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# A Common $q$ -Analogue of Two Supercongruences

Victor J. W. Guo and Wadim Zudilin

**Abstract.** We give a  $q$ -congruence whose specializations  $q = -1$  and  $q = 1$  correspond to supercongruences (B.2) and (H.2) on Van Hamme's list (in: *p*-Adic Functional Analysis (Nijmegen, 1996), Lecture Notes in Pure and Applied Mathematics, vol 192. Dekker, New York, pp 223–236, 1997):

$$\sum_{k=0}^{(p-1)/2} (-1)^k (4k+1) A_k \equiv p(-1)^{(p-1)/2} \pmod{p^3} \quad \text{and}$$
$$\sum_{k=0}^{(p-1)/2} A_k \equiv a(p) \pmod{p^2},$$

where  $p > 2$  is prime,

$$A_k = \prod_{j=0}^{k-1} \left( \frac{1/2 + j}{1 + j} \right)^3 = \frac{1}{2^{6k}} \binom{2k}{k}^3 \quad \text{for } k = 0, 1, 2, \dots,$$

and  $a(p)$  is the  $p$ th coefficient of the modular form  $q \prod_{j=1}^{\infty} (1 - q^{4j})^6$  (of weight 3). We complement our result with a general common  $q$ -congruence for related hypergeometric sums.

**Mathematics Subject Classification.** 33D15, 11A07, 11B65.

**Keywords.** Basic hypergeometric series,  $q$ -Dixon sum,  $q$ -congruence, supercongruence, creative microscoping.

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Victor J. W. Guo was supported by the National Natural Science Foundation of China (grant 11771175). Wadim Zudilin was supported by JSPS Invitational Fellowships for Research in Japan (fellowship S19126).

### 1. Introduction

The formula of Bauer [1] from 1859,

$$\sum_{k=0}^{\infty} (-1)^k (4k + 1) A_k = \frac{2}{\pi}, \quad \text{where } A_k = \frac{1}{2^{6k}} \binom{2k}{k}^3 \quad \text{for } k = 0, 1, 2, \dots, \tag{1.1}$$

is one of traditional targets for different methods of proofs of hypergeometric identities. Its special status is probably linked to the fact that it belongs to a family of series for  $1/\pi$  of Ramanujan type, after Ramanujan [21] brought to life in 1914 a long list of similar looking equalities for the constant but with a faster convergence. Identity (1.1) is a particular instance of  ${}_4F_3$  hypergeometric summation (known to Ramanujan) but there are several proofs of it, including the original one [1] of Bauer, that do not require any knowledge of hypergeometric functions. One notable—computer—proof of (1.1) was given in 1994 by Ekhad and Zeilberger [2] using the Wilf–Zeilberger (WZ) method of creative telescoping.

It was observed in 1997 by Van Hamme [28] that many Ramanujan’s and Ramanujan-like evaluations have nice  $p$ -adic analogues; for example, the congruence

$$\sum_{k=0}^{(p-1)/2} (-1)^k (4k + 1) A_k \equiv p(-1)^{(p-1)/2} \pmod{p^3} \tag{1.2}$$

(tagged (B.2) on Van Hamme’s list) is valid for any prime  $p > 2$  and corresponds to the equality (1.1). The congruence (1.2) was first proved by Mortenson [19] using a  ${}_6F_5$  hypergeometric transformation; it later received another proof by one of these authors [29] via the WZ method [in fact, using the very same ‘WZ certificate’ as in [2] for (1.1)]. Notice that (1.2) is an example of *supercongruence* meaning that it holds modulo a power of  $p$  greater than 1.

Another entry on Van Hamme’s 1997 list [28], tagged (H.2), is the congruence

$$\sum_{k=0}^{(p-1)/2} A_k \equiv \begin{cases} -\Gamma_p(1/4)^4 \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \end{cases} \tag{1.3}$$

again for any  $p > 2$  prime, and  $\Gamma_p(x)$  is the  $p$ -adic Gamma function. Van Hamme not only observed but also proved (1.3) in [28], and it was later generalized by Sun [23, 24, Theorem 2.5], Guo and Zeng [12, Corollary 1.2], Long and Ramakrishna [17], Liu [15, 16, Theorem 1.5] in different ways. For example, Long and Ramakrishna [17, Theorem 3] gave the following generalization of (1.3):

$$\sum_{k=0}^{(p-1)/2} A_k \equiv \begin{cases} -\Gamma_p(1/4)^4 \pmod{p^3} & \text{if } p \equiv 1 \pmod{4}, \\ -\frac{p^2}{16} \Gamma_p(1/4)^4 \pmod{p^3} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \tag{1.4}$$

Recently, these authors [14, Theorem 2] proved that, for any positive odd integer  $n$ , modulo  $\Phi_n(q)^2$ ,

$$\sum_{k=0}^{(n-1)/2} \frac{(q; q^2)_k^2 (q^2; q^4)_k}{(q^2; q^2)_k^2 (q^4; q^4)_k} q^{2k} \equiv \begin{cases} (q^2; q^4)_{(n-1)/4}^2 q^{(n-1)/2} & \text{if } n \equiv 1 \pmod{4}, \\ (q^4; q^4)_{(n-1)/4}^2 & \text{if } n \equiv 3 \pmod{4}. \\ 0 & \end{cases} \tag{1.5}$$

Here and in what follows,  $\Phi_n(q)$  denotes the  $n$ th *cyclotomic polynomial*; the  $q$ -shifted factorial is given by  $(a; q)_0 = 1$  and  $(a; q)_n = (1 - a)(1 - aq) \dots (1 - aq^{n-1})$  for  $n \geq 1$  or  $n = \infty$ , while  $[n] = [n]_q = 1 + q + \dots + q^{n-1}$  stands for the  $q$ -integer. Van Hamme [27, Theorem 3] also proved that

$$\binom{-1/2}{(p-1)/4} \equiv -\frac{\Gamma_p(1/4)^2}{\Gamma_p(1/2)} \pmod{p^2};$$

in view of  $\Gamma_p(1/2)^2 = -1$  for  $p \equiv 1 \pmod{4}$ , by letting  $q \rightarrow 1$  in (1.5) for  $n = p$  we immediately obtain (1.3).

One feature of (1.3) (not highlighted in [28]) is its connection with the coefficients

$$a(p) = \begin{cases} 2(a^2 - b^2) & \text{if } p = a^2 + b^2, a \text{ odd}, \\ 0 & \text{if } p \equiv 3 \pmod{4}, \end{cases} \tag{1.6}$$

of CM modular form  $q \prod_{j=1}^{\infty} (1 - q^{4j})^6$  of weight 3, namely, the congruence

$$a(p) \equiv -\Gamma_p(1/4)^4 \pmod{p^2} \quad \text{for primes } p \equiv 1 \pmod{4}.$$

This served as a main motivation in [14] for not only establishing (1.5) but also speculating on possible  $q$ -deformation of modular forms.

For some other recent progress on  $q$ -analogues of supercongruences, the reader is referred to [4, 5, 7–11, 13, 20, 22, 26, 29]. In particular, the authors [13] introduced and executed a new method of creative microscoping to prove (and reprove) many  $q$ -analogues of classical supercongruences and also raised some problems on  $q$ -congruences. Using this method, the first author [6] gave a refinement of (1.5) modulo  $\Phi_n(q)^3$  for  $n \equiv 3 \pmod{4}$ , in other words, a  $q$ -analogue of (1.4) for  $p \equiv 3 \pmod{4}$ .

A goal of this note is to present the following new  $q$ -analogue of Van Hamme’s supercongruence (1.3).

**Theorem 1.1.** *Let  $n$  be a positive odd integer. Then*

$$\begin{aligned} & \sum_{k=0}^{(n-1)/2} \frac{(1 + q^{4k+1})(q^2; q^4)_k^3}{(1 + q)(q^4; q^4)_k^3} q^k \\ & \equiv \frac{[n]_{q^2}(q^3; q^4)_{(n-1)/2}}{(q^5; q^4)_{(n-1)/2}} q^{(1-n)/2} \begin{cases} \pmod{\Phi_n(q)^2 \Phi_n(-q)^3} & \text{if } n \equiv 1 \pmod{4}, \\ \pmod{\Phi_n(q)^3 \Phi_n(-q)^3} & \text{if } n \equiv 3 \pmod{4}. \end{cases} \end{aligned} \tag{1.7}$$

Note that  $\Phi_n(q)\Phi_n(-q) = \Phi_n(q^2)$  for odd indices  $n$ .

The  $n \equiv 3 \pmod{4}$  case of Theorem 1.1 confirms a conjecture of these authors [13, Conjecture 4.13], which states that, for  $n \equiv 3 \pmod{4}$ ,

$$\sum_{k=0}^{(n-1)/2} \frac{(1 + q^{4k+1})(q^2; q^4)_k^3}{(1 + q)(q^4; q^4)_k^3} q^k \equiv 0 \pmod{\Phi_n(q)^2 \Phi_n(-q)}.$$

It is not difficult to verify that

$$\frac{(3/4)_{(p-1)/2}}{(5/4)_{(p-1)/2}} \equiv -\frac{p}{16} \Gamma_p(1/4)^4 \pmod{p^2}$$

for  $p \equiv 3 \pmod{4}$ , where  $(a)_n = a(a + 1) \dots (a + n - 1)$  denotes the rising factorial (also known as Pochhammer’s symbol). Therefore, the  $q$ -congruence (1.7) reduces to (1.4) for  $p \equiv 3 \pmod{4}$  when  $n = p$  and  $q \rightarrow 1$ , and it reduces to (1.3) for  $p \equiv 1 \pmod{4}$  when  $n = p$  and  $q \rightarrow 1$ . Moreover, letting  $n = p$  and  $q \rightarrow -1$  in (1.7), we immediately get (1.2). Thus, Theorem 1.1 presents a common  $q$ -analogue of supercongruences (1.2) and (1.3). We point out that other different  $q$ -analogues of (1.2) have been given in [7, 8].

Recently, Mao and Pan [18] (see also Sun [25, Theorem 1.3]) proved that, if  $p \equiv 1 \pmod{4}$  is a prime, then

$$\sum_{k=0}^{(p+1)/2} \frac{(-1/2)_k^3}{k!^3} \equiv 0 \pmod{p^2}. \tag{1.8}$$

In this note, we prove the following  $q$ -analogue of (1.8).

**Theorem 1.2.** *Let  $n > 1$  be an odd integer. Then*

$$\begin{aligned} & \sum_{k=0}^{(n+1)/2} \frac{(1 + q^{4k-1})(q^{-2}; q^4)_k^3}{(1 + q)(q^4; q^4)_k^3} q^{7k} \\ & \equiv \frac{[n]_{q^2}(q; q^4)_{(n-1)/2}}{(q^7; q^4)_{(n-1)/2}} q^{(n-3)/2} \begin{cases} (\text{mod } \Phi_n(q)^3 \Phi_n(-q)^3) & \text{if } n \equiv 1 \pmod{4}, \\ (\text{mod } \Phi_n(q)^2 \Phi_n(-q)^3) & \text{if } n \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

The  $n \equiv 1 \pmod{4}$  case of Theorem 1.2 also confirms a conjecture of the first author and Schlosser [11, Conjecture 10.2].

For  $n$  prime, letting  $q \rightarrow 1$  in Theorem 1.2 we obtain the following generalization of (1.8).

**Corollary 1.3.** *Let  $p$  be an odd prime. Then*

$$\sum_{k=0}^{(p+1)/2} \frac{(-1/2)_k^3}{k!^3} \equiv p \frac{(1/4)_{(p-1)/2}}{(7/4)_{(p-1)/2}} \begin{cases} (\text{mod } p^3) & \text{if } p \equiv 1 \pmod{4}, \\ (\text{mod } p^2) & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

On the other hand, for  $n$  prime and  $q \rightarrow -1$  in Theorem 1.2, we are led to the following result:

$$\sum_{k=0}^{(p+1)/2} (-1)^k (4k - 1) \frac{(-1/2)_k^3}{k!^3} \equiv p(-1)^{(p+1)/2} \pmod{p^3}. \tag{1.9}$$

It should be mentioned that a different  $q$ -analogue of (1.9) was given in [13, Theorem 4.9] with  $r = -1$ ,  $d = 2$  and  $a = 1$  (see also [11, Section 5]).

Moreover, for the summation formula

$$\sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k^3}{k!^3} = 12 \frac{\Gamma(3/4)^4}{\pi^3},$$

we have the following  $q$ -analogue.

**Theorem 1.4.** *We have*

$$\sum_{k=0}^{\infty} \frac{(1 + q^{4k-1})(q^{-2}; q^4)_k^3}{(1 + q^{-1})(q^4; q^4)_k^3} q^{7k} = \frac{(q^2; q^4)_{\infty}(q^5; q^4)_{\infty}^2(q^6; q^4)_{\infty}}{(q^3; q^4)_{\infty}(q^4; q^4)_{\infty}^2(q^7; q^4)_{\infty}}.$$

Both Theorems 1.1 and 1.2 are particular cases of a more general result, which we state and prove in the next section, while Theorem 1.4 follows from a classical  $q$ -identity.

## 2. A Family of $q$ -Congruences from the $q$ -Dixon Sum

In this section we establish the following family of one-parameter  $q$ -congruences.

**Theorem 2.1.** *Let  $n \geq 1$  be an odd integer and  $\ell$  an integer with  $0 \leq \ell \leq (n - 1)/2$ . Then*

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{(1 + q^{4k-2\ell+1})(q^{2-4\ell}; q^4)_k^3}{(1 + q^{1-2\ell})(q^4; q^4)_k^3} q^{(6\ell+1)k} \\ & \equiv \frac{(1 - q^{2n})(q^{3-6\ell}; q^4)_{(n-1)/2+\ell}}{(1 - q^{2-4\ell})(q^{5-2\ell}; q^4)_{(n-1)/2+\ell}} q^{(2\ell-1)((n-1)/2+\ell)} \begin{cases} (\text{mod } \Phi_n(q)^2 \Phi_n(-q)^3) \\ \text{if } n + 2\ell \equiv 1 \pmod{4}, \\ (\text{mod } \Phi_n(q)^3 \Phi_n(-q)^3) \\ \text{if } n + 2\ell \equiv 3 \pmod{4}. \end{cases} \end{aligned} \tag{2.1}$$

Note that the  $q$ -congruence (2.1) remains true when the sum is over  $k$  from 0 to  $(n - 1)/2 + \ell$ , since  $(q^{2-4\ell}; q^4)_k / (q^4; q^4)_k \equiv 0 \pmod{\Phi_n(q^2)}$  for  $(n - 1)/2 + \ell < k \leq n - 1$ . Furthermore, when  $\ell = 0$  and  $\ell = 1$  (hence  $n \geq 3$ ) the theorem reduces to Theorems 1.1 and 1.2, respectively.

The following easily proved  $q$ -congruence (see [11, Lemma 3.1]) is necessary in our derivation of Theorem 2.1.

**Lemma 2.2.** *Let  $n$  be a positive odd integer. Then, for  $0 \leq k \leq (n - 1)/2$ , we have*

$$\frac{(aq; q^2)_{(n-1)/2-k}}{(q^2/a; q^2)_{(n-1)/2-k}} \equiv (-a)^{(n-1)/2-2k} \frac{(aq; q^2)_k}{(q^2/a; q^2)_k} q^{(n-1)^2/4+k} \pmod{\Phi_n(q)}.$$

Like the proofs given in [13], we start with the following generalization of (1.7) with an extra parameter  $a$ .

**Theorem 2.3.** *Let  $n > 1$  be an odd integer and  $0 \leq \ell \leq (n - 1)/2$ . Then*

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{(1 + q^{4k-2\ell+1})(aq^{2-4\ell}; q^4)_k (q^{2-4\ell}/a; q^4)_k (q^{2-4\ell}; q^4)_k}{(1 + q^{1-2\ell})(aq^4; q^4)_k (q^4/a; q^4)_k (q^4; q^4)_k} q^{(6\ell+1)k} \\ & \equiv \frac{(1 - q^{2n})(q^{3-6\ell}; q^4)_{(n-1)/2+\ell}}{(1 - q^{2-4\ell})(q^{5-2\ell}; q^4)_{(n-1)/2+\ell}} \\ & \quad \times q^{(2\ell-1)((n-1)/2+\ell)} \begin{cases} (\text{mod } \Phi_n(-q)(1 - aq^{2n})(a - q^{2n})) \\ \text{if } n + 2\ell \equiv 1 \pmod{4}, \\ (\text{mod } \Phi_n(q^2)(1 - aq^{2n})(a - q^{2n})) \\ \text{if } n + 2\ell \equiv 3 \pmod{4}. \end{cases} \end{aligned} \tag{2.2}$$

*Proof.* Performing the parameter substitutions  $q \mapsto q^4$ ,  $a \mapsto q^{2-4\ell}$ ,  $b \mapsto bq^{2-4\ell}$  and  $c \mapsto cq^{2-4\ell}$  in the  $q$ -Dixon sum [3, Appendix (II.13)], we obtain

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{(1 + q^{4k-2\ell+1})(q^{2-4\ell}; q^4)_k (bq^{2-4\ell}; q^4)_k (cq^{2-4\ell}; q^4)_k}{(1 + q^{1-2\ell})(q^4/b; q^4)_k (q^4/c; q^4)_k (q^4; q^4)_k} \left( \frac{q^{6\ell+1}}{bc} \right)^k \\ & = \frac{(q^{6-4\ell}; q^4)_{\infty} (q^{2\ell+3}/b; q^4)_{\infty} (q^{2\ell+3}/c; q^4)_{\infty} (q^{4\ell+2}/bc; q^4)_{\infty}}{(q^4/b; q^4)_{\infty} (q^4/c; q^4)_{\infty} (q^{5-2\ell}; q^4)_{\infty} (q^{6\ell+1}/bc; q^4)_{\infty}}. \end{aligned} \tag{2.3}$$

Since  $n$  is odd, putting  $b = q^{-2n}$  and  $c = q^{2n}$  in (2.3) we see that the left-hand side terminates and is equal to

$$\begin{aligned} & \sum_{k=0}^{(n-1)/2+\ell} \frac{(1 + q^{4k-2\ell+1})(q^{2-4\ell-2n}; q^4)_k (q^{2-4\ell+2n}; q^4)_k (q^{2-4\ell}; q^4)_k}{(1 + q^{1-2\ell})(q^{4-2n}; q^4)_k (q^{4+2n}; q^4)_k (q^4; q^4)_k} q^{(6\ell+1)k} \\ & = \sum_{k=0}^{n-1} \frac{(1 + q^{4k-2\ell+1})(q^{2-4\ell-2n}; q^4)_k (q^{2-4\ell+2n}; q^4)_k (q^{2-4\ell}; q^4)_k}{(1 + q^{1-2\ell})(q^{4-2n}; q^4)_k (q^{4+2n}; q^4)_k (q^4; q^4)_k} q^{(6\ell+1)k}, \end{aligned}$$

while the right-hand side becomes

$$\begin{aligned} & \frac{(q^{2\ell-2n+3}; q^4)_{(n-1)/2+\ell} (q^{6-4\ell}; q^4)_{(n-1)/2+\ell}}{(q^{4-2n}; q^4)_{(n-1)/2+\ell} (q^{5-2\ell}; q^4)_{(n-1)/2+\ell}} \\ & = \frac{(1 - q^{2n})(q^{3-6\ell}; q^4)_{(n-1)/2+\ell}}{(1 - q^{2-4\ell})(q^{5-2\ell}; q^4)_{(n-1)/2+\ell}} q^{(2\ell-1)((n-1)/2+\ell)}. \end{aligned}$$

This proves that the  $q$ -congruence (2.2) holds modulo  $1 - aq^{2n}$  or  $a - q^{2n}$ .

On the other hand, by Lemma 2.2, for  $0 \leq k \leq (n - 1)/2 + \ell$ , modulo  $\Phi_n(q)$  we have

$$\begin{aligned}
 & \frac{(aq^{1-2\ell}; q^2)_{(n-1)/2+\ell-k}}{(q^2/a; q^2)_{(n-1)/2+\ell-k}} \\
 &= \frac{(aq^{1-2\ell}; q^2)_\ell (aq; q^2)_{(n-1)/2-k}}{(q^{n+1-2k}/a; q^2)_\ell (q^2/a; q^2)_{(n-1)/2-k}} \\
 &\equiv (-a)^{(n-1)/2-2k} \frac{(aq^{1-2\ell}; q^2)_\ell (aq; q^2)_k}{(q^{n+1-2k}/a; q^2)_\ell (q^2/a; q^2)_k} q^{(n-1)^2/4+k} \\
 &= (-a)^{(n-1)/2-2k} \frac{(aq^{1-2\ell}; q^2)_k (aq^{2k-2\ell+1}; q^2)_\ell}{(q^{n+1-2k}/a; q^2)_\ell (q^2/a; q^2)_k} q^{(n-1)^2/4+k} \\
 &\equiv (-a)^{(n-1)/2+\ell-2k} \frac{(aq^{1-2\ell}; q^2)_k}{(q^2/a; q^2)_k} q^{(n-1)^2/4+k+(2k-\ell)\ell},
 \end{aligned}$$

where we used  $q^n \equiv 1 \pmod{\Phi_n(q)}$  in the last step. Using the above  $q$ -congruence we can easily check that, for odd  $n > 1$  and  $0 \leq k \leq (n-1)/2 + \ell$ , sum of the  $k$ th and  $((n-1)/2 + \ell - k)$ th summands on the left-hand side of (2.2) is congruent to 0 modulo  $\Phi_n(-q)$  (or modulo  $\Phi_n(q^2)$  if  $n \equiv 3 - 2\ell \pmod{4}$ ). It follows that

$$\begin{aligned}
 & \sum_{k=0}^{(n-1)/2+\ell} \frac{(1 + q^{4k-2\ell+1}) (aq^{2-4\ell}; q^4)_k (q^{2-4\ell}/a; q^4)_k (q^{2-4\ell}; q^4)_k}{(1 + q^{1-2\ell}) (aq^4; q^4)_k (q^4/a; q^4)_k (q^4; q^4)_k} q^{(6\ell+1)k} \\
 &\equiv 0 \begin{cases} \pmod{\Phi_n(-q)} & \text{if } n + 2\ell \equiv 1 \pmod{4}, \\ \pmod{\Phi_n(q^2)} & \text{if } n + 2\ell \equiv 3 \pmod{4}. \end{cases}
 \end{aligned}$$

Clearly, the right-hand side of (2.1) is congruent to 0 modulo  $\Phi_n(-q)$  if  $n + 2\ell \equiv 1 \pmod{4}$  and modulo  $\Phi_n(q^2)$  if  $n + 2\ell \equiv 3 \pmod{4}$ . Therefore, the  $q$ -congruence (2.2) holds modulo  $\Phi_n(-q)$  if  $n + 2\ell \equiv 1 \pmod{4}$  and modulo  $\Phi_n(q^2)$  if  $n + 2\ell \equiv 3 \pmod{4}$ . Since the polynomials  $1 - aq^{2n}$ ,  $a - q^{2n}$  and  $\Phi_n(-q)$  (or  $\Phi_n(q^2)$ ) are pairwise coprime, we complete the proof of (2.2).  $\square$

*Proof of Theorem 2.1.* We assume that  $n > 1$ , since the  $n = 1$  case (making  $\ell = 0$  only possible) is trivial. The limits of the denominators on both sides of (2.2) as  $a \rightarrow 1$  are relatively prime to  $\Phi_n(q^2)$ , since  $k$  is in the range  $0 \leq k \leq (n-1)/2 + \ell$ . On the other hand, the limit of  $(1 - aq^{2n})(a - q^{2n})$  as  $a \rightarrow 1$  contains the factor  $\Phi_n(q^2)^2$ .  $\square$

*Proof of Theorem 1.4.* Take  $b = c = \ell = 1$  in Eq. (2.3).  $\square$

### 3. Discussion

The method of creative microscoping used in our proofs indicates the origin of  $q$ -congruences from infinite  $q$ -hypergeometric identities; for example, the  $q$ -congruence (1.7) corresponds to the identity

$$\sum_{k=0}^{\infty} \frac{(1 + q^{4k+1}) (q^2; q^4)_k^3}{(1 + q) (q^4; q^4)_k^3} q^k = \frac{(q^2; q^4)_{\infty}^2 (q^3; q^4)_{\infty}^2}{(1 + q) (q; q^4)_{\infty}^2 (q^4; q^4)_{\infty}^2}, \tag{3.1}$$



which is just a particular instance of (2.3). Note that the limiting cases as  $q \rightarrow -1$  and  $q \rightarrow 1$  of (3.1) give the formulas (1.1) and

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{k!^3} = \frac{\Gamma(1/4)^4}{4\pi^3} = \frac{8L(f, 1)}{\pi} = \frac{16L(f, 2)}{\pi^2} \tag{3.2}$$

where

$$f(\tau) = q \prod_{j=1}^{\infty} (1 - q^{4j})^6 = \sum_{n=1}^{\infty} a(n)q^n, \quad \text{with } q = \exp(2\pi i\tau),$$

is the CM modular form from the introduction and  $L(f, s)$  denotes its  $L$ -function. This means that the  $q$ -identity (3.1) presents a common  $q$ -extension of evaluations (1.1) and (3.2)—the fact that makes it less surprising that the  $q$ -congruence (1.7) simultaneously extends (1.2) and (1.3).

The intermediate use of *parametric*  $q$ -hypergeometric identities in our proof of Theorem 2.1 based on the  $q$ -Dixon sum suggests that different  $q$ -congruences underlying (3.1) are possible. This is indeed the case when we analyze the formula (3.1) as the  $a = 1$  specialization of

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{(1 + q^{4k+1})(aq; q^2)_k (q/a; q^2)_k (-q; q^2)_k^2 (q^2; q^4)_k}{(1 + q)(q^2; q^2)_k^2 (-aq^2; q^2)_k (-q^2/a; q^2)_k (q^4; q^4)_k} q^k \\ &= \frac{(-q; q^2)_{\infty}^2 (aq^3; q^4)_{\infty}^2 (q^3/a; q^4)_{\infty}^2}{(1 + q)(-aq^2; q^2)_{\infty} (-q^2/a; q^2)_{\infty} (q^2; q^2)_{\infty}^2} \end{aligned} \tag{3.3}$$

which originates from a  $q$ -analogue of Watson’s  ${}_3F_2$  sum [3, Appendix (II.16)]. When we choose  $a = q^n$  (or  $a = q^{-n}$ ) in (3.3), for  $n > 1$  odd, we get the sum terminating after  $(n - 1)/2$  terms on the left-hand side of (3.3), while the right-hand side vanishes if  $n$  is of the form  $4m + 3$  and it becomes equal to

$$\frac{(-q; q^2)_{\infty}^2 (q^{4m+4}; q^4)_{\infty}^2 (q^{2-4m}; q^4)_{\infty}^2}{(1 + q)(-q^{4m+3}; q^2)_{\infty} (-q^{1-4m}; q^2)_{\infty} (q^2, q^4; q^4)_{\infty}^2} = [4m + 1] \frac{(q^2; q^4)_m^2}{(q^4; q^4)_m^2}$$

if  $n = 4m + 1$ . This means that modulo  $(a - q^n)(1 - aq^n)$  we have

$$\begin{aligned} & \sum_{k=0}^N \frac{(1 + q^{4k+1})(aq; q^2)_k (q/a; q^2)_k (-q; q^2)_k^2 (q^2; q^4)_k}{(1 + q)(q^2; q^2)_k^2 (-aq^2; q^2)_k (-q^2/a; q^2)_k (q^4; q^4)_k} q^k \\ & \equiv \begin{cases} [4m + 1] \frac{(q^2; q^4)_m^2}{(q^4; q^4)_m^2} & \text{if } n = 4m + 1, \\ 0 & \text{if } n \equiv 3 \pmod{4}, \end{cases} \end{aligned}$$

for any  $N \geq (n - 1)/2$ . The limiting  $a \rightarrow 1$  case of the congruences can be shown to be

$$\sum_{k=0}^{(n-1)/2} \frac{(1 + q^{4k+1})(q^2; q^4)_k^3}{(1 + q)(q^4; q^4)_k^3} q^k \equiv \begin{cases} [4m + 1] \frac{(q^2; q^4)_m^2}{(q^4; q^4)_m^2} & \text{if } n = 4m + 1, \\ 0 & \text{if } n \equiv 3 \pmod{4}, \end{cases} \tag{3.4}$$

modulo  $\Phi_n(q)^2\Phi_n(-q)$ . This is quite similar in spirit to (1.5), though still far from constructing  $q$ -analogues for the coefficients  $a(p)$  in (1.6) of the modular form  $f(\tau)$ . The latter means that a hunt for  $q$ -rational functions, which equal the left-hand side of (1.5) or (3.4) modulo  $\Phi_n(q)^2$  and specialize to  $a(n)$  as  $q \rightarrow 1$  (at least for  $n$  prime), is still on its way. Such  $q$ -rational functions are also expected to be self-reciprocal, that is, invariant under the involution  $q \mapsto 1/q$ , as all the left- and right-hand sides in (1.5), (1.7), (3.4) and also (2.1) are.

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Received: October 30, 2019.

Accepted: February 3, 2020.

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.