J) \rightarrow (Z, \delta_Z)$ in $\text{OC}(C)_1$ such that $\mathcal{F}_1(j)$ is a kernel of $\mathcal{F}_1(q)$ in $\text{OM}(B)_1$. The uniqueness of kernels in $\text{OC}(C)_1$ will then imply that i and j are conjugate via an $\text{OC}(C)_1$ isomorphism $h : (I, \delta_I) \rightarrow (J, \delta_J)$, hence that $\mathcal{F}_1(i)$ and $\mathcal{F}_1(j)$ are conjugate via the $\text{OM}(B)_1$ isomorphism $\mathcal{F}_1(h)$, and hence that $\mathcal{F}_1(i)$ is a kernel of $\mathcal{F}_1(q)$ in $\text{OM}(B)_1$ as required.

Let J denote the operator B -submodule $\ker(q) \subseteq Z$, and let $j : J \rightarrow Z$ denote the inclusion map. Note that j is a kernel of q in $\text{OM}(B)_1$, by Lemma 2.4.

Because B is a C^* -algebra, Theorem 2.5 implies that the map $j \otimes \text{id}_C : J \otimes_B^h C \rightarrow Z \otimes_B^h C$ is a completely isometric embedding, which we shall use to regard $J \otimes_B^h C$ as a closed submodule of $Z \otimes_B^h C$. Since q is a map of C -comodules, the diagram

$$\begin{array}{ccc} J & \xrightarrow{\delta_Z|_J} & Z \otimes_B^h C \\ q|_J=0 \downarrow & & \downarrow q \otimes \text{id}_C \\ Q & \xrightarrow{\delta_Q} & Q \otimes_B^h C \end{array}$$

commutes, showing that $\delta_Z(J) \subseteq \ker(q \otimes \text{id}_C)$. Since the map $q : Z \rightarrow Q$ is a cokernel in $\text{OM}(B)_1$, Lemma 2.4 and Theorem 2.5 give $\ker(q \otimes \text{id}_C) = \text{image}(j \otimes \text{id}_C) = J \otimes_B^h C$. We may therefore regard $\delta_Z|_J$ as a completely contractive map $J \rightarrow J \otimes_B^h C$, furnishing J with the structure of a C -comodule such that the inclusion $j : J \rightarrow Z$ is a map of comodules.

It remains to check that $j : (J, \delta_Z|_J) \rightarrow (Z, \delta_Z)$ is a kernel of q in $\text{OC}(C)_1$, which is straightforward: if $f : (W, \delta_W) \rightarrow (Z, \delta_Z)$ is a morphism in $\text{OC}(C)_1$ with $q \circ f = 0$, then the image of f must lie in $J = \ker(q)$, and the restricted map $f : V \rightarrow J$ is still a morphism in $\text{OC}(C)_1$. \square

Corollary 6.19. *Let $({}_A L_B, {}_B R_A)$ be an adjoint pair of operator bimodules, where B is a C^* -algebra, and let $C = L \otimes_A^h R$ be the associated B -coalgebra. If the comparison functor $\mathcal{L}_C : \text{OM}(A) \rightarrow \text{OC}(C)$ is a completely isometric equivalence, then the functor $\mathcal{L} : \text{OM}(A) \rightarrow \text{OM}(B)$ preserves complete isometries.*

Proof. Let $i : X \rightarrow Y$ be a completely isometric map of operator A -modules. Letting $q : Y \rightarrow Y/i(X)$ be the quotient map, Lemma 2.4 ensures that (i, q) is a kernel-cokernel pair in $\text{OM}(A)_1$. If the functor \mathcal{L}_C is a completely isometric equivalence, then it restricts to an equivalence $\mathcal{L}_{C,1} : \text{OM}(A)_1 \rightarrow \text{OC}(C)_1$, and so $(\mathcal{L}_{C,1}(i), \mathcal{L}_{C,1}(q))$ is a kernel-cokernel pair in $\text{OC}(C)_1$. Now Lemma 6.18 implies that the map $\mathcal{F}_1(\mathcal{L}_{C,1}(i)) = \mathcal{L}(i)$ is a kernel in $\text{OM}(B)_1$ and hence, by Lemma 2.4, a complete isometry. \square

Remark 6.20. Replacing \mathcal{F}_1 by \mathcal{F} in the proof of Corollary 6.19 gives the following ‘completely bounded’ variant: if \mathcal{L}_C is an equivalence then \mathcal{L} preserves complete embeddings.

Theorem 6.17 follows easily from Corollary 6.19 and from a result of Blecher:

Proof of Theorem 6.17. If \mathcal{L}_C is a completely isometric equivalence then Corollary 6.19 implies that the dilation functor $\mathcal{L} : \text{OM}(A) \rightarrow \text{OM}(C^*A)$ preserves complete isometries. Blecher has shown that the latter property

holds (if and) only if A is a C^* -algebra: see [Ble99, Theorem 4.3]. Conversely, if A is a C^* -algebra then $C^*A = A$, and Theorem 4.1 ensures that the comparison functor $\mathcal{L}_C : \text{OM}(A) \rightarrow \text{OC}(A \otimes_A^h A)$ is an equivalence. \square

Remark 6.21. We can establish the following partial analogue of Theorem 6.17 in the completely bounded setting: if the comparison functor $\mathcal{L}_C : \text{OM}(A) \rightarrow \text{OC}(C)$ is a completely bounded equivalence, then the algebra A has the *module complementation property*: if $A \rightarrow B(H)$ is a completely contractive homomorphism and $H' \subseteq H$ is a closed A -invariant subspace, then H' is topologically complemented in H . (See [BLM04, Section 7.2] for a discussion of this property.) The proof is similar to the proof of Theorem 6.17: if \mathcal{L}_C is an equivalence then the dilation functor preserves complete embeddings (see Remark 6.20), and then replacing complete isometries by complete embeddings in Blecher’s proof of [Ble99, Theorem 4.2] shows that A has the module complementation property. It is known that for certain classes of operator algebras, the module complementation property is equivalent to being a C^* -algebra: see [BLM04, 7.2.7, 7.2.10] for example. On the other hand, [CFO14] provides an example of a non-self-adjoint operator algebra with the module complementation property.

Remark 6.22. The pair $({}_A C^* A_{C^* A}, {}_{C^* A} C^* A_A)$, where A is not a C^* -algebra, is an example of an adjoint pair of bimodules for which the left adjoint functor \mathcal{L} is not *comonadic* (that is to say, \mathcal{L}_C is not a completely isometric equivalence), while the right adjoint functor \mathcal{R} is *monadic*: indeed, the monad associated to this adjunction is the functor on $\text{OM}(A)$ of tensor product with the A -bimodule $K = C^* A \otimes_{C^* A}^h C^* A \cong C^* A$; the associated Eilenberg-Moore category is just the category $\text{OM}(C^* A)$; and the comparison functor

$$\mathcal{R}_K : \text{OM}(C^* A) \xrightarrow{Y \mapsto Y \otimes_{C^* A}^h C^* A} \text{OM}(C^* A)$$

is obviously a completely isometric equivalence.

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