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# Quantum spin systems versus Schrödinger operators: A case study in spontaneous symmetry breaking

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## Abstract

Spontaneous symmetry breaking (SSB) is mathematically tied to some limit, but must physically occur, approximately, *before* the limit. Approximate SSB has been independently understood for Schrödinger operators with double well potential in the *classical limit* [1, 2] and for quantum spin systems in the *thermodynamic limit* [3, 4]. We relate these to each other in the context of the Curie–Weiss model, establishing a remarkable relationship between this model (for finite  $N$ ) and a discretized Schrödinger operator with double well potential.



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## 19 1 Introduction

20 At first sight, spontaneous symmetry breaking (SSB) is a paradoxical phenomenon: in *Nature*,  
21 finite quantum systems, such as crystals, evidently display it, yet in *Theory* it seems forbidden  
22 in such systems. Indeed, for finite quantum systems the ground state of a generic Hamiltonian  
23 is unique and hence invariant under whatever symmetry group  $G$  it may have.<sup>1</sup> Hence SSB,  
24 in the sense of having a family of asymmetric ground states related by the action of  $G$ ,  
25 seems possible only in infinite quantum systems or in classical systems (for both of which  
26 the arguments proving uniqueness, typically based on the Perron–Frobenius Theorem break  
27 down, cf. Appendix A). However, both are idealizations, vulnerable to what we call *Earman’s*  
28 *Principle* from the philosophy of physics:

29 “While idealizations are useful and, perhaps, even essential to progress in physics,  
30 a sound principle of interpretation would seem to be that no effect can be counted  
31 as a genuine physical effect if it disappears when the idealizations are removed.”  
32 [5]

33 As argued in detail in Ref. [6–8], the solution to his paradox lies in Earman’s very principle  
34 itself, which (contrapositively) implies what we call *Butterfield’s Principle*:

35 “there is a weaker, yet still vivid, novel and robust behaviour that occurs before we  
36 get to the limit, i.e. for finite  $N$ . And it is this weaker behaviour which is physically  
37 real.” [9]

38 Applied to SSB in infinite quantum systems, this means that some approximate and robust  
39 form of symmetry breaking should already occur in large but finite systems, *despite the fact*  
40 *that uniqueness of the ground state seems to forbid this*. Similarly, SSB in a classical system  
41 should be foreshadowed in the quantum system whose classical limit it is, at least for tiny but  
42 positive values of Planck’s constant  $\hbar$ . To accomplish this, it must be shown that for finite  $N$   
43 or  $\hbar > 0$  the system is not in its ground state, but in some other state having the property  
44 that as  $N \rightarrow \infty$  or  $\hbar \rightarrow 0$ , it converges in a suitable sense (detailed in Ref. [7], Chapter 8  
45 and Chapter 7, respectively) to a symmetry-broken ground state of the limit system, which  
46 is either an infinite quantum system or a classical system. Since the symmetry of a state is  
47 preserved under the limits in question (provided these are taken correctly), this implies that  
48 *the actual physical state at finite  $N$  or  $\hbar > 0$  must already break the symmetry*. The mechanism  
49 to accomplish this, originating with Anderson [3], is based on forming symmetry-breaking  
50 linear combinations of low-lying states (sometimes called “Anderson’s tower of states”) whose  
51 energy difference vanishes in the pertinent limit.<sup>2</sup> This mechanism has been independently

<sup>1</sup>Similarly for equilibrium states at positive temperature, which are always (not just generically) unique.

<sup>2</sup>It must be admitted that the description in Anderson [3] and even in his textbook [10] is very brief and purely qualitative, and that in Ref. [10], which he calls “my most complete summary of the theory of broken symmetry in condensed matter systems” the idea is not even mentioned.

52 verified in two different contexts, namely quantum spin systems in the thermodynamic limit,<sup>3</sup>  
 53 and 1d Schrödinger operators with a symmetric double well potential in the classical limit.<sup>4</sup>  
 54 In the absence of cross-references so far, one of our contributions will be to map the specific  
 55 perturbations that play a key role in SSB for Schrödinger operators at  $\hbar > 0$  onto quantum spin  
 56 systems.

57 To this end, we now briefly recall the main point of Jona-Lasinio [1], later called the “flea on  
 58 the elephant” [2] or, in applications to the measurement problem, the “flea on Schrödinger’s  
 59 Cat” [8]. Consider the Schrödinger operator with symmetric double well, defined on suitable  
 60 domain in  $\mathcal{H} = L^2(\mathbb{R})$  by

$$h_{\hbar} = -\hbar^2 \frac{d^2}{dx^2} + \frac{1}{4} \lambda (x^2 - a^2)^2, \quad (1.1)$$

61 where  $\lambda > 0$  and  $a \neq 0$ . For any  $\hbar > 0$  the ground state of this Hamiltonian is unique and  
 62 hence invariant under the  $\mathbb{Z}_2$ -symmetry  $\psi(x) \mapsto \psi(-x)$ ; with an appropriate phase choice it  
 63 is real, strictly positive, and doubly peaked above  $x = \pm a$ . Yet the classical Hamiltonian

$$h_0(p, q) = p^2 + \frac{1}{4} \lambda (q^2 - a^2)^2, \quad (1.2)$$

64 defined on the classical phase space  $\mathbb{R}^2$ , has a two-fold degenerate ground state: the point(s)  
 65  $(p_0, q_0)$  in  $\mathbb{R}^2$  where  $h_0$  takes an absolute minimum are  $(p_0 = 0, q_0 = \pm a)$ . In the algebraic  
 66 formalism, where states are defined as normalized positive linear functionals on the C\*-algebra  
 67  $A_0 = C_0(\mathbb{R}^2)$ , the (pure) ground states are the *asymmetric* Dirac measures

$$\omega_{\pm}(f) = f(p = 0, q = \pm a). \quad (1.3)$$

68 From these, one may construct the mixed *symmetric* state

$$\omega_0 = \frac{1}{2}(\omega_+ + \omega_-), \quad (1.4)$$

69 which in fact is the limit of the (C\*-algebraic) ground state  $\omega_{\hbar}$  of (1.1) as  $\hbar \rightarrow 0$ , where

$$\omega_{\hbar}(a) = \langle \psi_{\hbar}^{(0)}, a \psi_{\hbar}^{(0)} \rangle, \quad (1.5)$$

70 in terms of the usual ground state  $\psi_{\hbar}^{(0)} \in L^2(\mathbb{R})$  of  $h_{\hbar}$  (assumed to be a unit vector).<sup>5</sup> In order  
 71 to have a quantum “ground-ish” state that converges to either one of the physical classical  
 72 ground states  $\omega_+$  or  $\omega_-$  rather than to the unphysical mixture  $\omega_0$ , we perturb (1.1) by  
 73 adding an asymmetric term  $\delta V$  (i.e., the “flea”), which, however small it is, under reasonable  
 74 assumptions localizes the ground state  $\psi_{\hbar}^{(\delta)}$  of the perturbed Hamiltonian in such a way that  
 75  $\omega_{\hbar}^{(\delta)} \rightarrow \omega_+$  or  $\omega_-$ , depending on the sign and location of  $\delta V$ .<sup>6</sup> In particular, the localization  
 76 of  $\psi_{\hbar}^{(\delta)}$  grows exponentially as  $\hbar \rightarrow 0$  (see §4 for details). In this paper, we adapt this scenario  
 77 to the *Curie–Weiss model* on a finite lattice  $\Lambda \subset \mathbb{Z}^d$ , with Hamiltonian

$$h_{\Lambda}^{\text{CW}} = -\frac{1}{2} |\Lambda|^{-1} \sum_{x, y \in \Lambda} \sigma_3(x) \sigma_3(y) - B \sum_{x \in \Lambda} \sigma_1(x). \quad (1.6)$$

<sup>3</sup>See e.g. the reviews [11–14], as well as the original papers [15–25], and, rigorously, [4].

<sup>4</sup>Three founding papers are [1, 2, 26]. Since in the context of Schrödinger operators the classical limit “ $\hbar \rightarrow 0$ ” typically means that  $m \rightarrow \infty$  at fixed  $\hbar$  (where  $m$  is the mass occurring in  $\hbar^2/2m$ ), one may physically see  $\hbar \rightarrow 0$  as a special case of  $N \rightarrow \infty$ .

<sup>5</sup>[7], §7.1. Here  $\omega_{\hbar}$  is defined as a normalized positive linear functional on the C\*-algebra  $A_{\hbar} = B_0(L^2(\mathbb{R}))$  of compact operators on  $L^2(\mathbb{R})$ . The algebraic formalism is particularly useful for combining classical and quantum expressions.

<sup>6</sup>For example, if  $\delta V$  is positive and is localized to the right, then the relative energy in the left-hand part of the double well is lowered, so that localization will be to the left. See §4 for details.

78 Here we take the spin-spin coupling to be  $J = 1$ , and  $B$  is an external magnetic field.<sup>7</sup>  
 79 This Hamiltonian has a  $\mathbb{Z}_2$ -symmetry  $(\sigma_1, \sigma_2, \sigma_3) \mapsto (\sigma_1, -\sigma_2, -\sigma_3)$ , which at each site  $x$  is  
 80 implemented by  $u(x) = \sigma_1(x)$ . The ground state of this model is unique for any  $|\Lambda| < \infty$   
 81 and any  $B \neq 0$ , and yet, as for the double well potential, in the thermodynamic limit it has  
 82 two degenerate ground states, provided  $0 < B < 1$ . As explained in Ref. [7, 29–31], perhaps  
 83 unexpectedly this limit actually defines a *classical* theory, with phase space  $B^3 \subset \mathbb{R}^3$ , i.e. the  
 84 three-sphere with unit radius (and boundary  $\partial B^3 = S^2$ ), seen as a Poisson manifold with  
 85 bracket  $\{x, y\} = z$  etc.), and Hamiltonian

$$h_0^{\text{CW}}(x, y, z) = -\frac{1}{2}z^2 - Bx. \tag{1.7}$$

86 The ground states of this Hamiltonian are simply its absolute minima, viz.  $(\vec{x} = (x, y, z))$ :

$$\vec{x}_{\pm} = (B, 0, \pm\sqrt{1-B^2}) \quad (0 \leq B < 1); \tag{1.8}$$

$$\vec{x} = (1, 0, 0) \quad (B \geq 1), \tag{1.9}$$

87 which lie on the boundary  $S^2$  of  $B^3$  (note that the points  $\vec{x}_{\pm}$  coalesce as  $B \rightarrow 1$ , where they  
 88 form a saddle point). Thus we seem to face a similar paradox as for the double well.  
 89 To address this, in §2 we first show that due to permutation invariance and strict positivity  
 90 of the ground state of the 1d Curie–Weiss Hamiltonian (for  $N < \infty$  and  $|\Lambda| = N$ ), which is  
 91 initially defined on the  $2^N$ -dimensional Hilbert space  $\mathcal{H}_N = \bigotimes_{n=1}^N \mathbb{C}^2$ , the ground state of this  
 92 Hamiltonian must lie in the range  $\text{ran}(S_N)$  of the appropriate symmetriser

$$S_N(v) = \frac{1}{N!} \sum_{\sigma \in S_n} L_{\sigma}(v) \tag{1.10}$$

93 on  $\mathcal{H}_N$ ; here  $v$  is a vector in the  $N$ -fold tensor product and  $L_{\sigma}$  is given by permuting the factors  
 94 of  $v$ , i.e.  $v_1 \otimes \dots \otimes v_n \mapsto v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}$ . Its range is  $(N+1)$ -dimensional, and we show that the  
 95 quantum Curie–Weiss Hamiltonian restricted to  $\text{ran}(S_N)$  becomes a *tridiagonal*  $(N+1) \times (N+1)$   
 96 matrix. Even for large  $N$ , this matrix is easy to diagonalize numerically. Using this tridiagonal  
 97 structure, in §3 we show that as  $N \rightarrow \infty$ , our restricted Curie–Weiss Hamiltonian (rescaled  
 98 by a factor  $1/N$ ) increasingly well approximates a 1d Schrödinger operator with a symmetric  
 99 double well potential defined on the interval  $[0, 1]$ , in which  $\hbar = 1/N$ . In §4 we use these ideas  
 100 to find the counterpart of the “flea” perturbation (symmetry breaking field) from the double  
 101 well potential for the Curie–Weiss model, which, analogously to the double well, localizes the  
 102 ground state of the perturbed Hamiltonian (this time, of course, in spin configuration space  
 103 rather than real space).

104 Although our physical mechanism for SSB in finite systems is the same as the one studied  
 105 in the condensed matter physics literature (namely Anderson’s), the mathematical form of  
 106 the flea perturbation is a bit different from the usual symmetry-breaking terms for quantum  
 107 spin systems, which in the double well would correspond to breaking the symmetry by simply  
 108 deepening one of the bottoms or changing its curvature, whereas the “flea” is typically localized  
 109 away from the minima, cf. [26]. In the mathematical framework in which we work, our  
 110 approach also has the advantage of making the limits  $\hbar \rightarrow 0$  and  $N \rightarrow \infty$  quite regular (i.e.,  
 111 continuous, properly understood), as opposed to the alternative view of regarding them as  
 112 singular (e.g. [13, 27, 28]). This is further discussed in our Conclusion, which also states some

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<sup>7</sup> Note that putting  $J = 1$  makes  $h_{\Lambda}^{\text{CW}}$  dimensionless. This model falls into the class of homogeneous mean-field theories, see [29–31], which differ from their short-range counterparts (which in this case would be the quantum Ising model) in that every spin now interacts with every other spin. This also makes the dimension  $d$  irrelevant (which marks a huge difference with short-range quantum spin models), and yet even the apparently simple Curie–Weiss model is extremely rich in its behaviour; see e.g. [32] for a detailed analysis (along quite different lines from our study), motivated by the measurement problem.

113 open problems and suggestions for further research. This is followed by an appendix on the  
 114 Perron–Frobenius Theorem, which plays a central role in our work, and another appendix  
 115 introducing the discretization techniques we use to non-specialists.

## 116 2 Reduction of the Curie–Weiss Hamiltonian

117 Since the spatial dimension is irrelevant, we may as well consider the Curie–Weiss Hamiltonian  
 118 (1.6) in  $d = 1$ , so that we may simply write  $|\Lambda| = N$ , and, with  $h_N \equiv h_N^{\text{CW}}$ ,

$$h_N = -\frac{1}{2}N^{-1} \sum_{x,y=1}^N \sigma_3(x)\sigma_3(y) - B \sum_{x=1}^N \sigma_1(x). \quad (2.1)$$

119 It seems folklore that the Perron-Frobenius theorem yields uniqueness and strict positivity of  
 120 the ground state  $\psi_N^{(0)}$  of  $h_N$  for any  $N < \infty$  and  $B \neq 0$ , but for completeness we provide the  
 121 details in Appendix A. It follows that  $\psi_N^{(0)}$  is  $\mathbb{Z}_2$ -invariant (see the Introduction), so that on a  
 122 first analysis (to be corrected in what follows!) there is no SSB for any finite  $N$ .

### 123 2.1 Tridiagonal form

124 Let  $S_N$  be the standard symmetriser (1.10) on the Hilbert space  $\mathcal{H}_N = \otimes_{n=1}^N \mathbb{C}^2$  on  
 125 which  $h_N$  acts, so that  $S_N$  projects onto the subspace  $\text{ran}(S_N) = \text{Sym}^N(\mathbb{C}^2)$  of totally  
 126 symmetrised tensors. An orthonormal basis for  $\text{Sym}^N(\mathbb{C}^2)$  is given by the vectors  
 127  $\{|n_+, n_-\rangle \mid n_+ = 0, \dots, N, n_+ + n_- = N\}$ , where  $|n_+, n_-\rangle$  is the totally symmetrised unit vector  
 128 in  $\otimes_{n=1}^N \mathbb{C}^2$ , with  $n_+$  spins up and  $n_- = N - n_+$  spins down. It follows that this space is  
 129  $(N + 1)$ -dimensional. Since  $h_N$  commutes with all permutations of  $\{1, \dots, N\}$ , in view of its  
 130 uniqueness the ground state  $\psi_N^{(0)}$  of  $h_N$  must be invariant under the permutation group and  
 131 hence under  $S_N$ . Hence we may expand  $\psi_N^{(0)}$  according to

$$\psi_N^{(0)} = \sum_{n_+=0}^N c(n_+/N) |n_+, n_-\rangle, \quad (2.2)$$

132 where we conveniently introduce a function  $c : \{0, 1/N, 2/N, \dots, (N - 1)/N, 1\} \rightarrow [0, 1]$  that  
 133 satisfies  $c(n_+/N) = c(n_-/N)$  as well as  $\sum_{n_+=0}^N c^2(n_+/N) = 1$ .

134 **Theorem 2.1.** *In the basis  $\{|n_+\rangle\} \equiv \{|n_+, N - n_+\rangle\}$ , the operator (2.1) is an  $(N + 1) \times (N + 1)$   
 135 tridiagonal matrix:<sup>8</sup>*

$$-\frac{1}{2N}(2n_+ - N)^2 \text{ on the diagonal}; \quad (2.3)$$

$$-B\sqrt{(N - n_+)(n_+ + 1)} \text{ on the upper diagonal}; \quad (2.4)$$

$$-B\sqrt{(N - n_+ + 1)n_+} \text{ on the lower diagonal}. \quad (2.5)$$

136 *Proof.* Given two arbitrary vectors  $|n_+\rangle$  and  $|n'_+\rangle$ , we have to compute the expression

$$\langle n_+ | h_N^{\text{CW}} | n'_+ \rangle, \quad (n_+, n'_+ = 0, \dots, N); \quad (2.6)$$

<sup>8</sup>The following relations can also be derived from those in Appendix 2 of Ref. [33].

137 where we we have used the bra-ket notation in the above expression. By linearity, we may  
 138 separately compute this for the operators

$$\begin{aligned}
 h_N^{(1)} &= \sum_{x,y=1}^N \sigma_3(x)\sigma_3(y) = \sum_{x=1}^N \sigma_3(x) \cdot \sum_{y=1}^N \sigma_3(y), \\
 h_N^{(2)} &= \sum_{x=1}^N \sigma_1(x).
 \end{aligned}
 \tag{2.7}$$

139 In order to compute (2.6), we need to know how  $\sigma_3$  and  $\sigma_1$  act on the vectors  $|n_+\rangle$ . Consider  
 140 the standard basis  $\{e_1, e_2\}$  for  $\mathbb{C}^2$  over  $\mathbb{C}$ . Then  $\{e_{n_1} \otimes \dots \otimes e_{n_N}\}_{n_1=1, \dots, n_N=1}^2$  is the standard basis  
 141 for  $\otimes_{n=1}^N \mathbb{C}^2$ . Note that  $\sigma_3(x) = 1 \otimes \dots \otimes 1 \otimes \sigma_3 \otimes 1 \dots \otimes 1$ , where  $\sigma_3$  acts on the  $x^{th}$  position  
 142 and similarly for  $\sigma_1(x)$ . It follows that for all  $x, y \in \{1, \dots, N\}$ ,

$$\begin{aligned}
 &\sigma_3(x)(e_{n_1} \otimes \dots \otimes e_{n_N}) = \\
 &1(e_{n_1}) \otimes \dots \otimes 1(e_{n_{x-1}}) \otimes \sigma_3(e_{n_x}) \otimes 1(e_{n_{x+1}}) \otimes \dots \otimes 1(e_{n_N}) = \\
 &\begin{cases} +(e_{n_1} \otimes \dots \otimes e_{n_N}), & \text{if } e_{n_x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ -(e_{n_1} \otimes \dots \otimes e_{n_N}), & \text{if } e_{n_x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{cases}
 \end{aligned}
 \tag{2.8}$$

143 We have  $\sigma_3(y)\sigma_3(x)(e_{n_1} \otimes \dots \otimes e_{n_N}) = \pm(e_{n_1} \otimes \dots \otimes e_{n_N})$ , where the minus sign appears only  
 144 if  $e_{n_x} \neq e_{n_y}$ . We conclude that the standard basis for the  $N$ -fold tensor product is a set of  
 145 eigenvectors for  $\sigma_3(y)\sigma_3(x)$  with eigenvalues  $\pm 1$ . Thus we know that  $\sum_{x,y} \sigma_3(x)\sigma_3(y)$  is a  
 146 diagonal matrix with respect to this standard basis. Note that  $|n_+\rangle$  is a (normalized) sum of  
 147 permutations of such basis vectors, with  $n_+$  times the vector  $e_1$  and  $N - n_+$  times the vector  
 148  $e_2$ . Since  $\sum_{x,y} \sigma_3(y)\sigma_3(x)$  acts diagonally on any of these vectors, and it is also permutation  
 149 invariant, it follows that in the inner product any vector with itself yields the same contribution,  
 150 namely

$$\langle e_1 \otimes \dots \otimes e_1 \otimes e_2 \otimes \dots \otimes e_2 | \sum_{x,y} \sigma_3(y)\sigma_3(x) | e_1 \otimes \dots \otimes e_1 \otimes e_2 \otimes \dots \otimes e_2 \rangle,
 \tag{2.9}$$

where  $e_1$  occurs  $n_+$  times and  $e_2$  occurs  $N - n_+$  times. The above expression equals

$$(n_+ - (N - n_+))^2 = (2n_+ - N)^2,$$

151 since there are  $2n_+(N - n_+)$  minus signs and hence  $N^2 - 2n_+(N - n_+) = n_+^2 + (N - n_+)^2$  plus  
 152 signs, so that in total the correct value is indeed given by

$$n_+^2 + (N - n_+)^2 - 2n_+(N - n_+) = (2n_+ - N)^2.
 \tag{2.10}$$

153 This shows that the contribution to the diagonal is given by (2.3).  
 154

155 In order to compute the off-diagonal contribution (2.6) with (2.7), we use an explicit  
 156 expression for the symmetric basis vector  $|n_+\rangle$ . Using (1.10), it is easy to show that

$$|n_+\rangle = \frac{1}{\sqrt{\binom{N}{n_+}}} \sum_{l=1}^{\binom{N}{n_+}} \beta_{n_+,l},
 \tag{2.11}$$

157 where the subindex  $l$  in  $\beta_{n_+,l}$  labels the possible permutations of the factors in the basis vector  
 158  $\beta_{n_+,l}$ . Since there are  $\binom{N}{n_+}$  such permutations, the subindex indeed goes from 1 to  $\binom{N}{n_+}$ . We fix  
 159  $N$  and  $n_+$ , and put

$$\begin{aligned} W_{n_+}^1 &= \{y \in \{1, \dots, N\} \mid \beta_{n_+} \text{ has } e_1 \text{ on position } y\}, \text{ and} \\ W_{n_+}^2 &= \{y \in \{1, \dots, N\} \mid \beta_{n_+} \text{ has } e_2 \text{ on position } y\}. \end{aligned} \tag{2.12}$$

160 Then

$$\#W_{n_+}^1 + \#W_{n_+}^2 = n_+ + (N - n_+) = n_+ + n_- = N. \tag{2.13}$$

161 Both sets are clearly disjoint. Then we compute

$$\begin{aligned} & \frac{1}{\sqrt{\binom{N}{n_+}}} \frac{1}{\sqrt{\binom{N}{n'_+}}} \sum_{l=1}^{\binom{N}{n_+}} \sum_{k=1}^{\binom{N}{n'_+}} \langle \beta_{n_+,l} \mid h_N^{(2)} \mid \beta_{n'_+,k} \rangle = \frac{1}{\sqrt{\binom{N}{n_+}}} \frac{1}{\sqrt{\binom{N}{n'_+}}} \sum_{l=1}^{\binom{N}{n_+}} \sum_{k=1}^{\binom{N}{n'_+}} \langle \beta_{n_+,l} \mid \sum_{x=1}^N \sigma_1(x) \mid \beta_{n'_+,k} \rangle \\ &= \frac{1}{\sqrt{\binom{N}{n_+}}} \frac{1}{\sqrt{\binom{N}{n'_+}}} \sum_{l=1}^{\binom{N}{n_+}} \sum_{k=1}^{\binom{N}{n'_+}} \langle \beta_{n_+,l} \mid \left( \sum_{x \in W_{n'_+}^1} + \sum_{x \in W_{n'_+}^2} \sigma_1(x) \right) \mid \beta_{n'_+,k} \rangle = \\ & \frac{1}{\sqrt{\binom{N}{n_+}}} \frac{1}{\sqrt{\binom{N}{n'_+}}} \left( \binom{N}{n_+} (N - n_+) \langle \beta_{n_+,l} \mid \beta_{n'_+,-1,k} \rangle + \binom{N}{N - n_+} n_+ \langle \beta_{n_+,l} \mid \beta_{n'_+,+1,k} \rangle \right) = \\ & \sqrt{(N - n_+)(n_+ + 1)} \delta_{n_+,n'_+,-1} + \sqrt{n_+(n_- + 1)} \delta_{n_+,n'_+,+1}. \end{aligned} \tag{2.14}$$

162 We used the fact that the vectors  $\beta_{n'_+,l}$  are orthonormal, that

$$\frac{1}{\sqrt{\binom{N}{n_+}}} \frac{1}{\sqrt{\binom{N}{n'_+}}} \binom{N}{n_+} (N - n_+) = \sqrt{(N - n_+)(n_+ + 1)}, \tag{2.15}$$

163 with  $n'_+ - 1 = n_+$ , and that

$$\frac{1}{\sqrt{\binom{N}{n_+}}} \frac{1}{\sqrt{\binom{N}{n'_+}}} \binom{N}{N - n_+} n_+ = \sqrt{n_+(N - n_+ + 1)}, \tag{2.16}$$

164 with  $n'_+ + 1 = n_+$ . Hence the matrix entries of  $h_N^{(2)}$  written with respect to the symmetric  
 165 basis vectors  $|n_+, N - n_+\rangle$ , are given by  $\sqrt{(N - n_+)(n_+ + 1)}$  on the upper diagonal and by  
 166  $\sqrt{n_+(N - n_+ + 1)}$  on the lower diagonal (see also [34], §3.1).  $\square$

167 From now on we will denote the Curie–Weiss Hamiltonian (2.1) represented in the symmetric  
 168 basis by  $J_{N+1}$ .

## 169 2.2 Numerical simulations

170 In the next section we will argue that for  $0 < B < 1$  the above  $(N + 1)$ -dimensional matrix,  
 171 denoted by  $J_{N+1}$ , can be linked to a Schrödinger operator with a symmetric double well  
 172 on  $L^2([0, 1])$ , for  $N$  sufficiently large.<sup>9</sup> Since for a sufficiently high and broad enough

<sup>9</sup>Mapping quantum spin systems onto Schrödinger operators is not new, see e.g. [35,36]. Schrödinger operators and quantum spin systems also meet in the large research field of Anderson localization and more generally random Schrödinger operators, see e.g. [37] for a rigorous approach.



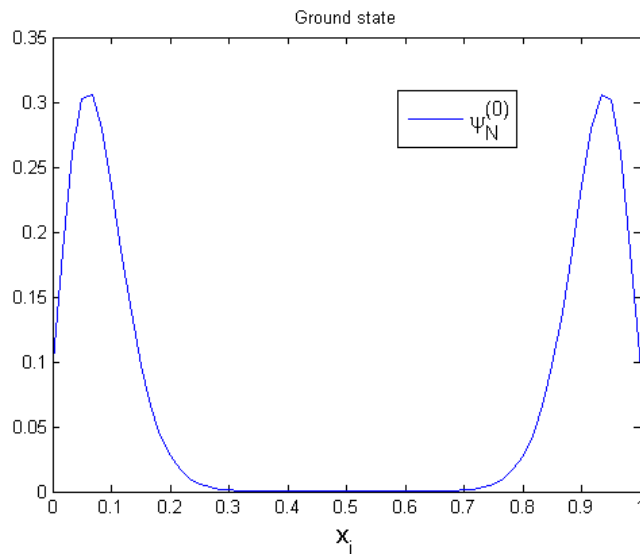


Figure 1: Ground state eigenfunction of  $h_N^{CW}$ , computed from the tridiagonal matrix  $J_{N+1}$  for  $N = 60$ ,  $J = 1$  and  $B = 1/2$ . The grid points on the horizontal axis are labeled by  $x_i = i/N$  for  $i = 0, \dots, N$ .

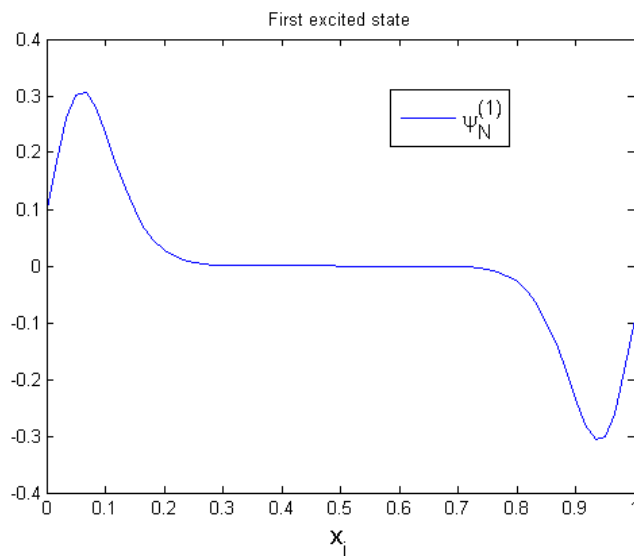


Figure 2: First excited state of  $h_N^{CW}$ , computed from the tridiagonal matrix  $J_{N+1}$  for  $N = 60$ ,  $J = 1$  and  $B = 1/2$ . The grid points on the horizontal axis are indicated by  $x_i = i/N$  for  $i = 0, \dots, N$ .

173 potential barrier the ground state of such a Schrödinger operator is approximately given by  
 174 two Gaussians, each of them located in one of the wells of the potential, we might expect the  
 175 same result for  $J_{N+1}$ . In fact, the first two eigenfunctions of this Schrödinger operator are  
 176 approximately given by

$$\psi^{(0)} \cong \frac{T_a(\varphi_0) + T_{-a}(\varphi_0)}{\sqrt{2}}; \quad \psi^{(1)} \cong \frac{T_a(\varphi_0) - T_{-a}(\varphi_0)}{\sqrt{2}}, \quad (2.17)$$

177 where  $T_{\pm a}$  is the translation operator over distance  $a$  (i.e.,  $(T_{\pm a}\varphi_0)(x) = \varphi_0(x \pm a)$ ),  
 178 where  $\pm a$  denotes the minima of the potential well. The functions  $\varphi_n$  are the weighted

179 Hermite polynomials given by  $\varphi_n(x) = e^{-x^2/2}H_n(x)$ , with  $H_n$  the Hermite polynomials. We  
 180 diagonalized the operator  $J_{N+1}$  and plotted the first two (discrete) eigenfunctions  $\psi_N^{(0)}$  and  
 181  $\psi_N^{(1)}$ . For convenience, we scaled the grid to unity. See Figures 1 and 2. From these two  
 182 plots, it is quite clear that both eigenvectors of  $h_N^{CW}$  are approximately given by (2.17). In the  
 183 following discussion about numerical (in)accuracy, we assume that not only the ground state  
 184  $\psi_N^{(0)}$  (for which the claim is a theorem) but also the first excited state  $\psi_N^{(1)}$  lies in the symmetric  
 subspace  $\text{Sym}^N(\mathbb{C}^2)$ .<sup>10</sup> For  $N \leq 60$  (a number that obviously depends on the machine precision

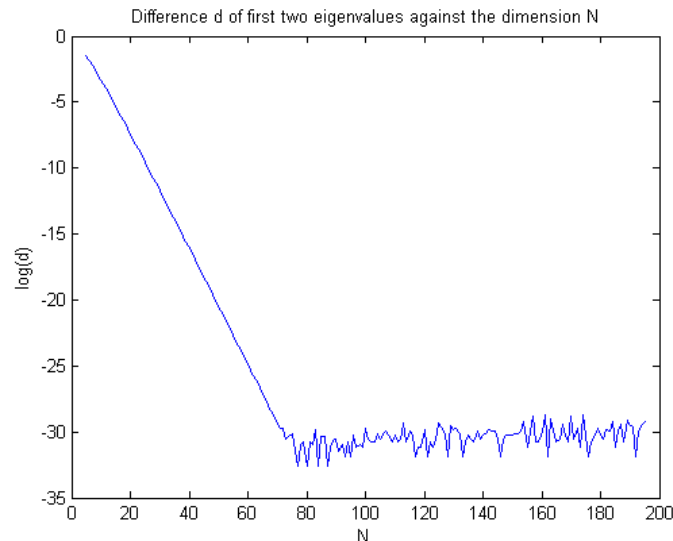


Figure 3: Energy splitting  $d = |\epsilon_N^{(0)} - \epsilon_N^{(1)}|$  between the first two eigenvalues  $\epsilon_N^{(0)}$  and  $\epsilon_N^{(1)}$  of the tridiagonal matrix  $J_{N+1}$ , for different values of  $N$  on a log scale. From about  $N = 80$  onwards, the energy splitting is in the order of the maximum achievable accuracy of the eigenvalues, so that the first two eigenvalues become numerically degenerate.

185 of the computer used to perform the numerical simulations) our simulations accurately reflect  
 186 the uniqueness of the ground state, proved from general principles. However, for larger values  
 187 of  $N$ , clearly visible from about  $N \geq 80$  (see Figure 3), up to numerical precision the exact  
 188 (and unique) ground state is joined by a degenerate state, and the selection of any specific  
 189 linear combination of those as the numerically obtained “ground state” is implicitly made  
 190 by numerical noise playing a role similar to some symmetry-breaking perturbation (be it the  
 191 flea perturbation in section 4 or a more traditional symmetry-breaking field typically used in  
 192 quantum spin systems); this noise then localizes the ground state purely due to the inaccuracy  
 193 of the computation.

195 Consequently, plotting the ground state and the first excited state of  $h_N^{CW}$  (at  $B = 1/2$  and  
 196  $J = 1$ ) for  $N \geq 80$  gives a Gaussian curve, located in one of the wells.<sup>11</sup> Specifically, the new

<sup>10</sup>This second claim is not essential for our results themselves but is helpful for the following explanation thereof. The point is that the first excited state computed from the tridiagonal matrix  $J_{N+1}$  might not be the same as the one from the original Hamiltonian  $h_N^{CW}$ , in which case Figure 2 would be misleading. Fortunately, we have shown numerically that up to  $N = 12$  the first excited state of  $h_N^{CW}$  represented as a matrix on  $\mathbb{C}^{2^N}$  is the same as the one corresponding to the tridiagonal matrix  $J_{N+1}$ . For  $N > 12$  this computation became unfeasible, as the dimension of the relevant subspace grows exponentially with  $N$ .

<sup>11</sup>We checked this numerically, but omitted the plots. Moreover, we observed that for increasing  $N$  these two numerical degenerate states were randomly localized in (one of the) both wells.

197 (numerically) degenerate ground state eigenvectors are given by the functions

$$\chi_+ = \frac{\psi_N^{(0)} + \psi_N^{(1)}}{\sqrt{2}}; \quad \chi_- = \frac{\psi_N^{(0)} - \psi_N^{(1)}}{\sqrt{2}}. \quad (2.18)$$

198 Using this result and equation (2.17), it follows by a simple calculation that

$$\chi_+ \cong T_a \varphi_0; \quad \chi_- \cong T_{-a} \varphi_0, \quad (2.19)$$

199 where the functions  $\varphi_n(x)$  now have to be understood as functions on a discrete grid. Once  
 200 again, for  $N = 60$ , the fact that  $\psi_N^{(0)}$  (rather than rather than  $\chi_{\pm}$ ) is the (doubly peaked)  
 201 ground state is confirmed by the numerical simulations summarized in Figure 1.

### 202 3 The Curie–Weiss Hamiltonian as a Schrödinger operator

203 Discretization is the process of approximating the derivatives in (partial) differential equations  
 204 by linear combinations of function values  $f$  at so-called *grid points*. The idea is to discretize  
 205 the domain, with  $N$  of such grid points, collectively called a *grid*. We give an example in one  
 206 dimension:

$$\Omega = [0, X], \quad f_i \approx f(x_i), \quad (i = 0, \dots, N), \quad (3.1)$$

207 with grid points  $x_i = i\Delta$  and grid size  $\Delta = X/N$ . The symbol  $\Delta$  is called the *grid spacing*. Note  
 208 this the grid spacing is chosen to be constant or uniform in this specific example. For the first  
 209 order derivatives we have

$$\frac{\partial f}{\partial x}(\bar{x}) = \lim_{\Delta x \rightarrow 0} \frac{f(\bar{x} + \Delta x) - f(\bar{x})}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(\bar{x}) - f(\bar{x} - \Delta x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(\bar{x} + \Delta x) - f(\bar{x} - \Delta x)}{2\Delta x}. \quad (3.2)$$

210 These derivatives are approximated with *finite differences*. There are basically three types of  
 211 such approximations:

$$\begin{aligned} \left(\frac{\partial f}{\partial x}\right)_i &\approx \frac{f_{i+1} - f_i}{\Delta x} \quad (\text{forward difference}) \\ \left(\frac{\partial f}{\partial x}\right)_i &\approx \frac{f_i - f_{i-1}}{\Delta x} \quad (\text{backward difference}) \\ \left(\frac{\partial f}{\partial x}\right)_i &\approx \frac{f_{i+1} - f_{i-1}}{2\Delta x} \quad (\text{central difference}). \end{aligned} \quad (3.3)$$

212 Since it is more accurate in our case, will focus on the central difference approximation method  
 213 and apply this to the second order differential operator  $d^2/dx^2$ .

#### 214 3.1 Locally uniform discretization

215 In the example above, the grid spacing was chosen to be uniform. Now reconsider this example  
 216 on the domain  $\Omega = [0, 1]$  with uniform grid spacing  $\Delta = 1/N$ . The second order derivative  
 217 operator is approximately given by

$$f_i'' = \frac{f_{i-1} - 2f_i + f_{i+1}}{\Delta^2} + O(\Delta^2) \quad (i = 1, \dots, N - 1). \quad (3.4)$$

218 By throwing away the error term  $O(\Delta^2)$  in the above equation, it follows that we can  
 219 approximate the second derivative operator in matrix form

$$\frac{1}{\Delta^2} \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & \mathbf{0} & \\ & \ddots & \ddots & \ddots & \\ \mathbf{0} & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}. \quad (3.5)$$

220 This matrix is the standard discretization of the second order derivative on a uniform grid  
 221 consisting of  $N$  points of length  $\Delta \cdot N$ , with uniform grid spacing  $\Delta$ . In this specific case, we  
 222 have  $\Delta = 1/N$ . We denote this matrix also by  $\frac{1}{\Delta^2}[\cdots 1 -2 1 \cdots]_N$ .

223 Suppose now that we are given a symmetric tridiagonal matrix  $A$  of dimension  $N$  with constant  
 224 off- and diagonal elements:

$$A = \begin{pmatrix} b & a & & & \\ a & b & a & \mathbf{0} & \\ & \ddots & \ddots & \ddots & \\ \mathbf{0} & & a & b & a \\ & & & a & b \end{pmatrix}. \quad (3.6)$$

225 We are going to extract a kinetic and a potential energy from this matrix. We write

$$A = a[\cdots 1 \frac{b}{a} 1 \cdots]_N = a[\cdots 1 -2 1 \cdots]_N + \text{diag}(b + 2a), \quad (3.7)$$

226 where the latter matrix is a diagonal matrix with the element  $b + 2a$  on the diagonal. It follows  
 227 that

$$A = T + V, \quad (3.8)$$

228 for  $T = a[\cdots 1 -2 1 \cdots]_N$ , and  $V = \text{diag}(b + 2a)$ . In view of the above, the matrix  $T$  corresponds  
 229 to a second order differential operator. This matrix plays the role of (3.5), but with uniform  
 230 grid spacing  $1/\sqrt{a}$  on the grid of length  $N/\sqrt{a}$ . Since the matrix  $V$  is diagonal, it can be seen  
 231 as a multiplication operator. Therefore, given such a symmetric tridiagonal matrix  $A$ , we can  
 232 derive an operator that is the sum of a discretization of a second order differential operator  
 233 and a multiplication operator. The latter operator is identified with the potential energy of the  
 234 system. Hence we can identify  $A$  with a discretization of a Schrödinger operator.<sup>12</sup>

235 The next step is to understand what happens in the case where we are given a symmetric  
 236 tridiagonal matrix with non-constant off- and on-diagonal elements. This is important as we  
 237 will see, since the Curie–Weiss Hamiltonian, written with respect to the canonical symmetric  
 238 basis for the subspace  $\text{Sym}^N(\mathbb{C}^2)$  of  $\mathbb{C}^{2^N} \simeq \bigotimes_{n=1}^N \mathbb{C}^2$ , is precisely an example of such a matrix.  
 239 The question we ask ourselves is if we can link such a matrix to a discretization of a Schrödinger  
 240 operator as well (see Appendix B for background).

241 Writing  $T = J_{N+1}$ , consider the ratios

$$\rho_j = \frac{h_{j-1}}{h_j} = \frac{T_{j+1}}{T_{j-1}} \quad (j = 1, \dots, N), \quad (3.9)$$

<sup>12</sup>Strictly speaking we have to put a minus sign in front of  $T$ , as the kinetic energy is defined as  $-\frac{d^2}{dx^2}$ .

242 with non-uniform grid spacing  $h_j$  and  $h_{j-1}$ . We divide the original tridiagonal matrix  $J_{N+1}$  by  
 243  $N$  for scaling. Thus, we consider  $J_{N+1}/N$ . If we then compute the distances  $h_j$ , we see that  
 244 they are almost all of  $O(1)$ , except at the boundaries. We will see later that the corresponding  
 245 Schrödinger operator analog of the matrix  $J_{N+1}/N$  will be an operator on a domain of length  
 246  $L = \frac{1}{N} \sum_{j=1}^N h_j$ .

247 First, we compute the ratios  $\rho_j$ :

$$\rho_j = \frac{T_{j+1}}{T_{j-1}} = \frac{\sqrt{(N-j)(j+1)}}{\sqrt{(N-j+1)j}} = \sqrt{\frac{N-j}{N-j+1}} \sqrt{\frac{j+1}{j}} = \sqrt{\frac{1}{1+\frac{1}{N-j}}} \sqrt{1+\frac{1}{j}}. \quad (3.10)$$

248 Use the following approximations

$$\sqrt{1+\frac{1}{j}} \approx 1 + O\left(\frac{1}{2j}\right) = 1 + O(1/j) \quad \text{and} \quad (3.11)$$

$$\sqrt{\frac{1}{1+1/(N-j)}} \approx 1 - O\left(\frac{1}{2(N-j)}\right) = 1 + O\left(\frac{1}{N-j}\right), \quad (3.12)$$

249 and observe that for  $j \gg 1$  and  $N-j \gg 1$ , we have

$$\sqrt{1+\frac{1}{j}} \approx O(1) \quad \text{and} \quad (3.13)$$

$$\sqrt{\frac{1}{1+1/(N-j)}} \approx O(1). \quad (3.14)$$

250 Moreover, we see that the ratio satisfies

$$\rho_j \approx 1 + O(1/j) + O\left(\frac{1}{N-j}\right), \quad (3.15)$$

251 using the fact that that the big-O notation respects the product, that  $O(\frac{1}{j} \frac{1}{N-j}) \leq O(1/j)$ , and  
 252 also  $O(\frac{1}{j} \frac{1}{N-j}) \leq O(\frac{1}{N-j})$ .

253 In the next subsection, we will see from numerical simulations that to a good approximation  
 254 the ground state eigenfunction is a double peaked Gaussian with maxima centered in the  
 255 minima of some double well potential that we are going to determine. This potential occurs in a  
 256 discrete Schrödinger operator analog of the matrix  $J_{N+1}/N$  for  $N$  large, i.e., in the semiclassical  
 257 limit. Furthermore, we showed by numerical simulations (Figure 4 below) that the width  $\sigma$   
 258 of each Gaussian-shaped<sup>13</sup> ground state of  $J_{N+1}$  located at one of minima of the potential  
 259 is of order  $\sqrt{N}$ , and hence that each peak rapidly decays to zero, so that the ground state  
 260 eigenfunction is approximately zero at both boundaries. In particular, the size of the domain  
 261 where the peak is non-zero contains  $O(\sqrt{N})$  grid points, as we clearly observe from the figure.  
 262 This is an approximation, since we neglect the (relatively small) function values of the Gaussian  
 263 that are more than  $O(\sqrt{N})$  away from the central maximum. However, this approximation is  
 264 highly accurate, as the Gaussian decays to zero exponentially. This observation is extremely  
 265 important, as we will now see.

266 Let us first focus on the left-located Gaussian. For a point  $x_j = j/N$ , clearly  $j \in O(N)$ .  
 267 Therefore, for  $N$  large enough,

$$\rho_j = 1 + O(1/N), \quad (3.16)$$

<sup>13</sup>We mean that if we plot the discrete points and draw a line through these points, then the corresponding graph has the shape of a Gaussian.

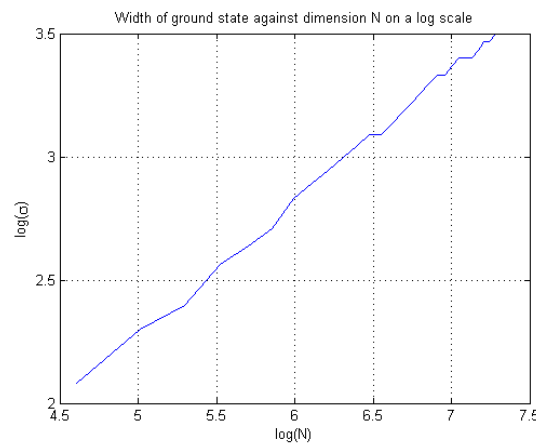


Figure 4: Width at half height of the ground state eigenvector of  $J_{N+1}$  ( $B = 1/2$  and  $J = 1$ ) against  $N$ , for  $N = 100 : 50 : 1500$  on a log scale. The slope of the line is about 0.5, which means that the width  $\sigma$  goes like  $\sqrt{N}$ .

268 since for these values of  $j < N - j$  we have  $O(\frac{1}{N-j}) \leq O(1/j)$ . For the right-located peak, we  
 269 have  $N - j < j$ , so that in this case  $O(1/j) \leq O(\frac{1}{N-j})$ , and we find

$$\rho_j = 1 + O\left(\frac{1}{N-j}\right). \tag{3.17}$$

270 We will now show that, in the present context where we work on a domain of order  $L$  (i.e.  
 271  $[0, L]$ ), we indeed have uniform discretization on a subinterval of this domain corresponding  
 272 to a matrix segment of  $O(\sqrt{N})$  entries. We start with the peak on the left. Since the error  
 273 per step that we make equals  $\rho_j$ , it follows that the error on a matrix segment of length of  
 274 order  $\sigma$  equals  $\rho_j^\sigma \approx (1 + \frac{1}{N})^\sigma$  for  $j < N - j$  and  $N$  large. Denoting the off-diagonal element  
 275 corresponding to the minimum  $x_{j_0}$  of the potential well by  $T_{j_0}$ , for the off-diagonal elements  
 276 within a range of order  $\sigma$ , we derive the next estimate:

$$|T_{j_0} - T_{j_0+\sigma}| \approx |T_{j_0} - O\left(1 + \frac{1}{N}\right)^\sigma T_{j_0}| = T_{j_0} \left|1 - O\left(1 + \frac{1}{N}\right)^\sigma\right| \leq C \frac{\sigma}{N}, \tag{3.18}$$

277 where we used the inequality  $(1 + 1/N)^\sigma \leq 1 + C \frac{\sigma}{N}$  as well as the fact that  $T_{j_0}$  is of order 1.  
 278 Here,  $C > 1$  is a constant independent of  $N$ . Since the left peak of the Gaussian eigenfunction  
 279 is approximately non-zero corresponding to a matrix segment of length of order  $\sqrt{N}$ , we  
 280 apply the above estimate to  $\sigma \approx \sqrt{N}$ . We see immediately that  $|T_{j_0} - T_{j_0+\sigma}|$  goes to zero.  
 281 Therefore, on matrix segment of length of order  $\sqrt{N}$  centered around the left minimum  $x_{j_0}$   
 282 of the potential, the off-diagonal elements coincide in the limit  $N \rightarrow \infty$ . This means that  
 283 the grid spacing becomes constant and hence that we have locally uniform discretization of  
 284 the domain. By symmetry, the same is true for the peak located on the right of the well. We  
 285 conclude that for large  $N$  the tridiagonal matrix locally behaves like a kinetic energy, and  
 286 therefore like a discretized Schrödinger operator. All this will be explained in more detail in  
 287 the next subsection.

### 288 3.2 Link with a Schrödinger operator

289 Our aim is to show that for  $N$  large enough, the matrix  $J_{N+1}/N$  obtained from the Curie–Weiss  
 290 Hamiltonian by reduction (see §2) locally approximates a discretization matrix representing  
 291 a Schrödinger operator describing a particle moving in a symmetric double well. This means

292 that there exists a sub-block of  $J_{N+1}/N$  that has a form approximately given by the sum of  
 293  $-\frac{1}{\hbar^2}[\cdots 1 -2 1 \cdots]_{N+1}$  (for a certain  $\hbar$ ) and a diagonal matrix playing the role of a potential.  
 294 We started with the symmetric tridiagonal matrix  $J_{N+1}/N$  with non-constant entries. In order  
 295 to link this matrix to a second derivative and a multiplication operator, we needed to apply  
 296 the non-uniform discretization procedure.

297 At first sight the off-diagonal matrix entries of  $J_{N+1}/N$  cannot immediately be identified  
 298 with a second order derivative operator, but we have seen that in the limit  $N \rightarrow \infty$  we do  
 299 have uniform discretization on some interval of a length scale corresponding to a segment  
 300 of  $O(\sqrt{N})$  matrix entries. Consequently, for sufficiently large  $N$  this discretization becomes  
 301 approximately uniform on this length scale. From this, we are now going to extract a matrix  
 302 of the form (3.6) corresponding to a Schrödinger operator on  $L^2([0, 1])$ . We first consider the  
 303  $(N \times N)$ -matrix  $H_N$ , defined by

$$H_N = T_N + V_N, \tag{3.19}$$

304 where at  $x = n_+/N$  we define

$$T_N(x) = -\frac{1}{d(x)^2}[\cdots 1 -2 1 \cdots]_N, \tag{3.20}$$

305 keeping in mind that  $d(x)$  varies per entry, and  $d(x)$  is defined by

$$d(x) = \frac{1}{\sqrt{B}} \frac{1}{((1-x)x)^{1/4}}. \tag{3.21}$$

306  $V_N$  is a diagonal matrix (and hence a multiplication operator, as in the continuum) given by

$$V_N(x) = -\frac{1}{2}(2x-1)^2 - B \left( \sqrt{(1-x)(x+\frac{1}{N})} + \sqrt{(1-x+\frac{1}{N})x} \right). \tag{3.22}$$

307 Note that  $d(x)$  corresponds to the non-uniform grid spacing. We rewrite:

$$H_N = -\frac{1}{N^2} \frac{1}{(d(x)/N)^2} [\cdots 1 -2 1 \cdots]_N + V_N \tag{3.23}$$

308 and hence the total length of the interval is given by  $\frac{1}{N} \sum_{n_+} d(n_+/N)$ . So if  $N \rightarrow \infty$ , it follows  
 309 that

$$L := \int_0^1 d(t) dt = \int_0^1 \frac{1}{\sqrt{B}} \frac{1}{((1-t)t)^{1/4}} dt = \frac{2\Gamma[3/4]^2}{\sqrt{B}\sqrt{\pi}}. \tag{3.24}$$

310 Moreover, the interval coordinate is then given by

$$D(x) = \int_0^x d(t) dt \in [0, L], \text{ for } x \in [0, 1]. \tag{3.25}$$

311 It follows that on each matrix segment of  $\sqrt{N}$  entries (for  $N$  large), the matrix  $H_N$  is  
 312 an approximation of the following Schrödinger operator on  $L^2([0, L])$  with the familiar  
 313 substitution  $\hbar = \frac{1}{N}$ :

$$h_1 = -\frac{1}{N^2} \Delta + V_N(x), \tag{3.26}$$

314 where  $x$  is chosen appropriately and the segment describes an interval of length  $\frac{d(x)\sqrt{N}}{N} = \frac{d(x)}{\sqrt{N}}$ ,  
 315 at location

$$D(x) = \int_0^x \frac{1}{\sqrt{B}} \frac{1}{((1-t)t)^{1/4}} dt, \quad x \in [0, 1]; \tag{3.27}$$

316 in the total interval of length  $L$ .<sup>14</sup> We saw in the previous section that on length scales of order  
 317  $d(x)/\sqrt{N}$ , the function  $d(x)$  is approximately constant so that  $T_N$  can be seen as a locally  
 318 uniform discretization of the second order derivative  $\Delta$  on the sub interval approximately  
 319 given by  $[D(x) - d(x)/\sqrt{N}, D(x) + d(x)/\sqrt{N}]$ . Therefore, on these length scales the operator  
 320  $\Delta$  indeed corresponds to a uniform part of  $\frac{1}{(d(x)/N)^2} [\cdots 1 -2 1 \cdots]_N$ .  
 321 We see in this section that to a very good approximation the spectral properties of both (a  
 322 priori different) matrices  $J_{N+1}/N$  and  $\tilde{H}_{N+1}$ , defined in (3.34) below, coincide, improving  
 323 with increasing  $N$ . Rescaling the interval  $[0, L]$  to unity yields a Schrödinger operator  $h$ , but  
 324 now defined on  $L^2([0, 1])$ . Hence  $h$  is given by

$$h = -\frac{1}{L^2 N^2} \Delta_y + \tilde{V}_N(y) \quad (y \in [0, 1]), \tag{3.28}$$

325 where, via the potential  $V_N$  defined in (3.22), the potential  $\tilde{V}_N$  is defined by

$$\tilde{V}_N(y) = V_N(D^{-1}(yL)) \quad y \in [0, 1]. \tag{3.29}$$

326 Let us now consider the scaled tridiagonal matrix  $J_{N+1}/N$ . By definition it follows that the  
 327 elements on the diagonal, the lower diagonal, and the upper diagonal are given by

$$-\frac{1}{2} \left(2 \frac{n_+}{N} - 1\right)^2, \quad -B \sqrt{\left(1 - \frac{n_+}{N} + \frac{1}{N}\right) \frac{n_+}{N}}, \quad -B \sqrt{\left(1 - \frac{n_+}{N}\right) \left(\frac{n_+}{N} + \frac{1}{N}\right)}, \tag{3.30}$$

328 respectively. The idea is to split the matrix  $J_{N+1}/N$  into two parts, one corresponding to the  
 329 kinetic energy and the other to the potential energy. However, since the off-diagonal elements  
 330 of  $J_{N+1}/N$  are non-constant and not an even function around  $x = 1/2$ , we cannot isolate these  
 331 elements and decompose  $J_{N+1}/N$  into two parts. Therefore, we approximate the off-diagonal  
 332 elements by the function (3.21), which is even in  $1/2$ . This approximation makes perfect sense  
 333 in the semi-classical limit, and from this it is clear that  $J_{N+1}/N \approx H_{N+1}$ , where  $H_N$  is defined  
 334 by (3.19). In the limit  $N \rightarrow \infty$  we pass to the continuum, so that the discrete points  $n_+/N$  are  
 335 understood as real numbers in  $[0, 1]$ . Therefore, equations (3.21) and (3.25) make sense on  
 336  $[0, 1]$ . The operator (3.26) is a Schrödinger operator defined on the space  $[0, L]$ , whereas the  
 337 operator (3.28) is obtained by rescaling to the unit interval. Therefore, scaling  $H_N$  to unity  
 338 yields the matrix

$$-\frac{1}{L^2 N^2} \frac{1}{\left(\frac{d(x)}{LN}\right)^2} [\cdots 1 -2 1 \cdots]_N + V_N(D^{-1}(yL)), \tag{3.31}$$

339 where each segment with order  $\sqrt{N}$  matrix entries now describes an interval of length  
 340  $d(x)/(L\sqrt{N})$  at location  $y = D(x)/L$  in  $[0, 1]$ , approximating the Schrödinger operator

$$-\frac{1}{L^2 N^2} \Delta + V_N(x) \quad \text{with } x = D^{-1}(yL), \tag{3.32}$$

341 which indeed coincides with (3.28). The matrix (3.31) corresponds to a non-uniform  
 342 discretization of  $h$ , in such a way that it is locally uniform, namely on length scales of order  
 343  $\sqrt{N}$ . Finally, discretizing  $\Delta_y$  on the unit interval with uniform grid spacing  $1/N$  yields the  
 344 following discretization matrix:

$$K_N = -\frac{1}{L^2 N^2} \frac{1}{\left(\frac{1}{N}\right)^2} [\cdots 1 -2 1 \cdots]_N = -\frac{1}{L^2} [\cdots 1 -2 1 \cdots]_N. \tag{3.33}$$

<sup>14</sup>Namely, if  $z \in [0, L]$ , then  $x = D^{-1}(z)$ . Hence the coordinate  $z$  corresponds to a location in the interval  $[0, L]$ , whilst  $x$  plays the role of the ‘argument’ of a matrix entry of  $V$ , e.g. if  $x = n_+/N$ , then  $V(x) = V(n_+/N) \equiv V_{n_+}$ .



345 Using this discretization of  $-\frac{1}{L^2 N^2} \Delta_y$ , we show that for  $N \rightarrow \infty$  the spectrum of the matrix  
 346  $J_{N+1}/N$  approximates the spectrum of the matrix

$$\tilde{H}_N = K_N + \tilde{V}_N. \tag{3.34}$$

347 Using the fact that  $\tilde{H}_N$  is a discretization of (3.28), this indeed establishes a link between the  
 348 compressed Curie–Weiss Hamiltonian and a Schrödinger operator.<sup>15</sup>

349 *Remark.* Consider the Schrödinger operator with a symmetric double well potential, given by  
 350 (1.1). Recall from §2.2 that for a sufficiently high and broad potential well, the ground state  
 351 of such a Schrödinger operator is approximately given by two Gaussians, each of them located  
 352 in one of the wells of the potential (see [38]). This fact will be useful for the next round of  
 353 observations.

354 We will now show that the Gaussian-shaped ground state of  $J_{N+1}/N$ , indeed localizes in both  
 355 minima of the potential well  $\tilde{V}_N$ . To this end, we have made a plot of the scaled potential  
 356  $\tilde{V}_N$  from equation (3.29) on a domain of length 1, for  $B = 1/2$  and  $J = 1$ . See Figure  
 357 5. We immediately recognize the shape of a symmetric double well potential. The points  
 358 in its domain are given by  $y_j = j/N$  for  $j = 0, \dots, N$ , and the argument of the potential  
 359 is given by  $D^{-1}(y_j L)$ , as explained before. Then we diagonalized the matrix  $J_{N+1}/N$  and  
 360 computed the ground state eigenvector. We plot this together with the potential in Figure  
 361 5. One should mention that only one Gaussian peak is visible, not two. As we have seen in  
 362 §2.2, this must be due to the finite precision of the computer i.e., the first two eigenvalues are  
 363 already numerically degenerate. Thus the computer picks a linear combination of the first two  
 364 eigenvectors as ground state (viz. (2.18)), even though we know from the Perron-Frobenius  
 365 Theorem (Appendix A) that the ground state is always unique for any finite  $N$ .<sup>16</sup> We also  
 366 observe that the maxima of the Gaussian ground state peaks are precisely centered in the  
 367 minima of these two wells (as should be the case). It is clear from this figure that the ground  
 368 state is localized in (one of) the minima of the double well.

369 One might suggest that there would be some critical value of  $N$  for which the eigenvalues are  
 370 not yet degenerate for the computer. We have seen in Figure 3 that this value of  $N$  (depending  
 371 on our machine) is about  $N = 80$ . Figure 6 is a similar plot for the ground state for  $N = 60$ , on  
 372 a par with Figure 1 in §2.2. We recognize the well-known doubly peaked Gaussian shape, but  
 373 now it is localized in both minima of the potential well. This is displayed in Figure 6. These  
 374 figures show that there is a convincing relation between the matrix  $J_{N+1}/N$  and a Schrödinger  
 375 operator describing a particle in a double well. The double well shaped potential is a result of  
 376 the choice  $B = 1/2$ . The value of the magnetic field needs to be within  $[0, 1)$  in order to get  
 377 spontaneous symmetry breaking of the ground state in the classical limit  $N \rightarrow \infty$ . For  $B \geq 1$   
 378 the Curie–Weiss model will not display SSB, not even in the classical limit. For these values of  
 379  $B$ , the well will be a single potential, as depicted in Figure 7 for  $B = 2$ .

380 In view of the corresponding Schrödinger operator, the ground state in the classical limit will  
 381 not break the symmetry for a single potential well, and is therefore also compatible with the  
 382 Curie–Weiss model for  $B \geq 1$ . We now return to the regime  $0 \leq B < 1$ . One can compute  
 383 the spectral properties of the matrix  $J_{N+1}/N$  and compare them with those of the matrix  $\tilde{H}_N$   
 384 corresponding to the sum of the uniform discretization  $K_N$  of the second order derivative (viz.  
 385 (3.33)) and  $\tilde{V}_N$  (viz. (3.29)). We will see that to a very good approximation the spectral  
 386 properties of both matrices coincide and get better with increasing  $N$ . We have programmed  
 387 the matrix  $\tilde{H}_N$  in MATLAB. The matrix has been diagonalized. The spectral properties have

<sup>15</sup>Note that the matrix  $\tilde{H}_N$  corresponding to (3.28) is by definition a discretization of the Schrödinger operator on the whole of  $[0, 1]$ , and not only for subintervals of length  $d(x)/L\sqrt{N}$ .

<sup>16</sup>Due to this degeneracy, the computer picks or the symmetric combination  $\chi_+$ , or the anti-symmetric combination  $\chi_-$ , which depends on the algorithm. We changed  $N$  and observed that the location of the peak changed as well. This suggests that random superpositions of the two degenerate states are formed.

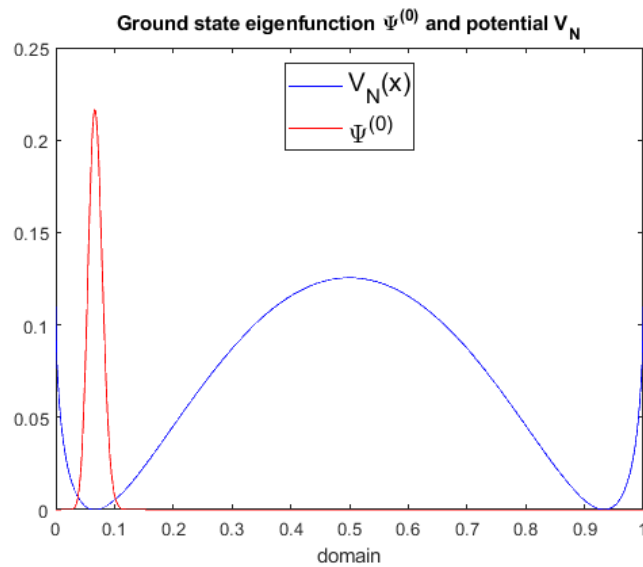


Figure 5: Scaled potential  $V_N := \tilde{V}_N$  and the ground state eigenfunction corresponding to  $J_{N+1}/N$  for  $N = 1000$  on the unit interval  $[0, 1]$ , plotted on uniform grid points. The potential is shifted so that its minimum is zero, and is plotted on the grid points  $x$  corresponding to the solution of  $y = D(x)/L$ , as explained in the main text.

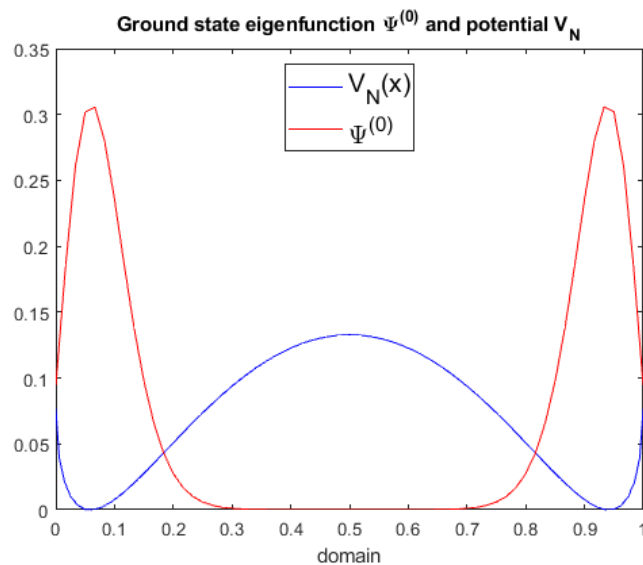


Figure 6: Scaled and shifted potential  $V_N := \tilde{V}_N$  for  $B = 1/2$  and  $J = 1$ , plotted on grid points that are solution of the equation  $y = D(x)/L$ , and the ground state eigenfunction corresponding to  $J_{N+1}/N$  for  $N = 60$ , plotted on uniform grid points.

388 been compared to those of  $J_{N+1}/N$  (Table 1). We computed the first ten eigenvalues of the  
 389 matrix  $J_{N+1}/N$ , denoted by  $\epsilon_n$ , and those of  $\tilde{H}_N$ , denoted by  $\lambda_n$ . In the left column the  
 390 eigenvalues  $\epsilon_n$  are displayed. In the right column the absolute difference  $|\lambda_n - \epsilon_n|$  is displayed.  
 391 The number  $N = 1000$  is fixed (the numerical eigenvalues in the table are dimensionless, since  
 392 we put  $J = 1$ , cf. footnote 7).

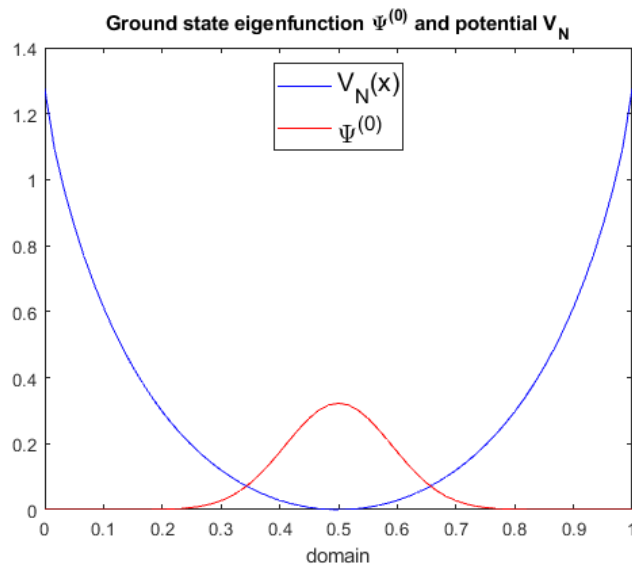


Figure 7: Scaled and shifted potential  $V_N := \tilde{V}_N$  for  $B = 2$  and  $J = 1$ , and the ground state eigenfunction corresponding to  $J_{N+1}/N$  for  $N = 60$ . The single well is clearly visible. The ground state is plotted on the uniform grid corresponding to  $[0, 1]$ , and also now the ground state eigenvector is normalized to 1.

Table 1. Eigenvalues and absolute differences ( $J = 1, B = 1/2$ )

n	$\epsilon_n$	$ \lambda_n - \epsilon_n $
0	-0.6251	$7.6038 \times 10^{-7}$
1	-0.6251	$7.6038 \times 10^{-7}$
2	-0.6234	$2.2446 \times 10^{-6}$
3	-0.6234	$2.2446 \times 10^{-6}$
4	-0.6217	$4.8378 \times 10^{-6}$
5	-0.6217	$4.8378 \times 10^{-6}$
6	-0.6200	$7.0222 \times 10^{-6}$
7	-0.6200	$7.0222 \times 10^{-6}$
8	-0.6183	$8.8010 \times 10^{-6}$
9	-0.6183	$8.8010 \times 10^{-6}$

393

394 We see that the first ten eigenvalues for both matrices are the same up to at least six decimals. It  
 395 is also clear that these eigenvalues are doubly degenerate, at least up to six decimals. Moreover,  
 396 we plotted all the eigenvalues  $\epsilon_n$  and  $\lambda_n$  corresponding to the bound states, i.e., the energy  
 397 levels within the well. This is depicted in Figure 8 below.

398 It follows that the energies of both systems are approximately the same. Moreover, we  
 399 compared a plot of the ground state eigenvector of  $J_{N+1}/N$  with the one corresponding to  
 400  $\tilde{H}_N$ . Completely analogously to  $J_{N+1}/N$ , we observed also now that the ground state of  $\tilde{H}_N$   
 401 is located in the minima of the potential well, only concentrates on length scales of order  
 402  $\sqrt{N}$ , and exponentially decays to zero. This is in agreement with the theory of Schrödinger  
 403 operators, since  $\tilde{H}_N$  represents a discretization of a Schrödinger operator as we have seen in  
 404 the beginning of this section. Table 1, Figure 8 and this observation show that we have strong  
 405 numerical evidence that the original tridiagonal matrix is related to  $\tilde{H}_N$ , which was *a priori*  
 406 not clear since the off-diagonal elements of  $J_{N+1}/N$  are non-constant.

407 *Remark.* It is well known that the ground state of the operator  $h$  for finite  $N$  looks approximately  
 408 like a doubly peaked Gaussian, where each peak is centered in one of the minima of the

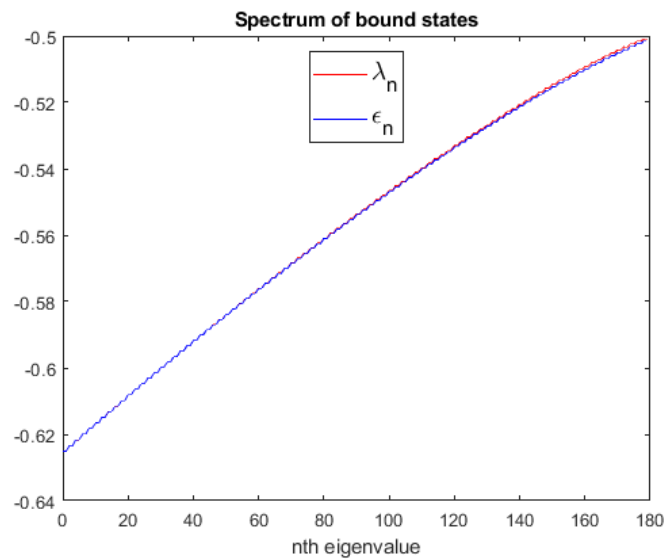


Figure 8: Spectrum of the bound states in the unscaled potential  $\tilde{V}_N$  for  $B = 1/2$ ,  $J = 1$ , and  $N = 1000$ ;  $\epsilon_n$  corresponds to the eigenvalues of  $J_{N+1}/N$  and  $\lambda_n$  to those of  $\tilde{H}_N$ .

409 potential. For infinite  $N$ , these peaks will behave like delta distributions. Moreover, numerical  
 410 simulations (Figure 4) show that the eigenfunctions of  $\tilde{H}_N$  live approximately on a grid of  
 411 order  $\sqrt{N}$  points on the interval  $[0, 1]$ . Using the above discretization, we then have about  
 412  $\sqrt{N}$  steps of  $1/N$  each, so that in particular the ground state Gaussian has a width of  $1/\sqrt{N}$ .  
 413 On the one hand, it is clear that this width will go to zero as  $N \rightarrow \infty$ . On the other hand, also  
 414 the unit interval depends on  $N$ , as the latter has to be discretized with  $N + 1$  points. Therefore,  
 415 the total number of grid points in the ground state peak living on a subset of order  $\sqrt{N}$  is

$$\frac{1/\sqrt{N}}{1/N} = \sqrt{N}. \tag{3.35}$$

416 In fact, due to the discretization of the grid we have a better approximation of the Gaussian  
 417 ground state when  $N$  increases.

418 We have computed the minimum of the potential, and subtracted this minimum from the  
 419 lowest eigenvalues. Then, we have set the potential minimum to zero. These shifted  
 420 eigenvalues then live in a positive potential well. For  $J_{N+1}/N$  with  $N = 1000$ , we now  
 421 consider its eigenvalues  $\epsilon_n$ . We have already seen above that the lowest eigenvalues of  $J_{N+1}/N$   
 422 become doubly degenerate. Therefore, we identify these approximately doubly degenerate  
 423 eigenstates with one single state that we denote by  $n$ . It follows that each  $n$  corresponds to  
 424 two (approximately) degenerate eigenvalues, e.g.,  $n = 0$  corresponds to the ground state as  
 425 well as the first excited state of  $J_{N+1}/N$ ,  $n = 1$  corresponds to the second and the third excited  
 426 state, and so on. This is displayed in table 2 below.

Table 2. Shifted eigenvalues for odd values of  $n$  ( $J = 1, B = 1/2$ )

$n$	$\epsilon_n$
0	0.000863
1	0.002591
2	0.004310
3	0.006013
4	0.007710

428 Using this table, we deduce that when  $N$  is large enough the energy splitting is approximately  
 429 given by  $\sqrt{3}/N$ . The ground state (shifted) eigenvalue (which is approximately doubly

430 degenerate) is then given by  $\frac{1/2\sqrt{3}}{N}$ , the first excited state (also approximately doubly  
 431 degenerate) is  $\frac{3/2\sqrt{3}}{N}$ , the second excited state is  $\frac{5/2\sqrt{3}}{N}$  etc. Therefore, there is excellent  
 432 numerical evidence that for sufficiently large  $N$  and  $J = 1, B = 1/2$  the (approximately) doubly  
 433 degenerate shifted spectrum of  $J_{N+1}/N$  is given by

$$\frac{(n + 1/2)\sqrt{3}}{N}, \quad (n = 0, 1, 2, \dots). \tag{3.36}$$

434 This firstly shows that for large  $N$  the wells approximately decouple since tunneling is  
 435 suppressed in the limit, and secondly that each well of the double well potential is locally  
 436 quadratic and therefore approximately has the spectrum of a harmonic oscillator (the latter  
 437 approximation increasingly breaks down at higher excitation energies, however).  
 438 Finally, both tables have been computed for fixed  $N$ , but also different values of  $N$  need to be  
 439 considered. Table 3 shows the ground state eigenvalue  $\epsilon_0^N$  of the matrix  $J_{N+1}/N$ .

Table 3.  $N\epsilon_0^N$  for increasing  $N$  ( $J = 1, B = 1/2$ )

N	$N\epsilon_0^N$
100	0.8473
1000	0.8633
2500	0.8653
5000	0.8655

441 Thus  $\epsilon_0^N$  will approximate  $\frac{1/2\sqrt{3}}{N}$  when  $N$  increases, which confirms (3.36).

## 442 4 Symmetry breaking in the Curie–Weiss model

443 In this section we introduce a perturbation in the quantum Curie–Weiss model  $h_N^{\text{CW}}$  such  
 444 that the symmetric and hence delocalized ground state as displayed in Figure 1 breaks the  
 445  $\mathbb{Z}_2$  symmetry and hence localizes already for finite (but large)  $N$ . To find the appropriate  
 446 perturbation of the Curie–Weiss Hamiltonian, let us first continue the discussion in the  
 447 Introduction by reviewing the flea perturbation of the symmetric double well potential,  
 448 following Ref. [1, 2, 26].

### 449 4.1 Review of the “flea” perturbation on the double well potential

450 Generalizing (1.2), consider a one-dimensional Schrödinger operator

$$h_{\hbar} = -\hbar^2 \frac{d^2}{dx^2} + V(x), \tag{4.1}$$

451 where the potential  $V$  is  $C^\infty$ , non-negative, strictly positive at  $\infty$ , zero at two points  $m_1$  and  
 452  $m_2 > m_1$ , and  $\mathbb{Z}_2$ -symmetric. Then consider the Agmon metric  $d_V$  on  $\mathbb{R}$ , defined by

$$d_V(x, y) = \int_x^y \sqrt{V(s)} ds. \tag{4.2}$$

453 A flea perturbation  $\delta V$  is  $C^\infty$ , non-negative, bounded, and such that  $\delta V(x) = 0$  in  
 454 neighbourhoods of the minima  $m_1$  and  $m_2$  of  $V$ . Nonetheless, its support should be close  
 455 to one of these minima, in the following sense. We use the following notation:

- 456 •  $d_0 = d_V(m_1, m_2)$  is the Agmon distance between the two minima of  $V$ ;

457 •  $d_1 = 2 \min\{d_V(m_1, \text{supp } \delta V), d_V(m_2, \text{supp } \delta V)\}$  is twice the Agmon distance between  
 458 the support of  $\delta V$  and the minimum that is closest to to this support;

459 •  $d_2 = 2 \max\{d_V(m_1, \text{supp } \delta V), d_V(m_2, \text{supp } \delta V)\}$  is twice the Agmon distance between  
 460 the support of  $\delta V$  and the minimum that is furthest away from this support.

461 Finally, and perhaps most crucially, as  $\hbar \rightarrow 0$  the perturbation should dominate the energy  
 462 difference  $\Delta(E)_\hbar = E_1(\hbar) - E_0(\hbar)$  between the ground state of the unperturbed Hamiltonian  
 463 (4.1) and the first excited state. But since  $\Delta(E)_\hbar \sim \exp(-d_0/\hbar)$ , satisfying this condition is a  
 464 piece of cake. Detailed analysis [2] then shows that:

- 465 • If  $d_0 < d_1 \leq d_2$  there is no localization of the ground state.
- 466 • If  $d_1 < d_0 \leq d_2$  the ground state localizes near the minimum  $m_i$  furthest from the support  
 467 of  $\delta V$  (if  $\delta V$  were negative, it would localize closest to the support of  $\delta V$ );
- 468 • If  $d_1 < d_2 < d_0$  the ground state localizes as in the previous case.

469 Though perhaps surprising at first sight, this is actually easy to understand, either from  
 470 energetic considerations or from a  $2 \times 2$  matrix analogy [2, 7]. First, the ground state tries  
 471 to minimize its energy according to the rules:

- 472 • The cost of localization (if  $\delta V = 0$ ) is  $\mathcal{O}(e^{-d_0/\hbar})$ .
- 473 • The cost of turning on  $\delta V$  is  $\mathcal{O}(e^{-d_1/\hbar})$  when the wave-function is delocalized.
- 474 • The cost of turning on  $\delta V$  is  $\mathcal{O}(e^{-d_2/\hbar})$  when the wave-function is localized in the well  
 475 around  $x_0 = m_i$  for which  $d_V(m_i, \text{supp } \delta V) = d_2$  ( $i = 1, 2$ ).

476 For the latter, define a 2-level Hamiltonian

$$h_\hbar^{(2)} = \frac{1}{2} \begin{pmatrix} 0 & -\Delta(E)_\hbar \\ -\Delta(E)_\hbar & 0 \end{pmatrix}. \tag{4.3}$$

477 The eigenvalues and eigenvectors of  $h_\hbar^{(2)}$ , respectively, are given by

$$E_0(\hbar) = -\frac{1}{2}\Delta(E)_\hbar \qquad \varphi^{(0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \tag{4.4}$$

$$E_1(\hbar) = \frac{1}{2}\Delta(E)_\hbar \qquad \varphi^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{4.5}$$

478 Hence  $E_1(\hbar) - E_0(\hbar) = \Delta(E)_\hbar$ , and the (metaphorically) localized states would be

$$\varphi^+ \equiv \frac{1}{2}(\varphi^{(0)} + \varphi^{(1)}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \varphi^- \equiv \frac{1}{2}(\varphi^{(0)} - \varphi^{(1)}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4.6}$$

479 Now take  $\delta > 0$  and introduce a “flea” perturbation by changing  $h_\hbar^{(2)}$  to  $h_\hbar^{(2)} + \delta^{(2)}V$ , where

$$\delta^{(2)}V = \begin{pmatrix} 0 & 0 \\ 0 & \delta \end{pmatrix}. \tag{4.7}$$

480 The eigenvalues of  $h_\hbar^{(2)} + \delta^{(2)}V$  then shift from  $(E_0(\hbar), E_1(\hbar))$  to  $(E_-(\hbar), E_+(\hbar))$ , where

$$E_\pm(\hbar) = \frac{1}{2} \left( \delta \pm \sqrt{\delta^2 + \Delta(E)_\hbar^2} \right), \tag{4.8}$$

481 with corresponding normalized eigenvectors  $(\psi_{\hbar}^-, \psi_{\hbar}^+)$  given by

$$\psi_{\hbar}^- = \frac{1}{\sqrt{2}} \left( \delta^2 + \Delta(E)_{\hbar}^2 + \delta \sqrt{\delta^2 + \Delta(E)_{\hbar}^2} \right)^{-1/2} \begin{pmatrix} \Delta(E)_{\hbar} \\ \delta + \sqrt{\delta^2 + \Delta(E)_{\hbar}^2} \end{pmatrix}; \quad (4.9)$$

$$\psi_{\hbar}^+ = \frac{1}{\sqrt{2}} \left( \delta^2 + \Delta(E)_{\hbar}^2 - \delta \sqrt{\delta^2 + \Delta(E)_{\hbar}^2} \right)^{-1/2} \begin{pmatrix} \Delta(E)_{\hbar} \\ \delta - \sqrt{\delta^2 + \Delta(E)_{\hbar}^2} \end{pmatrix}. \quad (4.10)$$

482 As long as  $\lim_{\hbar \rightarrow 0} \Delta(E)_{\hbar} / \delta = 0$ , we have

$$\lim_{\hbar \rightarrow 0} \psi_{\hbar}^{\pm} = \varphi^{\pm}, \quad (4.11)$$

483 so that under the influence of the flea perturbation the ground state localizes as  $\hbar \rightarrow 0$ . There  
484 is no need for a separate limit  $\delta \rightarrow 0$ , since it follows from the limit  $\hbar \rightarrow 0$ .

485 Returning to the real thing, a (mathematically) very natural flea-like perturbation  $\delta V$  for the  
486 Schrödinger operator  $h_{\hbar}$ , and the one we shall mimic for the Curie–Weiss model, is

$$\delta V_{b,c,d}(x) = \begin{cases} d \exp \left[ \frac{1}{c^2} - \frac{1}{c^2 - (x-b)^2} \right] & \text{if } |x-b| < c \\ 0 & \text{if } |x-b| \geq c \end{cases}, \quad (4.12)$$

487 where the parameters  $(b, c, d)$  represent the location of its center  $b$ , its width  $2c$  and its height  
488  $d$ , respectively. Tuning these, the conditions above can be satisfied in many ways: for example,  
489 if  $b > c > m_2$  the condition  $d_1 < d_0 \leq d_2$  for asymmetric localization reads

$$2 \int_{m_2}^{b-c} \sqrt{V(s)} < \int_{m_1}^{m_2} \sqrt{V(s)} ds \leq 2 \int_{m_1}^{b-c} \sqrt{V(s)}, \quad (4.13)$$

490 which can be satisfied by putting  $b$  close to  $m_2$  (depending on the central height of  $V$ ).

## 491 4.2 Perturbation of the Curie–Weiss Hamiltonian

492 The next step in our analysis, then, is to find an analogous perturbation to (4.12) but  
493 now for the Curie–Weiss Hamiltonian, using the fact that the Curie–Weiss model is related  
494 to a Schrödinger operator (see previous section). To this end, recall the symmetriser  $S_N$   
495 defined in (1.10), which is a projection onto the space of all totally symmetric vectors. As  
496 we have seen, a basis for the space of totally symmetric vectors is given by the vectors  
497  $\{|n_+, n_-| n_+ = 0, \dots, N\}$ , which spans the subspace  $\text{Sym}^N(\mathbb{C}^2)$ . In order to define the  
498 symmetry-breaking flea perturbation, again we may pick a basis for  $\mathcal{H}_N$  and define the  
499 perturbation on a basis for  $\mathcal{H}_N$ . Since the original Hamiltonian was defined on the standard  
500 basis  $\beta$ , we do the same for the perturbation. In the proof of Theorem 2.1, we have seen there  
501 is a bijection between the number of orbits and the dimension of  $\text{Sym}^N(\mathbb{C}^2)$ , namely

$$\mathcal{O}^k \leftrightarrow |N-k, k\rangle, \quad (4.14)$$

502 where  $k$  in  $|N-k, k\rangle$  labels the number of occurrences of the vector  $e_2$  in any of the basis  
503 vectors  $\beta_i \in \beta$ , and likewise  $N-k$  in  $|N-k, k\rangle$  labels the number of occurrences of the vector  
504  $e_1$  in  $\beta_i$ , so that  $N-k$  stands for the number of spins in the up direction whereas the second  
505 position  $k$  denotes the number of down spins. By definition,  $S_N$  maps any basis vector  $\beta_k \in \beta$   
506 in a given orbit  $\mathcal{O}^k$  to the same vector in  $\text{Sym}^N(\mathbb{C}^2)$ , which equals

$$\frac{1}{\sqrt{\binom{N}{k}}} \sum_{l=1}^{\binom{N}{k}} \beta_{k_l}. \quad (4.15)$$



507 Here the suffix  $l$  in  $\beta_{k_l}$  labels the basis vector  $\beta_k \in \beta$  within the same orbit  $\mathcal{O}^k$ . So for each  
 508 orbit  $\mathcal{O}^k$ , we have  $\binom{N}{k}$  vectors  $\beta_k$ . Hence for each  $l = 1, \dots, \binom{N}{k}$  the image  $S_N(\beta_{k_l})$  under  $S_N$  is  
 509 always the same, namely the coordinate vector written with respect to  $\beta$ .  
 510 The perturbation we are going to define is very similar to the symmetriser  $S_N$ . Of course, since  
 511 we have expressed our original Curie–Weiss Hamiltonian with respect to this  $|n_+, n_-\rangle$  basis,  
 512 we need to do the same for the perturbation. Since we have a partition of our  $2^N$ -dimensional  
 513 basis  $\beta$  into  $N + 1$  orbits, we define our perturbation  $\Delta V_N$  by

$$\Delta V_N(\beta) = \delta V_{b,c,d} \left( \frac{k}{N} \right) S_N(\beta), \tag{4.16}$$

514 where  $k \in \{0, \dots, N\}$  is the unique number such that  $\beta \in \mathcal{O}^k$ , and  $\delta V_{b,c,d}$  is defined by (4.12).  
 515 We will see that a specific choice of parameters results in localization of the ground state  
 516 as  $N \rightarrow \infty$ . First, note that when we transform the matrix  $[\Delta V_N]_\beta$  in the  $\beta$  basis to the  
 517 corresponding matrix in the  $|n_+, n_-\rangle$  basis, it is obvious that it becomes a diagonal matrix with  
 518 the value  $\Delta V_k \equiv \delta V_{b,c,d}(k/N)$  at entry  $(k, k)$ , since all basis vectors within the same orbit are  
 519 mapped to the same vector under  $\Delta V_N$ . If we can show that

$$[S_N, \Delta V_N] = 0, \tag{4.17}$$

520 and that the ground state eigenfunction of the perturbed Hamiltonian  $h_N + \Delta V_N$  is unique  
 521 and positive,<sup>17</sup> then we may conclude that the ground state lies in the subspace  $\text{Sym}^N(\mathbb{C}^2)$ .  
 522 The reason for this is the same as for the unperturbed Curie–Weiss Hamiltonian: these  
 523 properties push this eigenvector into the subspace  $\text{ran}(S_N) = \text{Sym}^N(\mathbb{C}^2)$ , so that we may  
 524 diagonalize this Hamiltonian represented as a matrix that can be written with respect to  
 525 the symmetric subspace, which will be a tridiagonal matrix of dimension  $N + 1$  as well.  
 526 This makes computations much easier, and allows one to compare the unperturbed system  
 527 with the perturbed one. Similarly as for the Curie–Weiss model, a sufficient condition for  
 528 uniqueness and positivity of the ground state of the perturbed matrix, originally written with  
 529 respect to the standard basis for  $\mathcal{H}_N$ , is non-negativity and irreducibility, so that we can apply  
 530 the Perron–Frobenius Theorem, as explained in Appendix A. This depends, of course, on the  
 531 parameters of the perturbation  $\Delta V_N$ . We will come back to this later. In order to prove (4.17),  
 532 it suffices to do so on a basis, for which we take the standard basis  $\beta$  of the  $N$ -fold tensor  
 533 product. Fix a basis vector  $\beta$  in  $\mathcal{O}^k$ . Then we immediately find

$$\Delta V_N S_N(\beta) = \delta V_{b,c,d}(k) S_N^2(\beta) = S_N \delta V_{b,c,d}(k) S_N(\beta) = S_N \Delta V_N(\beta), \tag{4.18}$$

534 which proves (4.17). The last step is to show that the Hamiltonian  $-(h_N + \Delta V_N)$ , written  
 535 with respect to the standard basis  $\beta$  for  $\mathcal{H}_N$ , is a non-negative and irreducible matrix. Since  
 536 the off-diagonal elements are completely determined by the unperturbed Hamiltonian and are  
 537 never zero, the matrix can never be decomposed into two blocks, so that it remains irreducible.  
 538 Non-negativity is achieved when

$$\frac{J}{2N} (2n_+ - N)^2 - \Delta V_{n_+} \geq 0, \tag{4.19}$$

539 which is clearly satisfied for positive  $d$ . Therefore, taking  $d > 0$  together with the fact that  
 540  $h_N + \Delta V_N$  commutes with  $S_N$ , shows in the same way as for the unperturbed Curie–Weiss model

<sup>17</sup>Equivalently, without using positivity of the eigenfunction one would reach the same conclusion by showing that the ground state is unique and  $\Delta V_N$  commutes with the entire permutation group on  $N$  elements (like the unperturbed Hamiltonian). Given positivity, it is enough to check commutativity merely with the projection  $S_N$ , because all nontrivial permutations would transform the ground state wavefunction into a function that is no longer strictly positive. We are indebted to Valter Moretti for this comment.



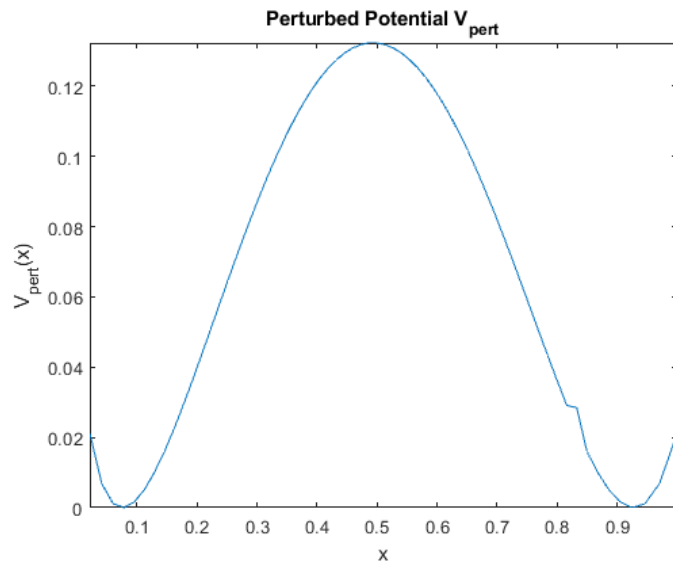


Figure 9: Perturbed potential computed from the tridiagonal matrix  $h_N^{CW} + \Delta V_N$  in the symmetric basis for  $N = 65$ ,  $b = (N - 9)/N$ ,  $c = 1/45$ ,  $d = 0.4$ ,  $J = 1$  and  $B = 1/2$ . This potential has a ‘flea’ on the right side of the well due the perturbation  $\Delta V_N$ .

541 that the ground state of the perturbed Hamiltonian is unique and positive, and therefore lies  
 542 in  $\text{ran}(S_N) = \text{Sym}^N(\mathbb{C}^2)$ , where it can be diagonalized.

543 Recall that in §2.2 the ground state  $\psi_N^{(0)}$  of the unperturbed Hamiltonian  $h_N^{CW}$  was  
 544 approximately given by two Gaussians (for  $N$  large), each of them located in one of the wells  
 545 of the potential, and was given by

$$\psi_N^{(0)} \cong \frac{T_a(\varphi_0) + T_{-a}(\varphi_0)}{\sqrt{2}}. \tag{4.20}$$

546 We now show numerically that our flea perturbation  $\Delta V_N$  forces the ground state to localize  
 547 for large  $N$ , leaving an analytic proof à la Simon (1985) to the future.

548 As in section 3, we extract the potential corresponding to the perturbed Hamiltonian  $h_N + \Delta V_N$ ,  
 549 written with respect to the symmetric basis, scale this Hamiltonian by  $1/N$ , and translate  
 550 the potential so that its minima are set to zero. We plot this perturbed potential on the unit  
 551 interval in Figure 9, where for convenience we scale the domain to the unit interval. Moreover,  
 552 we plot the ground state of this Hamiltonian and the one corresponding to the unperturbed  
 553 one in Figure 10, observing localization of the ground state in the left sided well. Numerical  
 554 simulations show that the eigenvalues of the perturbed Hamiltonian are non-degenerate, so  
 555 that the ground state is unique, and hence localization is not a result of numerical degeneracy  
 556 but is genuinely caused by the perturbation. A similar simulation for flea perturbation, but  
 557 now located on the left site of the barrier, is shown in Figure 11. As expected, we now see a  
 558 localization of the ground state to the right side of the barrier (Figure 12).

559 Our conclusion is that due to the flea perturbation, the ground state will localize in one of  
 560 the wells depending on where the flea is put. As in the continuous Schrödinger operator  
 561 case, this localization may be understood from energetic considerations: for example, if the  
 562 perturbation is located on the right, then the relative energy in the left-hand part of the double  
 563 well is lowered, so that localization will be to the left. This even happens if the perturbation  
 564 vanishes in the limit  $N \rightarrow \infty$ . We have seen that our (unscaled) tridiagonal matrix  $J_{N+1}$  to a

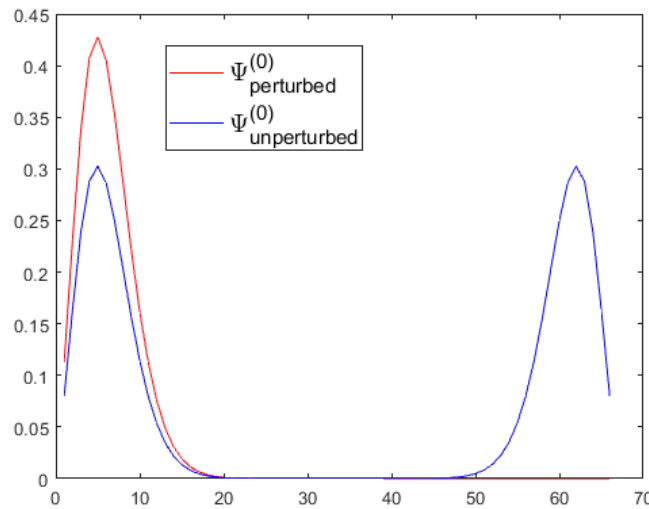


Figure 10: Corresponding ground state (in red) of the perturbed Hamiltonian  $h_N^{CW} + \Delta V_N$  is already localized for  $N = 65$ ,  $b = (N - 9)/N$ ,  $c = 1/45$ ,  $d = 0.4$ ,  $J = 1$  and  $B = 1/2$ . Localization takes place on the left side of the well, since the flea raises the potential on the right side.

565 very good approximation, is a discretization of the operator  $Nh$ , where

$$h = -\frac{1}{N^2 L^2} \Delta_y + V, \tag{4.21}$$

566 where  $\Delta_y = d^2/dy^2$  and  $V$  the double well potential. It follows that for the perturbed  
567 Hamiltonian (where  $\Delta V_N = O(1)$  fixed, as in our case)

$$\begin{aligned} J_N + \Delta V_N &\approx N \left( -\frac{1}{(NL)^2} \Delta_y + V \right) + \Delta V_N \\ &= N \left( -\frac{1}{(NL)^2} \Delta_y + V + \Delta V_N/N \right), \end{aligned} \tag{4.22}$$

568 which implies that the perturbation  $\Delta V_N/N$  effectively disappears as  $N \rightarrow \infty$ . According to  
569 Jona-Lasinio *et al* (1981) and Graffi *et al* (1984), we can even take

$$\Delta V_N = O(1/N); \tag{4.23}$$

$$\Delta V_N/N = O(1/N^2), \tag{4.24}$$

570 and also in this case a collapse of the ground state takes place. This is also clear from the  $2 \times 2$   
571 matrix argument given in the previous section.

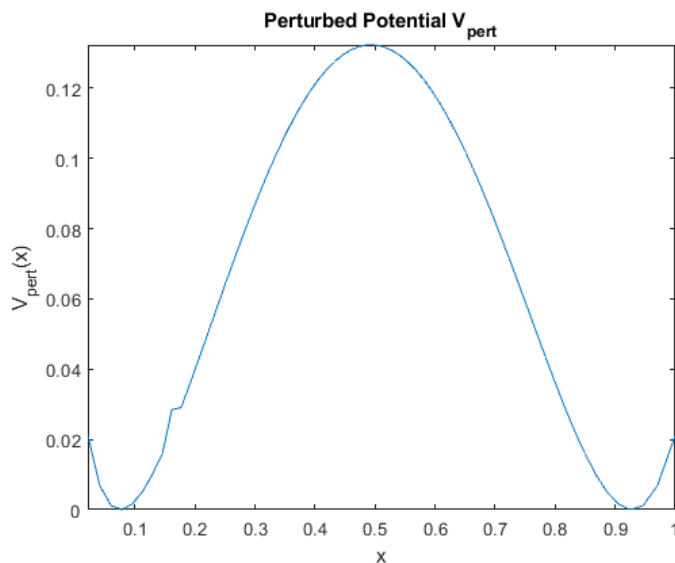


Figure 11: Perturbed potential computed from the tridiagonal matrix  $h_N^{CW} + \Delta V_N$  in the symmetric basis for  $N = 65$ ,  $b = 9/N$ ,  $c = 1/45$ ,  $d = 0.4$ ,  $J = 1$  and  $B = 1/2$ . This potential has a ‘flea’ on the left side of well due the perturbation  $\Delta V_N$ .

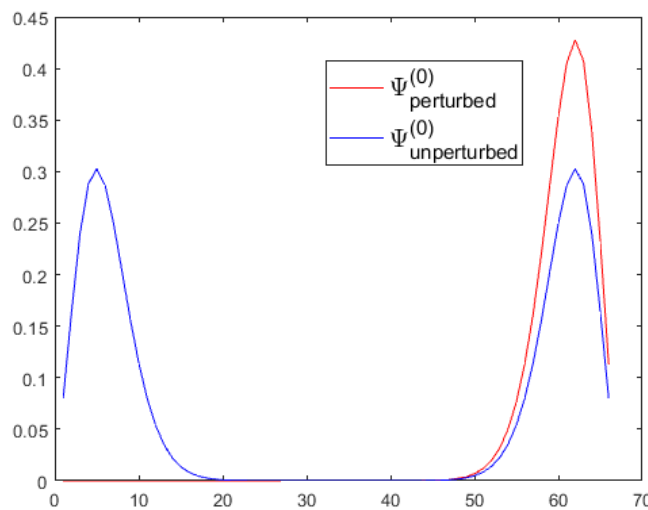


Figure 12: Corresponding ground state (in red) of the perturbed Hamiltonian  $h_N^{CW} + \Delta V_N$  is already localized for  $N = 65$ ,  $b = 9/N$ ,  $c = 1/45$ ,  $d = 0.4$ ,  $J = 1$  and  $B = 1/2$ . Localization takes place on the right side of the well, since the flea raises the potential on the left side.

572 **5 Conclusion**

573 We have established a link between the quantum Curie–Weiss Hamiltonian and a 1d  
 574 Schrödinger operator describing a particle in a symmetric double well potential for  $\hbar > 0$ ,  
 575 where  $\hbar = 1/N$ . We have shown that the scaled quantum Curie–Weiss Hamiltonian  
 576 restricted to the  $(N + 1)$ -dimensional subspace  $\text{Sym}^N(\mathbb{C}^2)$  approximates a discretization matrix  
 577 corresponding to this Schrödinger operator, defined on  $L^2([0, 1])$ . Subsequently, we have  
 578 shown that due to a small perturbation a  $\mathbb{Z}_2$ -symmetry of the Curie–Weiss model can already

579 be explicitly broken for finite  $N$ , resulting in a pure ground state in the classical limit. This  
 580 confirms Anderson’s mechanism for SSB in finite quantum systems (cf. the Introduction), but  
 581 our specific “flea” perturbation came from similar results for Schrödinger operators with a  
 582 symmetric double well potential in the classical limit  $\hbar \rightarrow 0$  [1, 2, 26]. The results in these  
 583 papers, which were obtained analytically, precisely match ours, obtained numerically, but this  
 584 still leaves the challenge of finding analytic proofs of our results. Furthermore, our approach,  
 585 and especially the specific perturbations we use, should be extended to the case of continuous  
 586 symmetries, where the relevant low-lying states (which for continuous symmetries are infinite  
 587 in number as  $N \rightarrow \infty$ ) now seem to be completely understood [4]. The dynamics of the  
 588 transition from a localized ground state of the unperturbed Hamiltonian to a delocalized  
 589 ground state of the perturbed Hamiltonian as  $N \rightarrow \infty$  remains to be understood (this is  
 590 an understatement); once achieved, it would perhaps also contribute to the solution of the  
 591 measurement problem or Schrödinger Cat problem along similar lines [7, 8].

592 We finally discuss some of the correspondences as well as differences between the  
 593 symmetry-breaking perturbations we used and those considered in the condensed matter  
 594 physics literature. The key physical idea is the same in both cases:<sup>18</sup>

595       “The general idea behind spontaneous symmetry breaking is easily formulated: as  
 596       a collection of quantum particles becomes larger, the symmetry of the system as a  
 597       whole becomes more unstable against small perturbations,”

598 where we add that the same is true as  $\hbar$  becomes smaller at fixed system size, cf. [7] and  
 599 references therein, and that the precise form of the instability is that for small  $N$  or large  $\hbar$  the  
 600 perturbation plays almost no role (in either the spectral properties of the Hamiltonian or in its  
 601 eigenfunctions), whereas for large  $N$  or small  $\hbar$  it metaphorically

602       “can irritate the elephant enough so that it shifts its weight, i.e., we will see  
 603       that the ground state, instead of being asymptotically in both wells, may reside  
 604       asymptotically in only one well”.

605 As explained in the Introduction, this accounts for the fact that real materials (which are  
 606 described by the quantum theory of finite systems) do display SSB, even though the theory  
 607 seems to forbid this. As we (and most condensed matter physicists) see it, these perturbations  
 608 should arise naturally and might correspond either to imperfections of the material or  
 609 contributions to the Hamiltonian from the (otherwise ignored) environment.

610 Mathematically, though, there are some differences between our mathematical physics  
 611 approach to SSB in finite quantum systems and the standard theoretical physics one.<sup>19</sup> These  
 612 differences are perhaps best explained by starting with the very *definition* of SSB. High  
 613 sensitivity to small perturbations in the relevant regime are common to both approaches  
 614 (as they are to the classical theory of critical phenomena and phase transitions, and even  
 615 to complexity theory, which is full of bifurcations and tipping points, cf. [39]. In the physics  
 616 literature this sensitivity to small perturbations is usually taken into account by adding an  
 617 “infinitesimal” symmetry-breaking term like

$$\delta h_N = \varepsilon \sum_{x=1}^N \sigma_3(x) \quad (5.25)$$

618 to the Curie–Weiss Hamiltonian (2.1) and arguing that the correct order of the limits in  
 619 question is  $\lim_{\varepsilon \rightarrow 0} \lim_{N \rightarrow \infty}$ , which gives SSB by one of the two pure ground states on the

<sup>18</sup>The first quotation is from Ref. [13] and the second one is from Ref. [2].

<sup>19</sup>See references in footnote 3.

620 limit algebra, the sign of  $\varepsilon$  determining the direction of symmetry breaking. In contrast, the  
 621 opposite order  $\lim_{N \rightarrow \infty} \lim_{\varepsilon \rightarrow 0}$  gives a symmetric but mixed and hence unstable or unphysical  
 622 ground state on the limit algebra.<sup>20</sup> Thus whenever there is difference

$$\lim_{\varepsilon \rightarrow 0} \lim_{N \rightarrow \infty} \neq \lim_{N \rightarrow \infty} \lim_{\varepsilon \rightarrow 0}, \quad (5.26)$$

623 this is taken to be a defining property SSB. This is valid, but it feeds the idea that, if SSB occurs,  
 624 the limit  $N \rightarrow \infty$  is “singular” (e.g. [13, 27, 28]), an idea that is increasingly challenged in  
 625 the philosophical literature [9] and has almost disappeared from the mathematical physics  
 626 literature.

627 Instead, we work with a definition of SSB that is standard in mathematical physics and applies  
 628 equally to finite and infinite systems (provided these are described correctly), and to classical  
 629 and quantum systems, namely that the ground state (suitably defined) of a system with  
 630  $G$ -invariant dynamics (where  $G$  is some group, typically a discrete group or a Lie group) is  
 631 either pure but not  $G$ -invariant, or  $G$ -invariant but mixed.<sup>21</sup> Accordingly, what is singular  
 632 about the thermodynamic limit of systems with SSB is the fact that the exact *pure* ground state  
 633 of a finite quantum system converges to a *mixed* state on the limit system.<sup>22</sup> However, this  
 634 singular behaviour is exactly what is *avoided* by Anderson’s tower of states triggered by the  
 635 right perturbations, where a single limit (i.e. either  $\hbar \rightarrow 0$  or  $N \rightarrow \infty$ ) suffices,<sup>23</sup> in which the  
 636 (still) *pure* ground state of the perturbed Hamiltonian (which is a symmetry-breaking linear  
 637 combination of low-lying states) converges to some symmetry-breaking *pure* ground state on  
 638 the limit system (be it a classical system or an infinite quantum system). The ensuing limit is  
 639 then duly continuous in an appropriate meaning of the word (which it would *not* be without  
 640 the perturbation mechanism).<sup>24</sup>

641 Of course, in order for this symmetry breaking to be *spontaneous* rather than *explicit*, the  
 642 perturbation should be small to begin with, and should disappear in the pertinent limit.<sup>25</sup>  
 643 As we have seen, it is easy to endow the “flea”  $\delta V$  with these properties; for symmetry breaking  
 644 in the double well potential all we need is that  $\Delta E \rightarrow 0$  more rapidly than  $\delta V \rightarrow 0$  as  $\hbar \rightarrow 0$ ,  
 645 cf. (4.22) - (4.24), and for symmetry breaking in the Curie–Weiss model the same holds for  
 646  $N \rightarrow \infty$ . Since in these models  $\Delta E$  vanishes exponentially in  $-1/\hbar$  or  $-N$  as  $\hbar \rightarrow 0$  or  $N \rightarrow \infty$   
 647 (a fact about the spectrum that has nothing to do with the perturbations), this can hardly go  
 648 wrong.

649 In this light, the following may help explain the relationship between our approach and the  
 650 traditional one based on the non-commuting limits (5.26). For the latter, consider the  $(x, y)$   
 651 plane with  $x = 1/N$  (or  $x = \hbar$ ) and  $y = \varepsilon$ , and for the argument in the previous paragraph,

<sup>20</sup>This procedure goes back at least to Bogoliubov [40], see also [41].

<sup>21</sup>See e.g. [7], Definition 10.3, page 379, and [42]. It may seem more natural to just require that the ground state fails to be  $G$ -invariant, but in the  $C^*$ -algebraic formalism we rely on ground states that are not necessarily pure, which leaves the possibility of forming  $G$ -invariant mixtures of non-invariant states that lose the purity properties one expects physical ground states to have. Similarly for equilibrium states, where ‘pure’ is replaced by ‘primary’, which is a mathematical property of a pure thermodynamical phase. Order parameters *follow* from this definition, cf. §10.3, *loc. cit.*

<sup>22</sup>And similarly for the limit  $\hbar$  of the symmetric double well system, where, as pointed out in the Introduction, the  $\hbar = 0$  limit of the ground state is the mixed state (1.4), as opposed to, for example, a Dirac delta-function located between the wells (i.e. at  $q = 0$ ). See also Refs. [2, 43]. This also explains the fact that the flea perturbations even work if their support is localized away from the bottoms of the two wells (or, equivalently, from the two peaks of the unperturbed ground state); see especially Ref. [26] for a detailed explanation of this point.

<sup>23</sup>This can also be achieved by letting  $\varepsilon$  depend on  $N$  in some suitable way, but if one does so one might as well drop the factorized form of perturbations like (5.25) altogether and admit more general expressions similar to the flea. Some literature indeed seems to do this, though not in a mathematically precise way.

<sup>24</sup>This is described via continuous fields of  $C^*$ -algebras and states ([7], Chapters 7–10).

<sup>25</sup>This must be taken to be a purely mathematical criterion, for real systems have  $\hbar > 0$  and  $N < \infty$ , so that strictly speaking any form of symmetry breaking in Nature is explicit rather than spontaneous.

652 take  $x = \delta V$  and  $y = \Delta E$ . In case of SSB, certain physical quantities  $m(x, y)$  (like the  
 653 magnetization) are discontinuous at  $(0, 0)$  and hence their value at  $(0, 0)$  depends on the  
 654 path towards the origin. Eq. (5.26) expresses this path dependence in a crude way, which  
 655 is captured by perturbations like (5.25), but the perturbations we consider follow specific  
 656 *parametrized paths* towards  $(0, 0)$ . This explains why we are able to work with a single  
 657 limit  $\hbar \rightarrow 0$ , since in our models  $\Delta E = \Delta E(\hbar)$  is *given* (by the double well or Curie–Weiss  
 658 Hamiltonian) and  $\delta V = \delta V(\hbar)$  can be freely *chosen*, subject to the condition just mentioned,  
 659 i.e. that the approach  $\Delta E(\hbar) \rightarrow 0$  (as  $\hbar \rightarrow 0$ ) must be quicker than  $\delta V(\hbar)$ .

660 The wealth of possible flea perturbations, as opposed to the more straightforward  
 661 symmetry-breaking perturbations à la (5.25) considered in the condensed matter physics  
 662 literature (i.e. coupling to a small constant external magnetic field) also weakens the  
 663 critique expressed by Wallace [44], to the effect that cooling e.g. ferromagnets (but  
 664 also antiferromagnets, (anti)ferroelectrics, ferroelastics, and superconductors) does not,  
 665 experimentally, lead to a ground state in which the spins are all aligned with the external field,  
 666 but rather to a state with various domains in which the spins are merely aligned locally but  
 667 may differ quite randomly from domain to domain ([45], pp. 346–354). Obtaining the right  
 668 domain sizes admittedly requires fairly special fleas (as Wallace points out), but in the absence  
 669 of any dynamical theory of cooling any argument in this direction, including this critique, is  
 670 speculative. Wallace’s suggestion that the formation of domains with specific sizes has already  
 671 been explained by energetic considerations of the kind presented by Kittel (*loc. cit.*, attributed  
 672 to Landau and Lifshitz), which even require the *absence* of constant external magnetic fields,  
 673 is not effective against flea perturbations, which indeed are *required* to explain why *specific*  
 674 domains in a *given* specimen (as opposed to *general* domains in *generic* materials) are formed;  
 675 the problem is quite analogous to explaining SSB. Kittel also draws attention to the importance  
 676 of single-domain regions, e.g. for magnetic recording devices, but also in sedimentary rock  
 677 formations.

678 Another objection to our approach to SSB, which equally well applies to the standard approach  
 679 in condensed matter physics, is that a mechanism based on symmetry-breaking perturbations  
 680 cannot be applied to gauge theories and hence cannot explain the Higgs mechanism. However,  
 681 it is not local gauge symmetry that is broken in the Higgs mechanism, but a global symmetry,  
 682 and by choosing gauge-invariant observables it is even possible to describe it without any  
 683 reference to SSB, see Ref. [7], §10.10.

## 684 A Perron–Frobenius Theorem

685 In this appendix we provide the machinery for proving uniqueness and strict positivity of  
 686 the ground state of the Curie–Weiss model for any finite  $N$ , based on the Perron-Frobenius  
 687 Theorem. Though the result is well known, the precise combination of arguments is hard to  
 688 find in the literature.

689 We start with some definitions and basis facts.

690 **Definition A.1.** 1. A square matrix is called non-negative if all its entries are non-negative.  
 691 It is called strictly positive if all its entries are strictly positive.

692 2. A non-negative matrix  $a$  is called irreducible if for every pair indices  $i$  and  $j$  there exists a  
 693 natural number  $m$  such that  $(a^m)_{ij}$  is not equal to zero. If the matrix is not irreducible, it  
 694 is said to be reducible.

695 3. A directed graph is a graph  $G = (V, E)$  with vertices  $V$  and edges  $E$  such that the vertices  
 696 are connected by the edges, and where the edges have a direction. A directed graph is also  
 697 called a digraph.

698 4. A digraph is called strongly connected if there is a directed path  $x$  to  $y$  between any two  
 699 vertices  $x, y$ .

700 We use the notion of the directed graph or digraph of a square  $N$ -dimensional matrix  $a$ ,  
 701 denoted by  $G(a)$ . We say that the digraph of  $a$  is the digraph with

$$V = \{1, 2, \dots, N\},$$

$$E = \{(i, j) \mid a_{ij} \neq 0\}.$$

702 The following result links irreducibility of a non-negative matrix to strongly connectedness of  
 703 its corresponding digraph. The proof is easy and therefore omitted.

704 **Lemma A.2.** *A non-negative square matrix  $a$  is irreducible if and only if the digraph of  $a$  is*  
 705 *strongly connected.*

706 We now come to the Perron-Frobenius Theorem. There are two versions of this theorem: one  
 707 for *strictly positive* matrices, and the other for *irreducible* matrices. We use the version for  
 708 irreducible matrices since the Curie–Weiss Hamiltonian  $-\hbar_N^{CW}$ , represented with respect to the  
 709 standard basis for  $\bigotimes_{n=1}^N \mathbb{C}^2$ , is a non-negative and irreducible matrix of dimension  $2^N$ , as we  
 710 will see below.

711 **Theorem A.3.** *Let  $a$  be an  $N \times N$  real-valued non-negative matrix, and denote its spectral radius*  
 712 *by  $r(a) = \lambda$ . If  $a$  is irreducible, then  $\lambda = r(a)$  is an eigenvalue of  $a$ , which is positive, simple,*  
 713 *and corresponds to a strictly positive eigenvector.*

714 This theorem is based on properties of a matrix relative to some basis, so that the  
 715 Perron-Frobenius Theorem is valid if there exists a basis such that the matrix representation  
 716 of the operator in this basis satisfies the assumptions of the theorem. Note that multiplying  
 717  $-\hbar_N^{CW}$  by  $-1$ , the eigenvalues will change sign and we find instead that the smallest eigenvalue  
 718 (i.e. the ground state) of  $\hbar_N^{CW}$  is simple and corresponds to a strictly positive eigenvector. As a  
 719 case in point, we are now going to prove a statement about our Hamiltonian  $-\hbar_N^{CW}$ , relative to  
 720 the standard basis of  $\mathbb{C}^2$  extended to a basis of the tensor product  $\bigotimes_{n=1}^N \mathbb{C}^2$  in the usual way.

721 **Theorem A.4.** *The Curie–Weiss Hamiltonian  $-\hbar_N^{CW}$  from (1.6), represented in the standard basis*  
 722 *for  $\bigotimes_{n=1}^N \mathbb{C}^2$ , is non-negative and irreducible.*

723 *Proof.* Since all constant factors in  $-\hbar_N^{CW}$  are strictly positive, we only have to consider both  
 724 terms containing sums. We show that

$$\sum_{x, y \in \Lambda_N} \sigma_3(x) \sigma_3(y) \quad \text{and} \quad \sum_{x \in \Lambda_N} \sigma_1(x) \tag{A.1}$$

725 are non-negative. We have seen in the proof of Theorem 2.1 that the operator  
 726  $\sum_{x, y \in \Lambda_N} \sigma_3(x) \sigma_3(y)$  is a diagonal matrix with respect to the standard basis  
 727  $\{e_{n_1} \otimes \dots \otimes e_{n_N}\}_{n_1=1, \dots, n_N=1}^2$  for  $\bigotimes_{n=1}^N \mathbb{C}^2$ . For non-negativity we must prove, independently of  
 728 the basis vectors, that there are at least as many plus signs as there are minus signs, i.e., we  
 729 have to show that

$$N^2 - 2n_+(N - n_+) \geq 2n_+(N - n_+). \tag{A.2}$$

730 This gives  $N^2 - 4n_+(N - n_+) \geq 0$  if and only if  $N^2 - 4Nn_+ + 4n_+^2 \geq 0$ . The parabola

$$n_+ \mapsto N^2 - 4n_+N + 4n_+^2 \tag{A.3}$$



731 attains its minimum in  $n_+ = N/2$ , which is given by  $N^2 - 4\frac{N}{2}n + 4(\frac{N}{2})^2 = 0$ . So indeed, there  
 732 are at least as many plus signs as minus signs, so that the corresponding diagonal term is  
 733 non-negative. The other term  $\sum_{x \in \Lambda_N} \sigma_1(x)$  does not contain any negative entries at all, so if  
 734 we apply this to any basis vector  $\{e_{n_1} \otimes \dots \otimes e_{n_N}\}$ , we get a non-negative matrix. It follows that  
 735 both operators in (A.1) are non-negative in the basis under consideration.  
 736 Now we show that the matrix corresponding to the Curie–Weiss Hamiltonian is irreducible.  
 737 Note that irreducibility of a matrix does not depend on the basis in which the operator  
 738 is represented, since similar matrices define equivalent representations which preserve  
 739 irreducibility. We use Lemma A.2 to show that there is a direct path between any two vertices.  
 740 But this is obvious: the operator  $\sum_x \sigma_1(x)$  flips the spins one by one, and therefore the  
 741 associated digraph is clearly strongly connected as we can find a directed path between any  
 742 two vertices.<sup>26</sup>  $\square$

743 By the Perron-Frobenius Theorem, the largest eigenvalue of the Curie–Weiss Hamiltonian  
 744  $-h_N^{CW}$  is positive, simple and corresponds to a strictly positive eigenvector. This in turn implies  
 745 that the ground state eigenvalue of  $h_N^{CW}$  is positive, simple, and has a strictly positive  
 746 eigenvector.

## 747 B Discretization

748 This information provided in this appendix is based on Refs. [46–49]. These results are  
 749 used above in §3.1. Recall from calculus that the following approximations are valid for  
 750 the derivative of single-variable functions  $f(x)$ . The first one is called the *forward difference*  
 751 *approximation* and is an expression of the form

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h) \quad (h > 0). \tag{B.1}$$

752 The *backward difference approximation* is of the form

$$f'(x) = \frac{f(x) - f(x-h)}{h} + O(h) \quad (h > 0). \tag{B.2}$$

753 Furthermore, the *central difference approximation* is

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2) \quad (h > 0). \tag{B.3}$$

754 The approximations are obtained by neglecting the error terms indicated by the O-notation.  
 755 These formulas can be derived from a Taylor series expansion around  $x$ ,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \dots = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x), \tag{B.4}$$

756 and

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{h^n}{n!} f^{(n)}(x), \tag{B.5}$$

757 where  $f^{(n)}$  is the  $n^{\text{th}}$  order derivative of  $f$ . Subtracting  $f(x)$  from both sides of the above two  
 758 equations and dividing by  $h$  respectively  $-h$  leads to the forward difference respectively the

<sup>26</sup>A different proof is given in Ref. [34], §5.3, p.78.



759 backward difference. The central difference is obtained by subtracting equation (B.5) from  
 760 equation (B.4) and then dividing by  $2h$ .

761 The question is how small  $h$  has to be in order for the algebraic difference  $\frac{f(x+h)-f(x)}{h}$  (i.e. in  
 762 this case the forward difference approximation) to be good approximation of the derivative.  
 763 It is clear from the above formulas that the error for the central difference formula is  $O(h^2)$ .  
 764 Thus, central differences are significantly better than forward and backward differences.  
 765 Higher order derivatives can be approximated using the Taylor series about the value  $x$

$$f(x + 2h) = \sum_{n=0}^{\infty} \frac{(2h)^n}{n!} f^{(n)}(x) \tag{B.6}$$

766 and

$$f(x - 2h) = \sum_{n=0}^{\infty} (-1)^n \frac{(2h)^n}{n!} f^{(n)}(x). \tag{B.7}$$

767 A forward difference approximation to  $f''(x)$  is then

$$\frac{f(x + 2h) - 2f(x + h) + f(x)}{h^2} + O(h), \tag{B.8}$$

768 and a centered difference approximation is for example

$$\frac{f(x + h) - 2f(x) + f(x - h))}{h^2} + O(h^2). \tag{B.9}$$

769 Now we discretize the kinetic and potential energy operator. For simplicity, consider the  
 770 one-dimensional case. We first discretize the interval  $[0, 1]$  using a uniform grid of  $N$  points  
 771  $x_i = ih, h = \frac{1}{N}, i = 0, 1, \dots, N$ . It follows that  $f(x) \mapsto f(x_i) =: f_i$ . The Taylor series expansion  
 772 of a function about a point  $x_i$  becomes

$$f_{i+k} = f_i + \sum_{n=0}^{\infty} (-1)^n \frac{(kh)^n}{n!} f^{(n)}(x), \tag{B.10}$$

773 where  $k = \pm 1, \pm 2, \dots, \pm N$ . Similar as above, we can find central difference formulas for  $f'_j, f''_j$ ,  
 774 namely

$$f'_j = \frac{-f_{j-1} + f_{j+1}}{2h} + O(h^2) \tag{B.11}$$

$$f''_j = \frac{f_{j-1} - 2f_j + f_{j+1}}{h^2} + O(h^2). \tag{B.12}$$

775 The approximations are again obtained by neglecting the error terms.  
 776 Using this uniform grid with grid spacing  $h = 1/N$ , it follows that the second derivative  
 777 operator in one dimension is given by the tridiagonal matrix  $\frac{1}{h^2}[\dots -2 \ 1 \ \dots]_N$  and the  
 778 potential which acts as multiplication, is given by a diagonal matrix. With the notation  
 779  $\frac{1}{h^2}[\dots -2 \ 1 \ \dots]_N$ , we mean the  $N$ -dimensional matrix

780

$$\frac{1}{h^2} \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \ddots & \\ & & \mathbf{0} & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}.$$

781 Now suppose that the values of the kinetic energy operator  $T$  are non-uniformly dependent  
 782 of the positions in space. Then one needs to use a non-uniform grid in order to get a  
 783 good description of the second derivative. We use the central difference approximation and  
 784 approach  $f$  by a Taylor series.

785 Denote  $x_j$  by the  $j^{\text{th}}$  grid point and  $f_k = f(x_k)$ . Then the Taylor series of  $f$  at  $x_j$  can be written  
 786 as

$$f_k = f_j + \sum_{m=1}^{\infty} \frac{(x_k - x_j)^m}{m!} f_j^{(m)}. \quad (\text{B.13})$$

787 If we let  $h_j = x_{j+1} - x_j$ , then similarly as above, for a three-point finite-difference formula i.e.,  
 788 only  $f_{i+1}, f_i, f_{i-1}$  are used, we find that

$$f_{j+1} = f_j + h_j f_j' + \frac{h_j^2}{2} f_j'' + \frac{h_j^3}{6} f_j^{(3)} + \dots \quad (\text{B.14})$$

789 and similarly one can write  $x_{j-1} = x_j - h_{j-1}$ , so that we find

$$f_{j-1} = f_j - h_{j-1} f_j' + \frac{h_{j-1}^2}{2} f_j'' - \frac{h_{j-1}^3}{6} f_j^{(3)} + \dots \quad (\text{B.15})$$

790 Both expressions can be used to eliminate  $f_j'$  to derive an expression for the second derivative:

$$f_j'' = \frac{2f_{j-1}}{h_{j-1}(h_{j-1} + h_j)} - \frac{2f_j}{h_{j-1}h_j} + \frac{2f_{j+1}}{h_j(h_{j-1} + h_j)} + \frac{h_j - h_{j-1}}{3} f_j^{(3)} + \mathcal{O}(h^2). \quad (\text{B.16})$$

791 This is the central difference approximation for the non-uniform grid. If we assume that  
 792  $h_j - h_{j-1}$  is small, we may neglect the last term, and we get precisely that

$$\frac{2}{h_{j-1}(h_{j-1} + h_j)} = T_{j,j-1}, \quad (\text{B.17})$$

$$\frac{-2}{h_{j-1}h_j} = T_{j,j}, \quad (\text{B.18})$$

$$\frac{2}{h_j(h_{j-1} + h_j)} = T_{j,j+1}. \quad (\text{B.19})$$

793 Therefore we find that the ratio, say  $\rho_j$ , equals

$$\rho_j = \frac{T_{j,j+1}}{T_{j,j-1}} = \frac{h_{j-1}}{h_j}. \quad (\text{B.20})$$

794 Thus

$$h_{j-1} = \rho_j h_j. \quad (\text{B.21})$$

795 We derive from this combined with the above three equations that

$$h_j^2 = \frac{2}{T_{j,j-1} \rho_j (1 + \rho_j)}, \quad (\text{B.22})$$

$$\text{or } h_j^2 = \frac{2}{T_{j,j+1} (1 + \rho_j)}. \quad (\text{B.23})$$

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805 **References**

- 806 [1] G. Jona-Lasinio, F. Martinelli and E. Scoppola, *New approach to the semiclassical limit of*  
807 *quantum mechanics*, Commun. Math. Phys. **80**, 223 (1981), doi:[10.1007/BF01213012](https://doi.org/10.1007/BF01213012).
- 808 [2] B. Simon, *Semiclassical analysis of low lying eigenvalues. IV. The flea on the elephant*, J.  
809 *Funct. Anal.* **63**, 123 (1985), doi:[10.1016/0022-1236\(85\)90101-6](https://doi.org/10.1016/0022-1236(85)90101-6).
- 810 [3] P. W. Anderson, *An approximate quantum theory of the antiferromagnetic ground state*,  
811 *Phys. Rev.* **86**, 694 (1952), doi:[10.1103/PhysRev.86.694](https://doi.org/10.1103/PhysRev.86.694).
- 812 [4] H. Tasaki, *Long-range order, “tower” of states, and symmetry breaking in lattice quantum*  
813 *systems*, J. Stat. Phys. **174**, 735 (2019), doi:[10.1007/s10955-018-2193-8](https://doi.org/10.1007/s10955-018-2193-8).
- 814 [5] J. Earman, *Curie’s principle and spontaneous symmetry breaking*, *Int. Stud. Philos. Sci.* **18**,  
815 173 (2004), doi:[10.1080/0269859042000311299](https://doi.org/10.1080/0269859042000311299).
- 816 [6] N. P. Landsman, *Spontaneous symmetry breaking in quantum systems: Emergence or*  
817 *reduction?*, *Stud. Hist. Philos. Sci. B* **44**, 379 (2013), doi:[10.1016/j.shpsb.2013.07.003](https://doi.org/10.1016/j.shpsb.2013.07.003).
- 818 [7] N. P. Landsman, *Foundations of quantum theory*, Springer International Publishing  
819 (2017), doi:[10.1007/978-3-319-51777-3](https://doi.org/10.1007/978-3-319-51777-3).
- 820 [8] N. P. Landsman and R. Reuvers, *A flea on Schrödinger’s cat*, *Found. Phys.* **43**, 373 (2013),  
821 doi:[10.1007/s10701-013-9700-1](https://doi.org/10.1007/s10701-013-9700-1).
- 822 [9] J. Butterfield, *Less is different: Emergence and reduction reconciled*, *Found. Phys.* **41**, 1065  
823 (2011), doi:[10.1007/s10701-010-9516-1](https://doi.org/10.1007/s10701-010-9516-1).
- 824 [10] P. W. Anderson, *Some general thoughts about broken symmetry*, World Scientific (2005),  
825 doi:[10.1142/5558](https://doi.org/10.1142/5558).
- 826 [11] J. L. Birman, R. G. Nazmitdinov and V. I. Yukalov, *Effects of symmetry breaking in finite*  
827 *quantum systems*, *Phys. Rep.* **526**, 1 (2013), doi:[10.1016/j.physrep.2012.11.005](https://doi.org/10.1016/j.physrep.2012.11.005).
- 828 [12] B. A. Malomed (ed.), *Spontaneous symmetry breaking, self-trapping, and Josephson*  
829 *oscillations*, Springer Berlin Heidelberg (2013), doi:[10.1007/978-3-642-21207-9](https://doi.org/10.1007/978-3-642-21207-9).
- 830 [13] J. van Wezel and J. van den Brink, *Spontaneous symmetry breaking in quantum mechanics*,  
831 *Am. J. Phys.* **75**, 635 (2007), doi:[10.1119/1.2730839](https://doi.org/10.1119/1.2730839).
- 832 [14] C. Yannouleas and U. Landman, *Symmetry breaking and quantum correlations in finite*  
833 *systems: studies of quantum dots and ultracold Bose gases and related nuclear and chemical*  
834 *methods*, *Rep. Prog. Phys.* **70**, 2067 (2007), doi:[10.1088/0034-4885/70/12/R02](https://doi.org/10.1088/0034-4885/70/12/R02).

- 835 [15] B. Bernu, C. Lhuillier and L. Pierre, *Signature of Néel order in exact spectra*  
836 *of quantum antiferromagnets on finite lattices*, Phys. Rev. Lett. **69**, 2590 (1992),  
837 doi:[10.1103/PhysRevLett.69.2590](https://doi.org/10.1103/PhysRevLett.69.2590).
- 838 [16] P. Horsch and W. von der Linden, *Spin-correlations and low lying excited states of the*  
839 *spin-1/2 Heisenberg antiferromagnet on a square lattice*, Z. Physik B **72**, 181 (1988),  
840 doi:[10.1007/BF01312134](https://doi.org/10.1007/BF01312134).
- 841 [17] T. A. Kaplan, P. Horsch and W. von der Linden, *Order parameter in quantum*  
842 *antiferromagnets*, J. Phys. Soc. Jpn. **58**, 3894 (1989), doi:[10.1143/JPSJ.58.3894](https://doi.org/10.1143/JPSJ.58.3894).
- 843 [18] T. A. Kaplan, W. von der Linden and P. Horsch, *Spontaneous symmetry breaking*  
844 *in the Lieb-Mattis model of antiferromagnetism*, Phys. Rev. B **42**, 4663 (1990),  
845 doi:[10.1103/PhysRevB.42.4663](https://doi.org/10.1103/PhysRevB.42.4663).
- 846 [19] T. Koma and H. Tasaki, *Symmetry breaking and finite-size effects in quantum many-body*  
847 *systems*, J. Stat. Phys. **76**, 745 (1994), doi:[10.1007/BF02188685](https://doi.org/10.1007/BF02188685).
- 848 [20] T. Papenbrock and H. A. Weidenmüller, *Effective field theory for finite systems*  
849 *with spontaneously broken symmetry*, Phys. Rev. C **89**, 014334 (2014),  
850 doi:[10.1103/PhysRevC.89.014334](https://doi.org/10.1103/PhysRevC.89.014334).
- 851 [21] E. Shamriz, N. Dror and B. A. Malomed, *Spontaneous symmetry breaking in a split*  
852 *potential box*, Phys. Rev. E **94**, 022211 (2016), doi:[10.1103/PhysRevE.94.022211](https://doi.org/10.1103/PhysRevE.94.022211).
- 853 [22] A. Shimizu and T. Miyadera, *Stability of quantum states of finite macroscopic systems*  
854 *against classical noises, perturbations from environments, and local measurements*, Phys.  
855 Rev. Lett. **89**, 270403 (2002), doi:[10.1103/PhysRevLett.89.270403](https://doi.org/10.1103/PhysRevLett.89.270403).
- 856 [23] J. van Wezel, *Quantum mechanics and the big world* (2007), PhD Thesis Leiden University,  
857 <https://openaccess.leidenuniv.nl/handle/1887/11468>.
- 858 [24] J. van Wezel, *Quantum dynamics in the thermodynamic limit*, Phys. Rev. B **78**, 054301  
859 (2008), doi:[10.1103/PhysRevB.78.054301](https://doi.org/10.1103/PhysRevB.78.054301).
- 860 [25] A. J. Beekman, L. Rademaker, and J. van Wezel, *An introduction to spontaneous symmetry*  
861 *breaking*, SciPost Phys. Lect. Notes **11** (2019), doi:[10.21468/SciPostPhysLectNotes.11](https://doi.org/10.21468/SciPostPhysLectNotes.11).
- 862 [26] S. Graffi, V. Grecchi and G. Jona-Lasinio, *Tunnelling instability via perturbation theory*, J.  
863 Phys. A: Math. Gen. **17**, 2935 (1984), doi:[10.1088/0305-4470/17/15/011](https://doi.org/10.1088/0305-4470/17/15/011).
- 864 [27] R. W. Batterman, *The devil in the details*, Oxford University Press (2001),  
865 doi:[10.1093/0195146476.001.0001](https://doi.org/10.1093/0195146476.001.0001).
- 866 [28] M. Berry, *Singular limits*, Phys. Today **55**, 10 (2002), doi:[10.1063/1.1485555](https://doi.org/10.1063/1.1485555).
- 867 [29] P. Bóna, *The dynamics of a class of quantum mean-field theories*, J. Math. Phys. **29**, 2223  
868 (1988), doi:[10.1063/1.528152](https://doi.org/10.1063/1.528152).
- 869 [30] N. G. Duffield and R. F. Werner, *Local dynamics of mean-field quantum systems*, Helv. Phys.  
870 Acta **65**, 1016 (1992), doi:[10.5169/seals-116521](https://doi.org/10.5169/seals-116521).
- 871 [31] G. A. Raggio and R. F. Werner, *Quantum statistical mechanics of general mean field systems*,  
872 Helv. Phys. Acta **62**, 980 (1989), doi:[10.5169/seals-116175](https://doi.org/10.5169/seals-116175).

- 873 [32] A. E. Allahverdyan, R. Balian and T. M. Nieuwenhuizen, *Understanding quantum*  
874 *measurement from the solution of dynamical models*, Phys. Rep. **525**, 1 (2013),  
875 doi:[10.1016/j.physrep.2012.11.001](https://doi.org/10.1016/j.physrep.2012.11.001).
- 876 [33] J. van Wezel, J. Zaanen and J. van den Brink, *Relation between decoherence and*  
877 *spontaneous symmetry breaking in many-particle qubits*, Phys. Rev. B **74**, 094430 (2006),  
878 doi:[10.1103/PhysRevB.74.094430](https://doi.org/10.1103/PhysRevB.74.094430).
- 879 [34] C. J. F. van de Ven, *Properties of quantum spin systems and their classical limit* (2018),  
880 M.Sc. Thesis, Radboud University, <http://www.math.ru.nl/~landsman/Chris2018.pdf>.
- 881 [35] G. Scharf, W. F. Wreszinski and J. L. van Hemmen, *Tunnelling of a large spin:*  
882 *mapping onto a particle problem*, J. Phys. A: Math. Gen. **20**, 4309 (1987),  
883 doi:[10.1088/0305-4470/20/13/032](https://doi.org/10.1088/0305-4470/20/13/032).
- 884 [36] O. B. Zaslavskii, *Spin tunneling and the effective potential method*, Phys. Lett. A **145**, 471  
885 (1990), doi:[10.1016/0375-9601\(90\)90317-H](https://doi.org/10.1016/0375-9601(90)90317-H).
- 886 [37] F. Martinelli and E. Scoppola, *Introduction to the mathematical theory of Anderson*  
887 *localization*, Riv. Nuovo Cim. **10**, 1 (1987), doi:[10.1007/BF02740933](https://doi.org/10.1007/BF02740933).
- 888 [38] B. Helffer and J. Sjöstrand, *Puits multiples en limite semi-classique. II. Interaction*  
889 *moléculaire. Symétries. Perturbation*, Annales de l'Institut Henri Poincaré Physique  
890 Théorique **42**, 127 (1985), <http://eudml.org/doc/76277>.
- 891 [39] M. Scheffer, *Critical transitions in nature and society*, Princeton University Press (2009).
- 892 [40] N. N. Bogoliubov, *Lectures on quantum statistics, Volume 2: quasi-averages*, Gordon and  
893 Breach Sci. Publ. (1970).
- 894 [41] W. F. Wreszinski and V. A. Zagrebnov, *Bogoliubov quasiaverages: Spontaneous symmetry*  
895 *breaking and the algebra of fluctuations*, Theor. Math. Phys. **194**, 157 (2018),  
896 doi:[10.1134/S0040577918020010](https://doi.org/10.1134/S0040577918020010).
- 897 [42] C. Liu and G. G. Emch, *Explaining quantum spontaneous symmetry breaking*, Stud. Hist.  
898 Philos. Sci. B **36**, 137 (2005), doi:[10.1016/j.shpsb.2004.12.003](https://doi.org/10.1016/j.shpsb.2004.12.003).
- 899 [43] E. M. Harrell, *Double wells*, Commun. Math. Phys. **75**, 239 (1980),  
900 doi:[10.1007/BF01212711](https://doi.org/10.1007/BF01212711).
- 901 [44] D. Wallace, *Spontaneous symmetry breaking in finite quantum systems: a*  
902 *decoherent-histories approach* (2018), [arXiv:1808.09547](https://arxiv.org/abs/1808.09547).
- 903 [45] C. Kittel, *Introduction to Solid State Physics*, 8th edition, Wiley (2005).
- 904 [46] G. C. Groenenboom and H. M. Buck, *Solving the discretized time-independent*  
905 *Schrödinger equation with the Lanczos procedure*, J. Chem. Phys. **92**, 4374 (1990),  
906 doi:[10.1063/1.458575](https://doi.org/10.1063/1.458575).
- 907 [47] T. Kajishima and K. Taira, *Computational fluid dynamics*, Springer International  
908 Publishing (2017), doi:[10.1007/978-3-319-45304-0](https://doi.org/10.1007/978-3-319-45304-0).
- 909 [48] D. Kuzmin, *Introduction to CFD, lecture 4: Finite difference method* (2017), [www.mathematik.uni-dortmund.de/~kuzmin/cfdintro/lecture4.pdf](http://www.mathematik.uni-dortmund.de/~kuzmin/cfdintro/lecture4.pdf).
- 910  
911 [49] H. Sundqvist and G. Veronis, *A simple finite-difference grid with non-constant intervals*,  
912 *Tellus* **22**, 26 (1970), doi:[10.1111/j.2153-3490.1970.tb01933.x](https://doi.org/10.1111/j.2153-3490.1970.tb01933.x).